ON REGULARITY BOUNDS AND LINEAR RESOLUTIONS OF TORIC ALGEBRAS OF GRAPHS

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Abstract. Let $G$ be a simple graph. In this article we show that if $G$ is connected and $R(I(G))$ is normal, then $\text{reg}(R(I(G))) \leq \alpha_0(G)$, where $\alpha_0(G)$ the vertex cover number of $G$. As a consequence, every normal König connected graph $G$, $\text{reg}(R(I(G))) = \text{mat}(G)$, the matching number of $G$. For a gap-free graph $G$, we give various combinatorial upper bounds for $\text{reg}(R(I(G)))$. As a consequence we give various sufficient conditions for the equality of $\text{reg}(R(I(G)))$ and $\text{mat}(G)$. Finally we show that if $G$ is a chordal graph such that $K[G]$ has $q$-linear resolution ($q \geq 4$), then $K[G]$ is a hypersurface, which proves the conjecture of Hibi-Matsuda-Tsuchiya [12, Conjecture 0.2], affirmatively for chordal graphs.

1. Introduction

Let $G$ be a finite simple graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $S = K[x_1, \ldots, x_n]$ and $R = K[T_1, \ldots, T_m]$ be polynomial rings, where $K$ is a field. Define a homomorphism $\phi : S \to R$, $\phi(T_i) = x_{i1}x_{i2}$ where $e_i = \{x_{i1}, x_{i2}\} \in E(G)$.

Then the ker($\phi$) is known as the toric ideal of $G$ and denote by $I_G$ and $K[G] = R/I_G$ is known as the toric algebra of $G$. It is well-known that $I_G$ is a homogeneous ideal of $S$ generated by the binomials corresponding to the even closed walks of the graph $G$ (see [19 Proposition 3.1]). For the edge ideal $I(G)$ in $S$, the Rees algebra of $I(G)$ is defined as $R(I(G)) := \bigoplus_{j \geq 0} I(G)^j t^j$. Then $R(I(G))$ is a standard graded $K$-algebra with $\text{deg}(x_i) = 1$ and $\text{deg}(t) = -1$. Finding the Castelnuovo-Mumford regularity (simply regularity), minimal free resolutions, Betti numbers of $K[G]$ and $R(I(G))$ in terms of the combinatorial data of $G$ is an active area of research in recent times. Many authors gave various bounds

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for the regularities $\text{reg}(K[G]), \text{reg}(R(I(G)))$ and Betti numbers $\beta_{i,j}(K[G]), \beta_{i,j}(R(I(G)))$ in terms of the combinatorial invariants of $G$. See for example [19, 16, 17, 8, 5, 2, 4, 9, 10, 12, 14, 15, 18]. In [10], J. Herzog and T. Hibi showed that if $R(I(G))$ is normal, then $\text{mat}(G) \leq \text{reg}(R(I(G))) \leq \text{mat}(G) + 1$, where $\text{mat}(G)$ the matching number of $G$ and they raised a question to characterize all the graphs $G$, such that $\text{reg}(R(I(G))) = \text{mat}(G)$. In this work, we give various sufficient conditions to have the equality of $\text{reg}(R(I(G)))$ and $\text{mat}(G)$. We show that if $G$ is connected and $R(I(G))$ is normal, then $\text{reg}(R(I(G))) \leq \alpha_0(G)$, where $\alpha_0(G)$ is the vertex cover number of $G$(Theorem 2.1). This gives that if $G$ is König, then $\text{reg}(R(I(G))) = \text{mat}(G)$, which recovers a result of Y. Cid-Ruiz [4, Theorem 4.2]. In the section 3, we give various upper bounds for the regularity of Rees algebras of gap-free graphs. In [13], D. Meister gave various characterizations of minimal triangulations of a gap-free graph $G$, where minimal triangulation means the smallest chordal graph on the vertex set $V(G)$ and contains $G$ as a subgraph. Using this minimal triangulation, we show that

$$\text{reg}(R(I(G))) \leq \min \left\{ \left\lceil \left\lfloor \frac{|V(G) \setminus U| + 1}{2} \right\rfloor \right\rfloor + 1 \mid U \in \mathcal{U}_G \right\},$$

see the section 3 for the definition of $\mathcal{U}_G$. As a consequence of this sharp upper bound we give various sufficient conditions for the equality of $\text{reg}(R(I(G)))$ and $\text{mat}(G)$. If $U_0$ is a maximal independent set in $G$ such that $V(G) \setminus U_0$ is a minimal vertex cover of maximal cardinality, then we show that

$$\text{reg}(R(I(G))) \leq \left\lceil \frac{|V(G) \setminus U_0| + 1}{2} \right\rceil + 1.$$

H. Ohsugi and T. Hibi gave necessary and sufficient conditions for a simple connected graph $G$, $K[G]$ has 2-linear resolution, see [17, Theorem 4.6]. Later in [12], T. Hibi, K. Matsuda and A. Tsuchiya gave a conjecture about the linear resolutions of $K[G]$.

**Conjecture 1.1.** [12, Conjecture 0.2] The toric algebra of a finite connected simple graph with a $q$-linear resolution, where $q \geq 3$, is a hypersurface.

They proved this conjecture positively for any graph $G$ with $q = 3$, see [12, Theorem 0.1]. Recently, A. Tsuchiya, in [18] proved the above conjecture affirmatively for connected bipartite graphs. Tsuchiya achieved this by studying the $h^*$-polynomial of the edge polytope $\mathcal{P}_G$ of $G$, $h^*(\mathcal{P}_G, t)$(see section 2) and the degree of $h^*(\mathcal{P}_G, t)$ is a lower bound for the $\text{reg}(K[G])$. In this work, we prove the above conjecture affirmatively for
the chordal connected graphs. First we identify the primitive walks in $G$, if $K[G]$ has $q$-linear resolution, where $q \geq 4$. Then by using the degree of $h^*(\mathcal{P}_G, t)$, we achieve that $K[G]$ is a hypersurface.

Now we give sectionwise description. In the section 2 we recall all the definitions and notations needed for the rest of the article. In section 3, for a gap-free graph $G$, we give various upper bounds for $\text{reg}(R(I(G)))$ and sufficient conditions for the equality of $\text{reg}(R(I(G)))$ and $\text{mat}(G)$. In the final section 4 we prove the Hibi-Matsuda-Tsuchiya conjecture for the chordal graphs.

2. Preliminaries

In this section we recall some definitions and notations required throughout the paper. Let $G$ be a graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. We say that $G$ is a gap-free graph if its complement $G^c$ does not contain any induced 4-cycle. The graph $G$ is said to be chordal, if $G$ has no induced cycle of length $\geq 4$. A subset $M \subseteq E(G)$, is called a matching if for any $e, e' \in M$, we have $e \cap e' = \emptyset$. The matching number of $G$ denoted by $\text{mat}(G)$, is defined as the cardinality of the maximal matching in $G$. Recall that a set $S \subseteq V(G)$, is called a vertex cover of $G$, if for every $e \in E(G)$, there exists a $v \in S$ such that $v \in e$. A vertex cover $S$ is called a minimal vertex cover, if no proper subset of $S$ is a vertex cover of $G$. The vertex cover number of $G$, denoted by $\alpha_0(G)$, is the minimal cardinality of minimal vertex cover of $G$.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field $K$ and $I$ be a homogeneous ideal of $S$. It is known that $S/I$ has a minimal graded free resolution over $S$ (see [3])

$$0 \to \oplus_j S(-j)^{\beta_{p,j}(S/I)} \to \cdots \to \oplus_j S(-j)^{\beta_{1,j}(S/I)} \to S \to S/I \to 0,$$

where $\beta_{i,j}(S/I)$ is known as $(i,j)^{th}$ graded Betti number of $S/I$. We say that $S/I$ has a $q$-linear resolution if $\beta_{i,j}(S/I) = 0$ for all $1 \leq i \leq p$ and for all $j \neq q + i - 1$. That is, the minimal graded free resolution is of the shape:

$$0 \to S(-q - p + 1)^{\beta_{p,q+p-1}(S/I)} \to \cdots \to S(-q)^{\beta_{1,q}(S/I)} \to S^{\beta_{0,0}(S/I)} \to S/I \to 0,$$

The Castelnuovo-Mumford regularity or simply regularity of $S/I$ is denoted and defined as

$$\text{reg}(S/I) := \max\{j - i \mid \beta_{i,j}(S/I) \neq 0\}.$$
For $e = \{v_i, v_j\} \in E(G)$, we set $\rho(e) := e_i + e_j \in \mathbb{R}^n$. The edge polytope of $G$, denote by $\mathcal{P}_G$, is defined as the convex hull of $\{\rho(e_1), \rho(e_2), \ldots, \rho(e_q)\}$, where $e_1, \ldots, e_n$ denote the standard unit vectors in $\mathbb{R}^n$. Recall the following definitions from [11]. The $h$-polynomial of $\mathcal{P}_G$, denoted by $h^*(\mathcal{P}_G,t)$, is defined as the polynomial

$$h^*(\mathcal{P}_G,t) := (1 - t)^{d+1} \left(1 + \sum_{i=1}^{\infty} |i\mathcal{P}_G \cap \mathbb{Z}^n| t^i\right),$$

where $d = \dim(\mathcal{P}_G)$, $i\mathcal{P}_G = \{ia \mid a \in \mathcal{P}_G\}$. The degree of $h^*(\mathcal{P}_G,t)$ is simply denoted as $\deg(\mathcal{P}_G)$. The codegree of $\mathcal{P}_G$ is defined as $\text{codeg}\mathcal{P}_G := d + 1 - \deg(\mathcal{P}_G)$. Note that $\text{codeg}(\mathcal{P}_G) = \min\{r \in \mathbb{N} : \text{int}(r\mathcal{P}_G) \cap \mathbb{Z}^n \neq \emptyset\}$, where $\text{int}(\mathcal{P}_G)$ is the relative interior of $\mathcal{P}_G$ in $\mathbb{R}^n$.

Now recall the following definitions from [20] or [8]. Let

$$\mathcal{A}' = \{e_i + e_j + e_{n+1} \mid v_i \text{ is adjacent to } v_j\} \cup \{e_i \mid 1 \leq i \leq n\} \subset \mathbb{R}^{n+1},$$

where $e_1, e_2, \ldots, e_n, e_{n+1}$ are the standard unit vectors in $\mathbb{R}^{n+1}$. The Rees cone, $\mathbb{R}_+\mathcal{A}'$ of $G$, is the cone generated by $\mathcal{A}'$, that is

$$\mathbb{R}_+\mathcal{A}' = \left\{ \sum_i a_i \alpha_i \mid a_i \in \mathbb{R}_+, \alpha_i \in \mathcal{A}' \right\}.$$ 

Note that $\mathbb{R}_+\mathcal{A}'$ is a finitely generated rational cone of dimension $n + 1$(see [6]). Also by [21 Theorem 4.11], it has a unique irreducible representation:

\begin{equation}
\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap H_{e_2}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{l_1}^+ \cap \cdots \cap H_{l_r}^+
\end{equation}

where each $l_k$ is in $\mathbb{Z}^{n+1}$, the nonzero entries of each $l_k$ are relatively prime and none of the closed half spaces mentioned above can be omitted from the intersection, and $H_l^+ = \{\alpha \in \mathbb{R}^{n+1} \mid \langle l, \alpha \rangle \geq 0\}$ for any $l \in \mathbb{R}^{n+1}$, $\langle , \rangle$ is a standard inner product on $\mathbb{R}^{n+1}$. We can always assume $l_k = -e_{n+1} + \sum_{v_i \in C_k} e_i, 1 \leq k \leq s \leq r$, where $C_1, \ldots, C_s$ are the all minimal vertex covers of $G$ (see [7 Lemma 3.1]).

Now we prove that the vertex cover number of $G$ is an upper bound for the regularity of $R(I(G))$.

**Theorem 2.1.** Let $G$ be a connected graph and $\alpha_0(G)$ denotes the vertex cover number of $G$. If $R(I(G))$ is normal, then $\text{reg}(R(I(G))) \leq \alpha_0(G)$.
Proof. We have \( R(I(G)) \) standard graded \( K \)-algebra with the grading \( \deg(x_i) = 1 \) and \( \deg(t) = -1 \). As \( R(I(G)) \) is normal therefore by the Danilov-Stanley [3, Theorem 6.3.5] formula its canonical module is given by

\[
\omega_{R(I(G))} = \left\{ x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} t^{a_{n+1}} \mid a = (a_i) \in (\mathbb{R}_+ A')^0 \cap \mathbb{Z}^{n+1} \right\}.
\]

Let \( a = (a_1, \ldots, a_{n+1}) \) be an arbitrary element in \((\mathbb{R}_+ A')^0 \cap \mathbb{Z}^{n+1}\). Then by the equation (1) we have

\[
a_i \geq 1 \text{ and } -a_{n+1} + \sum_{v_i \in C_k} a_i \geq 1
\]

where \( C_k \) is a minimal vertex cover of \( G \).

Suppose \( m = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} t^{a_{n+1}} \) be an element in \( \omega_{R(I(G))} \), then

\[
\deg(m) = a_1 + a_2 + \cdots + a_n - a_{n+1}
\]

\[
= \sum_{v_i \in V \setminus C_k} a_i + \sum_{v_i \in C_k} a_i - a_{n+1}
\]

\[
\geq \sum_{v_i \in V \setminus C_k} a_i + 1,
\]

\[
= |V(G) \setminus C_k| + 1.
\]

Thus \( \deg(m) \geq |V(G) \setminus C_k| + 1 \) for all \( k = 1, \ldots, s \). This implies that

\[
\deg(m) \geq \max_{1 \leq k \leq s} |V(G) \setminus C_k| + 1
\]

\[
= |V(G)| - \min_{1 \leq k \leq s} |C_k| + 1
\]

\[
= |V(G)| - \alpha_0(G) + 1, \quad \text{where } \alpha_0(G) = \min_{1 \leq k \leq s} |C_k|, \text{ the vertex cover number of } G.
\]

This gives that \( a(R(I(G))) = -\min\{j \mid (\omega_{R(I)})_j \neq 0\} \leq -(|V(G)| - \alpha_0(G) + 1) \). Since \( R(I(G)) \) is normal and hence \( R(I(G)) \) is Cohen-Macaulay, therefore

\[
\reg(R(I(G))) = a(R(I(G))) + \dim(R(I(G))) \leq -|V(G)| + \alpha_0(G) - 1 + |V(G)| + 1 = \alpha_0(G).
\]

\[\square\]

Corollary 2.2. Suppose \( G \) is a connected graph with at least two edges and \( R(I(G)) \) is normal. If \( G \) is König, then \( \reg(R(I(G))) = \mat(G) = \alpha_0(G) \).

Proof. From [10, Theorem 2.2], we have \( \mat(G) \leq \reg(R(I(G))) \). Now apply Theorem 2.1 \[\square\]
As a consequence of the Theorem 2.1, we can recover a result of Y. Cid-Ruiz.

**Corollary 2.3** ([4], Theorem 4.2). Let $G$ be a bipartite graph. Then $\text{reg}(R(I(G))) = \text{mat}(G)$.

**Proof.** The result follows from the Corollary 2.2 and the fact that every bipartite graph is König. □

Now we give an example of a graph $G$ such that $G$ is not König and $\text{reg}(R(I(G))) \neq \text{mat}(G)$.

**Example 2.4.** Let $G$ be the graph on 5 vertices $\{x_1, x_2, x_3, x_4, x_5\}$ and edges $E(G) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_4, x_5\}\}$. Then $G$ is normal and $\text{mat}(G) = 2$. The vertex cover number of $G$, $\alpha_0(G) = 3$. Thus $G$ is not König. Now by using CoCoA [1], we can compute the regularity, $\text{reg}(R(I(G))) = 3 \neq \text{mat}(G)$.

Now we give an example of a graph $G$ such that $G$ is König and $\text{reg}(R(I(G))) = \text{mat}(G)$.

**Example 2.5.** Let $G$ be the graph on 6 vertices $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and edges $E(G) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_5\}, \{x_2, x_6\}\}$. Note that $G$ is normal, $\text{mat}(G) = 2 = \alpha_0(G)$. Thus $G$ is König. By using CoCoA [1], we have $\text{reg}(R(I(G))) = 2$.

### 3. Bounds for the regularity of Rees algebras of gap-free graphs

In this section we give new bounds for the regularity of Rees algebras of a gap-free graph. As a consequence we give some sufficient conditions for the equality of regularity of Rees algebra and the matching number of $G$.

Recall the following results form [13]. Define a family $\mathcal{U}_G$ of maximal independent sets of $G$ in the following way.

$U \in \mathcal{U}_G$ if and only if $U$ satisfies the following three conditions:

1. $U$ is a maximal independent set in $G$.
2. $\{v \in V(G) \mid d_G(v) \leq 1\} \subseteq U$, where $d_G(v)$ denotes the degree of $v$ in $G$.
3. If there is a vertex $w \in V(G) \setminus U$, satisfying $|N_G(w) \cap U| = 1$ and $V(G) \setminus U \subseteq N_G[w]$, then there is no vertex $v \in U$ such that $N_G(v) = V(G) \setminus U$, where $N_G(w) = \{u \in V(G) \mid \{u, w\} \in E(G)\}$ and $N_G[w] = N_G(w) \cup \{w\}$.

**Theorem 3.1** ([13] Lemma 8, Theorem 10). Let $G$ be a gap-free graph with at least two edges and $U \in \mathcal{U}_G$. Then
(1) $H_U := G \cup F_U$ is a minimal chordal graph (known as the minimal triangulation of $G$), where $F_U = \{uv \notin E(G) \mid u, v \in V(G) \setminus U\}$. Moreover $H_U$ is co-chordal and gap-free.

(2) A graph $H$ on the vertex set $V(G)$ is a minimal triangulation of $G$ if and only if there is some $U \in \mathcal{U}_G$ such that $H = H_U$.

(3) $H_U$ is a split graph and $(U, V(G) \setminus U)$ is a split partition for $H_U$.

Now we prove a new upper bound for the Rees algebra of a gap-free graph.

**Proposition 3.2.** Let $G = (V(G), E(G))$ be a gap-free graph and $|E(G)| > 1$. Then

(i) $\text{reg}(R(I(G)))) \leq \min \left\{ \left\lceil \frac{|V(G)\setminus U| + 1}{2} \right\rceil + 1 \mid U \in \mathcal{U}_G \right\}$.

(ii) Suppose $U$ is a maximal independent set in $G$ such that $V(G) \setminus U$ is a maximal clique in $H_U$. Then $\text{reg}(R(I(G)))) \leq \left\lceil \frac{|V(G)\setminus U|}{2} \right\rceil + 1$.

**Proof.** (i) Let $U \in \mathcal{U}_G$. We show that $\text{reg}(R(I(G)))) \leq \frac{|V(G)\setminus U|}{2} + 1$. We have $G$ is gap-free implies that both $R(I(G))$ and $R(I(H_U))$ are normal. We have $R(I(G)) \subseteq R(I(H_U))$. Therefore by [10] Corollary 1.3, we have $\text{reg}(R(I(G)))) \leq \text{reg}(R(I(H_U))))$. By [10] Theorem 2.2, $\text{reg}(R(I(H_U)))) \leq \text{mat}(H_U) + 1$. Thus $\text{reg}(R(I(G)))) \leq \text{mat}(H_U) + 1$. By using the Theorem 3.1, we have $H_U$ is a split graph with split partition $(U, V(G) \setminus U)$. This implies that $\text{mat}(H_U) = \left\lceil \frac{|V(G)\setminus U|}{2} \right\rceil$ or $\left\lceil \frac{|V(G)\setminus U| + 1}{2} \right\rceil$ accordingly $K_{V(G)\setminus U}$ is a maximal clique or not. Thus we have

$$\text{reg}(R(I(G)))) \leq \text{mat}(H_U) + 1 \leq \left\lceil \frac{|V(G)\setminus U| + 1}{2} \right\rceil + 1.$$ 

(ii) By the assumption we get $U \in \mathcal{U}_G$. Since $V(G) \setminus U$ is a maximal clique in $H_U$, we have $\text{mat}(H_U) = \left\lceil \frac{|V(G)\setminus U|}{2} \right\rceil$. Now by using [10] Corollary 1.3, Theorem 2.2, we get

$$\text{reg}(R(I(H_U)))) \leq \text{reg}(R(I(H_U)))) \leq \text{mat}(H_U) + 1 \leq \left\lceil \frac{|V(G)\setminus U|}{2} \right\rceil + 1,$$

as required. \qed

The following corollary is an immediate consequence of the above proposition.

**Corollary 3.3.** Let $G = (V(G), E(G))$ be a gap-free graph and $|E(G)| > 1$. Let $U_0$ be a maximal independent set in $G$ such that $V(G) \setminus U_0$ is a minimal vertex cover of maximal cardinality. Then

$$\text{reg}(R(I(G)))) \leq \left\lceil \frac{|V(G)\setminus U_0| + 1}{2} \right\rceil + 1.$$
Corollary 3.4. Let $G = (V(G), E(G))$ be a gap-free graph and $|E(G)| > 1$. If

$$\min \{|V(G) \setminus U| : U \in \mathcal{U}_G\} \leq 2 \text{mat}(G) - 3,$$

then $\text{reg}(R(I(G))) = \text{mat}(G)$.

Proof. The assumption implies that $\min \left\{\left\lceil \frac{|V(G) \setminus U| + 1}{2} \right\rceil + 1 \mid U \in \mathcal{U}_G\right\} \leq \text{mat}(G)$. Therefore form the Proposition 3.2, we have $\text{reg}(R(I(G))) \leq \text{mat}(G)$. Now the assertion follows from [10, Theorem 2.2]. □

Corollary 3.5. Let $G$ be a gap-free graph and $|E(G)| > 1$ and $\alpha_0(G)$ denotes the vertex cover number of $G$. Suppose $I(G)$ is unmixed. Then $\text{reg}(R(I(G))) \leq \left\lceil \frac{\alpha_0(G) + 1}{2} \right\rceil + 1$. Furthermore, if $\alpha_0(G) \leq 2 \text{mat}(G) - 3$, then $\text{reg}(R(I(G))) = \left\lceil \frac{\alpha_0(G) + 1}{2} \right\rceil + 1 = \text{mat}(G)$.

The following example shows that the bound in the Proposition 3.2 is sharp.

Example 3.6. Let $G$ be a graph with $V(G) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and

$$E(G) = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_6\}, \{x_4, x_6\}, \{x_3, x_6\}, \{x_2, x_4\}, \{x_2, x_6\}\}.$$

Then $G$ is a gap-free graph with $\text{mat}(G) = 3$ and vertex cover number $\alpha_0(G) = 3$. Note that $U = \{x_1, x_3, x_5\}$ is a maximal independent set in $G$ and $U \in \mathcal{U}_G$. Also $V(G) \setminus U$ is a minimal vertex cover of minimal cardinality. Therefore $\left\lceil \frac{|V(G) \setminus U| + 1}{2} \right\rceil + 1 = 3$. Now by using the CoCoA [1], we get $\text{reg}(R(I(G))) = 3$. Thus

$$\text{reg}(R(I(G))) = \left\lceil \frac{|V(G) \setminus U| + 1}{2} \right\rceil + 1 = \min \left\{\left\lceil \frac{|V(G) \setminus W| + 1}{2} \right\rceil + 1 \mid W \in \mathcal{U}_G\right\} = 3.$$

Therefore the upper bound in the Proposition 3.2 is sharp.

4. Linear resolution of toric algebras of chordal graphs

Throughout this section we will assume that $G = (V(G), E(G))$ is a chordal graph. Let $I_G$ be the toric ideal associated to the graph $G$. From [5] we have the characterization of primitive walks of a graph $G$, which generates the ideal $I_G$.

Lemma 4.1. ([5, Lemma 3.1]) An even walk of length $2q$ is primitive if it is one of the following:

1. An even cycle
2. $\{C_1, C_2\}$, where $C_1$ and $C_2$ are odd cycles with exactly one common vertex,
(3) \( \{C_1, p_1, C_2, p_2, \ldots, C_h, P_h\} \) where \( p_i \)'s are paths of length \( \geq 1 \) and \( C_i \)'s are odd cycles such that \( C_i(\mod h) \) and \( C_{i+1}(\mod h) \) are vertex disjoint for each\( i. \)

We will show that if \( I_G \) has a \( q \)-linear resolution for \( q \geq 4 \) then all the primitive walks of \( G \) are of type (3).

**Lemma 4.2.** Let \( G \) be a chordal graph. Suppose \( I_G \) has \( q \)-linear resolution. Then all the primitive walks of \( G \) are type (3) only.

**Proof.** We show that \( G \) does not have a primitive walk of type (1) and type (2). Let \( W_{2q} \) be an even closed walk of length \( 2q \). Suppose \( W_{2q} \) is of type (1), that is \( W_{2q} = C_{2q} \). But \( G \) is chordal, so it can not contain any induced cycle of length \( \geq 4 \), which forces existence of some chord \( e \in E(G) \) of \( C_{2q} \) such that \( e \) will divide \( C_{2q} \) into two cycles, \( C_1 \) and \( C_2 \) of length \( < 2q \).

(a) If \( C_1, C_2 \) both are even cycles, then they will be primitive walk of smaller length, which will be present in the minimal generating set of \( I_G \), which implies that \( I_G \) is generated by elements of degree smaller than \( q \), which is a contradiction, as \( I_G \) has \( q \)-linear resolution.

(b) Suppose \( C_1, C_2 \) both are odd cycles. We can assume that \( C_1 \) is larger than \( C_2 \). Then length of \( C_1 \geq q + 1 \), if \( q \) is even and \( \geq q + 2 \) if \( q \) is odd. As \( q \geq 4 \), length of \( C_1 \geq 5 \). But \( G \) is chordal, so \( C_1 \) should have some chord, say \( e' \), which will divide \( C_1 \) into two even cycles of length \( < 2q \). Therefore using the same argument as (a), we can conclude.

Next suppose that \( W_{2q} \) is of type (2), which implies that \( W_{2q} \) is the union of two odd cycles \( C_3 \) and \( C_4 \) of length \( q \geq 5 \), having exactly one common vertex. As \( G \) is chordal so both \( C_3 \) and \( C_4 \) should have chords \( e_1 \) and \( e_2 \), which will divide them into smaller cycles of even length. Then using case (b), we can derive our result. \( \square \)

**Lemma 4.3.** Let \( G \) be a chordal graph. Suppose \( I_G \) has \( q \)-linear resolution. Then all the primitive walks of \( G \) are of the form \( C_1 p C_2 \), where \( C_1, C_2 \) are 3-cycles and \( p \) is a path in \( G \).

**Proof.** Let \( W_{2q} \) be an even closed walk of length \( 2q \). From the proof of the Lemma 4.2, if \( I_G \) has \( q \)-linear resolution, then \( G \) has primitive walks of type (3) only. That is \( W_{2q} \) is of the form \( C_1 p_1 C_2 p_2 \ldots C_h p_h \), where \( C_i \)'s are odd cycles of length \( \geq 3 \). But if \( C_i \) is a cycle of length \( \geq 4 \), then we can proceed as in the proof of the Lemma 4.2, we can conclude.
that $C_i$ is a 3-cycle only. Next we will show that in each such $W_{2q}$ only two 3-cycle will be present. That is, we show that $h = 2$.

If possible, suppose there are more than two 3-cycle in $W_{2q}$, then $h \geq 3$ and $p_h$ connects $C_1$ and $C_h$. Then the paths $p_1, p_2, \ldots, p_h$ will form another cycle $C'$ of length $< 2q$ as shown in the above figure. If $C'$ is of length 3, which implies that $h = 3$, then $C'$ and $C_i$ share exactly one common vertex for each $i = 1, 2, 3$. Then $\{C', C_i\}$ generates a primitive walk of length 6 $< 2q$, which is contradiction. Suppose $C'$ is of length $\geq 4$. By the construction, length of $(C') < 2q$. As $G$ is chordal, $C'$ should have a chord. Then using same kind of argument as in Lemma 4.3, we can get another primitive walk of length $< 2q$. Again we get a contradiction. Thus we can conclude that $I_G$ has $q$-linear resolution then $I_G$ will be generated by primitive binomials coming from the even walk $W_{2q}$, where $W_{2q}$ is of the form $C_1p_1C_2$. □

**Lemma 4.4.** Let $G$ be a connected graph. If $G$ contains two disjoint even walks of length $2q$ of the form $C_1p_1C_2$ and $C'_1p'_1C'_2$, where, $C'_1, C'_2, C_1, C_2$ are 3-cycles, then $\deg \mathcal{R}_G \geq q$.

**Proof.** Suppose $G_{2q}$ and $G'_{2q}$ denote the walks $C_1p_1C_2$ and $C'_1p'_1C'_2$ respectively. Let $E(C_1) = \{e_1, e_2, e_3\}, E(p_1) = \{e_4, e_5, \ldots, e_q\}, E(C_2) = \{e_{q+1}, e_{q+2}, e_{q+3}\}$ and $E(C'_1) = \{e'_1, e'_2, e'_3\}, E(p_1) = \{e'_4, e'_5, \ldots, e'_{q}\}, E(C'_2) = \{e'_{q+1}, e'_{q+2}, e'_{q+3}\}$, where $e_i = \{v_{i-1}, v_i\}$ and $e'_i = \{v'_{i-1}, v'_i\}$ for $4 \leq i \leq q$. 

![Diagram](image-url)
Let $G'$ be a graph on the vertex set $V(G') = \{v_1, v_2, \ldots, v_{q+2}, v'_1, v'_2, \ldots, v'_{q+2}\}$ with edge set $E(G') = E(G_{2q}) \cup E(G'_{2q})$. Then by [20, Proposition 10.4.1], we have

$$\dim P_G = |V(G')| - C_0(G') - 1 = 2q + 4 - 1 = 2q + 3,$$

where $C_0(G')$ is the number of bipartite components of $G'$ which is zero.

Suppose that $q$ is even. Consider

$$v = \frac{1}{3} \left[ \rho(e_1) + \rho(e_3) + \rho(e'_1) + \rho(e'_3) + \rho(e_{q+1}) + \rho(e_{q+3}) + \rho(e'_{q+1}) + \rho(e'_{q+3}) \right] +$$

$$\frac{2}{3} \left[ \rho(e_2) + \rho(e'_2) + \rho(e_{q+2}) + \rho(e'_{q+2}) \right] +$$

$$\frac{1}{3} \left[ \rho(e_4) + \rho(e_6) + \cdots + \rho(e_q) + \rho(e'_4) + \rho(e'_6) + \cdots + \rho(e'_q) \right] +$$

$$\frac{2}{3} \left[ \rho(e_5) + \rho(e_7) + \cdots + \rho(e_{q-1}) + \rho(e'_5) + \rho(e'_7) + \cdots + \rho(e'_{q-1}) \right]$$

$$= e_1 + e_2 + \cdots + e_{2q+4}$$

$$\in \text{int}((q + 3)P_{G'}) \cap \mathbb{Z}^{2q+4}.$$

Therefore we get $\text{codeg}(P_{G'}) \leq q + 3$, which implies that

$$\deg(P_{G'}) = \dim(P_{G'}) + 1 - \text{codeg}(P_{G'}) \geq 2q + 4 - q - 3 = q + 1 \geq q.$$

Now assume that $q$ is odd. Consider

$$v = \frac{1}{3} \left[ \rho(e_1) + \rho(e_3) + \rho(e'_1) + \rho(e'_3) \right] +$$

$$\frac{2}{3} \left[ \rho(e_2) + \rho(e'_2) \right] +$$

$$\frac{1}{3} \left[ \rho(e_4) + \rho(e_6) + \cdots + \rho(e_q) + \rho(e'_4) + \rho(e'_6) + \cdots + \rho(e'_q) \right] +$$

$$\frac{2}{3} \left[ \rho(e_5) + \rho(e_7) + \cdots + \rho(e_{q-1}) + \rho(e'_5) + \rho(e'_7) + \cdots + \rho(e'_{q-1}) \right] +$$

$$\frac{1}{6} \left[ \rho(e_{q+1}) + \rho(e_{q+3}) + \rho(e'_{q+1}) + \rho(e'_{q+3}) \right] +$$

$$\frac{5}{6} \left[ \rho(e_{q+2}) + \rho(e'_{q+2}) \right]$$

$$= e_1 + e_2 + \cdots + e_{2q+4}$$

$$\in \text{int}((q + 2)P_{G'}) \cap \mathbb{Z}^{2q+4}.$$

Thus we get $\text{codeg}(P_{G'}) \leq q + 2$, which gives that

$$\deg(P_{G'}) = \dim(P_{G'}) + 1 - \text{codeg}(P_{G'}) \geq 2q + 4 - q - 2 = q + 2 \geq q.$$
As $G'$ is a subgraph of $G$, therefore by [15, Lemma 1.3] we have $\deg(P_G) \geq \deg(P_{G'}) \geq q$. \hfill \Box

Now we prove our main theorem.

**Theorem 4.5.** Let $G$ be a connected chordal graph. If $I_G$ has $q$-linear resolution, then $G$ has exactly one primitive walk of length $2q$ of type $C_1pC_2$, where $C_1, C_2$ are 3-cycles and $p$ is a path in $G$.

**Proof.** Suppose $G$ contains two even walk of length $2q$ of the form $G_{2q} = C_1p_1C_2$ and $G'_{2q} = C'_1p'_1C'_2$. Suppose $G_{2q}$ and $G'_{2q}$ are disjoint. Then by the Lemma 4.4 we have $\deg(P_G) \geq q$. Now by [12, Corollary 3.4] this implies that $\text{reg}(K[G]) \geq q$. This is a contradiction because $\text{reg}(K[G]) = q - 1$. Therefore $G_{2q}$ and $G'_{2q}$ are not disjoint.

Case 1: Assume $G_{2q}$ and $G'_{2q}$ share exactly one common vertex, $v$.

(i) $v \in C_1/C_2/C'_1/C'_2$

Without loss of generality we can assume that $v \in C_1$. Then $v$ divides the path $p'_1$ into two parts $p_2$ and $p_3$ (as shown in the above figure), then we get another primitive walk $C'_1p_2C_1$ of length $< 2q$, which gives a contradiction. Note that the same argument is true if the length of the path $p_2$ or $p_3$ is zero.

(ii) Assume $v \in P_1$ and $v \in P'_1$.

Suppose $v$ divides the path $P_1$ into $P_2, P_3$ and $P'_1$ into $P'_2, P'_3$. Then we get 4 other primitive walks:

$$W_1 = C_1p_2p'_3C'_2, \quad W_2 = C_1p_2p'_2C'_1, \quad W_3 = C_2p_3p'_3C'_2, \quad W_4 = C_2p_3p'_2C'_1.$$  

As $I_G$ has $q$-linear resolution, all these four walks should be of length $2q$. Let $\ell(p)$ denotes the length of a path $p$. 

Thus we obtain

\[ \ell(p_2) + \ell(p'_2) = \ell(p_2) + \ell(p'_2) = \ell(p_3) + \ell(p'_3) = \ell(p_1) = q - 3, \]

which implies that \( \ell(p_2) = \ell(p_3) = \ell(p'_2) = \ell(p'_3) = \frac{q-2}{2} \). This implies that \( q \) should be an odd number and \( v \) should divide the paths \( p_1 \) and \( p'_1 \) into two equal parts, as shown in the above figure.

Now consider the graph \( G' \) with vertex set \( V(G') = \{v_1, v_2, \ldots, v_{q-3} = v, \ldots, v_{q+2}, v'_1, v'_2, \ldots, v'_q = v'_1, \ldots, v'_{q+2}\} \) and edge set \( E(G') = E(W_1) \cup E(W_2) \cup E(W_3) \cup E(W_4) \). Then \( G' \) is a connected subgraph of \( G \) with \( \dim(G') = 2q + 3 - 1 = 2q + 2 \). Note that the vertices of \( G' \) are labelled as in the same pattern as in the figure in the proof of the Lemma 4.4.

Consider the vector

\[
\begin{align*}
\mathbf{w} &= \frac{1}{2} \sum_{i=4}^{q} [\rho(e_i) + \rho(e'_i)] + \frac{3}{4} [\rho(e_2) + \rho(e_2) + \rho(e_{q+2}) + \rho(e'_{q+2})] + \\
&\quad \frac{1}{4} [\rho(e_1) + \rho(e'_1) + \rho(e_3) + \rho(e'_3) + \rho(e_{q+1}) + \rho(e'_{q+1}) + \rho(e_{q+3}) + \rho(e'_{q+3})] \\
&= e_1 + e_2 + \cdots + e_{2q+3} \\
&\in \text{int} \left( (q+2) \mathcal{P}_{G'} \right) \cap \mathbb{Z}^{2q+3}.
\end{align*}
\]
This implies that \( \text{codeg}(\mathcal{P}_{G'}) \leq q+2 \) and hence \( \text{deg}(\mathcal{P}_G) \geq \text{deg}(\mathcal{P}_{G'}) \geq 2q+2-q-2 = q \). This is a contradiction because \( \text{reg}(K[G]) = q-1 \).

Case 2: \( G_{2q} \) and \( G'_{2q} \) share more than one vertex.

Suppose they share at least two common vertices say \( v_1, v_2 \). Then we have four possibilities:

(a) \( v_1 \in C_1/C_2/C'_1/C'_2 \) and \( v_2 \in p_1, p'_1 \),
(b) \( v_2 \in C_1/C_2/C'_1/C'_2 \) and \( v_1 \in p_1, p'_1 \),
(c) \( v_1, v_2 \in C_1/C_2/C'_1/C'_2 \),
(d) \( v_1, v_2 \in p_1, p'_1 \).

The possibilities (a), (b), (c) fall under case 1(i) above. So we can use the same argument.

Now suppose \( v_1, v_2 \in p_1, p'_1 \), then by using the same argument as in (ii), we can say that these two vertices should divide \( p_1 \) and \( p'_1 \) into two equal paths, which is absurd. \( \square \)

**Corollary 4.6.** Suppose \( G \) is a chordal graph. If \( I_G \) has \( q \)-linear resolution, then \( K[G] \) is a hypersurface.

**Proof.** Let \( G_1, \ldots, G_s \) be the connected components of \( G \). Then each \( G_i \) is a chordal connected graph. If \( I_G \) has \( q \)-linear resolution, then there exists only one \( j \) such that \( I_{G_j} \neq 0 \). That is, \( K[G_i] \) are polynomial algebras for all \( i \neq j \). As \( K[G] \cong K[G_1] \otimes_K K[G_2] \otimes_K \cdots \otimes_K K[G_s] \), this implies that \( K[G_j] \) has \( q \)-linear resolution. Therefore by the Theorem \( \square \) we get that \( K[G_j] \) is a hypersurface. This implies that \( K[G] \) is a hypersurface. \( \square \)

**References**


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