Japan J. Indust. Appl. Math., 23 (2006), 63–73 Area ⟨2⟩

On a Generalization of the Euler-Chebyshev Method for Simultaneous Extraction of Only a Part of All Roots of Polynomials

Anton Iliev[†], Nikolay Kyurkchiev[††] and Qing Fang[‡]

[†] *Department of Computer Science, Faculty of Mathematics and Informatics, Plovdiv University, Plovdiv 4000, Bulgaria & Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia 1113, Bulgaria*
E-mail: aii@pu.acad.bg

[††] *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia 1113, Bulgaria*
E-mail: nkyurk@math.bas.bg

[‡] *Department of Mathematical Sciences, Faculty of Science, Yamagata University, Yamagata 990-8560, Japan*
E-mail: fang@sci.kj.yamagata-u.ac.jp

Received November 17, 2003
Revised September 8, 2005

We propose a method with raised speed of convergence for simultaneous extraction of a part of all roots of polynomials. The method is efficient for the polynomials which have well separated real roots. The proof of local convergence is shown and numerical results are given.

Key words: total-step method, single-step procedure, zeros of polynomials, local convergence theorem

1. Introduction

We consider a polynomial of n-th degree

$$A_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n. \tag{1}$$

Wide area of problems and practical tasks in economics, biology, chemistry, and physics are reduced to the problem of finding only a part of all roots of (1). We set the task in this paper to build iteration methods with raised speed of convergence and at the same time they give opportunities for searching only one part of all roots of (1) (real, complex, lying in given area, satisfying given conditions). Many methods for specifying number of zeros and their approximation are displayed in [15], [10], [16], [1], [11], [17] and [12].

Let us denote the approximations of the k-th iteration to the zeros $x_1, x_2, \ldots, x_n$ of (1) by $x_1^k, x_2^k, \ldots, x_n^k$. The iteration formula for simultaneous inclusion of all roots of (1)

$$x_i^{k+1} = x_i^k - \frac{A_n(x_i^k)}{\prod_{j=1,\, j\neq i}^{n} (x_i^k - x_j^k)}, \quad i = 1, 2, \ldots, n,\ k = 0, 1, 2, \ldots$$

was derived by Weierstrass [18], Durand [4], Dochev [3], Prešić [13], etc., in different way, provided that x_i are distinct.

In 1971 Prešić [14] published the following iteration formula

$$x_i^{k+1} = x_i^k - \frac{A_n(x_i^k)}{\prod_{j=1,\, j\neq i}^{m} (x_i^k - x_j^k).A_{x^k}(x_i^k)} \tag{2}$$

for finding $m\,(\leq n)$ roots simultaneously. He obtained it by using the presentation

$$\begin{aligned} A_n(x) \;&=\; (x - x_i^{k+1}) \prod_{j=1,\, j\neq i}^{m} (x - x_j^k).A_{x^k}(x) \\ &\quad + \sum_{l=1,\, l\neq i}^{m} A_n(x_l^k) \prod_{j=1,\, j\neq l}^{m} \frac{x - x_j^k}{x_l^k - x_j^k}, \quad i = 1, 2, \ldots, m, \end{aligned} \tag{3}$$

where $A_{x^k}(x)$ denotes the m-th divided difference $A_n[x_1^k, \ldots, x_m^k, x]$. It is remarked in [19] that the equality (3) is not an identity and that the iteration formula (2) follows not from (3) but from the identity

$$\begin{aligned} A_n(x) \;&=\; (x - x_i^{k+1}) \prod_{j=1,\, j\neq i}^{m} (x - x_j^k).A_{x^k}(x) + \sum_{l=1,\, l\neq i}^{m} A_n(x_l^k) \prod_{j=1,\, j\neq l}^{m} \frac{x - x_j^k}{x_l^k - x_j^k} \\ &\quad - \left[(x_i^k - x_i^{k+1})A_{x^k}(x) - \frac{A_n(x_i^k)}{\prod_{j=1,\, j\neq i}^{m} (x_i^k - x_j^k)} \right] \prod_{j=1,\, j\neq i}^{m} (x - x_j^k). \end{aligned}$$

The polynomial (1) can be written in the way [6]

$$A_n(x) = Q_m(x)\,T_{n-m}(x),$$

where $Q_m(x)$ is a polynomial of zeros, which we desire to find and $T_{n-m}(x)$ is a polynomial whose zeros we drop off. We assume that the roots of $Q_m(x)$ are distinct, but do not assume it for the roots of $T_{n-m}(x)$. Let us define

$$\begin{aligned} Q_m(x) \;&=\; x^m + b_1 x^{m-1} + \cdots + b_m, \\ T_{n-m}(x) \;&=\; x^{n-m} + c_1 x^{n-m-1} + \cdots + c_{n-m}. \end{aligned} \tag{4}$$

Between (1) and (4) there exist the following relations

$$
\begin{aligned}
a_1 &= c_1 + b_1, \\
a_2 &= c_2 + b_2 + c_1 b_1, \\
\cdots & \quad \cdots \\
a_l &= c_l + b_l + c_1 b_{l-1} + c_2 b_{l-2} + \cdots + c_{l-1} b_1, \\
\cdots & \quad \cdots \\
a_{n-m} &= c_{n-m} + b_{n-m} + c_1 b_{n-m-1} + c_2 b_{n-m-2} + \cdots + c_{n-m-1} b_1, \\
\cdots & \quad \cdots \\
a_n &= c_{n-m} b_m.
\end{aligned}
$$

We define polynomials

$$
\begin{aligned}
Q_m^k(x) &= x^m + b_1^k x^{m-1} + \cdots + b_m^k, \\
T_{n-m}^k(x) &= x^{n-m} + c_1^k x^{n-m-1} + \cdots + c_{n-m}^k.
\end{aligned} \tag{5}
$$

Then it is fulfilled evidently from (5) that

$$
\begin{aligned}
b_1^k &= -\sum_{j=1}^{m} x_j^k, \\
b_2^k &= \sum_{j=1}^{m-1} \left(x_j^k \sum_{s=j+1}^{m} x_s^k \right), \\
\cdots & \quad \cdots \\
b_m^k &= (-1)^m \prod_{j=1}^{m} x_j^k
\end{aligned} \tag{6}
$$

hold.

We define c_j^k, $j = 1, \ldots, n-m$, by formulas

$$
\begin{aligned}
c_1^k &= a_1 - b_1^k, \\
c_2^k &= a_2 - b_2^k - (a_1 - b_1^k) b_1^k = a_2 - b_2^k - c_1^k b_1^k, \\
\cdots & \quad \cdots \\
c_{n-m}^k &= a_{n-m} - b_{n-m}^k - \sum_{j=1}^{n-m-1} c_j^k b_{n-m-j}^k.
\end{aligned} \tag{7}
$$

In [14] the Brouwer theorem is not used to prove convergence, and it only gives requisite conditions that ensure convergence to the exact zeros. In [6] and [7], Iliev and Kyurkchiev wrote the method (2) in the following way

$$
x_i^{k+1} = x_i^k - \frac{A_n(x_i^k)}{\prod_{j=1, j \neq i}^{m} \left(x_i^k - x_j^k \right) . T_{n-m}^k \left(x_i^k \right)}, \quad i = 1, \ldots, m, \; k = 0, 1, \ldots .
$$

for finding $m\,(\leq n)$ zeros of (1) simultaneously and, in [6], proved a theorem which gives sufficient conditions for certain convergence to the part of roots of (1). It is stated as follows:

Theorem 1.1. *Let the polynomial* (1) *have real as well as complex roots. Decomposition* $A_n(x) = Q_m(x)\,T_{n-m}(x)$ *is valid, where the polynomials* $Q_m(x)$ *and* $T_{n-m}(x)$ *have for their roots the real roots and the complex roots of* (1), *respectively. Let* $c > 0$, $1 > q > 0$ *be real numbers such that*

$$c\,[A_1 g + [A_2 + gc]\,z]\,U^{-1} < 1,$$

where A_1, A_2, g, z *and* U *are some appropriate positive constants. If initial approximations* $x_1^0, x_2^0, \ldots, x_m^0$ *to the real roots of* (1) *satisfy the inequalities* $\left|x_i^0 - x_i\right| < cq$, $i = 1, 2, \ldots, m$, *then for every natural* k *the following inequalities are satisfied*

$$\left|x_i^k - x_i\right| < cq^{2^k}, \quad i = 1, 2, \ldots, m.$$

From computational point of view, the coefficients (6) and (7) can be calculated easily and they are also convenient for computer programming. In comparison with it, in [14] the polynomial $T_{n-m}^k(x)$ is derived with a technique based on divided differences which are calculated at each iteration step. In [9] we discussed Gauss-Seidel modification of method (2).

2. Main Result

The iterative method

$$x_i^{k+1} = x_i^k - \sigma_i^k \left(1 + \sigma_i^k \sum_{j \neq i} \frac{1}{x_i^k - x_j^k}\right), \quad i = 1, 2, \ldots, n;\ k = 0, 1, 2, \ldots \tag{8}$$

with

$$\sigma_i^k = \frac{A_n(x_i^k)}{A_n'(x_i^k)}$$

is due to Euler [5] and also considered independently by Tanabe [16] in an equivalent form

$$x_i^{k+1} = x_i^k - \sigma_i^k \left(1 - \sum_{j \neq i} \frac{\sigma_j^k}{x_i^k - x_j^k}\right), \quad i = 1, 2, \ldots, n;\ k = 0, 1, 2, \ldots . \tag{9}$$

On the other hand, the Chebyshev method for solving a nonlinear equation

$$F(x) = 0, \quad x \in \mathbb{C}^n$$

is defined by

$$x^{k+1} = x^k - \left[I + \frac{1}{2}\Gamma_k F''(x^k)\Gamma_k F(x^k)\right]\Gamma_k F(x^k), \quad k = 0, 1, 2, \ldots \tag{10}$$

with $\Gamma_k = F'(x^k)^{-1}$.

It is known (e.g. [8]) that (9) is the Chebyshev method applied to a system of nonlinear equations

$$f_i = (-1)^i \varphi_i(x_1, x_2, \dots, x_n) - a_i = 0, \quad i = 1, 2, \dots, n,$$

where φ_i denote the i-th elementary symmetric functions.

Hence, we shall call (8) the Euler-Chebyshev method or Euler-Chebyshev-Tanabe method. On the basis of (8), we construct an iterative formula

$$x_i^{k+1} = x_i^k - \sigma_i^k \left[1 + \sigma_i^k \left(\sum_{j \neq i}^{m} \frac{1}{x_i^k - x_j^k} + \frac{{T_{n-m}^k}'(x_i^k)}{T_{n-m}^k(x_i^k)} \right) \right], \qquad (11)$$
$$i = 1, 2, \dots, m; \ k = 0, 1, 2, \dots,$$

where $T_{n-m}^k(x)$ is as defined in Section 1, which is a generalization of the classical Euler-Chebyshev method (8) since (11) reduces to (8) in the case $m = n$.

We prove the following theorem which asserts that the order of convergence of the method (11) is locally cubic.

THEOREM 2.1. *Let $d = \min_{i \neq j} |x_i - x_j| > 0$ where $i, j = 1, 2, \dots, m$ and q be real numbers with $1 > q > 0$. Then there is a positive constant σ satisfying the following property* (P).

(P) *If we take a number $c > 0$ which is so small that*

$$d - 2c > c(n-1) \ (> 0),$$
$$c^2 \left[\sigma (d - 2c)^2 + (n-1)^2 + m - 1 \right] < \left[d - 2c - c(n-1) \right]^2, \qquad (12)$$

then the inequalities

$$|x_i^k - x_i| < cq^{3^k}, \quad i = 1, 2, \dots, m \qquad (13)$$

hold for every $k \in N$ provided that initial approximations $x_1^0, x_2^0, \dots, x_m^0$ to the real roots of (1) *satisfy inequalities $|x_i^0 - x_i| < cq$, $i = 1, 2, \dots, m$.*

Proof. Suppose that for some $k \in N \bigcup \{0\}$ inequalities (13) are fulfilled. The equalities

$$b_j^k = b_j + R_j, \quad j = 1, \dots, m, \qquad (14)$$

where $|R_j| \leq \rho_j c q^{3^k}$ and where ρ_j is independent of iteration number k, are valid. For c_s^k, it is true that

$$c_j^k = c_j + R_j^*, \quad j = 1, \dots, m \qquad (15)$$

hold, where $|R_j^*| \leq z_j c q^{3^k}$ and where z_j is independent of iteration number k. Here, we prove (14) and (15) for $j = 1$ and $j = 2$ only, since the proofs of (14) and (15) for other j are similar.

In fact,

$$b_1^k = -\sum_{j=1}^{m} x_j^k = -\sum_{j=1}^{m} x_j + \sum_{j=1}^{m}(x_j - x_j^k) = b_1 + R_1,$$

where

$$\begin{aligned} |R_1| &= \left|\sum_{j=1}^{m}(x_j - x_j^k)\right| = |b_1^k - b_1| \\ &\le |x_1^k - x_1| + \cdots + |x_m^k - x_m| \le mcq^{3^k} \end{aligned}$$

and therefore $|R_1| \le \rho_1 c q^{3^k}$ with $\rho_1 = m$ independent of iteration number k. For b_2^k, we have

$$\begin{aligned} b_2^k &= \sum_{j=1}^{m-1}\left(x_j^k \sum_{s=j+1}^{m} x_s^k\right) = \sum_{j=1}^{m-1}\left(x_j \sum_{s=j+1}^{m} x_s\right) \\ &\quad + \sum_{j=1}^{m-1}\sum_{s=j+1}^{m}(x_j^k x_s^k - x_j x_s) \\ &= b_2 + R_2 \end{aligned}$$

where

$$\begin{aligned} R_2 &= \sum_{j=1}^{m-1}\sum_{s=j+1}^{m}(x_j^k x_s^k - x_j x_s) \\ &= \sum_{j=1}^{m-1}\sum_{s=j+1}^{m}\big((x_j^k - x_j)(x_s^k - x_s) - 2x_j x_s + x_j^k x_s + x_s^k x_j\big) \\ &= \sum_{j=1}^{m-1}\sum_{s=j+1}^{m}\big((x_j^k - x_j)(x_s^k - x_s) - x_j(x_s - x_s^k) - x_s(x_j - x_j^k)\big) \end{aligned}$$

and therefore

$$|R_2| \le Ac^2 q^{3^{k+1}} + Bcq^{3^k} = cq^{3^k}\big(Acq^{3^k} + B\big) \le cq^{3^k}(Ac + B),$$

where A and B are some positive constants independent of iteration number k. That is, $|R_2| \le \rho_2 c q^{3^k}$. For c_1^k, we know

$$c_1^k = a_1 - b_1^k = a_1 - b_1 - R_1$$

i.e., $c_1^k - c_1 = -R_1 = R_1^*$. For c_2^k, we have

$$\begin{aligned} c_2^k &= a_2 - b_2^k - (a_1 - b_1^k)b_1^k = a_2 - b_2 - R_2 - (a_1 - b_1 - R_1) \times (b_1 + R_1) \\ &= c_2 - R_2 + b_1 R_1 - (a_1 - b_1)R_1 + R_1^2 = c_2 + R_2^*, \end{aligned}$$

where $R_2^* = -R_2 + R_1^2 - (a_1 - 2b_1)R_1$. Hence

$$\begin{aligned}|R_2^*| &\le |R_2| + |R_1^2| + |a_1 - 2b_1||R_1| \\ &\le \left(\rho_2 + \rho_1^2 c + |a_1 - 2b_1|\rho_1\right) c q^{3^k} = z_2 c q^{3^k}\end{aligned}$$

and $z_2 = \rho_2 + \rho_1^2 c + |a_1 - 2b_1|\rho_1$ is independent of iteration number k.

Now, we will show the inequalities (13). Evidently

$$A_n^{'}(x_i^k)/A_n(x_i^k) = \sum_{j=1}^{n} 1/(x_i^k - x_j), \tag{16}$$

and

$$T_{n-m}^{'}(x_i^k)/T_{n-m}(x_i^k) = \sum_{j=m+1}^{n} 1/(x_i^k - x_j), \quad i = 1, \ldots, m. \tag{17}$$

Using (16) and (17), and after removing the parentheses in the right side of the equality (11), we have

$$\begin{aligned} x_i^{k+1} &= x_i^k - \left(\frac{1}{x_i^k - x_i} + \sum_{j\neq i}^{n} \frac{1}{x_i^k - x_j}\right)^{-1} - \left(\sum_{j\neq i}^{m} \frac{1}{x_i^k - x_j^k} + \frac{T_{n-m}^{k\,'}(x_i^k)}{T_{n-m}^k(x_i^k)}\right) \\ &\quad \times \left(\frac{1}{x_i^k - x_i} + \sum_{j\neq i}^{n} \frac{1}{x_i^k - x_j}\right)^{-2}, \quad i = 1, \ldots, m. \end{aligned} \tag{18}$$

We subtract x_i from both sides of (18) and after some transformation we arrive to

$$\begin{aligned} x_i^{k+1} - x_i = (x_i^k - x_i)^2 \Bigg[&(x_i^k - x_i)\left(\sum_{j\neq i}^{n} \frac{1}{x_i^k - x_j}\right)^2 + \sum_{j\neq i}^{m} \frac{x_j - x_j^k}{(x_i^k - x_j)(x_i^k - x_j^k)} \\ &+ \frac{T_{n-m}^{'}(x_i^k)}{T_{n-m}(x_i^k)} - \frac{T_{n-m}^{k\,'}(x_i^k)}{T_{n-m}^k(x_i^k)}\Bigg] \Bigg/ \left(1 + (x_i^k - x_i)\sum_{j\neq i}^{n} \frac{1}{x_i^k - x_j}\right)^2 \end{aligned} \tag{19}$$

Thus, using (5) and (15), we get

$$\begin{aligned} T_{n-m}^k(x_i^k) &= (x_i^k)^{n-m} + (c_1 + R_1^*)(x_i^k)^{n-m-1} + (c_2 + R_2^*)(x_i^k)^{n-m-2} \\ &\quad + \cdots + c_{n-m} + R_{n-m}^* \\ &= T_{n-m}(x_i^k) + R_1^*(x_i^k)^{n-m-1} + R_2^*(x_i^k)^{n-m-2} + \cdots + R_{n-m}^* \\ &\equiv T_{n-m}(x_i^k) + M_1. \end{aligned} \tag{20}$$

For $|M_1|$, we have the estimation

$$|M_1| \le g c q^{3^k},$$

where g is a positive number which is independent of the iteration number k.

For $T_{n-m}^{k\,'}(x_i^k)$, we get

$$\begin{aligned} T_{n-m}^{k\,'}(x_i^k) &= (n-m)(x_i^k)^{n-m-1} + (c_1 + R_1^*)(n-m-1)(x_i^k)^{n-m-2} \\ &\quad + (c_2 + R_2^*)(n-m-2)(x_i^k)^{n-m-3} + \cdots + c_{n-m-1} + R_{n-m-1}^* \\ &= T_{n-m}^{'}(x_i^k) + R_1^*(n-m-1)(x_i^k)^{n-m-2} \\ &\quad + R_2^*(n-m-2)(x_i^k)^{n-m-3} + \cdots + R_{n-m-1}^* \\ &\equiv T_{n-m}^{'}(x_i^k) + M_2, \end{aligned} \tag{21}$$

where

$$|M_2| \le ycq^{3^k}$$

and y is independent of the iteration number k. Evidently

$$\begin{aligned} \frac{T_{n-m}^{'}(x_i^k)}{T_{n-m}(x_i^k)} - \frac{T_{n-m}^{k\,'}(x_i^k)}{T_{n-m}^{k}(x_i^k)} &= \frac{T_{n-m}^{'}(x_i^k)}{T_{n-m}(x_i^k)} - \frac{T_{n-m}^{'}(x_i^k) + M_2}{T_{n-m}(x_i^k) + M_1} \\ &= \frac{M_1 T_{n-m}^{'}(x_i^k) - M_2 T_{n-m}(x_i^k)}{T_{n-m}(x_i^k)\big(T_{n-m}(x_i^k) + M_1\big)}. \end{aligned} \tag{22}$$

We examine the functions $T_{n-m}^{'}(x)$ and $T_{n-m}(x)$ when $x \ne x_i$, $i = m+1, \dots, n$. They are restricted within domain of considerations and consequently there exist real positive constants L_1, F_1 and F_2 independent of the iteration number and such that

$$\begin{aligned} &L_1 \le |T_{n-m}(x)| \le F_1, \\ &|T_{n-m}^{'}(x)| \le F_2. \end{aligned} \tag{23}$$

It follows from (22) and (23) that

$$\left| \frac{T_{n-m}^{'}(x_i^k)}{T_{n-m}(x_i^k)} - \frac{T_{n-m}^{k\,'}(x_i^k)}{T_{n-m}^{k}(x_i^k)} \right| \le \frac{|M_1|F_2 + |M_2|F_1}{L_1(L_1 + |M_1|)} \le \frac{gF_2 + yF_1}{L_1^2} cq^{3^k} \tag{24}$$

holds, and also the inequalities

$$\begin{aligned} &|x_i^k - x_j| \ge |x_i - x_j| - |x_i - x_i^k| \ge d - cq^{3^k} > d - c > d - 2c \\ &|x_i^k - x_j^k| \ge |x_i^k - x_j| - |x_j - x_j^k| \ge d - cq^{3^k} - cq^{3^k} > d - 2c, \quad i \ne j \end{aligned} \tag{25}$$

are true. Then, using (24), (25) and (12), it can be obtained from (19) that

$$\begin{aligned} |x_i^{k+1} - x_i| &\le (cq^{3^k})^2 \Big(cq^{3^k}(n-1)^2/(d-2c)^2 + (gF_2 + yF_1)cq^{3^k}/L_1^2 \\ &\qquad + cq^{3^k}(m-1)/(d-2c)^2 \Big) / \left(1 - c(n-1)/(d-2c)\right)^2 \\ &< cq^{3^{k+1}}. \end{aligned} \tag{26}$$

Therefore, it follows from (26) that iteration process (11) is of locally cubic convergence i.e.,

$$|x_i^{k+1} - x_i| < cq^{3^{k+1}}$$

if we set $\sigma = (yF_1 + gF_2)/L_1^2$. Thus we prove Theorem 2.1. □

3. A Numerical Example

In this section, we show results of some numerical experiments for the algorithm (11).

Example 3.1. The equation

$$\begin{aligned} A_{10}(x) &= (x-1)(x+3)(x+8)(x-5)(x+6)(x-4)(x^2+6)(x^2+7) \\ &= x^{10} + 7\,x^9 - 38\,x^8 - 192\,x^7 + 209\,x^6 - 1009\,x^5 + 5768\,x^4 + 19002\,x^3 \\ &\quad - 2580\,x^2 + 99792\,x - 120960 \end{aligned}$$

and the initial approximations

$$x_1^0 = 0.8,\ x_2^0 = -2.7,\ x_3^0 = -8.2,\ x_4^0 = 5.2,\ x_5^0 = -5.7,\ x_6^0 = 3.8$$

are considered.

Using the formula (11), we get the real roots $x_1 = 1$, $x_2 = -3$, $x_3 = -8$, $x_4 = 5$, $x_5^0 = -6$ and $x_6 = 4$ with accuracy 18 decimal digits (except x_5^k) after only 4 iterations, which are shown in Table 3.1.

Table 3.1. Numerical results for Example 3.1 by (11).

	x_1^k	x_2^k	x_3^k
$k=1$	1.006184091337086300	−2.989695413032682900	−8.010609186020062100
$k=2$	0.999998802480556730	−2.999998189633442900	−8.000003178452360000
$k=3$	1.000000000000000000	−3.000000000000000000	−8.000000000000000000
$k=4$	1.000000000000000000	−3.000000000000000000	−8.000000000000000000
$k=5$	1.000000000000000000	−3.000000000000000000	−8.000000000000000000
$k=6$	1.000000000000000000	−3.000000000000000000	−8.000000000000000000

	x_4^k	x_5^k	x_6^k
$k=1$	5.019153162232133700	−5.963283139087074900	3.994780877313887300
$k=2$	5.000032475564413700	−5.999963456891165900	3.999999537421087500
$k=3$	5.000000000000167000	−5.999999999999966200	4.000000000000000000
$k=4$	5.000000000000000000	−5.999999999999999100	4.000000000000000000
$k=5$	5.000000000000000000	−5.999999999999999100	4.000000000000000000
$k=6$	5.000000000000000000	−5.999999999999999100	4.000000000000000000

We note that our methods are efficient to get high accuracy within few iterations when the polynomial has well separated real roots. For the equation in which some real roots are close, however, the iterations may not get so high accuracy roots.

The computational cost of $T_{n-m}^{k}{}'(x_i^k)/T_{n-m}^k(x_i^k)$, which are carried out by (6) and (7), is not small. But this is the price that we pay for getting the convergence method of third order. Index of effectiveness, in the Ostrowski-Traub sense [11], is $3^{1/4}$. In Table 3.2, we show the numerical results for the terms $T_{n-m}^{k}{}'(x_i^k)/T_{n-m}^k(x_i^k)$ in Example 3.1.

Table 3.2. Numerical results for $T_{n-m}^{k}{}'(x_i^k)/T_{n-m}^k(x_i^k)$ in Example 3.1.

	$T_{n-m}^{k}{}'(x_1^k)/T_{n-m}^k(x_1^k)$	$T_{n-m}^{k}{}'(x_2^k)/T_{n-m}^k(x_2^k)$	$T_{n-m}^{k}{}'(x_3^k)/T_{n-m}^k(x_3^k)$
$k=1$	0.268171607354484420	−0.642229504036485040	−0.447216825363411340
$k=2$	0.569035172588895910	−0.790483420255177480	−0.455254554486917430
$k=3$	0.535765638319055550	−0.775023028374614010	−0.453925768688997980
$k=4$	0.535714285714167130	−0.774999999999965490	−0.453923541247486960
$k=5$	0.535714285714287700	−0.775000000000000690	−0.453923541247484910
$k=6$	0.535714285714287700	−0.775000000000000690	−0.453923541247484910

	$T_{n-m}^{k}{}'(x_4^k)/T_{n-m}^k(x_4^k)$	$T_{n-m}^{k}{}'(x_5^k)/T_{n-m}^k(x_5^k)$	$T_{n-m}^{k}{}'(x_6^k)/T_{n-m}^k(x_6^k)$
$k=1$	0.594462209284029820	−0.580425811397484950	0.656042724146176810
$k=2$	0.630901188828704380	−0.571096551453553760	0.708230185527435110
$k=3$	0.635073842629359840	−0.564791704983316430	0.711456842023672880
$k=4$	0.635080645161241520	−0.564784053156150770	0.711462450592822090
$k=5$	0.635080645161290370	−0.564784053156146550	0.711462450592885490
$k=6$	0.635080645161290370	−0.564784053156146550	0.711462450592885490

Acknowledgements. The authors wish to express their gratitude to Prof. T. Yamamoto of Waseda University for his helpful suggestions and valuable comments. The authors also thank the referee and area editor for valuable comments.

References

[1] G. Alefeld and J. Herzberger, Introduction to Interval Computations. Academic Press, New York, 1983.
[2] I. Berezin and N. Zhidkov, Computing Methods II. Pergamon Press, Oxford, 1965.
[3] K. Dochev, An alternative method of Newton for simultaneous calculation of all the roots of a given algebraic equation (Bulgarian). Phys.-Math. J., **5** (1962), 136–139.
[4] E. Durand, Solutions numérique des équations algébriques. Tome I: Équations du type $F(x) = 0$. Racines d'une Polynôme, Masson, Paris, 1960, 279–281.
[5] L. Euler, Opera Omnia. Ser. I, Vol. X (eds. F. Engel and L. Schlesinger). Birkhäuser, 1913, 422–455.
[6] A. Iliev and N. Kyurkchiev, On a generalization of Weierstrass-Dochev method for simultaneous extraction of only a part of all roots of algebraic polynomials: part I. C. R. Acad. Bulg. Sci., **55** (2002), 23–26.
[7] A. Iliev and N. Kyurkchiev, Some methods for simultaneous extraction of only a part of all roots of algebraic polynomials part II. C. R. Acad. Bulg. Sci., **55** (2002), 17–22.
[8] S. Kanno, N. Kjurkchiev and T. Yamamoto, On some methods for the simultaneous deter-

mination of polynomial zeros. Japan J. Indust. Appl. Math., **13**, No.2 (1996), 267–288.

[9] N. Kyurkchiev and A. Iliev, A family of methods for simultaneous extraction of only a part of all roots of algebraic polynomials. BIT, **42** (2002), 879–885.

[10] N. Kyurkchiev, Initial Approximation and Root Finding Methods. WILEY-VCH Verlag Berlin GmbH, **Vol. 104**, Berlin 1998.

[11] J. Ortega and W. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York, 1970.

[12] M. Petković, Iterative methods for simultaneous inclusion of polynomial zeros. Lecture Notes in Mathematics **1387** (eds. A. Dold and B. Eckmann), Springer-Verlag, Berlin-Heidelberg, 1989.

[13] S. Prešić, Un procédé itératif pour la factorisation des polynomes. C. R. Acad. Sci. Paris, **262** (1966), 862–863.

[14] M. Prešić, Un procédé itératif pour déterminer k zéros d'un polynome. C. R. Acad. Sci. Paris, **273** (1971), 446–449.

[15] B. Sendov, A. Andreev and N. Kyurkchiev, Numerical solution of polynomial equations. Handbook of Numerical Analysis III (eds. P. Ciarlet and J. Lions), Elsevier Sci. Publ., Amsterdam, 1994.

[16] K. Tanabe, Behavior of the sequences around multiple zeroes generated by some simultaneous methods for solving algebraic equations (Japanese). Tech. Rep. Inf. Process. Numer. Anal., **4**, No.2 (1983), 1–6.

[17] J. Traub, Iterative Methods for the Solution of Equations. Prentice-Hall, Englewood Cliffs, NJ, 1964.

[18] K. Weierstrass, Neuer Beweis des Satzes, dass jede ganze rationale Funktion einer Veränderlichen dargestellt werden kann als ein Product aus Linearen Functionen derselben Veränderlichen. Gesammelte Werke, **3** (1903), Johnson, New York, 251–269.

[19] T. Yamamoto, S. Kanno and L. Atanassova, Validated computation of polynomial zeros by the Durand-Kerner method. Topics in Validated Computations, North Holland, Amsterdam, 1994.

[20] T. Yamamoto, SOR-like methods for the simultaneous determination of polynomial zeros. Numerical Methods and Error Bounds (eds. G. Alefeld and J. Herzberger), Akademie Verlag, Berlin, 1996, 287–296.