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We consider optimal transportation *with constraint*, as did Korman and McCann (2013, 2015), provide simplifications and generalizations of their examples and results, and provide some new examples and results.

1. Introduction

The classical problem of optimal transportation seeks the least-cost way to move material between two locations in $\mathbb{R}^n$. Monge [1781] sought an optimal mapping. A more general problem, introduced in [Kantorovitch 1942], see also [Villani 2009, Theorem 3.1], seeks a cost-minimizing coupling between two measure spaces. If the coupling is absolutely continuous, it is given by a density H on the product. Recently optimal transportation has been used to better understand Riemannian manifolds and extend concepts such as Ricci curvature to more general spaces; [Cordero-Erausquin et al. 2006; Villani 2009].

Korman and McCann [2015] studied a constraint on the amount of material that can be transported between any two locations, an upper bound $h(x, y)$ on the density H, which goes back at least to [Levin 1984]. It is easy to show (Proposition 2.2) that if h is not prohibitively small, there is an optimal density H which equals 0 or h almost everywhere.

In this paper we specialize to the case of constant h. We assume $h \geq 1$, which is necessary and sufficient for existence (Proposition 2.2). Focusing on the solutions of the form 0 or h almost everywhere, for this paper we define a transportation plan as a map F from X to subsets of Y with measure $1/h$.

Our main section, Section 3, recognizes that many old and new examples of optimal transportation have the stronger "universal" property of minimizing the cost at each point separately. This leads to simplified proofs for many of the results and examples of [Korman and McCann 2013; 2015], as well as explicit examples of optimal transportation plans for all constraints $h \geq 1$. For instance, Example 3.7, due to Korman and McCann [2015, Example 1.1], provides a very short proof

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that optimal transportation from the unit interval to itself with cost $(x-y)^2$ with constraint $h=2$ maps each point to whichever half of the unit interval it lies in. Proposition 3.13 proves that the intersection of two optimal transportation plans is optimal under certain conditions. Proposition 3.18 shows that in the torus or any Lie group, every admissible translation-invariant transportation plan is optimal for some continuous cost.

Proposition 4.3 presents a simplified approach to the surprising symmetries for dual cost constraints found by Korman and McCann [2013, Section 4].

Section 5 relates the case of finite spaces to some known combinatorial computations and asymptotic estimates.

2. Existence and uniqueness of optimal transportation plans

Proposition 2.2 provides existence of an optimal transportation plan F for admissible (constant) constraint h.

Definitions 2.1. Let X and Y be smooth manifolds, not necessarily compact, complete, or connected. Let f and g be nonnegative densities on X and Y, yielding probability measures on X and Y. A *transportation plan* F with constant constraint $h \geq 1$ is a measurable map from X to the power set $\mathscr{P}(Y)$ such that $F(x)$ has measure $1/h$ in Y for almost all $x \in X$ and such that $\{x \in X \mid y \in F(x)\}$ has measure $1/h$ in X for almost all $y \in Y$. (By F measurable we mean that the associated density $H(x, y)$, defined as the characteristic function of $F(x)$, is measurable.) For a cost function $c(x, y) \in L^\infty(X \times Y)$, the *total cost of transportation* is defined as

$$c[F] = \int_X \int_{F(x)} c(x, y)\, dy\, dx.$$

A transportation plan F is *optimal* if it minimizes cost.

Proposition 2.2. *Let X and Y be smooth* (*positive-dimensional*) *manifolds with nonnegative densities f and g respectively and total measure* 1. *There exists an optimal transportation plan $F(x)$ if and only if the* (*constant*) *constraint h is at least* 1.

Proof. If $h < 1$, $F(x)$ cannot have measure $1/h$. On the other hand, if $h \geq 1$, the set of transportation densities $\Gamma(X, Y)$ is nonempty, since it includes $H(x, y) = 1$, and an optimal transportation density exists by standard compactness arguments; see [Korman and McCann 2015, Theorem 3.1].

Because $L^\infty(X, Y)$ is the dual of $L^1(X, Y)$, by Alaoglu's theorem, see [Rudin 1991, Section 3.15], the unit ball is compact in the weak-$*$ topology. Thus the set of transportation densities $\Gamma(X, Y)$ is compact as well as convex. By the Krein–Milman theorem, every compact convex set has an extreme point, see [Wikipedia 2014a], and thus $\Gamma(X, Y)$ has an extreme point. The set of optimal transportation densities is a convex face of $\Gamma(X, Y)$ which contains an extreme point H, which is also an extreme point of $\Gamma(X, Y)$. Such an extreme H must equal 0 or h almost everywhere, i.e., it must be a transportation plan F, see [Korman and McCann 2013, Proposition 3.2], although this can fail for finite sets of points; see Remark 2.3 below. □

Remark 2.3. Of course there is an optimal transportation plan between finite sets (because there are only finitely many transportation plans), but the same proof does not work because there might be a better transportation density. For example the optimal transportation density from $\{0, 1\}$ to $\left\{0, \frac{1}{2}, 1\right\}$ with $h = \frac{3}{2}$ maps 0 to 1 with density $\frac{2}{3}$ and $\frac{1}{2}$ with density $\frac{1}{3}$ and maps 1 to $\frac{1}{2}$ with density $\frac{1}{3}$ and 2 with density $\frac{2}{3}$, and there is no equally good transportation plan $F(x)$; the only transportation plan maps each point to all three points. Actually Proposition 2.2 and its proof work as long as one of the two manifolds is positive-dimensional.

Although we will not need it, we provide the following uniqueness theorem of Korman and McCann.

Proposition 2.4 [Korman and McCann 2013, Theorem 3.3]. *Let X and Y be smooth manifolds with nonnegative densities f and g respectively and total measure* 1. *If the cost $c(x, y)$ is bounded, twice differentiable, and nondegenerate, i.e.,* $\det[D^2_{x^i y^j} c(x, y)] \neq 0$ *for almost all $(x, y) \in X \times Y$, then an optimal transportation plan $F(x)$ is unique* (*up to measure* 0).

Proof. Theorem 3.3 in [Korman and McCann 2013] gives a unique optimal density H. Since at least one optimal transportation density is an extreme point of Γ, H must be an extreme point of Γ and thus a transportation plan F. □

Additionally, we give necessary and sufficient conditions for a map F from X to subsets of Y to be a transportation plan.

Proposition 2.5. *Let F be a measurable map from X to subsets of Y with constant constraint $h \geq 1$ such that $F(x)$ has measure $1/h$ in Y for almost all $x \in X$. Then F is a transportation plan if and only if for every $A \subset X$ of measure greater than $1/h$, $\bigcap_{x \in A} F(x)$ has measure* 0.

Proof. If F is a transportation plan, the condition holds. Suppose that F is not a transportation plan. Then it is not true that $\{x \in X \mid y \in F(x)\}$ has measure $1/h$ for almost all y. Since by Fubini's theorem the average satisfies

$$\int_Y f(\{x \in X \mid y \in F(x)\})\, dy = \int_X g(F(x))\, dx = 1/h$$

for some nontrivial subset of Y, we have $\{x \in X \mid y \in F(x)\}$ has measure greater than $1/h$, and the condition fails. □

3. Universally optimal transportation

Finding optimal transportation plans for a given cost and constraint is hard. For example, the problem of optimal transportation from the unit interval $I = [0, 1]$ to itself with cost $c(x, y) = (x - y)^2$ is still open for $h \neq 2$; see [Korman and McCann 2013, Figure 1; 2015, Example 1.2]. In certain cases, however, it is possible to minimize the cost at each point separately. Further, for every optimal density, the cost function can be adjusted so that the same optimal density is also minimal at each point separately; see Remark 3.2 below.

Definition 3.1. For two smooth manifolds X and Y, a transportation plan F for the cost function c under constant constraint $h \geq 1$ is *universally optimal* if for almost every $x \in X$ it minimizes

$$\int_{F(x)} c(x, y)\, dy.$$

It follows immediately that F is optimal.

Remark 3.2. Korman, McCann, and Seis [Korman et al. 2015, Theorem 4.2] showed that for continuous densities f, g and $h > 1$, every optimal density is universally optimal for some equivalent cost $c(x, y) + u(x) + v(y)$ and hence for $c(x, y) + v(y)$; by Proposition 2.2 and its proof, this applies to optimal plans in the positive-dimensional case.

Morgan uses this concept of universal optimality to generalize and give shorter proofs of some of the examples of Korman and McCann.

Proposition 3.3 [Korman and McCann 2015, Example 1.3; Morgan 2013, Proposition 1]. *Let X be a Riemannian manifold of unit volume, with a transitive group of measure-preserving isometries, with cost of transportation $c(x, y)$ increasing in distance with constant constraint h. Then unique (universally) optimal transportation is that which maps each $x \in X$ to a geodesic ball about x of volume $1/h$.*

Proof. An optimal transportation plan F with constraint h must map a point $x \in X$ to a set of volume at least $1/h$, and the geodesic ball minimizes cost among such. By the symmetry assumption, all balls of the same radius have the same volume, so the set mapped to a target point $y \in Y$ is the ball about x with volume $1/h$ and the map satisfies the definition of a transportation plan and is clearly uniquely optimal (up to sets of measure 0). □

Proposition 3.4 [Korman and McCann 2015, Example 1.1; Morgan 2013, Proposition 2]. *Let X and Y be two Riemannian manifolds of unit volume with cost of transportation $c(x, y)$ and constant constraint $h \geq 1$. Suppose that for almost all $x \in X$, $c(x, y)$ is negative for $1/h$ of the y's in Y and nonnegative for the rest, and for almost all $y \in Y$, $c(x, y)$ is negative for $1/h$ of the x's in X and nonnegative for* *the rest. Then unique (universally) optimal transportation maps each $x \in X$ to the subset of Y with negative cost.*

Proof. By hypothesis, both $F(x)$ and $\{x \in X \mid y \in F(x)\}$ have measure $1/h$ for almost all $x \in X$ and $y \in Y$ respectively, and F is clearly universally and uniquely optimal (up to sets of measure 0). □

Proposition 3.5. *Every transportation plan for which all images and inverse images have measure $1/h$ is optimal for some cost.*

Proof. Let $c(x, y) = -\chi_{F(x)}(y)$. Then F is optimal by Proposition 3.4. □

Example 3.6 [Korman and McCann 2015, Example 1.1; Morgan 2013, Example 2.1]. For $h \geq 2$ an integer, let X consist of h equal-volume regions in $\mathbb{R}^n$ such that the maximum diameter of a region is less than the minimum distance between regions. Let $c(x, y)$ be a cost function on $X \times X$ increasing in distance. Then optimal transportation from X to itself with constant constraint h maps the points of each region to itself. (To apply Proposition 3.4, subtract a constant from the cost.)

Example 3.7 [Korman and McCann 2015, Example 1.1; Morgan 2013, Example 2.2]. Let X be a centrally symmetric body in $\mathbb{R}^n$. For cost $c(x, y) = -2x \cdot y$, which is equivalent to $(x - y)^2$ because its integral differs by a constant, and for constraint $h = 2$, (universally) optimal transportation from X to itself is that which maps x to y with $x \cdot y$ positive (see Figure 1). In $\mathbb{R}^1$ central symmetry is unnecessary as long as the origin is the median. A similar result holds for any cost having the same sign at each point as $-x \cdot y$. The analysis generalizes to any centrally symmetric probability measure on $\mathbb{R}^n$ for which hyperplanes through the origin have measure 0 and to any probability measure on $\mathbb{R}^1$. Optimal transportation from X to itself with cost $-2x \cdot y$ is still open for constraint $h \neq 2$, though numerical estimates from some cases are given in [Korman and McCann 2013, Figure 1; 2015, Example 1.2].

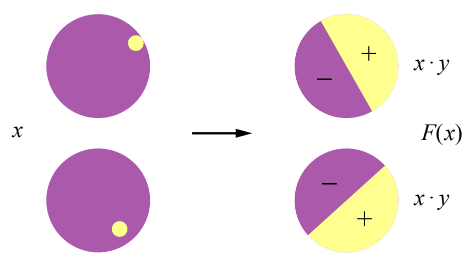

Figure 1. Optimal transportation F from the unit ball in $\mathbb{R}^2$ to itself with cost $c(x, y) = (x - y)^2$ and constraint $h = 2$ maps each x to the half-ball $\{x \cdot y > 0\}$.

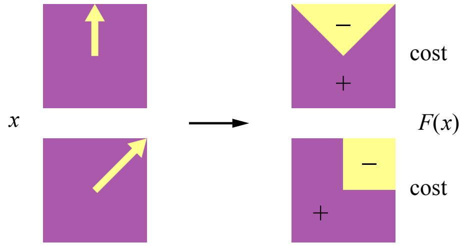

Figure 2. Optimal transportation F maps each x on a ray from the origin to a cone about that ray.

Example 3.8. Unique (universally) optimal transportation from the sphere $\mathbb{S}^n$ to the ball $\mathbb{B}^{n+1}$ with cost $c(x, y) = (x - y)^2$ and constraint $h = 2$ maps a point x to the half-ball $\{x \cdot y > 0\}$.

Proof. As in Example 3.6, the cost is equivalent to $-2x \cdot y$, which is negative precisely on the asserted half-ball, proving the asserted map uniquely universally optimal. □

Example 3.9 [Morgan 2013, Example 2.3]. Let X be a planar region with h-fold rotational symmetry, such as a square ($h = 4$) as in Figure 2. For cost

$$c(x, y) = \cos(\pi/h)|x||y| - x \cdot y,$$

and constant constraint $h \geq 1$, (universally) optimal transportation maps all points on a ray from the origin to a cone of angle π/h about that ray.

Remark 3.10 [Morgan 2013, Example 2.4]. Such examples of universally optimal transportation plans from X_i to Y_i extend to universally optimal transportation plans from $\prod X_i$ to $\prod Y_i$ with a cost which is negative if and only if the costs of the projections are all negative: optimal transportation with constraint $h = \prod h_i$ maps to points of negative cost. In particular, Example 3.9 generalizes to a product of such actions on $\mathbb{R}^{2n}$ with negative cost if and only if $x_i \cdot y_i \geq (\cos \pi/h_i)|x_i||y_i|$ for all i: optimal transportation with constant constraint $h = \prod h_i$ maps all points with projections on rays from the origin to a product of cones of angle π/h_i about the ray.

Remark 3.11 [Morgan 2013, Example 2.5]. Such examples of universally optimal transportation plans from X to Y with cost $c(x, y)$ extend to universally optimal transportation plans on warped products $A \times X$, $A \times Y$, as long as the cost $c'(a, x, a, y)$ has the same sign as $c(x, y)$. For example, for any $h \geq 1$, Proposition 3.4 on the sphere, with cost $c(x, y) = a|x||y| - x \cdot y$, with a chosen so that optimal transportation maps to points of negative cost, extends to the ball, with points on a ray from the origin mapped to a cone of negative cost about that ray.

Remark 3.12. Although universally optimal transportation plans are by definition optimal transportation plans, the converse is not true in general. Consider transportation from the unit interval to itself with cost of transportation increasing with distance and constant constraint $h = 2$. Minimizing cost for each x does not even give a valid transportation plan because points near 0 and 1 are mapped to by less than half of the interval.

Given two universally optimal transportation plans for two different costs, we seek ways to generate a third cost and a related universally optimal transportation plan.

Proposition 3.13. *Let F_1 and F_2 be optimal transportation plans from X to Y with costs c_1 and c_2 and constant constraints h_1 and h_2 respectively. Suppose that for almost all x, we have $F_i(x) = \{y \in Y \mid c_i(x, y) < 0\}$. If for some $1 \leq h < \infty$, for almost all $x \in X$, $F_1(x) \cap F_2(x)$ has measure $1/h$, and for almost all $y \in Y$, $\{x \in X \mid y \in F_1(x)\} \cap \{x \in X \mid y \in F_2(x)\}$ has measure $1/h$, then $F(x) = F_1(x) \cap F_2(x)$ is a universally optimal transportation plan from X to Y with cost $c(x, y) = \max(c_1, c_2)$ and constraint h.*

Proof. It suffices to show that for almost all $x \in X$, $c(x, y)$ is negative for $1/h$ of the $y \in Y$ and nonnegative for the rest and for almost all $y \in Y$, $c(x, y)$ is negative for $1/h$ of the $x \in X$ and nonnegative for the rest. By hypothesis on F, for almost all $x \in X$, $c(x, y)$ is negative for $1/h$ of the $y \in Y$. It is nonnegative for the rest because $x \notin F(x)$ implies some $c_i(x, y)$ must be nonnegative; thus $c(x, y)$ must also be nonnegative. The reverse condition holds by a similar argument. □

Corollary 3.14. *Let X be a region with 4-fold rotational symmetry in $\mathbb{R}^2$ with cost of transportation from X to X given by $c(x, y) = \max((x \cdot y), \det[x \mid y])$, where $\det[x \mid y]$ is the determinant of the matrix with x and y as its column vectors. Mapping each point to the region of negative cost uniquely gives (universally) optimal transportation for $h = 4$ (see Figure 3).*

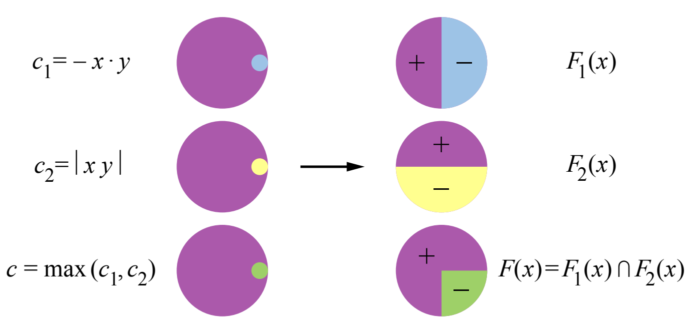

Figure 3. The intersection of optimal transportation plans yields a new optimal transportation plan under certain hypotheses.

Proof. The map $F_1(x) = \{y \in X \mid x \cdot y > 0\}$ is an optimal transportation plan from X to itself with cost $c_1(x, y) = -x \cdot y$ and constraint $h = 2$ (see Example 3.6). Similarly the cost $c_2(x, y) = \det[x \mid y]$ with constraint $h = 2$ satisfies the hypotheses of Proposition 3.4 and thus the map $F_2(x) = \{y \in X \mid \det[x \mid y] < 0\}$ is an optimal transportation plan from X to itself with cost $c_2(x, y)$ and constraint $h = 2$. By Proposition 3.13, if for almost all $x, y \in X$, $F_1(x) \cap F_2(x)$ and $\{x \in X \mid y \in F_1(x)\} \cap \{x \in X | y \in F_2(x)\}$ both have constant measure $1/h$ for some $h > 1$, then $F(x) = F_1(x) \cap F_2(x)$ is an optimal transportation plan from X to itself with cost $c(x, y) = \max(c_1, c_2)$ and constraint $1/h$. For almost all $x \in X$, $\partial F_1(x)$ is the line through the origin normal to the line through x and the origin and $\partial F_2(x)$ is the line through x and the origin. Because two normal lines both through the origin partition a region with 4-fold rotational symmetry centered on the origin in $\mathbb{R}^2$ into four congruent regions, and exactly one of these regions is equivalent to $F_1(x) \cap F_2(x)$, it follows that $F_1(x) \cap F_2(x)$ has constant measure $\frac{1}{4}$. Similarly, the boundary of the set $\{x \in X \mid y \in F_1(x)\}$ is the line through the origin normal to the line through y and the origin and the boundary of the set $\{x \in X \mid y \in F_2(x)\}$ is a line through y and the origin. Thus, by the same argument as above, $\{x \in X \mid y \in F_1(x)\} \cap \{x \in X \mid y \in F_2(x)\}$ has measure $\frac{1}{4}$. By Proposition 3.13, the asserted map is optimal. □

Remark 3.15. If the hypotheses of Proposition 3.13 hold, then the maps $F(x) = F_1(x) \cup F_2(x)$ and $F(x) = F_1(x) \triangle F_2(x)$ are optimal transportation plans for costs $c = \min(c_1, c_2)$ and $c' = c_1 \cdot c_2$ and some constraints h and h' respectively (the symbol $\triangle$ denotes the symmetric difference of two sets).

Example 3.16. Let X be a region with 4-fold rotational symmetry in $\mathbb{R}^2$. Then an optimal transportation plan F for cost

$$c(x, y) = ((\cos 3\pi/4h)|x||y| - x \cdot y)((\cos \pi/4h)|x||y| - x \cdot y)$$

and constraint $h = 2$ maps points on a ray from the origin to two cones (see Figure 4).

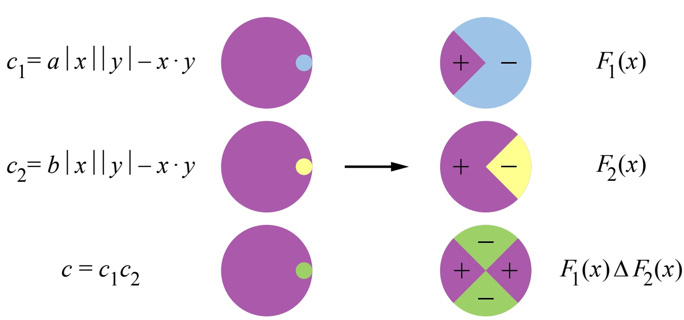

Figure 4. Other set operations yield even more examples of optimal transportation.

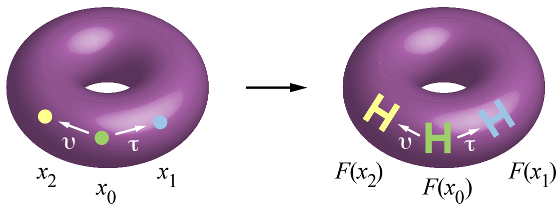

Figure 5. Optimal transportation F maps each x to an H-shaped region.

The condition in Proposition 3.13 that $F_1(x) \cap F_2(x)$ have constant measure $1/h$ for almost all $x \in X$ and some h is independent of the condition that $\{x \in X \mid y \in F_1(x)\} \cap \{x \in X \mid y \in F_2(x)\}$ have constant measure $1/h$ for almost all $y \in Y$.

Example 3.17. Let X be $\{1, 2, 3, 4\}$ or equivalently the unit interval divided into four quarters. Consider transportation F_1, F_2 from X to X such that

$$F_1(1) = \{3, 4\}, \quad F_1(2) = \{2, 3\}, \quad F_1(3) = \{1, 2\}, \quad F_1(4) = \{1, 4\},$$
$$F_2(1) = \{1, 4\}, \quad F_2(2) = \{1, 2\}, \quad F_2(3) = \{2, 3\}, \quad F_2(4) = \{3, 4\}.$$

Then $F(x) = F_1(x) \cap F_2(x)$ has constant measure $\frac{1}{4}$ but

$$\{x \in X \mid y \in F(x)\} = \{x \in X \mid y \in F_1(x)\} \cap \{x \in X \mid y \in F_2(x)\}$$

has measure $\frac{1}{2}$ for $\{2, 4\}$ and measure 0 for $\{1, 3\}$.

Proposition 3.18. *Let X be a Lie group. Given an open subset A of X with measure $1/h$, there exists a continuous cost function $c(x, y)$ such that the unique (universally) optimal transportation plan F from X to itself with constant constraint h maps the identity to the set A and maps each element $x \in X$ to the set $x \cdot A = xA$.*

Proof. Let the cost $c(x, y)$ equal the distance from y to the boundary of $x \cdot A$, with negative cost on the interior of $x \cdot A$ and nonnegative cost on the complement of $x \cdot A$. By Proposition 3.4, the asserted map is optimal. □

Example 3.19. Let $X = \mathbb{S}^1 \times \mathbb{S}^1$ with unit area. Given an open subset $A \subset X$ with measure $1/h$, such as the H-shaped region in Figure 5, there exists a continuous cost function $c(x, y)$ such that the unique (universally) optimal transportation plan F from X to itself with constant constraint h maps the origin to the set A and maps almost every x to the set $\tau_x(A)$, where τ_x is the translation that takes the origin to x.

Example 3.20. Optimal transportation from a flat rectangular torus to itself with cost $c(x, y) = \min(d(x_i, y_i))$ and constraint h maps each point to a neighborhood around the coordinate axis centered at that point; see Figure 6.

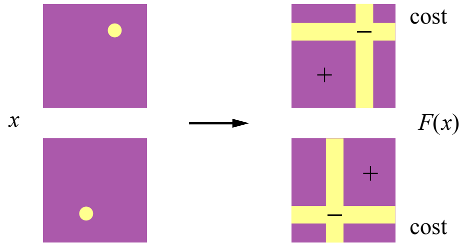

Figure 6. Optimal transportation on the flat rectangular torus maps each x to a small neighborhood around the coordinate axis centered at that point.

4. Transportation and symmetry

Korman and McCann [2013, Section 4] found surprising symmetries between optimal transportation plans with dual constraints. Proposition 4.3 presents a simplified approach.

Definition 4.1. A map f from X' to X is called *measure preserving* if the measure of any $A \subset X$ equals the measure of $f^{-1}(A) \subset X'$.

Proposition 4.2. *Let F be an optimal transportation plan from X to Y with cost $c(x, y)$ and constraint h. Let f and g be measure-preserving maps from X' to X and from Y' to Y respectively. Then $G(x') = g^{-1}(F(f(x')))$ provides optimal transportation from X' to Y' with cost $c \circ (f, g)$ and constraint h.*

Proof. We need to show that $G(x')$ and $G^{-1}(y')$ both have measure $1/h$ and that the total cost of transportation is minimal. For $x' \in X'$, $G(x') = g^{-1}(F(f(x')))$ must have the same measure as $F(f(x'))$, which is $1/h$ by hypothesis. Similarly, for y' in Y', $G^{-1}(y') = f^{-1}(F^{-1}(g(y')))$ must have the same measure as $F^{-1}(g(y'))$, which is also $1/h$ by hypothesis. To show that G is optimal, we will show that G and F have the same cost and that any other transportation plan G_2 from X' to Y' has the same cost as an analogous transportation plan F_2 from X to Y and therefore must be of greater total cost than G. The cost of transportation from x' to y' is equal to $c(f(x'), F(f(x')))$; thus G and F have the same total cost of transportation. Let G_2 be another transportation map from X' to Y'. Let $F_2(f(x'))$ = $g(G_2)$. Then $G_2 = g^{-1}(F_2(f(x')))$ and the result follows. □

Proposition 4.3 [Korman and McCann 2013, Lemma 4.1; Morgan 2013, Proposition 3]. *Let M_1, M_2 be subsets of $\mathbb{R}^n$ or Riemannian manifolds with boundary or metric measure spaces of volume V. Let T_i be a measure-preserving map from M_i to itself and let $T = T_1 \times T_2$. Let $c(x, y)$ be a cost satisfying $c \circ T = -c$. If the map F is an optimal transportation plan from M_1 to M_2 with cost $c(x, y)$ with* *constraint h, then the map $T_2(F' \circ T_1)$ is an optimal transportation plan from M_1 to M_2 with cost c and constraint h', where $1/h + 1/h' = 1$ and $F'(x) = F(x)^C$.*

Proof. If F is an optimal transportation plan for cost c and constraint h, then $F(x)' = F(x)^C$ is the most expensive transportation plan for cost c with constraint h', and hence an optimal transportation plan for cost $-c$. Therefore $T_2(F' \circ T_1)$ is an optimal transportation plan for cost $-c \circ T = c$ and constraint h'. $\square$

Example 4.4 [Morgan 2013, Example after Proposition 3; Korman and McCann 2013, Lemma 4.1]. Let M_1 and M_2 be subsets of $\mathbb{R}^n$, with M_1 centrally symmetric, and let $c(x, y) = -x \cdot y$ (which is equivalent to $(x - y)^2$). Then central inversion in x carries optimal transportation with constraint h to optimal transportation with constraint h'.

5. Transportation plans on finite sets

Consider the case where X and Y are finite sets, say $X = \{1, 2, \dots, m\}$ and $Y = \{1, 2, \dots, n\}$. In this case we may assume that the constraint h is a common divisor of m and n. A map F from X to Y is equivalent to the $n \times m$ matrix of 0's and 1's with entry $a_{ij} = 1$ if and only if $i \in F(j)$; see [Wikipedia 2014b; Weisstein]. Such a matrix gives a transportation plan if and only if the matrix has m/h 1's in each column and n/h 1's in each row. Thus the number of transportation plans is equal to the number of $n \times m$ binary matrices with constant column sums n/h and constant row sums m/h. Asymptotic estimates exist for large m, n; see [Canfield and McKay 2005; McKay and Wang 2003].

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