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Some properties of the twisted Grassmann graphs

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Abstract

In this note we determine the full automorphism group of the twisted Grassmann graph. Further we show that twisted Grassmann graphs do not have antipodal distance-regular covers. At last, we show that the twisted Grassmann graphs are not the halved graphs of bipartite distance-regular graphs.

Keywords: automorphism group twisted Grassmann graph, antipodal covers, bipartite

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1 Introduction

In November 2004, E. van Dam and J. Koolen [4] constructed the twisted Grassmann Graphs. These graphs have the same parameters as the Grassmann graphs $J_q(2e+1, e)$, but are not vertex-transitive. In this note we will determine the full automorphism groups of these graphs, and also show that they do not have distance-regular antipodal covers. Furthermore we show that they are not the halved graphs of bipartite distance-regular graphs.
In Section 2 we will give the preliminaries and definitions, in Section 3 we recall the twisted Grassmann graphs and their maximal cliques, in Section 4 we determine the automorphism group and in Section 5 and 6, respectively we show that they do not have distance-regular antipodal covers and are not the halved graphs of bipartite distance-regular graphs.

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2 Definitions and preliminaries

We begin this section by recalling some facts concerning distance-regular graphs (for more details see [1]). Suppose that Γ is a connected graph. The distance $d(u,v)$ between any two vertices u,v in the vertex set $V\Gamma$ of Γ is the length of a shortest path between u and v in Γ. For any $v \in V\Gamma$, define $\Gamma_i(v)$ to be the set of vertices in Γ at distance precisely i from v, where i is any non-negative integer not exceeding the diameter D of Γ. In addition, define $\Gamma_{-1}(v) = \Gamma_{D+1}(v) := \emptyset$.

Following [1], we call Γ *distance-regular* if there are integers b_i, c_i, $0 \le i \le D$, such that for any two vertices $u,v \in V\Gamma$ at distance $i = d(u,v)$, there are precisely c_i neighbors of v in $\Gamma_{i-1}(u)$ and b_i neighbors of v in $\Gamma_{i+1}(u)$. In particular, Γ is regular with valency $k := b_0$. The numbers c_i, b_i and

$$a_i := k - b_i - c_i \quad (i = 0, \dots, D),$$

the number of neighbors of v in $\Gamma_i(u)$ for $d(u,v) = i$, are called the *intersection numbers* of Γ.

Let Γ be a distance-regular graph, with diameter D and n vertices. A partition $\Pi = P_1, P_2, \dots, P_f$ of the vertex set $V\Gamma$ is called *equitable* if there are constants α_{ij} $(i,j \in \{1,\dots,f\})$ such that for any $x \in P_i$ the number of neighbours of x in P_j equals α_{ij}.

A *code* C in Γ is just a subset of $V\Gamma$. For a vertex x and a code C define $d(x,C) = \min\{d(x,y) \mid y \in C\}$. For $i \le D$ define $C_i = \{x \in V\Gamma \mid d(x,C) = i\}$. The *covering radius* of C, ρ is defined as

$$\rho = \max\{i \mid C_i \neq \emptyset\}.$$

A code C is called *completely regular* if $\{C_i \mid 0 \le i \le \rho\}$ is an equitable partition of Γ.

Let Γ be a distance-regular graph. Let $M_{V\Gamma}(\mathbb{C})$ be the matrix algebra indexed by $V\Gamma$ over $\mathbb{C}$. The matrix A_i denotes the i-th adjacency matrix of Γ, that is to say that, A_i is the matrix in $M_{V\Gamma}(\mathbb{C})$ whose (x,y)-entry is 1 if $d(x,y) = i$, and 0 otherwise.

Let $\mathfrak{A}$ be the subalgebra of $M_{V\Gamma}(\mathbb{C})$ generated by the adjacency matrix A_1. Then, for all i, the matrix $A_i \in \mathfrak{A}$. Since $A_0, \dots, A_D$ are pairwise commutative normal matrices, they can be diagonalized simultaneously. It is well-known that the number of the maximal common eigenspaces of $A_0, \dots, A_D$ is $D+1$ and span$(1,1,\dots,1)$ is one of the maximal common eigenspaces. Denote the maximal common eigenspaces by $V_0 = \langle (1,1,\dots,1) \rangle$, $V_1, \dots, V_D$ and let E_i be the orthogonal projection $\mathbb{C}^{|V\Gamma|} \to V_i$ expressed in the matrix form with respect to the unit vectors. Then $E_0, \dots, E_D$ are the primitive idempotents of $\mathfrak{A}$. Let

C be a code in Γ, and let χ be its characteristic vector. The *width* of C, w, is defined by $w = \max\{d(x, y) \mid x, y \in C\}$. The *dual degree* s^* of C is defined by $s^* := \#\{i \geq 1 \mid \chi^T E_i \chi \neq 0\}$. If C is a completely regular code of Γ with covering radius ρ, then it is known that $s^* = \rho$, cf. [1, Theorem 11.1.1(ii)].

3 Twisted Grassmann graphs

Let us first recall the twisted Grassmann graphs $\tilde{J}_q(2e + 1, e)$, q a prime power and $e \geq 2$ integer, the distance-regular graphs constructed by Van Dam and Koolen [4].
Let $e \geq 2$ be an integer and q a prime power. Let V be a $(2e + 1)$-dimensional vector space over the finite field $\mathbb{F}_q$, and let H be a fixed hyperplane of V. We define the sets $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}$ as follows:

$$\begin{aligned}\mathcal{B}_1 &:= \{W \text{ subspace of } V \mid \dim W = e + 1,\ W \not\subseteq H\},\\ \mathcal{B}_2 &:= \{W \text{ subspace of } V \mid \dim W = e - 1,\ W \subseteq H\},\\ \mathcal{B} &:= \mathcal{B}_1 \cup \mathcal{B}_2\,.\end{aligned}$$

The twisted Grassmann graph $\tilde{J}_q(2e + 1, e)$ has as vertex set $\mathcal{B}$, and vertices $B_1, B_2 \in \mathcal{B}$ are adjacent as follows:

$$B_1 \sim B_2 \iff \dim(B_1) + \dim(B_2) - 2\dim(B_1 \cap B_2) = 2.$$

3.1 Maximal cliques

In this subsection we recall the maximal cliques of the twisted Grassmann graph as determined by [4]:

(I) Fix an e-dimensional subspace S. Then

$$\mathcal{C}_I(S) := \{B \in \mathcal{B}_1 \mid S \subseteq B\} \cup \{B \in \mathcal{B}_2 \mid B \subseteq S \cap H\}$$

is a maximal clique of type I. The size of $\mathcal{C}_I(S)$ is as follows:

$$\#\,\mathcal{C}_I(S) = \begin{cases} \left[{e+1 \atop 1}\right] + 1 & \text{if } S \not\subseteq H,\\ \left[{e+1 \atop 1}\right] & \text{if } S \subseteq H.\end{cases}$$

Here $\left[{n \atop m}\right]$ denotes the q-ary Gaussian binomial coefficient.

(II) Fix an $(e + 2)$-dimensional subspace S which is not contained in H. Then

$$\mathcal{C}_{II}(S) := \{B \in \mathcal{B}_1 \mid B \subseteq S\}$$

is a maximal clique of type II. Its size is: $\#\,\mathcal{C}_{II}(S) = \left[{e+2 \atop 1}\right] - 1$.

(III) Fix an $(e+2)$-dimensional subspace S which is not contained in H, and also fix an $(e-1)$-dimensional subspace S' in $S \cap H$. Then

$$\mathcal{C}_{III}(S,S') := \{B \in \mathcal{B}_1 \mid S' \subseteq B \subseteq S\} \cup \{S'\}$$

is a maximal clique of type III. For its size we have $\#\mathcal{C}_{III}(S,S') = \left[{3 \atop 1}\right]$.

(IV) Fix an $(e-2)$-dimensional subspace S in H. Then

$$\mathcal{C}_{IV}(S) := \{B \in \mathcal{B}_2 \mid S \subseteq B\}$$

is a maximal clique of type IV. For its size we have $\#\mathcal{C}_{IV}(S) = \left[{e+2 \atop 1}\right]$.

4 Automorphism group

By the definition of the twisted Grassmann graph, it follows easily that the group $\mathsf{P\Gamma L}(V)_H$ acts as an automorphism group on $\tilde{J}_q(2e+1,e)$. Van Dam and Koolen conjectured that this is the full automorphism group. In this section we show that this is indeed the case, by showing the following theorem.

Theorem 4.1. *Let $e \geq 2$ be an integer and q a prime power. The full automorphism group of $\tilde{J}_q(2e+1,e)$ equals $\mathsf{P\Gamma L}(V)_H$, where V is the $(2e+1)$-dimensional vector space over $\mathbb{F}_q$ and H the fixed hyperplane in V, as in the definition of the twisted Grassmann graph.*

Before we show this theorem we recall the automorphism group of the Grassmann graphs.

Let V be a n-dimensional vector space over $\mathbb{F}_q$ and $1 \leq e \leq n-1$. The Grassmann graph $J_q(n,e)$ is the graph whose vertices are e-dimensional subspaces of V, and whose adjacency relation is defined as follows: for vertices W_1 and W_2, we have $W_1 \sim W_2$ if and only if $\dim(W_1 \cap W_2) = e-1$.

Theorem 4.2. *(Chow [3], cf. [1, Thm 9.3.1]) Let Γ be the Grassmann graph $J_q(n,e)$, and suppose that Γ is not complete, i.e., $1 < e < n-1$. Then*

$$\operatorname{Aut}\Gamma \cong \begin{cases} \mathsf{P\Gamma L}(V) & \text{if } n \neq 2e\,, \\ \mathsf{P\Gamma L}(V).2 & \text{if } n = 2e\,. \end{cases}$$

Proof of Theorem 4.1. Let G be the full automorphism group of $\tilde{J}_q(2e+1,e)$. First we will show the following claim.

Claim. (i) *$\mathcal{B}_1$ and $\mathcal{B}_2$ are the orbits of $\mathcal{B}$ under G.*

(ii) *For each maximal clique, G preserves its type.*

Proof of Claim. Clearly, G preserves the size of maximal cliques, and hence it preserves the types if $e > 2$, as each type has a different size. For $e = 2$, this implies that the types II and IV are preserved. The group $\mathsf{P\Gamma L}(V)_H$ has orbits $\mathcal{B}_1$ and $\mathcal{B}_2$ on $\mathcal{B}$. As cliques of type IV only contain vertices in $\mathcal{B}_1$, it follows that $\mathcal{B}_1$ and $\mathcal{B}_2$ are the orbits of $\mathcal{B}$ under G. This shows (i). In order to finish the proof for this step we need to show that for $e = 2$ and a maximal clique $\mathcal{C}$ of size $\begin{bmatrix}3\\1\end{bmatrix}$ the type of this clique is preserved. If $\mathcal{C}$ is of type I then it contains exactly $\begin{bmatrix}e\\1\end{bmatrix}$ vertices in $\mathcal{B}_2$, whereas if $\mathcal{C}$ is of type III then it contains exactly one element of $\mathcal{B}_2$. This shows that (ii) is true for $e = 2$. This concludes the proof of the claim. □

Define the graph Δ on the maximal cliques of type I. If S, S' are e-dimensional subspaces of V, then $\mathcal{C}_I(S) \sim \mathcal{C}_I(S')$ in Δ, if $\mathcal{C}_I(S) \cap \mathcal{C}_I(S') \neq \emptyset$. Note that $\mathcal{C}_I(S) \sim \mathcal{C}_I(S')$ if and only if $\dim(S \cap S') = e - 1$. Therefore, the graph Δ is isomorphic to the Grassmann graph $J_q(2e+1, e)$, so its automorphism group $\overline{G}$ equals $\mathsf{P\Gamma L}(2e+1, q)$, by Chow's Theorem.

Any automorphism of Γ induces naturally to an automorphism of Δ. Let ψ be an automorphism of Γ whose induced automorphism equals the identity. Then for $B_1 \in \mathcal{B}_1$, take S, S' two e-dimensional subspaces of V, both not in H, such that $B_1 = S + S'$. As $\mathcal{C}_I(S) \cap \mathcal{C}_I(S') = \{B_1\}$, it follows that ψ fixes $\mathcal{B}_1$ pointwise. Now let $B_2 \in \mathcal{B}_2$. Then, let S and S' be e-dimensional subspaces of H such that $S \cap S' = B_2$. Then $\mathcal{C}_I(S) \cap \mathcal{C}_I(S') \cap \mathcal{B}_2 = \{B_2\}$. This implies that ψ fixes $\mathcal{B}_2$ pointwise and hence ψ is the identity. This shows that $\#G \leq \#\,\mathsf{P\Gamma L}(V)_H$ (as an induced automorphism has to fix the hyperplane H) and hence $G = \mathsf{P\Gamma L}(V)_H$. □

5 Antipodal covers

In this section we show that the twisted Grassmann graphs can not have antipodal distance-regular covers.

Theorem 5.1. *For $e \geq 2$, the twisted Grassmann graphs $\tilde{J}_q(2e+1, e)$ do not have any antipodal distance-regular covers.*

Proof. As $\{\mathcal{B}_1, \mathcal{B}_2\}$ is an equitable partition of $\Gamma := \tilde{J}_q(2e+1, e)$ it follows that $\mathcal{B}_1$ is a completely regular code of Γ with covering radius 1 and hence $s^* = 1$. Moreover, as its width w equals $e - 1$, and e is the diameter of Γ, it follows that $w + s^* = e$. Now by [2, Corollary 2], the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$ has no antipodal distance-regular cover of diameter at least 5.

In order to show the theorem, we only need to show that $\tilde{J}_q(5,2)$ has no antipodal distance-regular cover of diameter 4. For $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that $d(B_1, B_2) = 2$ it is easy to check that the induced subgraph on the common neighbours is connected. This shows that $\tilde{J}_q(5,2)$ can not have any antipodal distance-regular cover of diameter 4. This concludes the proof. □

Remark 5.2. That the twisted Grassmann graphs do not have antipodal distance-regular covers of diameter at least 7, also follows from [5]

6 Halved graphs

In this section we show that, in contrast to the Grassmann graph $J_q(2e+1, e)$, any twisted Grassmann graph is not the halved graph of a bipartite distance-regular graph:

Theorem 6.1. *For q a prime power and e an integer at least 2, the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$ is not the halved graph of a bipartite distance-regular graph.*

Proof. We show the statement by contradiction. To this end, suppose that there exists a bipartite graph Δ with colour classes V_R and V_B such that the halved graph with vertex set V_B is the twisted Grassman graph $\Gamma = \tilde{J}_q(2e+1, e)$. As the diameter of Γ equals $e \geq 2$, it follows that the diameter of Δ is at least 5. Let $x \in V_R$, and let $\Delta(x)$ be the set of neighbours in Δ. Then it is easy to see that $\Delta(x)$ has the following properties:

(i) $\Delta(x)$ forms a completely regular code in Γ;

(ii) The subgraph of Γ induced on $\Delta(x)$ forms a maximal clique in Γ.

The maximal cliques in Γ are known, see Subsection 3.1. We first show that a maximal clique of type (III) is not possible. Let S be a subspace of V of dimension $e+2$ not contained in H and let S' be an $(e-1)$-dimensional subspace of $S \cap H$. Then $\mathcal{C}_{III}(S, S') := \{B \in \mathcal{B}_1 \mid S' \subseteq B \subseteq S\} \cup \{S'\}$ is a maximal clique of type III. We now show that $C_{III}(S, S')$ is not completely regular. In order to show this let $U \in \mathcal{B}_2$ such that $U \subseteq H \cap S$ and $\dim(U \cap S') = e-2$. Now U has more than one neighbour in $C_{III}(S, S')$. On the other hand, let $W \in \mathcal{B}_2$ such that $W \not\subseteq H \cap S$ and $\dim(W \cap S') = e-2$. Then W has exactly one neighbour in $C_{III}(S, S')$. This shows that no clique of type III is completely regular.
Hence we have

$$k(\Delta) \in \left\{ \begin{bmatrix} e+2 \\ 1 \end{bmatrix}, \begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1, \begin{bmatrix} e+1 \\ 1 \end{bmatrix} + 1, \begin{bmatrix} e+1 \\ 1 \end{bmatrix} \right\}$$

If $k(\Delta) = \begin{bmatrix} e+2 \\ 1 \end{bmatrix}$ then $\Delta(x)$ is maximal clique of type IV for all $x \in V_R$. But this is impossible as cliques of type IV only contain vertices in $\mathcal{B}_2$. Similarly $k(\Delta) \neq \begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1$. As the total number of maximal cliques of type I equals $\begin{bmatrix} 2e+1 \\ e \end{bmatrix}$, and there are two sizes for type I, depending whether $S \subseteq H$ or not, and both sizes occur, it follows that $\Delta(x)$ can not be a clique of type I. This completes the proof. □

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