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A characterization of the sets of periods within shifts of finite type

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We characterize precisely the possible sets of periods and least periods for the periodic points of a shift of finite type (SFT). We prove that a set is the set of least periods of some mixing SFT if and only if it is either $\{1\}$ or cofinite, and the set of periods of some mixing SFT if and only if it is cofinite and closed under multiplication by arbitrary natural numbers. We then use these results to derive similar characterizations for the class of irreducible SFTs and the class of all SFTs. Specifically, a set is the set of (least) periods for some irreducible SFT if and only if it can be written as a natural number times the set of (least) periods for some mixing SFT, and a set is the set of (least) periods for an SFT if and only if it can be written as the finite union of the sets of (least) periods for some irreducible SFTs. Finally, we prove that the possible sets of (least) periods of mixing sofic shifts are exactly the same as for mixing SFTs, and that the same is not true for the class of nonmixing sofic shifts.

1. Introduction

Modern dynamical systems theory has a relatively short history, though scientists from many disciplines have begun to use nonlinear dynamics techniques to describe problems ranging from physics and chemistry to ecology and economics. Fundamentally, a dynamical system is a set or space with structure, usually denoted by X, partnered with a function or map, usually denoted by f, that preserves that structure through repeated iterations. This function f can then be applied arbitrarily many times to subsets or elements of X, which incites certain possible patterns. One of the simplest is when a point returns to itself after some number (say n) of iterations of f; such a point is said to be periodic with period n. Different points of the system can have different periods, and so a simple natural object of study is the set of periods of points of a given dynamical system. The celebrated Sharkovsky's

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theorem gives some surprising information about this set of periods for dynamical systems given by continuous self-maps of intervals.

Sharkovsky's theorem [1964]. *For any interval I in $\mathbb{R}$, if $f : I \to I$ is continuous and has a point of least period k, then there exist points of all least periods less than k in the Sharkovsky ordering, where the ordering is as follows*:

$$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec 2\cdot 9 \prec 2\cdot 11 \prec \cdots \\ \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec 2^2\cdot 7 \prec 2^2\cdot 9 \prec 2^2\cdot 11 \prec \cdots \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

In particular, Sharkovsky's theorem implies that for any such f, the set of natural numbers which are least periods of periodic points for f is a downward closed set with respect to the Sharkovsky ordering. In fact, examples are also constructed in [Sharkovsky 1964] which, given any such (nonempty) downward closed set, yield an f which realizes that set as the least periods of periodic points. This then yields a complete characterization of which sets can appear as the sets of least periods for such f. The goal of the present work is to obtain such a characterization for a completely different class of dynamical systems, called the shifts of finite type. In the process, we also prove some results about the more general class of so-called sofic shifts.

Here we step into the realm of symbolic dynamics. For symbolic dynamical systems, one begins with a finite set of symbols called the alphabet, denoted by A. Elements of A are called letters and can be combined to form "words" or "blocks". A symbolic dynamical system, or shift space, is a subset of all possible biinfinite sequences created with the alphabet A based on a collection of "forbidden blocks" $\mathcal{F}$, essentially rules on what words or symbols can and cannot appear in these biinfinite sequences. For shift spaces, the dynamics are always given by the shift map σ, which shifts a sequence in the space one unit to the left. A shift space described by a finite set of forbidden blocks is called a shift of finite type (SFT). An example of an SFT would be where X is the set of all binary sequences with no two 1s next to each other, induced by $\mathcal{F} = \{11\}$. This is known as the golden mean shift, because its so-called topological entropy is equal to the logarithm of the golden mean. Because they have a simple representation using a finite, directed graph (see Section 3), SFTs are attractive to study, as questions about the SFT can typically be phrased as questions about the graph which can be translated back to the original shift.

A periodic point of a shift space is just a biinfinite sequence made only of a word w of length p repeated biinfinitely with no additional words, which is then said to have period p. In this work, we study periodic points in SFTs as, though they are in some sense the "simplest" shifts, they play an important role in dynamical systems by facilitating the study of more complex systems. We prove a characterization for shifts of finite type analogous to Sharkovsky's theorem, along with a corresponding characterization for the sets of (not necessarily least) periods for shifts of finite type. Unlike the $f : I \to I$ case above, our characterizations do not come from any ordering of $\mathbb{N}$, but rather from structural properties of the sets. Our main results are the following.

Theorem 1.1. *A set S is closed under $\mathbb{N}$-multiples and cofinite if and only if there exists a topologically mixing SFT such that S is the set of periods of its periodic points.*

Theorem 1.2. *A set R can be written as $p \cdot S$, where $p \in \mathbb{N}$ and S is a cofinite set which is closed under $\mathbb{N}$-multiples, if and only if there exists an irreducible SFT such that R is the set of periods of its periodic points.*

Theorem 1.3. *A set Q can be written as $\bigcup_{i=1}^{n} p_i \cdot S_i$ for some $p_i \in \mathbb{N}$ and cofinite sets S_i which are closed under $\mathbb{N}$-multiples if and only if there exists an SFT such that Q is the set of periods of its periodic points.*

Theorem 1.4. *A set S is either $\{1\}$ or cofinite if and only if there exists a topologically mixing SFT such that S is the set of least periods of its periodic points.*

Theorem 1.5. *A set R is either a singleton or can be written as $p \cdot S$, where $p \in \mathbb{N}$ and S is a cofinite set, if and only if there exists an irreducible SFT such that R is the set of least periods of its periodic points.*

Theorem 1.6. *A set Q can be written as $U \cup \bigcup_{i=1}^{n} p_i \cdot S_i$ for some finite set U, $p_i \in \mathbb{N}$, and cofinite sets S_i, if and only if there exists an SFT such that Q is the set of least periods of its periodic points.*

In addition to the relation to Sharkovsky's theorem already outlined, these results also connect to other characterizations of various other important objects for SFTs, most notably topological entropy [Lind 1983] and the Artin–Mazur zeta function [Kim et al. 2000]. The latter is most relevant to our work, due to the connection of the zeta function to periodic points. The zeta function is a formal power series defined by

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} p_n \frac{z^n}{n}\right),$$

where p_n is the number of points of period n in the system. For SFTs, the zeta function always has the form $1/p(z)$ for some polynomial p, see [Bowen and Lanford 1970], and the classification from [Kim et al. 2000] is in terms of these $p(z)$, more specifically in terms of the sets of nonzero complex numbers (with multiplicity) which can be realized as the roots of such $p(z)$. Relevant for this work is the fact that knowledge of the zeta function is theoretically equivalent to knowledge of p_n for all n, and the set of periods is the set of exponents with positive coefficients. Therefore, theoretically speaking, the classification from [Kim et al. 2000] contains enough information to derive a classification of the sets of periods for SFTs. However, practically speaking, it is not at all simple to turn information about roots of $p(z)$ into information about the set of exponents with positive coefficients for the power series expansion of $1/p(z)$.

Finally, we note that the possible sets for a generalized notion of least periods for multidimensional SFTs (which consist of $\mathbb{Z}^d$-indexed arrays of letters rather than sequences) were recently characterized in [Jeandel and Vanier 2010]. As is often the case for multidimensional SFTs, their characterization is in recursion-theoretic terms and much more complicated than the ones we derive in one dimension. It is noted in [Jeandel and Vanier 2010] that the set of least periods for a (one-dimensional) SFT must be semilinear, i.e., a finite union of sets of the form $\left\{a_0 + \sum_{i=1}^{k} n_i a_i : n_i \in \mathbb{N}\right\}$ for fixed $a_i \in \mathbb{N}$. However, as our results show, not all semilinear sets are realizable in this way; for instance, the set of positive odd integers is semilinear and yet is not the set of least periods of any (one-dimensional) SFT. It is strange that the much more complicated and difficult results of [Jeandel and Vanier 2010] appeared even though the one-dimensional characterization does not seem to be present anywhere in the literature; we hope that our results fill this gap.

In addition to the previous theorems, our results yield a characterization of the sets of (least) periods for mixing sofic shifts. Sofic shifts are the shift spaces which are so-called factors of SFTs, i.e., images (of SFTs) under continuous shift-commuting maps, and are a significantly larger class. They can be alternatively defined using labeled graphs; see Definitions 24 and 25.

Theorem 1.7. *A set S is closed under $\mathbb{N}$-multiples and cofinite if and only if there exists a topologically mixing sofic shift such that S is the set of periods of its periodic points.*

Theorem 1.8. *A set S is either* $\{1\}$ *or cofinite if and only if there exists a topologically mixing sofic shift such that S is the set of least periods of its periodic points.*

The problem of finding similar characterizations for irreducible and general sofic shifts is quite interesting, but seems to be more difficult; in particular, it is not true that the same characterizations as in Theorems 1.2, 1.3, 1.5, and 1.6 hold. We comment further on this in Section 6.

2. Definitions

Definition 1. A *topological dynamical system* is a pair (X, T) where X is a compact metric space and $T : X \to X$ is a continuous map.

Definition 2. For any finite set of symbols A (which we call an alphabet), the *full A-shift* is the collection $A^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} : x_i \in A \text{ for all } i \in \mathbb{Z}\}$ of all biinfinite sequences of symbols from A.

Definition 3. A *word* on the alphabet A is a finite sequence of elements of A.

Definition 4. The *shift map* σ on the full shift $A^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate y_i is x_{i+1}, the $(i+1)$-th coordinate of x.

Definition 5. A point x is *periodic* for σ if $\sigma^n(x) = x$ for some $n \geq 1$. The point x is then said to have period n under σ.

Definition 6. For a point x that is periodic, the smallest positive integer n for which $\sigma^n(x) = x$ is the *least period* of x.

Definition 7. Let (X, T) and (Y, S) be two topological dynamical systems. The systems (X, T) and (Y, S) are *conjugate* if there exists between them a homeomorphism $h : X \to Y$ such that $h(T(x)) = S(h(x))$ for all $x \in X$.

We note that if (X, T) and (Y, S) are conjugate via $h : X \to Y$, then $T^n x = x$ if and only if $S^n h(x) = h(x)$, and so conjugacy preserves the number of points of (least) period n in any dynamical system.

Definition 8. A *shift space* is a subset X of a full shift $A^{\mathbb{Z}}$ such that, for some collection $\mathcal{F}$ of forbidden blocks over A, the subset X is equal to $X(\mathcal{F})$, the set of all possible biinfinite sequences that do not contain any blocks from $\mathcal{F}$.

Whenever X is a shift space, (X, σ) is a topological dynamical system when X is given the induced product topology from $A^{\mathbb{Z}}$.

Definition 9. A *shift of finite type* (or SFT) is a shift space X that is equal to $X(\mathcal{F})$ for some finite collection $\mathcal{F}$ of forbidden blocks.

Definition 10. A *sofic shift* is any shift space which is the image of an SFT under a continuous map which commutes with the shift σ.

Definition 11. The *language* is the set of all possible blocks of length $n \in \mathbb{N}$ of a shift space X, denoted by $B(X)$.

Definition 12. A shift space X is *irreducible* if for every ordered pair of blocks $u, v \in B(X)$ there exists $w \in B(X)$ such that $uwv \in B(X)$.

Definition 13. A shift space X is *mixing* if, for every ordered pair $u, v \in B(X)$, there is an N such that for each $n \geq N$ there exists $w \in B_n(X)$ such that $uwv \in B(X)$.

Definition 14. A *graph* G consists of a finite set $V = V(G)$ of vertices (or states) together with a finite set $E = E(G)$ of edges. All of the graphs discussed in this paper are directed graphs, meaning that each edge points from one vertex, called the initial vertex, to another, called the terminal vertex.

Definition 15. A *path* in a graph G is a finite sequence of edges such that the terminal vertex of each is the initial vertex of the next.

Definition 16. A *cycle* is a path which begins and ends at the same vertex. A cycle is *simple* if each vertex along it is visited exactly once.

Definition 17. A cycle C of an arbitrary graph G is called *nonelementary* if C is composed of a single smaller cycle followed two or more times. A cycle is called *elementary* if it is not nonelementary.

Definition 18. A graph G is *irreducible* if for every ordered pair of vertices I and J there is a path in G starting at I and terminating at J.

Definition 19. Given a graph G, the *period of* G, denoted $\mathrm{per}(G)$, is the greatest common divisor of its cycle lengths.

Definition 20. A graph is *aperiodic* if $\mathrm{per}(G) = 1$.

Definition 21. A graph is *primitive* if it is irreducible and aperiodic.

Definition 22. A set S is *closed under* $\mathbb{N}$*-multiples* if for all $n \in S$, mn is also in S for all $m \in \mathbb{N}$.

3. Preliminaries

The following theorems, definitions, and descriptions are used extensively in the proofs of our results. We will see that any SFT can be studied by way of an associated graph, every graph can be broken down into primitive pieces, and from these primitive graphs we can build our results.

Definition 23. For an arbitrary graph G with set of edges $E(G)$, the *edge shift* χ_G is the shift space over the alphabet $A = E(G)$ consisting of all biinfinite sequences of edges which are connected end-to-end in G.

By Proposition 2.2.6 from [Lind and Marcus 1995], for any graph G, the associated edge shift χ_G is an SFT. Surprisingly, every SFT can also be depicted as a graph.

Proposition 3.1. *For any SFT X, there exists a graph G such that X is conjugate to the edge shift χ_G. In addition, if X is irreducible, then G can be taken to be irreducible, and if X is mixing, then G can be taken to be primitive.*

Proof. The first sentence follows from Theorem 2.3.2 in [Lind and Marcus 1995]. The reader may check that the remaining statements hold for the construction done there. □

This then allows a connection to be made between the periodic points of an SFT and the cycle lengths of its associated graph G:

Proposition 3.2. *For any SFT X, there exists a graph G such that for all $p \in \mathbb{Z}$, the number of points of (least) period p in X equals the number of (elementary) cycles of length p in G.*

Proposition 3.3. *For any irreducible SFT X, there exists an irreducible graph G such that for all $p \in \mathbb{Z}$, the number of points of (least) period p in X equals the number of (elementary) cycles of length p in G.*

Proposition 3.4. *For any topologically mixing SFT X, there exists a primitive graph G such that for all $p \in \mathbb{Z}$, the number of points of (least) period p in X equals the number of (elementary) cycles of length p in G.*

Proof of Propositions 3.2–3.4. By Proposition 3.1, for any SFT X we can find a graph G such that $X \cong \chi_G$, where G is irreducible if X is irreducible and G is primitive if X is mixing. Then by Proposition 2.2.12 from [Lind and Marcus 1995], the number of cycles of length m in G is equal to the number of points in the edge shift χ_G with period m. (To prove this, Lind and Marcus use an object called the adjacency matrix, but we will not need this object in our work.) X is conjugate to χ_G, and as we defined, conjugacy preserves periodic points, completing the proof. □

Thus, in light of Propositions 3.1–3.4, the following six theorems are equivalent to Theorems 1.1–1.6:

Theorem 3.5. *A set S is closed under $\mathbb{N}$-multiples and cofinite if and only if there exists a primitive graph G such that S is the set of cycle lengths in G.*

Theorem 3.6. *A set R can be written as $p \cdot S$, where $p \in \mathbb{N}$ and S is a cofinite set which is closed under $\mathbb{N}$-multiples, if and only if there exists an irreducible graph H such that R is the set of cycle lengths in H.*

Theorem 3.7. *A set Q can be written as $\bigcup_{i=1}^{n} p_i \cdot S_i$ for some $p_i \in \mathbb{N}$ and cofinite sets S_i which are closed under $\mathbb{N}$-multiples if and only if there exists a graph F such that Q is the set of cycle lengths in F.*

Theorem 3.8. *A set S is either $\{1\}$ or cofinite if and only if there exists a primitive graph G such that S is the set of elementary cycle lengths in G.*

Theorem 3.9. *A set R is either a singleton or can be written as $p \cdot S$, where $p \in \mathbb{N}$ and S is a cofinite set, if and only if there exists an irreducible graph H such that R is the set of elementary cycle lengths in H.*

Theorem 3.10. *A set Q can be written as $U \cup \bigcup_{i=1}^{n} p_i \cdot S_i$ for some finite set U, $p_i \in \mathbb{N}$, and cofinite sets S_i, if and only if there exists a graph F such that Q is the set of elementary cycle lengths in F.*

Now, the graphs themselves can be decomposed into irreducible and primitive components, the consequences of which will be used extensively in the proofs of our results.

Proposition 3.11. *For every graph G, there exist irreducible subgraphs G_1, G_2, $\ldots$, G_k such that the set of (elementary) cycles that appear in G is the disjoint union of the sets of (elementary) cycles that appear in the G_i.*

Proof. Begin by separating G into communicating classes $C_i \subset V(G)$ defined by the collections of vertices such that, for each pair of vertices I and J within a collection, there exists a path I to J and J to I. Let G_i be the subgraph $G|_{C_i}$. We claim that no cycles of G can contain vertices from two different communicating classes. To see this, let C, B be two communicating classes. Then suppose for a contradiction there exists an edge connecting a vertex of C to a vertex of B and another edge connecting a vertex of B to a vertex of C. This would create a larger communicating class, which is a contradiction by definition. By again considering the definition of communicating classes, we see the G_i are irreducible. It is clear that every cycle of G is part of some G_i; thus the set of cycles that appear in G is the disjoint union of the sets of cycles that appear in the G_i. □

Proposition 3.12. *Any irreducible graph G has an associated primitive graph G' for which the set of* (*elementary*) *cycles of G is $p \cdot S$, where $p = \operatorname{per}(G)$ and S is the set of* (*elementary*) *cycle lengths of G'.*

Proof. Let G be irreducible. By Proposition 4.5.6 from [Lind and Marcus 1995], $V(G)$ can be grouped into exactly p period classes which can be ordered as $D_0, D_1, \ldots, D_{p-1}$ so that every edge that starts in D_i terminates in D_{i+1} (or in D_0 if $i = p-1$). The comment following Proposition 4.5.6 states that there is an associated graph G^p, called the higher power graph, that consists of p primitive (aperiodic and irreducible), disjoint subgraphs $G_1, \ldots, G_p$. In Exercise 4.5.6 from [Lind and Marcus 1995], it is shown that the edge shifts χ_{G_i} associated to each G_i are conjugate to each other, and therefore contain the same numbers of points with (least) period n for every n. Then by Propositions 3.1 and 3.2, the G_i all contain the same (elementary) cycle lengths.

Since all G_i have the same (elementary) cycle lengths, we consider any G_i. By definition of the higher power graph (not given here), the set of (elementary) cycle lengths in G is p-times the set of (elementary) cycle lengths in G_i; thus there exists a cycle in G_i of length k if and only if there exists a cycle in G of length pk. □

Any SFT X can therefore be represented by an associated graph G. Through use of the irreducible components and higher power graph, this G can be reduced to a primitive graph which is far simpler to work with; this fact will be useful in the proofs of the following results.

Remark. We should remark that those directions of our results which guarantee cofiniteness of sets of (least) periods for mixing SFTs can be alternately proven by using some more advanced results involving so-called topological entropy. Extremely roughly speaking, entropy measures the exponential growth rate of the number of words with n-letters in an SFT/paths with n edges in a graph. It is well known, see Theorem 4.3.6 and Corollary 4.3.8 in [Lind and Marcus 1995], that in a primitive graph, the number of cycles/elementary cycles of length n grow exponentially with the same growth rate, and so in particular that this number is eventually positive, yielding the desired cofiniteness. We elected here to instead give direct proofs in order to avoid introducing entropy and keep proofs at an elementary level.

4. Results on general cycle lengths

4.1. ***Primitive graphs.*** The proof of Theorem 3.5, which as noted in Section 3 is equivalent to Theorem 1.1, now follows. Most of the work is devoted to, for any set S which is closed under $\mathbb{N}$-multiples and is cofinite, constructing a graph G whose set of cycle lengths is S. Interestingly, even when S is not necessarily closed under $\mathbb{N}$-multiples, the set of elementary cycle lengths of G will still be S; this will be addressed in the later proof of Theorem 3.8.

Proof of Theorem 3.5. "$\Rightarrow$" Let S be a set that is closed under $\mathbb{N}$-multiples and is cofinite. We will now construct a graph G with the set S as the set of cycle lengths of G. Since S is cofinite, there exists $N \in S$ such that for all $n \geq N$, we have $n \in S$. Build a simple cycle of length N. For all $s \in S$ such that $s < N$, build a simple cycle of length s such that it shares a vertex with the N-cycle, but does not share a vertex with any other cycle of length less than N. This is possible as the N-cycle has N vertices and there exist at most $N-1$ elements $s \in S$ such that $s < N$. Call these cycles of length less than N the "small cycles". Then, on the smallest cycle k, build simple cycles of length $N+i$, with $i \in \{1, \ldots, k-1\}$, each sharing a unique vertex with the k-cycle. Call these cycles of length at least N the "large cycles". Call the resulting graph G. (See Figure 1.)

Every cycle shares a vertex with either the N-cycle or the k-cycle, the N-cycle and the k-cycle themselves sharing a vertex. Thus, there exists a path between vertices in any two cycles, and so G is irreducible. The gcd of the lengths of cycles in G is 1 since G contains cycles of lengths N and $N+1$. Thus G is aperiodic. Therefore, G is primitive.

Let P be the set of cycle lengths of G. Then let s be any element of S. By construction, for all $s \in S$ such that $s \leq N$, there exist cycles of length s. Thus if $s \leq N$, the s-cycle already exists within G by construction. Else, $s > N$. Then there exists i, $0 \leq i \leq k-1$, such that $s \equiv N+i \pmod k$ since the $N+i$ cover all k residue classes. Thus $s = N+i+mk$. If $m < 0$, then $s < N+i$ and because $i \in \{1, \ldots, k-1\}$, we have $s < N$. This violates the assumption of $s > N$; hence $m \geq 0$. Thus if $s > N$, the s-cycle can be achieved by going around the $(N+i)$-cycle once and the k-cycle m times. Therefore, $s \in P$ and since $s \in S$ was arbitrary, $S \subseteq P$.

Let c be any element of P, where C is an associated cycle of G with length c.

There are two cases we consider:

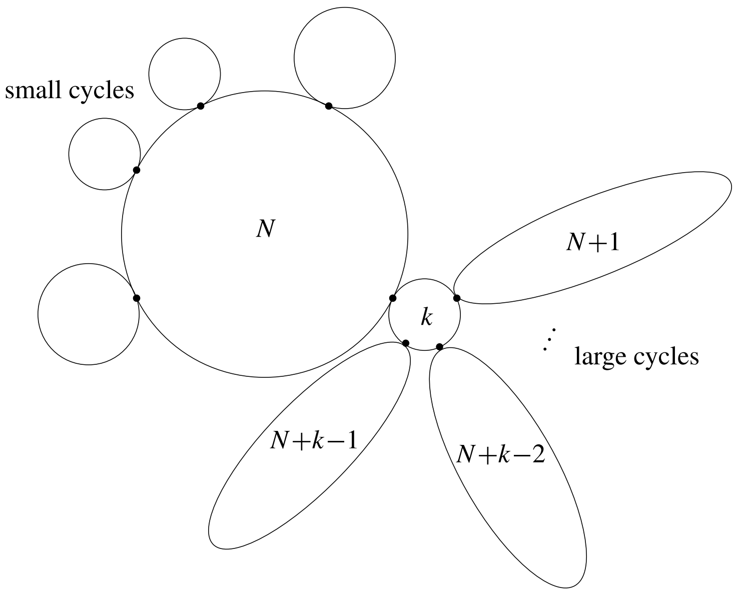

Figure 1. The graph G. All labels refer to lengths of the corresponding cycles. The lengths of small cycles are all elements of $S \cap \{1, \ldots, N-1\}$.

(1) *The cycle C contains at least one edge from a large cycle.* Then by construction C must contain the entire large cycle. If C contains even one large cycle, $c \geq N$ and thus $c \in S$ since S contains all integers greater than or equal to N.

(2) *The cycle C contains no edges from any large cycle.* Then C must be made up entirely of small cycles. By construction no small cycles share a vertex; thus C is a cycle of length $c = ms$, where $m \in \mathbb{N}$ and $s \in S$. Hence $c \in S$ as S is closed under $\mathbb{N}$-multiples.

Thus, since C was arbitrary, $P \subseteq S$, and therefore S represents the set of cycles of G.

"$\Leftarrow$" First, it is clear by definition that the set of cycle lengths of any graph is closed under $\mathbb{N}$-multiples. It will be shown in the proof of Theorem 3.8 that the set of elementary cycle lengths of a primitive graph is cofinite, which trivially implies the same of the even larger set of cycle lengths. □

4.2. ***Irreducible graphs.***

Proof of Theorem 3.6. "$\Rightarrow$" Let $R = p \cdot S$, where $p \in \mathbb{N}$ and S is a set that is closed under $\mathbb{N}$-multiples and is cofinite. By Theorem 3.5, there exists a primitive graph G such that the set of cycle lengths of G is S. Take this graph G and, for every directed edge between two vertices I and J in G, create a path of p directed edges and $p-1$ vertices beginning at I and ending at J; all such sets of newly created vertices are disjoint. Call the new graph G_p.

Each cycle length of G has been multiplied by p in G_p; thus $\mathrm{per}(G_p) = p$ since $\mathrm{per}(G) = 1$ as G is primitive. Then take I and J, two vertices of G_p. There are three cases:

(1) *Both exist in G.* Then there exists a path in G starting at I and terminating at J. Such a path then also exists in G_p, but its length has been multiplied by a factor of p.

(2) *One vertex exists in G.* Assume I exists in G and J exists only in G_p. By construction, there exist $p-1$ directed edges forming a path starting at a vertex V existing in G and terminating at J. By case (1) there exists a path in G_p starting in I and terminating in V. Then V is at most $p-1$ directed edges away from J; thus there is a path from I to J consisting of the path I to V then V to J. The case where I exists only in G_p and J exists in G is similar.

(3) *Both exist only in G_p.* Then by construction there exist at most $p-1$ directed edges forming a path starting at I and terminating at a vertex existing in G, call it A, and there exist at most $p-1$ directed edges forming a path starting at a vertex existing in G, call it B, and terminating at J. By case (1) there exists a path starting at A and terminating at B; thus there exists a path from I to J.

In each case, for any two vertices I and J there exists a path in G_p starting at I and terminating at J. Therefore G_p is irreducible.

"$\Leftarrow$" Let H be an irreducible graph with period p. By Proposition 3.12, H can be associated to a primitive graph G with set of cycle lengths T so that the set of cycle lengths of H is $p \cdot T$ where, by Theorem 3.5, T is a set that is closed under $\mathbb{N}$-multiples and is cofinite. □

4.3. *Arbitrary graphs.*

Proof of Theorem 3.7. "$\Rightarrow$" Let $Q = \bigcup_{i=1}^{n} R_i$, with $R_i = p_i \cdot S_i$, where $p_i \in \mathbb{N}$ and S_i is closed under $\mathbb{N}$-multiples and is cofinite. By Theorem 3.6, for each $i \in \{1, \dots, n\}$, there exists an irreducible graph G_i such that each G_i has set of cycle lengths R_i. Place them together, with no edges connecting any G_i to any other, and call the resulting graph G. As there do not exist any edges connecting vertices from different G_i, the cycles of G are only the cycles of the individual G_i; thus the set of cycles of G is $Q = \bigcup_{i=1}^{n} R_i$.

"$\Leftarrow$" Let F be an arbitrary graph. By Proposition 3.11, F can be broken down into irreducible subgraphs F_i for $1 \le i \le n$. By Theorem 3.6, the set of cycle lengths of each F_i can be written as $R_i = p_i \cdot T_i$, where T_i is a set that is closed under $\mathbb{N}$-multiples and is cofinite and p_i is the period of F_i. Hence the set of cycle lengths of F is $\bigcup_{i=1}^{n} R_i$. □

5. Results on elementary cycle lengths

5.1. *Primitive graphs.*

Proof of Theorem 3.8. "$\Rightarrow$" Let S' be either $\{1\}$ or cofinite. If S' is $\{1\}$, then we can create the primitive graph consisting of a single vertex with a self-loop; this graph clearly has only one elementary cycle, with length 1. We then assume S' is a cofinite set. By the same construction found in the proof of Theorem 3.5, use S' to construct a primitive graph G'. Let P' be the set of elementary cycle lengths of G'. Note that, since $S \subset P$ in Theorem 3.5 and all cycles created are elementary cycles, $S' \subset P'$.

Then let c' be an element of P', where C' is an elementary cycle of G' of length c'. The analysis from the proof of the forward direction of Theorem 3.5 can be repeated with only one change: S was closed under $\mathbb{N}$-multiples, which may not be true of S'. However, the only place this was used was to treat cycles composed of multiple traversals of a small cycle, which C' cannot be since it is elementary. Therefore, as before, $P' \subseteq S'$.

"$\Leftarrow$" Let G be a primitive graph. By definition, as G is primitive, G is irreducible and aperiodic. First, we consider the case where G contains only a single elementary cycle. Since G is aperiodic, this must be a self-loop and G must be the graph consisting of a single vertex v with a self-loop. Thus, in this case, the set of elementary cycle lengths is $\{1\}$. We can then assume that for the remainder of the proof, G contains multiple elementary cycles.

By Lemma 4.5.6 from [Lind and Marcus 1995], since G is irreducible, for an arbitrary $v \in V(G)$, the gcd of all lengths of cycles starting and ending at v is $\text{per}(G)$. Since $\text{per}(G) = 1$, there exist cycles $D_1, D_2, \ldots, D_l$ beginning and ending at v such that $\gcd(|D_j|) = 1$.

If a cycle D_i visits v more than only at the beginning and end, it can be written as a concatenation of cycles $C_i^1, C_i^2, \ldots, C_i^m$, which each visit v only at the beginning and end. Then, $|D_i| = |C_i^1| + \cdots + |C_i^m|$, and we claim that

$$\gcd(|C_i^1|, \ldots, |C_i^m|, |D_1|, \ldots, |D_{i-1}|, |D_{i+1}|, \ldots, |D_l|) = 1.$$

Assume for a contradiction that

$$\gcd(|C_i^1|, \ldots, |C_i^m|, |D_1|, \ldots, |D_{i-1}|, |D_{i+1}|, \ldots, |D_l|) \neq 1.$$

Then there exists a factor q such that

$$\gcd(|C_i^1|, \ldots, |C_i^m|, |D_1|, \ldots, |D_{i-1}|, |D_{i+1}|, \ldots, |D_l|) = q.$$

Since $|C_i^1| + \cdots + |C_i^m| = |D_i|$ and q is a factor of all the $|C_i^j|$, we know q is a factor of $|D_i|$. Thus $\gcd(|D_j|) = q$, but this is a contradiction since $\gcd(|D_j|) = 1$.

Continuing in this way, we have shown that we can decompose $D_1, \dots, D_l$ into cycles $C_1, \dots, C_k$ which only contain v at their beginning and end, and whose lengths still have a gcd of 1. In addition, we can assume without loss of generality that $|C_1| < |C_2| < \cdots < |C_k|$ by removing cycles with repeated lengths and reordering.

We then break into two cases: either $|C_1| = 1$ or $|C_1| > 1$. Assume first that $|C_1| > 1$, which implies that $k > 1$, and define

$$S_1 = \{n_1|C_1| + \cdots + n_k|C_k| : n_i \geq 0\}.$$

It is well known that S_1 is cofinite in $\mathbb{N}$ and so there exists N such that for all $n \geq N$, $n \in S_1$. Let

$$S_2 = \{n_1|C_1| + \cdots + n_k|C_k| : \text{ there exist } j, j' \text{ such that } n_j, n_{j'} > 0\};$$

we will show that S_2 is also cofinite. Choose an n bigger than N and all possible $|C_i||C_j|$ for $1 \leq i, j \leq k$. Then there exists $n_i \geq 0$ such that $n = n_1|C_1| + \cdots + n_k|C_k|$. We then break into subcases:

(1) If at least two of the n_i are positive, $n \in S_2$ by definition.

(2) Else, since $k > 1$, there exists i such that for all $j \neq i$, we have $n_j = 0$. Then choose any $j \neq i$. Thus we have $n = |C_j||C_i| + |C_i|(n_i - |C_j|)$. Recall $n > |C_i||C_j|$. As $n = n_i|C_i|$, this implies $n_i > |C_j|$. Thus $n_i - |C_j| > 0$. Then $n_j = |C_i|$ and $n \in S_2$.

In either case, $n \in S_2$; therefore S_2 is cofinite.

For all $n \in S_2$, where $n = n_1|C_1| + \cdots + n_k|C_k|$, we now construct a cycle of length n by beginning at v and following C_1 n_1-many times, then C_2 n_2-many times, ..., and finally C_k n_k-many times. Call this cycle C. We can then write $C = C_{i_1}^{t_1} C_{i_2}^{t_2} \cdots C_{i_\ell}^{t_\ell}$ where $\ell > 1$, $i_1 < i_2 < \cdots < i_\ell$, and all $t_i > 0$. We claim that any such C is elementary. To see this, consider the sequence of numbers of edges between visits to the vertex v as C is traversed. This sequence is $|C_{i_1}|, \dots, |C_{i_\ell}|$, which is nonconstant and nondecreasing since $|C_1| < \cdots < |C_k|$. It is clear that this precludes C being multiple traversals of a smaller cycle.

Hence, for every $n \in S_2$, we have an elementary cycle C in G, and so in this case the set of elementary cycle lengths of G is cofinite.

Our only remaining case is $|C_1| = 1$, indicating that a self-loop exists at v. As we have already dealt with the case where G consists of a single cycle, G must contain a different elementary cycle, C. Then by irreducibility of G, a cycle exists that begins at v and traverses C before returning to v; this path is not just a repeated traversal of the self-loop since we assumed C to be elementary. Then, every cycle consisting of C followed by any number of traversals of the self-loop is elementary, yielding a cofinite set of elementary cycle lengths, and completing the proof. $\square$

5.2. *Irreducible graphs.*

Proof of Theorem 3.9. "$\Rightarrow$" Let $R = p \cdot S$, where $p \in \mathbb{N}$ and S is either $\{1\}$ or a cofinite set. In both cases, we can use Theorem 3.8 and the same construction as in the proof of Theorem 3.6 to construct an irreducible graph H whose set of elementary cycle lengths is R.

"$\Leftarrow$" Let G be an irreducible graph where R is the set of elementary cycle lengths of G. Let $p = \mathrm{per}(G)$. By Proposition 3.12, G has an associated primitive graph G' such that the set of elementary cycle lengths of G is given by $p \cdot S$, where S is the set of elementary cycle lengths of G'. By Theorem 3.8, S is $\{1\}$ or cofinite. Since $R = p \cdot S$, either $R = \{p\}$ or $R = p \cdot S$ for S cofinite. □

5.3. *Arbitrary graphs.*

Proof of Theorem 3.10. "$\Rightarrow$" Consider any set Q which can be written as $U \cup \bigcup_{i=1}^{n} p_i \cdot S_i$ for some finite set U, $p_i \in \mathbb{N}$, and cofinite sets S_i. Then Q can be written as a finite union of singletons and sets of the form $p_i \cdot S_i$, each of which is the set of elementary cycle lengths of an irreducible graph by Theorem 3.9. We can use the same construction as in the proof of Theorem 3.7 to construct a graph F whose set of elementary cycle lengths is Q.

"$\Leftarrow$" Let F be an arbitrary graph. By Proposition 3.11, F can be broken down into irreducible subgraphs F_i. By Theorem 3.9, the set of elementary cycle lengths of each F_i can be written as $p_i \cdot T_i$, where T_i is a cofinite set or a singleton and $p_i = \mathrm{per}(F_i)$. Hence the set of elementary cycle lengths of F is the union $U \cup \bigcup_{i=1}^{n} p_i \cdot S_i$, where U is the finite union of the singletons and $\{S_i\}_{i=1}^{n}$ is the collection of all T_i which are cofinite. □

6. Results on sofic shifts

The forward directions of Theorems 1.7 and 1.8 follow immediately from those of Theorems 1.1 and 1.4, since all mixing SFTs are mixing sofic shifts. For the reverse directions, we will adapt the proof of the reverse direction of Theorem 3.8, which will require some well-known structural facts about sofic shifts.

We begin by introducing an alternative definition of sofic shift which will be far more useful for our purposes.

Definition 24. A *labeled graph* consists of a graph G along with a (usually not injective) map ℓ on $E(G)$, called a labeling.

Definition 25. For a labeled graph (G, ℓ), the *labeled edge shift* $\chi_{G,\ell}$ is the shift space over the alphabet $A = \ell(E(G))$ consisting of all labels of biinfinite sequences of edges which are connected end-to-end in G.

Theorem 6.1 [Lind and Marcus 1995, Theorem 3.2.1 and Proposition 3.3.11]. *A shift space X is sofic if and only if it is a labeled edge shift $\chi_{G,\ell}$ for some labeled graph (G, ℓ). If G is irreducible, then $\chi_{G,\ell}$ is as well.*

In fact more can be said about this graph when X is irreducible.

Definition 26. For a labeled graph (G, ℓ), a word w on the alphabet $\ell(E(G))$ *synchronizes to* a vertex v if every path in G which is labeled by w has terminal vertex v.

Theorem 6.2 [Lind and Marcus 1995, Theorem 3.3.2 and Propositions 3.3.9, 3.3.11, and 3.3.17]. *For any irreducible sofic shift X, there exists a labeled graph (G, ℓ) where G is irreducible, (G, ℓ) has a word w synchronizing to some vertex v, and $X = \chi_{G,\ell}$.*

We may now prove the following.

Theorem 6.3. *For any mixing sofic shift X, the set of least periods of X is either $\{1\}$ or cofinite.*

Proof. Assume that X is a mixing sofic shift. By Theorem 6.2, there exists a labeled graph (G, ℓ) where G is irreducible, (G, ℓ) has a word w synchronizing to some vertex v, and $X = \chi_{G,\ell}$. Denote by L the length of w. We will use a similar argument to that of Theorem 3.8 to create points in X with least periods in a cofinite set.

First, we note that by the definition of topological mixing, there exist N and words u, u' of lengths N and $N + 1$ respectively where $wuw, wu'w \in B(X)$. Since $X = \chi_{G,\ell}$, this means that there exist paths P_1 and P_2 labeled by wuw and $wu'w$ respectively. Since w synchronizes to v, we may remove the first L letters from P_1 and P_2 to get cycles D_1 and D_2 beginning and ending at the vertex v, and for which $\ell(P_1)$ and $\ell(P_2)$ end with w. In addition, $\gcd(|D_1|, |D_2|) = 1$ since $|D_1|$ and $|D_2|$ differ by 1.

We can now proceed as in the proof of Theorem 3.8; exactly as was done there, we can break D_1 and D_2 into cycles $C_1, C_2, \dots, C_k$ which start and end at v, and whose labels end with w and contain no other w. For exactly the same reasons as before, we can assume without loss of generality that $|C_1| < |C_2| < \cdots < |C_k|$, and that $\gcd(|C_1|, \dots, |C_k|) = 1$.

As before, we break into cases $|C_1| = 1$ and $|C_1| > 1$. If $|C_1| > 1$, then $k > 1$, and again the set

$$S_2 = \{n_1|C_1| + \cdots + n_k|C_k| : \text{there exist } j, j' \text{ such that } n_j, n_{j'} > 0\}$$

is cofinite. For every $s \in S_2$ we may construct a cycle in G of length s made by concatenating the cycles $C_1, \dots, C_k$ traversed $n_1, \dots, n_k$ times respectively; denote this cycle by $C^{(s)}$ and its label by $w^{(s)}$. The same argument as was used in the proof of Theorem 3.8 (where "visits to the vertex v" is replaced by "occurrences of the subword w") shows that no $w^{(s)}$ is made up of multiple concatenated copies of a single word, and so the biinfinite point $\cdots w^{(s)}w^{(s)}w^{(s)}\cdots$ has least period s. In addition, this point is in $X = \chi_{G,\ell}$ since $C^{(s)}$ is a cycle. We have then shown that every s in the cofinite set S_2 is in the least period set of X, and so the least period set of X is cofinite in this case.

Finally, we consider the case $|C_1| = 1$. First, this means that w is a single letter, labeling a self-loop at v. If every edge of G is labeled by w, then clearly the set of least periods in X is $\{1\}$. If instead G has an edge not labeled by w, then by irreducibility, that edge is part of a cycle C starting and ending at v, with label $\ell(C)$ containing a non-w letter. For every $k > 0$, repeating a cycle composed of C followed by k traversals of the self-loop at v yields a point $\cdots \ell(C)w^k\ell(C)w^k\cdots$ in X with least period $|C|+k$. This shows that the least period set of X is cofinite in this case, completing the proof. □

We may now prove Theorems 1.7 and 1.8.

Proofs of Theorems 1.7 and 1.8. As noted above, the forward directions of each of these results follow immediately from those of Theorems 1.1 and 1.4. The reverse direction of Theorem 1.8 is precisely Theorem 6.3. For the reverse direction of Theorem 1.7, we need only note that the period set of any shift space is closed by definition under $\mathbb{N}$-multiples, and that if the set of least periods is $\{1\}$, then the set of periods is forced to be $\mathbb{N}$ and therefore cofinite. □

Finally, we will briefly describe the issues in trying to find characterizations of (least) period sets for irreducible and general sofic shifts. The general obstacle is simple to describe: for a labeled graph (G, ℓ), it is not necessarily true that the (elementary) cycle lengths of G are the same as the (least) periods of $X = \chi_{G,\ell}$. It is true that every cycle gives rise to a periodic point, but the least period of that point can now be strictly smaller than the cycle length, even when the cycle is elementary. For instance, suppose that G consists of a single cycle of length 4, and that the labels given to its edges are 0, 1, 0, 1 (in order). Then, though the smallest cycle in X has length 4, the point of X induced by it has least period 2.

If X is irreducible, then by Theorem 6.2, we can find (G, ℓ) so that $X = \chi_{G,\ell}$, and G has some period $p > 1$. The same argument used to prove Theorem 6.3 can be used to show that the least period set of X contains a cofinite subset of $p\mathbb{N}$. However, it is no longer the case that this least period set must be a subset of $p\mathbb{N}$ at all, and this is what makes the problem of characterization more difficult.

Example. Consider an irreducible graph G with 11 vertices and 12 edges, composed of two 6-cycles which share a single vertex. Define a labeling ℓ which labels the edges of the first cycle (starting from the shared vertex) by A, B, A, B, A, B, and which labels the edges of the second cycle (starting from the shared vertex) by

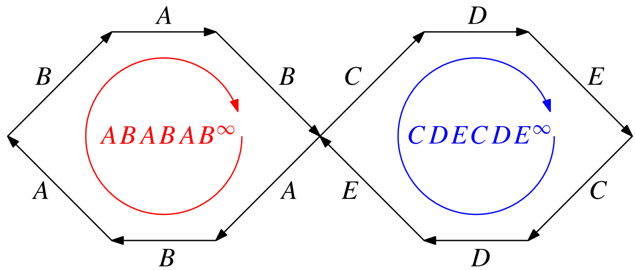

Figure 2. The labeled graph (G, ℓ) inducing the irreducible sofic shift X. All cycles of G are concatenations of the cycles on the left and right halves of G.

C, D, E, C, D, E. (See Figure 2.) Define X to be the labeled edge shift $\chi_{G,\ell}$, which is an irreducible sofic shift by Theorem 6.1.

Clearly the only biinfinite paths in G consist of concatenations of traversals of the two cycles. This means that there are three types of periodic points in X: the shifts of $\cdots ABABAB \cdots$, the shifts of $\cdots CDECDE \cdots$, and points with least period containing both cycle labels $ABABAB$ and $CDECDE$ as subwords. It's clear that the first type has least period 2 and the second has least period 3. Points of the third type clearly must have least period which is a multiple of 6 and strictly greater than 6, and all such least periods occur in X; for $k > 1$, the point $\cdots (ABABAB)^{k-1}CDECDE(ABABAB)^{k-1}CDECDE \cdots$ is clearly in X and has least period $6k$.

So, the set of least periods of X is $\{2\} \cup \{3\} \cup 6(\mathbb{N}+1)$, and the set of periods of X is $2\mathbb{N} \cup 3\mathbb{N}$, the set of all positive integer multiples of the set of least periods. Note that the period of the graph G is 6, and yet there are least periods of points in X which are not multiples of 6. $\square$

The reader may note that though neither the period set nor least period set of X above satisfies the conditions of Theorems 1.2 or 1.5 respectively, and so the characterizations from those theorems are definitely not correct for irreducible sofic shifts. However, the period set $2\mathbb{N} \cup 3\mathbb{N}$ is the finite union of two sets as described in Theorem 1.2, and the least period set $\{2\} \cup \{3\} \cup 6(\mathbb{N}+1)$ is just the union of a finite set with a set as described in Theorem 1.5. This gives some indication that slight changes to our results could give characterizations of sets of periods/least periods for irreducible (and maybe general) sofic shifts, but at the moment we are not able to prove this.

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