THE LIE ALGEBRAS WITH A QUOTIENT TRACE FORM

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1. Introduction

Suppose that F is an algebraically closed field of characteristic $p > 3$. The main purpose of this paper is the determination of all Lie algebras of the form $L/L^{\perp}$, where L is a Lie algebra over F and $L^{\perp}$ is the radical of a trace form on L. By a *trace form* on L is meant a bilinear form f on L for which there is a representation Δ of L such that $f(a, b) = \operatorname{tr}(\Delta(a)\Delta(b))$ for all a, b in L. If f is a trace form on a Lie algebra L, then $L^{\perp}$ (the set of all a in L for which $f(a, b) = 0$ for all b in L) is an ideal of L and f induces a bilinear form $\bar{f}$ on the quotient algebra $\bar{L} = L/L^{\perp}$. By a *quotient trace form* is meant any bilinear form $\bar{f}$ on a Lie algebra $\bar{L}$ arising in the above way from a trace form f on an algebra L such that $\bar{L} = L/L^{\perp}$; a quotient trace form is in particular a nondegenerate symmetric invariant form.

It has been shown by the author in [1] that if $\bar{L}$ is a simple Lie algebra over F with a quotient trace form, then $\bar{L}$ is of classical type, as defined by Mills and Seligman [4], and so is a simple analogue over F of a simple Lie algebra over the complex numbers. Among these simple algebras with a quotient trace form are the algebras of *type PA*—a Lie algebra over F is said to be of type PA if for some multiple n of p, $L \cong PSM(n, F)$, the Lie algebra of all $n \times n$ matrices of trace 0, modulo scalar matrices.

Zassenhaus [7] has examined the structure of arbitrary Lie algebras over F with a quotient trace form. He showed that if $\bar{L}$ is such an algebra, then it is a direct sum of mutually orthogonal, orthogonally indecomposable algebras, each with a quotient trace form; and that if moreover $\bar{L}$ is orthogonally indecomposable but is neither 0, 1-dimensional nor simple, then the center $z\bar{L}$ has the same dimension as the quotient $\bar{L}/D\bar{L}$ of $\bar{L}$ by the derived algebra; $0 \subset z\bar{L} \subset D\bar{L}$, and $D\bar{L}$ is the sum of mutually orthogonal perfect ideals $\bar{L}_1, \cdots, \bar{L}_m$ of $\bar{L}$ such that there is the decomposition

$$D\bar{L}/z\bar{L} = \textstyle\sum_{i=1}^{m} (\bar{L}_i + z\bar{L})/z\bar{L}$$

of $D\bar{L}/z\bar{L}$ into the direct sum of the m ideals $(\bar{L}_i + z\bar{L})/z\bar{L}$, each of which is simple with a quotient trace form.

It has not been known whether it is ever possible that $m > 1$. Zassenhaus and the author [3] have shown that this cannot happen if the trace form is nondegenerate (that is, if $L^{\perp} = 0$), and indeed have shown that if $L = \bar{L}$ is orthogonally indecomposable but is neither abelian nor simple, then L is isomorphic to a total matrix algebra $M(n, F)$ with p dividing n. In Section 2 below we shall construct a class of indecomposable algebras over F with a

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quotient trace form, such that $m > 1$; indeed for an arbitrary family $\bar{L}_1, \cdots, \bar{L}_m$ of simple algebras of type PA, there are indecomposable algebras $\bar{L} = \bar{L}(q_1, \cdots, q_m, Y, n_1, \cdots, n_m)$ of this class such that $D\bar{L}/z\bar{L}$ is the direct sum of $\bar{L}_1, \cdots, \bar{L}_m$. These algebras generalize the algebras $PSM(n, F)$ and $M(n, F)$ (with $p \mid n$); for an uncomplicated example, let L be represented by all $2p \times 2p$ matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where A and B are $p \times p$ and $\operatorname{tr} B = \alpha \operatorname{tr} A$ for a fixed scalar α; then $L^{\perp}$ is 1-dimensional, $\bar{L} = L/L^{\perp}$ is indecomposable, but $D\bar{L}/z\bar{L}$ is not simple.

It will be shown below that every Lie algebra over F with a quotient trace form is a direct sum of algebras which are either 1-dimensional, simple of classical type, or isomorphic to one of the algebras

$$\bar{L}(q_1, \cdots, q_m, Y, n_1, \cdots, n_m)$$

constructed in Section 2. All of these algebras do have a quotient trace form, except possibly for the simple algebra of type E_8 when $p = 5$.

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2. A class of nonsimple indecomposable algebras with quotient trace form

Let F be a field of characteristic $p > 2$, and let $q_1, \cdots, q_m$ be m positive integers, not necessarily distinct, each of which is a multiple of p. For $i = 1, \cdots, m$, let N_i be the Lie algebra $M(q_i, F)$ of all $q_i \times q_i$ matrices, and let N be the direct sum

$$N = N_1 \oplus \cdots \oplus N_m .$$

For $i = 1, \cdots, m$, define elements z_i and x_i in N by writing

$$z_i = (0, \cdots, 0, I_i, 0, \cdots, 0), \qquad x_i = (0, \cdots, 0, (E_{11})_i, 0, \cdots, 0),$$

where I_i and $(E_{11})_i$ denote elements of N_i which are, respectively, the identity matrix and the matrix with 1 in the $(1, 1)$ place and 0 elsewhere. Also for any a in N, denote the component of a in N_i by a^i:

$$a = (a^1, \cdots, a^i, \cdots, a^n).$$

Now let Y be a subspace of the m-dimentional space spanned by $x_1, \cdots, x_m$, and let L be the Lie subalgebra of N generated by Y and the derived algebra DN of N; we shall write $L = L(q_1, \cdots, q_m, Y)$.

We shall now determine some trace forms on L. For $i = 1, \cdots, m$, let n_i be an integer such that $1 \leq n_i < p$, and let Δ_i be the representation of N which maps each element of N onto its component in N_i. Let Δ be the direct sum of $n_1 + \cdots + n_m$ representations of N, n_i of which are Δ_i $(i = 1, \cdots, m)$.

Thus Δ is a faithful fully reducible representation of N of degree $\sum_{i=1}^{m} n_i q_i$. Let f denote the trace form induced on L by Δ. Thus

$$f(a, b) = \sum_{i=1}^{m} n_i \operatorname{tr}(a^i b^i).$$

Write $L^{\perp}$ for the trace radical of L with respect to f. Denote $L/L^{\perp}$ by $\bar{L} = \bar{L}(q_1, \cdots, q_m, Y, n_1, \cdots, n_m)$, and write π for the natural mapping of L onto $\bar{L}$.

Here are some properties of L and $\bar{L}$, property (a) being obvious:

(a) *The algebra L, as a vector space, is the direct sum of Y and DN; $DL = DN = DN_1 \oplus \cdots \oplus DN_m$; the restriction Δ^L of Δ to L is faithful and fully reducible; and for each irreducible constituent Γ of Δ^L, $L \neq L_{\Gamma}^{\perp}$ (the radical of the trace form of Γ).*

(b) *$L^{\perp}$ consists of all linear combinations $\sum_{i=1}^{m} \alpha_i z_i$ such that $\sum_{i=1}^{m} n_i \alpha_i \beta_i = 0$ whenever $\sum_{i=1}^{m} \beta_i x_i \in Y$ ($\alpha_i, \beta_i \in F$).*

Proof. If $a \in L$ and if, for some i, $a^i \notin (I_i)$, then there is a b in DN_i such that $f(a, b) = n_i \operatorname{tr}(a^i b) \neq 0$. Hence every element of $L^{\perp}$ is a linear combination of $z_1, \cdots, z_m$. But for each i, $f(z_i, DN) = 0$. Thus $a = \sum_{i=1}^{m} \alpha_i z_i \in L^{\perp}$ if and only if $f(a, Y) = 0$. Since $f(\sum_i \alpha_i z_i, \sum_i \beta_i x_i) = \sum_i n_i \alpha_i \beta_i$, (b) holds.

It follows that if α is a nonzero element of the prime field F_p, and if each n_i is replaced by n_i', where, for the residue classes modulo p, $(n_i')_p = \alpha(n_i)_p$, then $L^{\perp}$ and $\bar{L}$ remain unchanged.

(c) *The elements $z_1, \cdots, z_m$ form a basis for zL; the radical of L is zL; and $z\bar{L} = \pi(zL)$.*

Proof. Suppose that $a \in zL$. If $a^i \in DN_i$ then $a^i \in z(DN_i)$, while if $a^i \notin DN_i$ then a^i and DN_i span N_i so that $a^i \in zN_i$. Since $z(DN_i) = zN_i = (I_i)$, the first statement follows. Now if K is a solvable ideal of L and if $a \in K$, one sees easily that $a^i \in (I_i)$ ($i = 1, \cdots, m$), so that $K \subseteq zL$. Since $L^{\perp}$ is abelian, $\pi^{-1}(z\bar{L})$ is solvable and hence equals zL, and (c) holds.

(d) *If Y has dimension k, then $L^{\perp}$ has dimension $m - k$, and $\bar{L}/D\bar{L}$ and $z\bar{L}$ have dimension k.*

Proof. The form f induces a bilinear mapping of $(zL) \times Y$ into F with right annihilator 0. By (b), the left annihilator of this mapping is $L^{\perp}$, and (d) follows from this.

(e) *$D\bar{L}/z\bar{L}$ is the direct sum of m simple algebras of type PA, isomorphic to $PSM(q_1, F), \cdots, PSM(q_m, F)$, respectively.*

Proof. This follows from (c) and the second statement of (a).

THEOREM 2.1. *The algebra $\bar{L} = \bar{L}(q_1, \cdots, q_m, Y, n_1, \cdots, n_m)$ defined above is the direct sum of two proper ideals if and only if there is a nonempty* *proper subset S of the set $\{1, \cdots, m\}$ such that, for every y in Y,*

$$\sum_{i \in S} y^i \in Y \tag{2.1}$$

(where y^i denotes the component of y in N_i). If this condition holds then L, as well as $\bar{L}$, is the direct sum of two proper orthogonal ideals.

Proof. Suppose that $\bar{L}$ is the direct sum of proper ideals $\bar{K}_0$ and $\bar{K}_1$. For $j = 0, 1$, let $K_j = \pi^{-1}(\bar{K}_j)$; thus $K_0 + K_1 = L$ and $K_0 \cap K_1 = L^{\perp}$. For any i, if there is an a in K_j such that $a^i \notin (I_i)$, then by multiplication by a suitable element of DN_i we obtain an element b in $K_j \cap DN_i$ such that $b \notin (I_i)$. Since (I_i) is the only proper nonzero ideal of DN_i, it follows that for any i, either $DN_i \subseteq K_0$ or $DN_i \subseteq K_1$. Now let $S = \{i \mid DN_i \subseteq K_0\}$. If S were empty, we would have $K_0 \subseteq zL \subseteq DL \subseteq K_1$, $K_0 = L^{\perp}$ and $\bar{K}_0 = 0$. Similarly if $S = \{1, \cdots, m\}$ then $\bar{K}_1 = 0$. Hence S is a proper nonempty subset of $\{1, \cdots, m\}$. Now suppose that $y = \sum_{i=1}^{m} \beta_i x_i \in Y$. Then $y = y_0 + y_1$, where $y_j \in K_j$ $(j = 0, 1)$. If $i \notin S$, then $y_0^i \in (I_i) \subseteq DN_i$. It follows that there is an a in DN such that $y_0 - a = \sum_{i \in S} \beta_i x_i$. Since $DN \subseteq L$, $a \in L$, so that $y_0 - a \in L$, and, by (a), (2.1) holds.

Conversely if (2.1) holds, then clearly L (respectively $\bar{L}$) is the orthogonal direct sum of proper ideals K_0 and K_1 (respectively, $\pi(K_0)$ and $\pi(K_1)$) where

$$K_0 = L \cap \sum_{i \in S} N_i, \qquad K_1 = L \cap \sum_{i \in S} N_i,$$

and the theorem is proved.

Corollary 2.1. *For any positive integers m and k, with $k < m$, there are indecomposable algebras $\bar{L}$ over F with a quotient trace form such that $D\bar{L}/z\bar{L}$ is a direct sum of m simple algebras and* $\dim z\bar{L}(= \dim \bar{L}/D\bar{L}) = k$.

Proof. With the above N, and k such that $1 \leq k < m$, let

$$Y = \{\textstyle\sum_{i=1}^{m} \beta_i x_i \mid \beta_{k+1} = \cdots = \beta_m = \sum_{i=1}^{k} \beta_i\}.$$

Then Y is k-dimensional and it may be seen that the decomposability condition of Theorem 2.1 does not hold. By (d) and (e) the corollary follows.

3. Determination of DL

A Lie algebra L over a field F of characteristic p will be called a *nonsimple algebra of type A* if there is a multiple n of p such that $L \cong SM(n, F)$ (the $n \times n$ matrices of trace 0).

Lemma 3.1. *Let L be a Lie algebra over an algebraically closed field of characteristic $p \neq 2, 3$. Let Δ be a representation of L such that $\bar{L} = L/L^{\perp}$ does not decompose into an orthogonal direct sum, and such that $L^{\perp} \subseteq zL$. Then*

(a) *If $\bar{L}$ is abelian then L is abelian.*

(b) *If $\bar{L}$ is simple but not of type PA then $\bar{L}$ is of classical type and L is the direct sum of $L^{\perp}$ and an algebra isomorphic to $\bar{L}$.*

(c) *If $\bar{L}$ is simple of type PA then L is the direct sum of a subalgebra of $L^\perp$ and a nonsimple algebra of type A, with center contained in $L^\perp$.*

(d) *If $\bar{L}$ is neither simple nor abelian then DL is the direct sum of a subalgebra of $L^\perp$ and mutually orthogonal ideals of L which are nonsimple algebras of type A.*

Proof. (a) $\bar{L}$ is 1-dimensional since it is indecomposable. If L were not abelian then zL would have codimension one, which is impossible.

(b) Since $\bar{L}$ is a simple algebra with a quotient trace form, it is of classical type by [1]. It is proved in [2] (see also [5]) that if M is an extension of an algebra $\bar{M}$ by a kernel contained in zM, and if $\bar{M}$ is either simple of classical type or a nonsimple algebra of type A, then either the extension is trivial, or else $\bar{M} \cong PSM(n, F)$ $(p \mid n)$ and M is a trivial extension of a copy of $SM(n, F)$. The conclusion of (b) follows since L is a central extension of $\bar{L}$.

(c) It is not possible for L to be the direct sum of $L^\perp$ and an algebra of type PA, since algebras of type PA do not have a nondegenerate trace form by [1, Theorem 6.2]. The conclusion of (c) now follows by the result of [2] stated above.

(d) By [7, Theorem 3], $0 \neq z\bar{L} \subset D\bar{L}$, and $D\bar{L}$ is the sum of mutually orthogonal perfect ideals $\bar{K}_1, \cdots, \bar{K}_m$ of $\bar{L}$ such that there is a decomposition

$$D\bar{L}/z\bar{L} = \sum_{i=1}^{m} (\bar{K}_i + z\bar{L})/z\bar{L}$$

of $D\bar{L}/z\bar{L}$ into the direct sum of the ideals $(\bar{K}_i + z\bar{L})/z\bar{L}$, each of which is simple. Let π denote the natural homomorphism of L onto $\bar{L}$, and, for each i, write $K_i^* = \pi^{-1}(z\bar{K}_i)$. Thus K_i^* is a solvable ideal of L. By [7, p. 66], for each irreducible constituent Γ of Δ, either $L = L_\Gamma^\perp$ (the radical of the trace form of Γ) or $\Gamma(K_i^* L) = 0$. Hence $K_i^* L \subseteq L_\Gamma^\perp$, so that $K_i^* L \subseteq L^\perp$, and $z\bar{K}_i \subseteq z\bar{L}$. But obviously $\bar{K}_i \cap z\bar{L} \subseteq z\bar{K}_i$, so that $z\bar{K}_i = \bar{K}_i \cap z\bar{L}$. Thus

$$\bar{K}_i / z\bar{K}_i = \bar{K}_i/(\bar{K}_i \cap z\bar{L}) \cong (\bar{K}_i + z\bar{L})/z\bar{L},$$

so that $\bar{K}_i/z\bar{K}_i$ is simple $(i = 1, \cdots, m)$.

Denote by f the restriction to $D\bar{L}$ of the quotient trace form on $\bar{L}$. Since $D\bar{L}$ is perfect, the radical of f is $z\bar{L}$. For each i, $\bar{K}_i \not\subseteq z\bar{L}$; since the $\bar{K}_i$ are orthogonal, it follows that f does not vanish identically on $\bar{K}_i$. But $\bar{K}_i$ is perfect and $\bar{K}_i/z\bar{K}_i$ is simple, and therefore the restriction of f to $\bar{K}_i$ has radical $z\bar{K}_i$. Thus $\bar{K}_i/z\bar{K}_i$ is a simple algebra with a quotient trace form and hence by [1] is of classical type. Moreover $z\bar{K}_i \neq 0$, since otherwise $\bar{L}$ (which is assumed to be orthogonally indecomposable) would be the direct sum of $\bar{K}_i$ and the orthogonal complement of $\bar{K}_i$ in $\bar{L}$. Since each $\bar{K}_i$ is perfect, it now follows from the result of [2] stated above that each $\bar{K}_i$ is nonsimple of type A.

Now for $i = 1, \cdots, m$, let $K_i = \pi^{-1}(\bar{K}_i)$. Then, again by [2], K_i is the direct sum of DK_i and $L^\perp$, and DK_i is nonsimple of type A. Also

$$DL + L^\perp = K_1 + \cdots + K_m = L^\perp + DK_1 + \cdots + DK_m .$$

Now write $\bar{L}_1 = \bar{K}_i$ and $\bar{L}_2 = \sum_{j \neq i} \bar{K}_j$ for some i. Then $\bar{L}_1$ and $\bar{L}_2$ are orthogonal perfect ideals of $\bar{L}$ whose sum is $D\bar{L}$, so that [3, Lemma 4.1] is applicable, and shows that $DL_1 \cap DL_2 \subseteq L^{\perp}$ (where $L_j = \pi^{-1}(\bar{L}_j), j = 1, 2$). But $DL_1 = DK_i$, $DK_i \cap L^{\perp} = 0$, and $(\sum_{j\neq i} DK_j) \subseteq DL_2$, and consequently $DK_i \cap (\sum_{j\neq i} DK_j) = 0$. Since this holds for $i = 1, \cdots, m$, the sum $DK_1 + \cdots + DK_m$ is direct. The ideals DK_i are obviously mutually orthogonal, and since

$$DK_1 + \cdots + DK_m \subseteq DL \subseteq L^{\perp} + DK_1 + \cdots + DK_m,$$

it follows that DL is the direct sum of $DK_1, \cdots, DK_m$ and a subalgebra of $L^{\perp}$, and the lemma is proved.

4. Determination of L when $\bar{L}$ is indecomposable

LEMMA 4.1. *Let L be a Lie algebra over a field F of characteristic $p > 3$. Suppose that DL is the direct sum of algebras $K_0, K_1, \cdots, K_m$, where $K_0 \subseteq zL$ and $K_i \cong SM(q_i, F)$, with q_i a multiple of p, for $i = 1, \cdots, m$. Suppose moreover that every Cartan subalgebra of L is abelian. Then $K_0 = 0$, and L is the direct sum of an abelian algebra and an algebra isomorphic to one of the algebras $L(q_1, \cdots, q_m, Y)$ of Section 2.*

Proof. For $i = 1, \cdots, m$ we assume for convenience that $K_i = SM(q_i, F)$, and write σ_i for the derivation of K_i induced by right multiplication by the matrix E_{11} in $M(q_i, F)$. Thus [6, Satz 20, p. 60] every derivation of K_i differs by an inner derivation from a scalar multiple of σ_i. Now take a basis $\{b_1, \cdots, b_r, c_1, \cdots, c_s, a_1, \cdots, a_k\}$ of L such that $b_1, \cdots, b_r$ span DL, $c_1, \cdots, c_s$ are in zL, and $b_1, \cdots b_r, c_1, \cdots, c_s$ span $DL + zL$. By subtracting suitable elements of DL from the a_i's we may suppose that each ad a_i induces a scalar multiple of σ_j on K_j:

$$((\text{ad } a_i) \mid K_j) = \alpha_{ij}\,\sigma_j, \qquad i = 1, \cdots, k; \;\; j = 1, \cdots, m.$$

Now let H be the subspace of L spanned by $c_1 \cdots, c_s, a_1, \cdots, a_k$ together with all elements of DL whose components in $K_1, \cdots, K_m$ are diagonal matrices. Since all $a_i\, a_j$ are in $z(DL)$ and since σ_j annihilates all diagonal matrices in K_j, $DH \subseteq zL \subseteq H$. Hence H is a subalgebra of L and is nilpotent. It is easy to see that H is a Cartan subalgebra of L—indeed, each nondiagonal matrix unit in K_j $(j = 1, \cdots, m)$ spans a 1-dimensional root space for a nonzero root. By hypothesis H must be abelian; therefore $a_i\, a_j = 0$ $(i, j = 1, \cdots, k)$, $DL = K_1 + \cdots + K_m$ and $K_0 = 0$.

Now take the direct sum M of an s-dimensional abelian algebra A and the algebra $L(q_1, \cdots, q_m, Y)$ where Y is the space spanned by the k elements $y_i = \sum_{j=1}^{m} \alpha_{ij}\, x_j$, $i = 1, \cdots, k$. Map each element of DL onto itself, considered as an element of $DL(q_1, \cdots, q_m, Y)$, and extend this to a linear mapping τ of L onto M by letting τ map $c_1, \cdots, c_s$ onto a basis of A and setting $\tau(a_i) = y_i$ $(i = 1, \cdots, k)$. Since ad y_i acts on $DL(q_1, \cdots, q_m, Y)$ in the same way as ad a_i acts on DL and since all $a_i\, a_j$ vanish, τ preserves multiplication. To show that τ is an isomorphism, it remains to prove that its restriction to the space $(a_1, \cdots, a_k)$ is one-to-one. But if $\tau(a) = 0$, where a is a linear combination of $a_1, \cdots, a_k$, then ad a vanishes on DL, so that $a \in zL \subseteq (b_1, \cdots, b_r, c_1, \cdots, c_s)$, and hence $a = 0$. Therefore τ is an isomorphism of L onto M, and the lemma is proved.

We now put together the information of the preceding lemmas.

LEMMA 4.2. *Let L be a nonabelian Lie algebra over an algebraically closed field of characteristic $p > 3$, with a faithful fully reducible representation Δ. Suppose that $\bar{L} = L/L_{\Delta}^{\perp}$ does not decompose into a direct sum of mutually orthogonal ideals, that $L_{\Delta}^{\perp} \subseteq zL$, and that for each irreducible constituent Γ of Δ, $L_{\Gamma}^{\perp} \neq L$. Then L is the direct sum of an abelian algebra and an algebra K, where either K is simple of classical type other than PA or K is isomorphic to one of the algebras $L(q_1, \cdots, q_m, Y)$ of Section 2.*

Proof. Suppose the hypotheses hold. Then [7, Lemma 4, pp. 70–71] every Cartan subalgebra of L is abelian. By Lemma 3.1(b) and (c), if $\bar{L}$ is simple then the conclusion of the present lemma holds, since the nonsimple algebras of type A are just the algebras $L(q_1, Y)$ with $Y = 0$. By Lemma 3.1(d) if $\bar{L}$ is not simple then the hypotheses of Lemma 4.1 hold, so that again L has the specified form, and the proof is complete.

5. Determination of $\bar{L}$

LEMMA 5.1. *Let L be the Lie algebra of all $r \times r$ matrices of trace 0 over a field F of characteristic $\neq 2$, and let Δ be a representation of L. Then there exists a nonnegative integer n, with $n < p$ if the characteristic is a prime p, such that*

$$\operatorname{tr}(\Delta(a)\Delta(b)) = n \operatorname{tr}(ab), \qquad a, b \in L.$$

Proof. We may assume that $r > 1$, and, by extending the base field, that F is algebraically closed. Let f denote the trace form of Δ. Since L is perfect, f induces an invariant form on L/zL; since L/zL is simple, an invariant form on it is unique up to scalar multiple. Hence there is an α in F such that $f(a, b) = \alpha \operatorname{tr}(ab)$ for all a, b in L. Take the 3-dimensional subalgebra L_1 of L spanned by E_{12}, $-E_{21}$ and $h = E_{22} - E_{11}$. Let g denote the trace form of the given natural representation of L_1, let $\Delta_1, \cdots, \Delta_k$ be the irreducible constituents of the restriction of Δ to L_1, and let f_i be the trace form of Δ_i. Then $f_i = \alpha_i g$ for some α_i in F, and $\alpha = \sum_{i=1}^{k} \alpha_i$. If F has characteristic 0, then for each i, Δ_i is one of the well known representations of L_1, for each of which $f_i(h, h)$ is a nonnegative even integer, so that α is a nonnegative integer. If F has characteristic p (> 2) and if Δ_i is restricted, then $\Delta_i(h)$ acts diagonally with characteristic roots in F_p, so that $f_i(h, h) \in F_p$ and hence $\alpha_i \in F_p$, while if Δ_i is not restricted, then, by Lemma 5.1 of [1] (or by its proof if $p = 3$), again $\alpha_i \in F_p$ (indeed, $\alpha_i = 0$ if $p > 3$). Hence $\alpha \in F_p$, and the conclusion of the lemma follows immediately.

An extension of the above proof also shows that if t is a nondegenerate trace form on a simple Lie algebra L of characteristic $\neq 2$ then any trace form on L is a multiple of t by an element of the prime field.

We now prove our main result.

THEOREM 5.1. *Let $\bar{L}$ be a Lie algebra with a quotient trace form, over an algebraically closed field of characteristic $p > 3$. Then $\bar{L}$ is the direct sum of indecomposable algebras which are either abelian, simple of classical type, or isomorphic to one of the algebras $\bar{L}(q_1, \cdots, q_m, Y, n_1, \cdots, n_{m-1}, 1)$ of Section 2. Conversely, each of these algebras has a quotient trace form, provided either that $p > 5$ or that none of the direct summands is if type E_8.*

Proof. Since $\bar{L}$ is a direct sum of mutually orthogonal ideals, each of which has a quotient trace form and is orthogonally indecomposable, we may assume, without loss of generality, that $\bar{L}$ does not decompose into a direct sum of mutually orthogonal proper ideals. By Theorem 2 of [7] and its proof, we may choose L and Δ such that $\bar{L} = L/L_\Delta^\perp$ and such that the hypotheses of Lemma 4.2 hold; in particular, $L_\Delta^\perp \subseteq zL$. Now if L is the direct sum of an abelian algebra and a simple algebra of classical type then by the invariance of the trace form these algebras are orthogonal and hence $\bar{L}$ would be simple of classical type. Thus by Lemma 4.2, there only remains to be considered the case in which L is the direct sum of an abelian algebra A and an algebra $M = L(q_1, \cdots, q_m, Y)$. We shall write $N_1, \cdots, N_m$ for the total matrix algebras of degrees $q_1, \cdots, q_m$ used in the construction of M in Section 2, and similarly x_i, z_i will have the same meaning as in Section 2. For $i = 1, \cdots, m$, let f_i be the trace form induced by Δ on DN_i. By Lemma 5.1, there are integers n_i, where $0 \leq n_i < p$, such that f_i is n_i times the natural trace form on DN_i. If some f_i vanished, then by the invariance of the trace form and the perfectness of the ideal DN_i, we would have

$$DN_i \subseteq L_\Delta^\perp \subseteq zL,$$

a contradiction. Hence $1 \leq n_i < p$ $(i = 1, \cdots, m)$.

Let f denote the trace form of Δ on L. If $y = \beta_1 x_1 + \cdots + \beta_m x_m \in Y$ and $a, b \in DN_i$ then

$$f([a, b], y) = f(a, [b, y]) = n_i \operatorname{tr}(a[b, y]) = n_i \beta_i \operatorname{tr}(a[b, x_i]) = n_i \beta_i \operatorname{tr}([a, b]x_i);$$

since $z_i \in DN_i$ and DN_i is perfect

$$f(z_i, y) = n_i \beta_i \operatorname{tr}(z_i x_i) = n_i \beta_i . \tag{5.1}$$

Therefore no nonzero element of Y is orthogonal to zM. It follows that if g is a linear functional on Y then there exists an element w in zM such that $f(w, y) = g(y)$ for all y in Y.

Now let $a_1, \cdots, a_k$ be a basis of A, and for $i = 1, \cdots, m$, let w_i be an element of zM such that $f(a_i, y) = f(w_i, y)$ for all y in Y. Then $a_1 - w_1, \cdots, a_k - w_k$ span an abelian algebra B such that L is the direct sum of B and M, and B is orthogonal to Y. Since DM is perfect, B is also orthogonal to DM and hence to M. Therefore $L^{\perp}$ is the direct sum of $L^{\perp} \cap B$ and $L^{\perp} \cap M$, and (since $\bar{L}$ is assumed to be orthogonally indecomposable) $\bar{L} \cong M/L^{\perp} \cap M$. Every element of $L^{\perp} \cap M$ is a linear combination of $z_1, \cdots, z_m$; since $f(z_i, DN) = 0$ for each i, $a = \sum_i^m \alpha_i z_i \epsilon L^{\perp} \cap M$ if and only if $f(a, Y) = 0$. But if $y = \sum_{i=1}^m \beta_i x_i \epsilon Y$, then, by (5.1),

$$f(\textstyle\sum_{i=1}^m \alpha_i z_i, y) = \sum_{i=1}^m n_i \alpha_i \beta_i .$$

Hence $L^{\perp} \cap M$ is the space described in (b) of Section 2, and

$$\bar{L} \cong \bar{L}(q_1, \cdots, q_m, Y, n_1, \cdots, n_m).$$

By the remark in Section 2 following (b), we may assume that $n_m = 1$. This completes the determination of $\bar{L}$; the converse part of the theorem follows from Section 2 and the known results on quotient trace forms on simple algebras.

We remark that since $PSM(n) \cong \bar{L}(n; 0; 1)$ (when $p \mid n$), and since

$$\begin{aligned}\bar{L}(q_1, \cdots, q_m, Y, n_1, \cdots, n_m) &\oplus \bar{L}(q_1', \cdots, q_s', Y', n_1', \cdots, n_s') \\ &= \bar{L}(q_1, \cdots, q_m, q_1', \cdots, q_s', Y \oplus Y', n_1, \cdots, n_m, n_1', \cdots n_s'),\end{aligned}$$

we may replace the indecomposability condition on the direct summands of Theorem 5.1 by the condition that no more than one of them is of type PA or isomorphic to one of the algebras of Section 2.

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