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Truncated path algebras and Betti numbers of polynomial growth

Ryan Coopergard and Marju Purin

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We investigate a class of truncated path algebras in which the Betti numbers of a simple module satisfy a polynomial of arbitrarily large degree. We produce truncated path algebras where the i-th Betti number of a simple module S is $\beta_i(S) = i^k$ for $2 \leq k \leq 4$ and provide a result of the existence of algebras where $\beta_i(S)$ is a polynomial of degree 4 or less with nonnegative integer coefficients. In particular, we prove that this class of truncated path algebras produces Betti numbers corresponding to any polynomial in a certain family.

1. Introduction

We consider finite-dimensional algebras Λ over an algebraically closed field with $\operatorname{rad}^2 \Lambda = 0$, where $\operatorname{rad} \Lambda$ denotes the Jacobson radical of the algebra. We work with these algebras by representing them as quotients of path algebras. The motivation behind investigating these algebras lies in the universality of path algebras. Namely, any finite-dimensional algebra over an algebraically closed field is a quotient of a path algebra. We use quivers (directed graphs) to write down these algebras and provide numerous examples along the way.

We study modules by means of their projective resolutions. Betti numbers are of particular interest in examining the projective resolutions of modules as they provide a method of describing the growth of resolutions. Such growth was examined in the groundbreaking paper [Tate 1957] in the setting of commutative rings and in [Alperin and Evens 1981] for group algebras. Since then the growth of resolutions has been shown to be related to many fundamental properties of an algebra such as, for example, the representation type of an algebra [Diveris and Purin 2014; Erdmann et al. 2004] or codimension of a commutative ring [Avramov 1998; Avramov and Buchweitz 2000; Avramov et al. 1997; Eisenbud 1980].

A fundamental question that is driving our work in this paper is to determine which polynomials are eventually realizable as sequences of Betti numbers. To this

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end we introduce a particular class of path algebras, namely those given by quivers of the form

$$
\begin{tikzcd}
 & & n+2 \arrow[loop above] & & \\
1 \arrow[r, "x_1"'] \arrow[urr, "y_1"] & 2 \arrow[ur, "y_2"] & \cdots & n \arrow[ul, "y_n"'] \arrow[r, "x_n"'] \arrow[loop below] & n+1 \arrow[ull, "y_{n+1}"'] \arrow[loop below]
\end{tikzcd}
$$

where the x_i and y_{n+1} are positive integers and y_i, $1 \le i \le n$, are nonnegative integers that represent the number of arrows between vertices. To clarify, the general $n = 0$ case is of the form

$$
\begin{tikzcd}
1 \arrow[r, "y_1"'] & 2 \arrow[loop below]
\end{tikzcd}
$$

and the general $n = 1$ case is of the form

$$
\begin{tikzcd}
 & 3 \arrow[loop above] & \\
1 \arrow[ur, "y_1"] \arrow[rr, "x_1"'] & & 2 \arrow[ul, "y_2"'] \arrow[loop below]
\end{tikzcd}
$$

From here on, we refer to these algebras as *pyramidal algebras*. Given a pyramidal algebra Λ, we refer to the quotient algebra $\Lambda/\operatorname{rad}^m \Lambda$ as an *m-pyramidal algebra*. For the majority of the paper, we consider only 2-pyramidal algebras and briefly discuss the more general version at the end.

A key result in the paper is to show that 2-pyramidal algebras have a simple module whose Betti numbers have polynomial growth of arbitrarily high degree. More precisely, the Betti numbers over algebras of pyramidal form satisfy $\beta_i(S_1) = p_n(i)$, where S_1 is the simple module at vertex 1 and p_n is a polynomial of degree n. In addition to proving this result, we provide examples of algebras with particularly interesting behavior of Betti numbers. We end with an application of our work to a question about the existence of algebras in which $\beta_i(S_1)$ is a polynomial of a specific form.

2. Preliminaries

A quiver is a set of vertices and arrows (an oriented graph). In this paper we work with finite quivers, that is, quivers with finitely many vertices and arrows. Furthermore, we assume that the quiver is connected, which means that the underlying graph is connected. We concatenate arrows to form paths in the quiver. In addition, there is a trivial path at each vertex, which we denote by e_i for vertex i.

A path algebra over a field k is the k-vector space that has as its basis the set of all paths. The multiplication of paths is given by concatenation of compatible paths. For incompatible paths the product is zero. With this operation the set of paths has a natural structure as a k-algebra. Furthermore, the Jacobson radical of the algebra is simply the ideal generated by the set of all arrows.

Example 2.1. We illustrate the above notions with the 2-pyramidal algebra where $y_1 = 1$:

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop below, "\beta"] \end{tikzcd}$$

The quiver above has two vertices and two arrows. As for paths, there are four nonzero paths: the two trivial paths $\{e_1, e_2\}$ and two arrows $\{\alpha, \beta\}$. Some examples of multiplication are: $e_1 \cdot \alpha = \alpha$, $\alpha \cdot e_2 = \alpha$, $\alpha \cdot \beta = \alpha\beta = 0$ (as the path lies in $\operatorname{rad}^2 \Lambda$), and $\beta \cdot \alpha = 0$ (as the arrows are incompatible).

In this paper we work with finitely generated right modules over finite-dimensional algebras. Every such module has a projective cover and consequently a minimal projective resolution over the algebra. For a path algebra, the number of indecomposable nonisomorphic projective modules corresponds to the number of vertices in the quiver of the algebra. In particular, there are only finitely many such projective modules, while there can be infinitely many indecomposable nonisomorphic modules over such algebras. Therefore projective modules, by means of resolutions, provide a method of studying any module over a finite-dimensional algebra.

We measure the complexity of an algebra by measuring the complexity of the projective resolutions of the modules over the algebra. We do this by examining the growth of the Betti sequence of the resolutions. For $m \geq 0$, the m-th term, the m-th Betti number, is the number of indecomposable projective modules at the m-th step of the resolution. Thus, faster growth of a Betti sequence corresponds to a higher-complexity module.

It suffices to examine the resolutions of the simple modules as the fastest growth rate is always realized by a simple module. The goal in this paper is precisely this — to examine the resolutions of simple modules.

Throughout the paper we use the following notation. We denote by S_n the simple module at vertex n. For $i \geq 0$, the i-th term in a projective resolution of a module M is denoted by $P_i(M)$ and the i-th Betti number is $\beta_i(M)$.

We also make use of dimension vectors of modules. The dimension vector of a module M represents the element $[M]$ in the Grothendieck group $K_0(\Lambda)$ corresponding to M, where $K_0(\Lambda)$ is the free abelian group on a set of isomorphism classes of the simple Λ-modules. As such, dimension vectors record the multiplicity of each composition factor in the composition series of the module. For ease of notation, k_i copies of S_i in the composition series of a module M are denoted by $1^{k_1}2^{k_2}\cdots t^{k_t}$. In particular, we are not tracking the radical layers in which the composition factors occur.

Example 2.2. The 2-pyramidal algebra in Example 2.1 has two nonisomorphic simple modules, one at each vertex, denoted by S_1 and S_2. The projective covers of the simple modules can be obtained by recording the maximal path starting at the corresponding vertex, keeping in mind that in a 2-pyramidal algebra the composite of any two arrows vanishes. Thus, the projective cover of the simple module $S_1 = 1$ is $P_0(S_1) = {}^{1}_{2} = 1\,2$, the projective cover of $S_2 = 2$ is $P_0(S_2) = {}^{2}_{2} = 2^2$.

Note that the zeroth Betti number, corresponding to the zeroth step in the projective resolution, is 1. This will always be the case, and for this reason we will ignore the zeroth Betti number and consider only β_k with $k \geq 1$ for the remainder of this paper. The first syzygy, denoted by $\Omega^1(S_1)$, in the projective resolution of S_1 is the kernel of the epimorphism $P_0(S_1) \to S_1$. It has dimension vector $\Omega^1(S_1) = 2$. A projective resolution of S_1 is obtained by iterating the process and finding a projective cover, denoted by $P_1(S_1)$, for the syzygy $\Omega^1(S_1) = 2$. We obtain the resolution

$$\cdots {}^{2}_{2} \to {}^{2}_{2} \to {}^{2}_{2} \to {}^{1}_{2} \to S_1 = 1.$$

In other words, we have $P_i(S_1) = {}^{2}_{2}$ and syzygies $\Omega^i(S_1) = 2$ for $i > 0$. The Betti sequence is the constant sequence $\beta_i(S_1) = 1$ for $i \geq 0$.

For more background on modules over path algebras we refer the reader to [Auslander et al. 1995; Assem et al. 2006].

We make frequent use of difference tables of polynomials. Given a polynomial $p(n)$ of degree n, the *difference table* of $p(n)$ is a table of rows and columns, $D = \{d_{i,j}\}$, $i \geq 1$, $j \geq 0$ such that $\{d_{i,0}\} = p(i)$ and the other entries are defined recursively as $d_{i,j} = d_{i+1,j-1} - d_{i,j-1}$. That is, the j-th column in the difference table of $p(n)$ is the difference between the elements in the $(j-1)$-th column. We then refer to the j-th column as the j-th difference of $p(n)$.

Example 2.3. The difference table for the polynomial $p(n) = n^2$ is

1	3	2	0	$\cdots$
4	5	2	0	$\cdots$
9	7	2	0	$\cdots$
16	9	2	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Note that each column produces a sequence that is polynomial of degree one less than the previous column, until we reach a column of zeros.

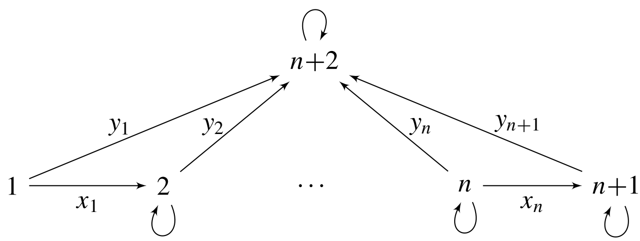
The difference tables of polynomials eventually reach a column of zeros. Thus, we will refer to the tables only up to, but not including, the first column of zeros.

3. Pyramidal algebras

In this section we examine the behaviour of projective resolutions over pyramidal algebras. We begin with a key observation that describes the syzygies of a projective resolution of a simple module.

Before proceeding with the results, we remark that for 2-pyramidal algebras all syzygies are semisimple. This is because these algebras have radical squared zero. Hence it is sufficient to work with dimension vectors when calculating the syzygies in a resolution.

Lemma 3.1. *In a* 2*-pyramidal algebra of the form,*



the multiplicity of S_k *as a direct summand in the syzygy* $\Omega^i(S_1),\ i \geq k-1$, *is*

$$\binom{i-1}{k-2} x_1 x_2 \cdots x_{k-1}$$

if $2 \leq k \leq n+1$ *and*

$$y_1 + \sum_{j=1}^{n} \binom{i-1}{j} x_1 x_2 \cdots x_j y_{j+1}$$

if $k = n+2$ *and* $i \geq 1$.

In the case where $k \neq n+2$ *and* $i \leq k-2$, *or* $k = n+2$ *and* $i \leq 1$, *the multiplicity of* S_k *in* $\Omega^i(S_1)$ *is zero.*

Proof. Note that S_k appears as a summand of $\Omega^i(S_1)$ if and only if there is a walk of length i from vertex 1 to vertex k in the underlying quiver. The final statement in the lemma is an immediate corollary of this fact.

We will prove the first case where $2 \leq k \leq n+1$ by double induction on the statement "the multiplicity of S_k as a direct summand in the syzygy $\Omega^i(S_1),\ i \geq k-1$, is

$$\binom{i-1}{k-2} x_1 x_2 \cdots x_{k-1}."$$

We will induct on i and k, in that order. When inducting on i, the base case is $k = 2$, $i = 1$, as this is the first syzygy in which S_2 appears. We then proceed by varying i

and fixing $k=2$ to complete the induction on i. When inducting on k, we must start with the base case $i=k-1$, as this is the smallest value of i in which S_k appears as a summand of $\Omega^i(S_1)$. Finally, we induct on k given an arbitrary fixed $i \geq k-1$.

For $k=2$ and $i \geq 1$ arbitrary, we see that the multiplicity of S_2 in $\Omega^i(S_1)$ is x_1 always. This is equal to $\binom{i-1}{0}x_1$, so this concludes the first part of the induction.

Assume the statement holds for $i=k-1$ and consider the multiplicity of S_k as a direct summand in $\Omega^{k-1}(S_1)$. Because there is no S_k in $\Omega^{k-2}(S_1)$, only the multiplicity of S_{k-1} in $\Omega^{k-1}(S_1)$ contributes to the multiplicity of S_k in $\Omega^{k-1}(S_1)$. By the induction hypothesis, there are

$$\binom{k-3}{k-3}x_1x_2\cdots x_{k-3}x_{k-2} = x_1x_2\cdots x_{k-3}x_{k-2}$$

copies of S_{k-1} in the $(k-2)$-th syzygy. Thus the multiplicity of S_k in the $(k-1)$-th syzygy is

$$x_1x_2\cdots x_{k-2}x_{k-1} = \binom{k-2}{k-2}x_1x_2\cdots x_{k-2}x_{k-1}.$$

Now assume the statement holds up to $k-1$ and $i-1$. By induction, the multiplicity of S_{k-1} in the $(i-1)$-th syzygy is given by

$$\binom{i-2}{k-3}x_1x_2\cdots x_{k-3}x_{k-2}.$$

Similarly the multiplicity of S_k in the $(i-1)$-th syzygy is

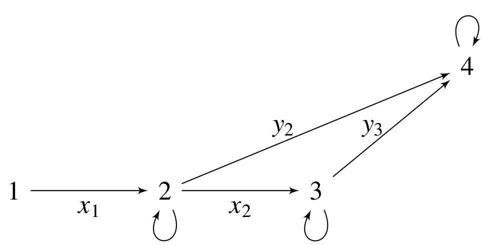
$$\binom{i-2}{k-2}x_1x_2\cdots x_{k-1}.$$

Therefore, the multiplicity of S_k in $\Omega^i(S_1)$ is

$$\begin{aligned} x_{k-1}\binom{i-2}{k-3}x_1x_2\cdots x_{k-2}+\binom{i-2}{k-2}x_1x_2\cdots x_{k-1} &= \left(\binom{i-2}{k-3}+\binom{i-2}{k-2}\right)x_1x_2\cdots x_{k-1} \\ &= \binom{i-1}{k-2}x_1x_2\cdots x_{k-1}, \end{aligned}$$

and the induction is complete. A similar argument can be made for the multiplicity of S_{n+2} as a direct summand of the i-th syzygy. $\square$

Example 3.2. To see an example of this lemma, consider the 2-pyramidal algebra



with $y_1 = 0$ and $x_1 = x_2 = y_2 = y_3 = 1$, i.e., there is one arrow from vertex 2 to 3, one from vertex 2 to 4, and one from vertex 3 to 4.

The projective resolution of S_1 is

$$\cdots P_2 \oplus P_3 \oplus P_4 \to P_2 \to P_1 \to S_1,$$

with syzygies

$$\Omega^1(S_1) = 2, \quad \Omega^2(S_1) = 234, \quad \Omega^3(S_1) = 23^2 4^3, \quad \Omega^4(S_1) = 23^3 4^6,$$

etc. The multiplicity of S_3 in the dimension vector of $\Omega^4(S_1)$ is 3, while the multiplicity of S_4 in $\Omega^4(S_1)$ is 6.

Using our formula to calculate the multiplicity of S_3 and S_4 in the dimension vector $\Omega^4(S_1)$ gives the following.

First, for $k = 3$ and $i = 4$ we obtain the multiplicity of S_3 as

$$\binom{3}{1} \cdot 1 \cdot 1 = 3.$$

Similarly for $k = 4$ and $i = 4$, we get the multiplicity of S_4 as

$$\sum_{j=1}^{2} \binom{3}{j} \cdot 1 = 3 + 3 = 6.$$

Note that we interpret $x_j = 0$ for $j \geq 3$ because their corresponding edges in the quiver are not present, so further sums do not appear.

Theorem 3.3. *Every 2-pyramidal algebra with $n+2$ vertices in the underlying quiver has Betti numbers*

$$\beta_i(S_1) = \begin{cases} 1 & \text{for } i = 0, \\ p_n(i) & \text{for } i \geq 1, \end{cases}$$

where p_n is a polynomial of degree n.

Proof. We proceed by induction on n. If $n = 0$, then we have an algebra of the form

$$1 \xrightarrow[y_1]{} 2 \circlearrowleft$$

Because $\Omega^i(S_1) = 2^{y_1}$ for all i, it follows that $\beta_i(S_1) = y_1$ for all i, so $\beta_i(S_1)$ is constant, and thus is a polynomial of degree 0.

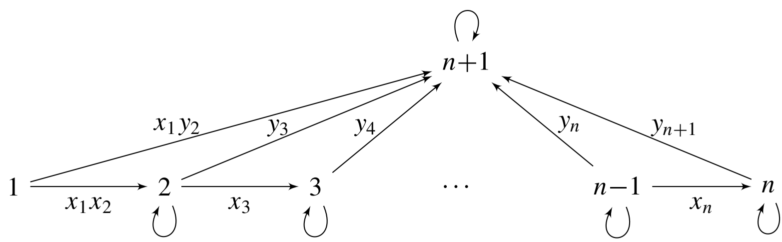
Suppose the statement holds for all values less than n, and consider an algebra of this form with $n+2$ vertices. The Betti numbers are calculated by adding the multiplicities of the various S_k together. These multiplicities were calculated in Lemma 3.1, so we see that the i-th Betti number is given by

$$\sum_{j=0}^{n-1} \binom{i-1}{j} x_1 \cdots x_{j+1} + y_1 + \sum_{j=1}^{n} \binom{i-1}{j} x_1 x_2 \cdots x_j y_{j+1}.$$

Now taking the $(i+1)$-th Betti number and subtracting the i-th Betti number yields

$$\sum_{j=0}^{n-1}\binom{i}{j}x_1\cdots x_{j+1}+y_1+\sum_{j=1}^{n}\binom{i}{j}x_1x_2\cdots x_j y_{j+1}$$
$$-\left(\sum_{j=0}^{n-1}\binom{i-1}{j}x_1\cdots x_{j+1}+y_1+\sum_{j=1}^{n}\binom{i-1}{j}x_1x_2\cdots x_j y_{j+1}\right)$$
$$=\sum_{j=0}^{n-1}\left(\binom{i}{j}-\binom{i-1}{j}\right)x_1\cdots x_{j+1}+\sum_{j=1}^{n}\left(\binom{i}{j}-\binom{i-1}{j}\right)x_1x_2\cdots x_j y_{j+1}$$
$$=\sum_{j=1}^{n-1}\binom{i-1}{j-1}x_1\cdots x_{j+1}+\sum_{j=1}^{n}\binom{i-1}{j-1}x_1x_2\cdots x_j y_{j+1}.$$

Observe that this is the i-th Betti number of the following algebra:



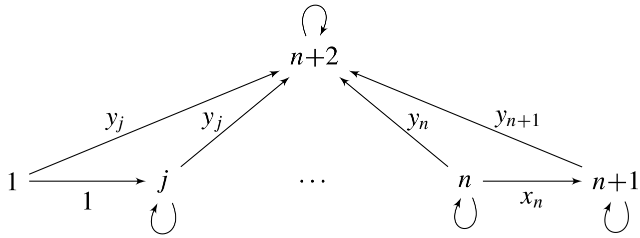
By the induction hypotheses, this 2-pyramidal algebra satisfies $\beta_i(S_1)=p_{n-1}(i)$, where p_{n-1} is a polynomial of degree $n-1$. Thus we see that the difference between the terms of the original algebra's Betti numbers is a polynomial of degree $n-1$, so the Betti numbers follow a polynomial of degree n, as desired. $\square$

It is interesting to mention an alternative approach to the above result, as was suggested by one of the referees. Namely, we may also analyze the Betti sequence by means of the action of the syzygy operator Ω. Because the syzygies of a module over a radical square zero algebra are semisimple, Ω acts as an endomorphism on the Grothendieck group $K_0(\Lambda)$. The action of Ω on S_i is evidently the dimension vector of $\Omega(S_i)$. Considering these vectors over all $n+2$ simple modules, the action of Ω is given by the matrix

$$\Omega=\begin{bmatrix}0&0&0&\cdots&0\\x_1&1&0&\cdots&0\\0&x_2&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\y_1&y_2&y_3&\cdots&1\end{bmatrix}.$$

Thus, $\Omega^m(S_1)$ is the first column of the m-th power of this matrix. Moreover, this matrix is the transpose of the adjacency matrix of the quiver, so the first column of Ω^m gives the number of paths starting at vertex 1 that have length m.

While we have only considered the Betti numbers for the projective resolution of S_1, we may also consider them for projective resolutions of any other simple module, say S_j. In this case, by restricting our quiver to the vertices $j, j+1, \ldots, n+2$, we get another algebra. The Betti numbers of the projective resolution of S_j are evidently the same as that of the 2-pyramidal algebra below:



By previous work, we see that the Betti numbers of S_j agree with a polynomial of degree $n+2-j$.

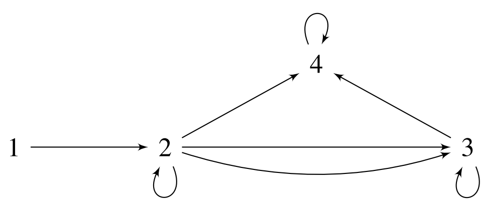
Theorem 3.3 is quite useful for the theorems in this paper due to the following corollary.

Corollary 3.4. *Let Λ be a 2-pyramidal algebra as in Lemma 3.1. If the first $n+1$ Betti numbers are known to fit a polynomial p of degree n, then $\beta_i(S_1) = p(i)$.*

Proof. By Theorem 3.3, we know that $\beta_i(S_1) = p_n(i)$ for some polynomial p_n of degree n. It is well known that given $n+1$ pairs of points $\{(x_j, y_j)\}_{j=1}^{n+1}$, there is a unique polynomial p of degree n such that $p(x_j) = y_j$ for $1 \leq j \leq n+1$. Because $\beta_i(S_1) = p(i)$ for $1 \leq i \leq n+1$, it follows that $\beta_i(S_1) = p(i)$ for all $i \geq 1$. $\square$

This theorem and its corollary will help us find algebras with Betti numbers of growth given by $\beta_i(S_1) = i^2$, $\beta_i(S_1) = i^3$ and $\beta_i(S_1) = i^4$. From this, we show that given any polynomial $p(i)$ of degree 4 or less with nonnegative integer coefficients, there exists an algebra such that $\beta_i(S_1) = p(i)$.

Lemma 3.5. *In the following 2-pyramidal algebra, $\beta_i(S_1) = i^2$ for $i \geq 1$:*

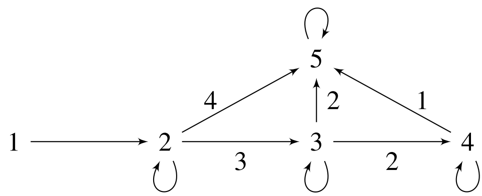


Proof. By Corollary 3.4, we need only show that the first three terms agree with $\beta_i(S_1) = i^2$. Indeed, we can calculate these quite easily:

$$\Omega^1(S_1) = 2, \quad \Omega^2(S_1) = 2\,3^2\,4, \quad \Omega^3(S_1) = 2\,3^4\,4^4.$$

Thus $\beta_i(S_1) = i^2$ for $1 \leq i \leq 3$. Therefore, $\beta_i(S_1) = i^2$ for all $i \geq 1$. $\square$

Lemma 3.6. *In the following* 2*-pyramidal algebra*, $\beta_i(S_1) = i^3$ *for* $i \geq 1$:

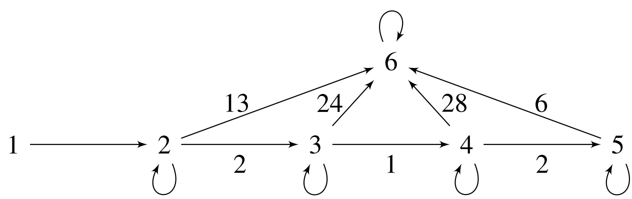


Proof. Again by Corollary 3.4, we need only show that $\beta_i(S_1) = i^3$ for $1 \leq i \leq 4$. We compute the syzygies directly as before. Here we obtain

$$\Omega^1(S_1) = 2, \quad \Omega^2(S_1) = 2\,3^3\,5^4, \quad \Omega^3(S_1) = 2\,3^6\,4^6\,5^{14}, \quad \Omega^4(S_1) = 2\,3^9\,4^{18}\,5^{36}.$$

We see that $\beta_i(S_1) = i^3$ for $1 \leq i \leq 4$. Therefore, $\beta_i(S_1) = i^3$ for all $i \geq 1$. □

Lemma 3.7. *In the following* 2*-pyramidal algebra*, $\beta_i(S_1) = i^4$ *for* $i \geq 1$:



Proof. We find the i-th syzygy of S_1 for $1 \leq i \leq 5$ to show that $\beta_i(S_1) = i^4$ for these values of i. Indeed, the syzygies are as follows:

$$\Omega^1(S_1) = 2, \quad \Omega^2(S_1) = 2\,3^2\,6^{13}, \quad \Omega^3(S_1) = 2\,3^4\,4^2\,6^74,$$
$$\Omega^4(S_1) = 2\,3^6\,4^6\,5^4\,6^{239}, \quad \Omega^5(S_1) = 2\,3^8\,4^{12}\,5^{16}\,6^{588}.$$

By examining the size of these syzygies, we find

$$\beta_1(S_1) = 1, \quad \beta_2(S_1) = 1+2+13 = 16 = 2^4, \quad \beta_3(S_1) = 1+4+2+74 = 81 = 3^4,$$
$$\beta_4(S_1) = 1+6+6+4+239 = 256 = 4^4, \quad \beta_5(S_1) = 1+8+12+16+588 = 625 = 5^4.$$

By Corollary 3.4, it follows that $\beta_i(S_1) = i^4$ for all $i \geq 1$. □

The next lemma will give us a method of constructing algebras with specific Betti numbers for a simple module.

Lemma 3.8. *Let* Λ_1 *and* Λ_2 *be truncated path algebras such that the projective resolution of the simple module at vertex* k *in* Λ_1 *follows* $\beta_i(S_k) = f(i)$ *and the projective resolution of the simple module at vertex* m *in* Λ_2 *follows* $\beta_i(S_m) = g(i)$ *for some functions* f *and* g*. Then there exists an algebra with Betti numbers given by* $\beta_i(S) = f(i) + g(i)$ *for some simple module* S *and all* $i \geq 1$.

Proof. Begin with the algebras Λ_1 and Λ_2 with underlying quivers Γ_1 and Γ_2. Let R_1 and R_2 be the set of relations in Λ_1 and Λ_2 respectively. Create a new algebra, Λ_3, whose underlying quiver, Γ_3, is obtained by taking the disjoint union of Γ_1 and Γ_2 and adding a new vertex $\tilde{1}$. Additionally, for each arrow $k \xrightarrow{\alpha} n$ in Λ_1, there is an arrow $\tilde{1} \xrightarrow{\tilde{\alpha}} n$ in Γ_3, and for each arrow $m \xrightarrow{\gamma} l$ in Λ_2, there is an arrow $\tilde{1} \xrightarrow{\tilde{\gamma}} l$ in Γ_3. The set of relations of Λ_3, denoted by R_3, is defined as

$$R_3 := R_1 \cup R_2 \cup \{\tilde{\alpha} w_1 \mid \alpha w_1 \in R_1\} \cup \{\tilde{\gamma} w_2 \mid \gamma w_2 \in R_2\},$$

where w_1 and w_2 could be paths of any length. Note that the elements in the last two sets of this union are nonzero because the target of $\tilde{\alpha}$ is the same as that of α, and the target of $\tilde{\gamma}$ is the same as that of γ. The addition of these relations ensures, for example, that if Λ_1 and Λ_2 are radical square zero algebras, then Λ_3 is as well.

By construction, there are bijections

$$\{\text{vertices in } \Gamma_1\} \cup \{\text{vertices in } \Gamma_2\} \Longleftrightarrow \{\text{vertices in } \Gamma_3\} \setminus \{\tilde{1}\},$$
$$\{\text{paths in } \Gamma_1\} \cup \{\text{paths in } \Gamma_2\} \Longleftrightarrow \{\text{paths in } \Gamma_3 \text{ not involving } \tilde{1}\},$$

both induced by inclusion of quivers. Moreover, the bijection of paths is compatible with the bijection of vertices. This, along with the choice of relations in Λ_3, gives a bijection

$$\{\text{projective } \Lambda_1-\text{modules}\} \cup \{\text{projective } \Lambda_2-\text{modules}\}$$
$$\Longleftrightarrow \{\text{projective } \Lambda_3-\text{modules}\} \setminus \{P_{\tilde{1}}\},$$

where $P_{\tilde{1}}$ is the indecomposable projective Λ_3-module at vertex $\tilde{1}$. This correspondence takes radical layers to radical layers bijectively in a manner compatible with the first two bijections. Let

$$\cdots \to Q_1 \to Q_0 \to S_k \to 0,$$
$$\cdots \to R_1 \to R_0 \to S_m \to 0,$$
$$\cdots \to F_1 \to F_0 \to S_{\tilde{1}} \to 0$$

be minimal projective resolutions of S_k, S_m, and $S_{\tilde{1}}$ respectively as Λ_3-modules. We will now show that for $i \geq 1$, we have $F_i \cong Q_i \oplus R_i$ and $\Omega^i(S_{\tilde{1}}) \cong \Omega^i(S_k) \oplus \Omega^i(S_m)$. Note that the bijections above imply that the minimal projective resolutions of S_k and S_m in Λ_3 correspond to those in Λ_1 and Λ_2, so proving this will yield the lemma.

We proceed by induction on i. We compute $\mathrm{rad}(F_0) = \mathrm{rad}(P_{\tilde{1}}) = \Omega^1(S_{\tilde{1}})$. The simple modules in the k-th radical layer of $P_{\tilde{1}}$ correspond to the vertices at the end of paths of length k from $\tilde{1}$ which do not lie in R_3. By the construction of Λ_3, this is precisely the union of the simple modules in the k-th radical layer of P_k and P_m. Also by construction, we in fact get $\mathrm{rad}(P_{\tilde{1}}) \cong \mathrm{rad}(P_k) \oplus \mathrm{rad}(P_m)$, and so

$\Omega^1(S_{\tilde{1}}) \cong \Omega^1(S_k) \oplus \Omega^1(S_m)$. Moreover, the projective cover of this syzygy is the direct sum of the covers of its summands, so $F_1 \cong Q_1 \oplus R_1$. Note that F_i does not have $P_{\tilde{1}}$ as a summand for any $i > 0$.

Suppose that $F_i \cong Q_i \oplus R_i$ and $\Omega^i(S_{\tilde{1}}) \cong \Omega^i(S_k) \oplus \Omega^i(S_m)$ for $i-1,\ i > 1$. The hypothesis implies that at the $(i-1)$-th step of the projective resolution for $S_{\tilde{1}}$, we have a projective cover $Q_{i-1} \oplus R_{i-1} \to \Omega^{i-1}(S_k) \oplus \Omega^{i-1}(S_m)$. By the bijection of projective modules and the fact that the radical layers are preserved under this bijection, we get

$$\ker[Q_{i-1} \oplus R_{i-1} \to \Omega^{i-1}(S_k) \oplus \Omega^{i-1}(S_m)] \cong \Omega^i(S_k) \oplus \Omega^i(S_m),$$

so $\Omega^i(S_{\tilde{1}}) \cong \Omega^i(S_k) \oplus \Omega^i(S_m)$. From this it also follows that $F_i \cong Q_i \oplus R_i$, and the induction is complete. Thus $\beta_i(S_{\tilde{1}}) = f(i) + g(i)$ for all $i \geq 1$. □

We apply this lemma to 2-pyramidal algebras to construct Betti sequences that realize desired polynomials.

Example 3.9. Let $p(i) = ai^4 + bi^3 + ci^2 + di + e$ for some nonnegative integers a, b, c, d, e. Then there exists an algebra Λ, where $\beta_i(S) = p(i)$ for a simple module S.

Proof. Begin by choosing algebras Λ_4, Λ_3, Λ_2, Λ_1 and Λ_0 and simple modules S_4, S_3, S_2, S_1, and S_0 satisfying

$$\beta_i^{\Lambda_4}(S_4) = i^4, \quad \beta_i^{\Lambda_3}(S_3) = i^3, \quad \beta_i^{\Lambda_2}(S_2) = i^2, \quad \beta_i^{\Lambda_1}(S_1) = i, \quad \beta_i^{\Lambda_0}(S_0) = 1,$$

respectively. Next, take a, b, c, d, and e copies of the algebras Λ_4, Λ_3, Λ_2, Λ_1 and Λ_0, respectively, and apply Lemma 3.8 to these algebras to obtain a new algebra Λ with

$$\beta_i(S) = ai^4 + bi^3 + ci^2 + di + e$$

for a simple Λ-module S. □

We will see in the following section that these polynomials can be realized as the Betti numbers of some 2-pyramidal algebra.

4. Characterizations

In this section we characterize Betti numbers over 2-pyramidal algebras. We start with some general statements and proceed to provide a characterization of the polynomials that give the growth of Betti sequences over these algebras.

Lemma 4.1. *Let p be a polynomial such that $p(1) \in \mathbb{Z}^+$, and let p' be the polynomial generating the first differences in the difference table of p. Then there exists a 2-pyramidal algebra in which $\beta_i(S_1) = p(i)$ if and only if there exists a 2-pyramidal algebra such that $\beta_i(S_1) = p'(i)$.*

Proof. The forward direction of this proof is made trivial by a fact in the proof of Theorem 3.3. In this proof, we saw that the n-th element of the first difference of the Betti numbers are the Betti numbers of S_1 over the algebra

$$\begin{tikzcd}
 & & n+1 \arrow[loop above] & & \\
1 \arrow[r, "x_1x_2"'] \arrow[urr, "x_1y_2"] & 2 \arrow[ur, "y_3"'] \arrow[loop below] & \cdots & n-1 \arrow[ul, "y_n"] \arrow[r, "x_n"'] \arrow[loop below] & n \arrow[ull, "y_{n+1}"'] \arrow[loop below]
\end{tikzcd}$$

This conclude the first part of the proof.

For the reverse direction, let $p(1) = k \in \mathbb{Z}^+$ and let p' correspond to the Betti numbers of

$$\begin{tikzcd}
 & & n+1 \arrow[loop above] & & \\
1 \arrow[r, "x_2"'] \arrow[urr, "y_2"] & 2 \arrow[ur, "y_3"'] \arrow[loop below] & \cdots & n-1 \arrow[ul, "y_n"] \arrow[r, "x_n"'] \arrow[loop below] & n \arrow[ull, "y_{n+1}"'] \arrow[loop below]
\end{tikzcd} \tag{4-1}$$

We now consider the algebra

$$\begin{tikzcd}
 & & & n+2 \arrow[loop above] & & \\
1 \arrow[r] \arrow[urrr, "k-1"] & 2 \arrow[r, "x_2"'] \arrow[urr, "y_2"] \arrow[loop below] & 3 \arrow[ur, "y_3"'] \arrow[loop below] & \cdots & n \arrow[ul, "y_n"] \arrow[r, "x_n"'] \arrow[loop below] & n+1 \arrow[ull, "y_{n+1}"'] \arrow[loop below]
\end{tikzcd}$$

Now this algebra has the property that $\beta_1(S_1) = k$ and the differences are the Betti numbers of (4-1). Because the Betti numbers of (4-1) are given by p', the differences are given by p', as desired. $\square$

We can now use this result to provide some necessary and sufficient conditions that a polynomial must meet in order to represent the Betti numbers of some 2-pyramidal algebra.

Theorem 4.2. *A polynomial p is such that $\beta_i(S_1) = p(i)$ for some 2-pyramidal algebra if and only if the difference table of p consists of only positive integers.*

Proof. We will prove the forward direction by induction on the columns of the difference table of p. Let p be a polynomial of degree n and let Λ be a 2-pyramidal algebra such that $\beta_i(S_1) = p(i)$. We first show that the zeroth difference, that is p, has all positive entries. Because x_1 is positive and there is an arrow from 2 to itself, it follows that $p(1) = \beta_1(S_1) \geq x_1$ and in fact, $p(i) \geq x_1$ for all i.

Suppose the statement holds for the k-th difference, and consider the $(k+1)$-th difference. The k-th difference is given by a polynomial of degree $n-k$ and gives the Betti numbers of some algebra. Because the first entry of the k-th column is positive, it follows from the forward direction of Lemma 4.1 that the $(k+1)$-th difference is also a polynomial of this form. By the first step, it follows that all entries for this polynomial are positive, and the induction is complete.

We now prove that every difference gives the Betti numbers over some 2-pyramidal algebra. We proceed by reverse induction on the columns of the difference table of p. Suppose p is a polynomial whose difference table contains only positive integers. In particular, the column of constants is some positive integer m. This polynomial represents $\beta_i(S_1)$ of the 2-pyramidal algebra,

$$\begin{tikzcd} 1 \arrow[r, "m"'] & 2 \arrow[loop below] \end{tikzcd}$$

so the base case holds.

Assume that the statement holds for the $(n-k)$-th column, and consider the $(n-(k+1))$-th column. Because the first entry of the $(n-(k+1))$-th column is positive, it follows from the reverse direction of Lemma 4.1 that this column gives the Betti numbers of some 2-pyramidal algebra. This completes the induction, and thus p gives the Betti numbers of some 2-pyramidal algebra. □

Note that in this proof, we only used the fact that the first entry in every column must be a positive integer. Indeed, this leads to a slightly stronger formulation of the theorem.

Corollary 4.3. *A polynomial p is such that $\beta_i(S_1) = p(i)$ for some 2-pyramidal algebra if and only if the first row of the difference table of p contains only positive integers.*

5. Producing pyramidal algebras given a polynomial

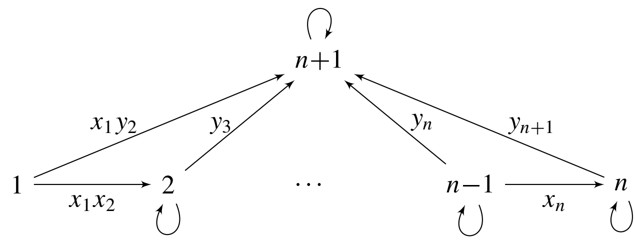
So far we have examined the types of polynomial growth possible for the Betti numbers of 2-pyramidal algebras. Another question that arises is: given a polynomial described in Corollary 4.3, can we produce a 2-pyramidal algebra whose Betti numbers follow this polynomial? Moreover, can we produce *all* algebras of this form that correspond to this polynomial?

We answer both of these questions in the affirmative. First, we need to define some notation. Let p be a polynomial. We then define $D_k(p)$ to be the k-th entry of the first row of the difference table for p. As before, we denote the columns starting at 0 and ending at n.

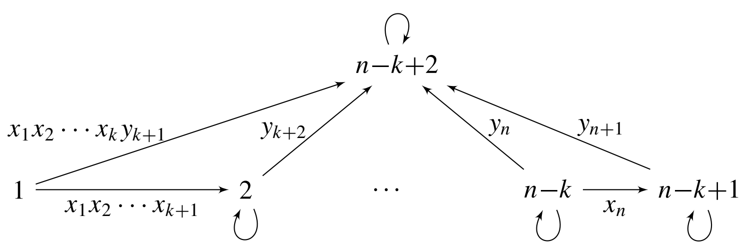
Theorem 5.1. *Let p be a polynomial of degree n such that $\beta_i(S_1) = p(i)$ for some 2-pyramidal algebra. Then*

$$D_i(p) = \begin{cases} x_1 + y_1 & \text{if } i = 0, \\ x_1 x_2 \cdots x_{i-1} x_i (x_{i+1} + y_{i+1}) & \text{if } 1 \le i \le n-1, \\ x_1 x_2 \cdots x_{i-1} x_i y_{i+1} & \text{if } i = n. \end{cases}$$

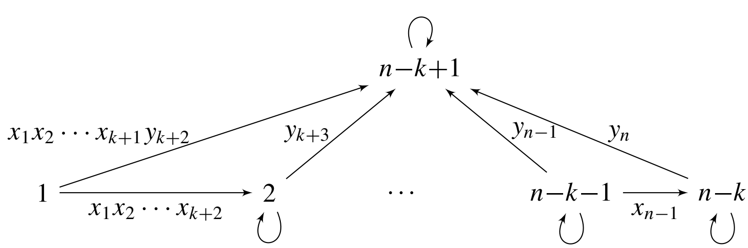
Proof. The first case is immediate. We prove the second case by induction on i by looking at the algebras associated with the differences of p. For $i = 1$, we know that the first difference of p gives the Betti numbers for the 2-pyramidal algebra



Hence, $D_1 = x_1 x_2 + x_1 y_2 = x_1(x_2 + y_2)$. For the induction step, we assume that the k-th difference of p produces the Betti numbers over Λ_k, shown below:



Then $D_k(p) = x_1 x_2 \cdots x_k (x_{k+1} + y_{k+1})$. By previous work, the first difference of the Betti numbers of the simple module S_1 over Λ_k are the Betti numbers of the simple module S_1 over



Note that this is also the algebra with the simple module S_1 whose Betti numbers are the $(k+1)$-th difference of p, and thus $D_{k+1} = x_1x_2\cdots x_kx_{k+1}(x_{k+2}+y_{k+2})$. This completes the induction for the second case.

For the last case, we know by the previous case that the $(n-1)$-th difference is given by the Betti numbers of the simple module S_1 over the 2-pyramidal algebra

$$\begin{tikzcd}
 & 3 \arrow[loop above] & \\
1 \arrow[ur, "x_1x_2\cdots x_{n-1}y_n"] \arrow[rr, "x_1x_2\cdots x_n"'] & & 2 \arrow[ul, "y_{n+1}"'] \arrow[loop below]
\end{tikzcd}$$

The difference of the Betti numbers of S_1 is given by the Betti numbers of the simple module S_1 over

$$\begin{tikzcd}
1 \arrow[r, "x_1x_2\cdots x_ny_{n+1}"'] & 2 \arrow[loop below]
\end{tikzcd}$$

which is clearly the constant $x_1x_2\cdots x_ny_{n+1}$. □

This theorem provides a way to determine restrictions on the x_i in order to produce a pyramidal algebra with a simple module S_1 whose Betti numbers follow a given polynomial. We now reformulate the previous theorem with added emphasis on the values of the x_i.

Corollary 5.2. *Let Λ be a 2-pyramidal algebra such that $\beta_i(S_1) = p(i)$ for some polynomial p. Then $x_1x_2\cdots x_k \mid D_k(p)$ and $x_k \le D_{k-1}(p)/(x_1x_2\cdots x_{k-1})$ for all $k \le n$.*

Theorem 5.3. *Let p be a polynomial of degree n and $x_1, x_2, \ldots, x_n$ be positive integers such that $x_1x_2\cdots x_k \mid D_k(p)$ and $x_k \le D_{k-1}(p)/(x_1x_2\cdots x_{k-1})$ for all $k \le n$. Then there exists a unique 2-pyramidal algebra such that $\beta_i(S_1) = p(i)$, and, for $1 \le k \le n$, the number of arrows between vertex k and vertex $k+1$ is x_k.*

Proof. We need only show that given these restrictions, we can choose the appropriate y_k such that $D_k(p)$ is the required value. For $k = 1$, simply choose $y_1 = D_1(p) - x_1$. Because $D_1(p)$ and x_1 are positive integers with $D_1(p) > x_1$, we know y_1 is a nonnegative integer as required.

Suppose that $2 \le k \le n-1$. Then choose $y_k = D_{k-1}(p)/(x_1x_2\cdots x_{k-1}) - x_k$. This value is a nonnegative integer by assumption.

Finally, choose $y_{n+1} = D_n/(x_1x_2\cdots x_{n-1}x_n)$ to ensure that we have the equality $x_1x_2\cdots x_{n-1}x_ny_{n+1} = D_n$.

At each step in this process, there is only one choice for the value of y_k. Thus the 2-pyramidal algebra exists and is unique. □

Given a polynomial p of degree n with $D_k(p) \in \mathbb{Z}$, this theorem allows us to construct a 2-pyramidal algebra with $\beta_i(S_1) = p(i)$. Simply choose the 2-pyramidal algebra on $n+2$ vertices with $x_k = 1$ and $y_k = D_{k-1}(p) - 1$ for all k. The existence and uniqueness of these algebras given the appropriate choice of $\{x_i\}_{i=1}^n$ also provides a method of finding the number of algebras of this form whose Betti numbers correspond to a given polynomial.

Corollary 5.4. *Let p be a polynomial of degree n. Then the number of 2-pyramidal algebras such that $\beta_i(S_1) = p(i)$ is equal to the number of n-tuples $\{(x_1, x_2, \ldots, x_n)\}$ such that $x_i \in \mathbb{Z}^+$ for all i and $x_1 x_2 \cdots x_k \mid D_k(p)$ for all k and $x_k \le D_{k-1}(p)/(x_1 x_2 \cdots x_{k-1})$ for all $k \le n$.*

6. Generalizing by changing the ideal

Up until now, we have been examining algebras with $\operatorname{rad}^2 \Lambda = 0$. We will now consider algebras with $\operatorname{rad}^m \Lambda = 0$ for $m > 2$ and provide results analogous to the $m = 2$ case.

We use the following notation throughout this section. Given an algebra Λ with $\operatorname{rad}^m \Lambda = 0$ for some $m > 2$, let Λ' be the algebra that has the same underlying quiver as Λ with the relations $\operatorname{rad}^2 \Lambda' = 0$. Denote by S_1' the simple Λ'-module at vertex 1, by $\beta_k(S_1')$ the i-th Betti number and by $\Omega^i(S_1')$ the i-th syzygy of the simple module S_1' over the algebra Λ'.

Lemma 6.1. *Let Λ be an m-pyramidal algebra with $m \ge 2$. Let*

$$Q : \cdots \to Q_2 \to Q_1 \to Q_0 \to S_1 \to 0$$

be a minimal projective resolution of S_1, and let

$$Q' : \cdots \to Q_2' \to Q_1' \to Q_0' \to S_1' \to 0$$

be a minimal projective resolution of S_1' over Λ'. Then the number of indecomposable projective summands of Q_i is equal to the number of projective summands of $Q'_{(i/2)m}$ if i is even, and $Q'_{((i-1)/2)m+1}$ if i is odd. Hence, the Betti numbers of the Λ-module S_1 are given by

$$\beta_i(S_1) = \begin{cases} \beta_{(i/2)m}(S_1') & i \text{ is even}, \\ \beta_{((i-1)/2)m+1}(S_1') & i \text{ is odd}. \end{cases}$$

Note that for $m = 2$, the number of indecomposable projective modules in Q_i and Q_i' are equal, and $\beta_i(S_1) = \beta_i(S_1')$ for all i.

Proof. The $m = 2$ case is trivial. Let $m > 2$ be fixed and let Λ be an m-pyramidal algebra. We construct a list representing simple modules as follows. For each walk of length j starting at vertex 1 in the underlying quiver of Λ, record the vertex at the end of the walk in row j of the list. We use the convention that the trivial walk from a vertex to itself along no edges is a walk of length 0, and the first written row, which is always a 1, is row 0.

For example, the 2-pyramidal algebra

$$\begin{tikzcd} 1 \arrow[loop above] \arrow[r] & 2 \arrow[loop above] \end{tikzcd}$$

generates the list

$$\begin{array}{c} 1 \\ 1\,2 \\ 1\,2\,2 \\ 1\,2\,2\,2 \\ \vdots \end{array}$$

Observe that the projective module appearing in step j of a minimal projective resolution of S_1' is precisely

$$\bigoplus_{k \in \text{row}\, j} P_k'.$$

With this in mind, we will prove the first statement by proving the following: for even i

$$Q_i = \bigoplus_{k \in \text{row}(i/2)m} P_k$$

and for odd i

$$Q_i = \bigoplus_{k \in \text{row}((i-1)/2)m+1} P_k.$$

We will prove this by induction on i. For $i = 0$ we have $Q_0 = P_1$. For $i = 1$, note that Q_1 is the projective cover of $\operatorname{rad} P_1$. This is equal to the projective cover of its top radical layer, which is precisely $\bigoplus_{k \in \text{row}\,1} S_k$, and this has projective cover $\bigoplus_{k \in \text{row}\,1} P_k$.

We examine the syzygies of Q. Note that for any projective Λ-module A,

$$\operatorname{soc} A \cong P(\operatorname{soc} A)/\operatorname{rad} P(\operatorname{soc} A),$$
$$\operatorname{rad} A \cong P(\operatorname{rad} A)/\operatorname{soc} P(\operatorname{rad} A).$$

We will show by induction that for even i

$$\Omega^i(Q) = \operatorname{soc} Q_{i-1}$$

and for odd i

$$\Omega^i(Q) = \operatorname{rad} Q_{i-1}.$$

For $i = 1$, we have

$$\Omega^1(Q) = \ker(Q_0 \to S_1) = \operatorname{rad} Q_0.$$

For $i = 2$

$$\Omega^2(Q) = \ker(Q_1 \to \operatorname{rad} Q_0) = \operatorname{soc} Q_1,$$

because $Q_1 = P(\operatorname{rad} Q_0)$ and

$$\operatorname{rad} Q_0 = P(\operatorname{rad} Q_0)/\operatorname{soc} P(\operatorname{rad} Q_0) = Q_1/\operatorname{soc} Q_1.$$

Assuming i is even and $\Omega^i(Q) = \operatorname{soc} Q_{i-1}$, we have

$$\begin{aligned}\Omega^{i+1}(Q) &= \ker(Q_i \to \Omega^i(Q))\\ &= \ker(Q_i \to \operatorname{soc} Q_{i-1})\\ &= \ker[P(\operatorname{soc} Q_{i-1}) \to P(\operatorname{soc} Q_{i-1})/\operatorname{rad} P(\operatorname{soc} Q_{i-1})]\\ &= \operatorname{rad} P(\operatorname{soc} Q_{i-1}) = \operatorname{rad} Q_i.\end{aligned}$$

Assuming i is odd and $\Omega^i(Q) = \operatorname{rad} Q_{i-1}$, we have

$$\begin{aligned}\Omega^{i+1}(Q) &= \ker(Q_i \to \Omega^i(Q))\\ &= \ker(Q_i \to \operatorname{rad} Q_{i-1})\\ &= \ker[P(\operatorname{rad} Q_{i-1}) \to P(\operatorname{rad} Q_{i-1})/\operatorname{soc} P(\operatorname{rad} Q_{i-1})]\\ &= \operatorname{soc} P(\operatorname{rad} Q_{i-1}) = \operatorname{soc} Q_i.\end{aligned}$$

We now return to the proof of the structure of the Q_i. Assume i is even. Then

$$Q_i = P(\Omega^i(Q)) = P(\operatorname{soc} Q_{i-1}).$$

By hypothesis,

$$\operatorname{soc} Q_{i-1} = \operatorname{soc} \bigoplus_{k\in \operatorname{row}((i-2)/2)m+1} P_k = \bigoplus_{k\in \operatorname{row}(i/2)m} S_k.$$

Because Q_i is the projective cover of $\operatorname{soc} Q_{i-1}$, it follows that

$$Q_i \cong \bigoplus_{k\in \operatorname{row}(i/2)m} P_k.$$

Assuming i is odd, we have

$$Q_i = P(\Omega^i(Q)) = P(\operatorname{rad} Q_{i-1}).$$

By hypothesis,

$$\operatorname{rad} Q_{i-1} = \operatorname{rad} \bigoplus_{k\in \operatorname{row}((i-1)/2)m} P_k.$$

Now Q_i is the projective cover of $\operatorname{rad} Q_{i-1}$, so it is the projective cover of $\operatorname{rad} Q_{i-1}/\operatorname{rad}^2 Q_{i-1}$ as well. Because the radical quotient of $\operatorname{rad} \bigoplus_{k\in \operatorname{row}((i-1)/2)m} P_k$

is $\bigoplus_{k\in\mathrm{row}((i-1)/2)m+1} S_k$, it follows that

$$Q_i \cong \bigoplus_{k\in\mathrm{row}((i-1)/2)m+1} P_k. \qquad \square$$

The next theorem gives us asymptotic information about the Betti numbers for m-pyramidal algebras for $m \geq 3$. We will be using the following notation.

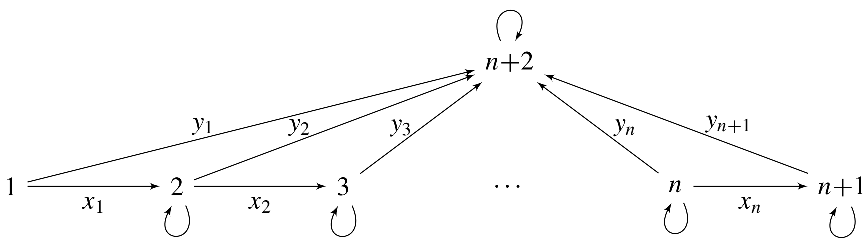
Definition 6.2. For a function $f(x)$, we write $f(x) = \Theta(g)$ if there exist positive constants M and N, $M \leq N$ and a real number x_0 such that

$$Mg(x) \leq f(x) \leq Ng(x)$$

for all $x \geq x_0$.

Theorem 6.3. *For all $m \geq 3$ and $n \geq 1$, there exists an m-pyramidal algebra such that $\beta_i(S_1) = \Theta(i^n)$.*

Proof. Let n be a fixed positive integer. Let Λ be the algebra



with $\mathrm{rad}^m \Lambda = 0$. It suffices to show that $\beta_i(S_1)$ is bounded above and below by polynomials of degree n. Using Lemma 6.1 and the fact that the Betti numbers are strictly increasing for all $m \geq 2$, we obtain the inequalities

$$\beta_i(S_1') \leq \beta_i(S_1) \leq \beta_{mi}(S_1').$$

By previous work, $\beta_i(S_1') = p(i)$ and $\beta_{mi}(S_1') = p(mi)$, where p is a polynomial of degree n. Because both $p(i)$ and $p(mi)$ are polynomials in i of degree n, we have $\beta_i(S_1) = \Theta(i^n)$. $\square$

Future work

This work prompts some natural questions. We currently have a class of algebras whose quotients have Betti numbers asymptotic to polynomials of arbitrarily high degree. When does there exist a path algebra Λ such that, for some $m \geq 3$, the quotient $\Lambda/\mathrm{rad}^m \Lambda$ has a simple module whose Betti numbers follow a polynomial exactly, not just asymptotically? Based on the proof of Lemma 6.1, it seems unlikely that there exists an algebra that satisfies this property for multiple m, but perhaps there exists such a path algebra for each m.

We showed that for polynomials of a certain type, we can construct an algebra whose Betti numbers at the simple module at vertex 1 satisfy that polynomial. However, the description of the number of such 2-pyramidal algebras, offered in Corollary 5.4, is complex. Perhaps there is a simpler description of the number of these algebras.

The Betti numbers of simple modules for a 2-pyramidal algebras are different at each vertex. A natural question is whether there exists an algebra where one of its quotients has the same polynomial Betti numbers at all of its simple modules. We can produce an algebra in which two simple modules have the same syzygies: starting with a 2-pyramidal path algebra, add a copy of vertex 1 called $\tilde{1}$, copy all of its arrows, and consider the new algebra modulo its radical squared. Then S_1 and $S_{\tilde{1}}$ have the same syzygies, and by repeating this process we can produce an algebra with arbitrarily many such simple modules. However, this process does not create a path algebra in which *all* simple modules have the same Betti numbers.

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