ON THE EMBEDDING OF COMPLEXES IN 3-SPACE[1]

BY

P. H. DOYLE

Introduction

This paper is a sequel to [5], [6], [7]. The example in [5] was discovered to be tame by R. H. Bing, but a similar and wild one can be found in [4].

By a complex we mean a finite geometric simplicial complex [11]. If K is such a complex and if h is a homeomorphism from K into a euclidean space E^n, $h(K)$ is said to be a topological complex with the understanding that the simplices of $h(K)$ are the topological images of the simplices of K under h. Similarly the i-skeleton of $h(K)$ is the image under h of the i-skeleton of K.

Our main result asserts that if K is a finite complex and if $h(K)$ is embedded in E^3, then $h(K)$ is tame [9] if and only if h carries the 1-skeleton of K onto a tame set while each 2-simplex of $h(K)$ is tame [2], [14]. Of course, if K is 1-dimensional the result applies only trivially; the 1-dimensional case is considered in [7].

The characterization of tame sets as locally tame in [2] and [14] permits a reduction of our problem to complexes which are stars; that is, to complexes which are the closed star of a vertex. In ¶1 certain special 2-complexes are shown to be tame. In ¶2, the special case of a 2-dimensional complex which is a star is studied. The theorem is then established in ¶3 by reducing the general 2- and 3-dimensional cases to the special case of ¶2.

1. Unions of disks and arcs

If S^2 is a 2-sphere in E^3, $\operatorname{ext}(S^2)$ denotes the component of $E^3 - S^2$ with non-compact closure, and $\operatorname{int}(S^2)$ is the other component. An n-frame is defined in [4] and is simply the topological image of a 1-complex which is the star of a vertex, the branch point of the n-frame. The 1-simplices of an n-frame are called its branches. If D is a simplex, Bd D is its boundary.

(1.1) LEMMA. *Let G be a tame n-frame and S^2 a tame 2-sphere in E^3. If two branches (one branch) of G lie in S^2 while the remainder of G lies in* $\operatorname{ext}(S^2)$, *then $G \cup S^2$ is a tame set.*

Proof. Let B_1 and B_2 be the branches of G which lie in S^2 while b is the branch point of G. Then $G_1 = (G - \bigcup_1^2 B_i) \cup b$ is a tame $(n-2)$-frame with branch point b and $G_1 \cap S^2 = b$. We show first that G_1 lies on a tame disk D, while $D \cap S^2 = b$.

Since G is tame, $G_1 \cup B_1$ is tame and there is a tame disk D_1 which contains

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$G_1 \cup B_1$ while $D_1 \cap B_2 = b$ and $B_1 \subset \text{Bd } D_1$. Then by the Schoenflies Theorem for E^2 there is a disk P in S^2 which contains $B_2 - b$ in its interior, $B_1 \cap \text{Bd } P = b$ and $P \cap D_1 = b$. Now let U be an open set in E^3 which contains $\text{int}(P)$, $U \cap D_1 = \square$. There is a homeomorphism g of E^3 onto E^3 which leaves $G_1 \cup B_2$ fixed and $g(S^2) \subset U \cup P$; this follows from the tameness of S^2. We note that $g(S^2) \cap D_1 = b$. Evidently the disk $D = g^{-1}(D_1)$ meets the conditions we required. The lemma will follow if we show that $D \cup S^2$ is tame.

If B_3 is a branch of G_1, then $B_3 \cup B_2$ is tame since G is tame. The branch B_2 lies on the boundary of a disk Q in $\overline{\text{int}(S^2)}$ such that Q is tame and $Q \cap S^2 = B_2$. Thus by selecting an arc J on Bd Q having b as an end point while $J - b \subset \text{int}(S^2)$ we see that $J \cup B_3$ is a tame arc piercing S^2 at b. Whence, by [13], $B_3 \cup S^2$ is tame.

There is no loss of generality in supposing that B_3 lies in the interior of D except for its two end points. We assume this is the case and let k be a homeomorphism of D onto a triangle T so that $k(b)$ is a vertex of T. Let $\{l_i'\}$ be a sequence of segments in T such that each l_i' is parallel to the side of T opposite $k(b)$ and spans Bd T; it is supposed that $k(B_3)$ is a segment and that $\{l_i'\}$ converges monotonically to $k(b)$. Then let $l_i = k^{-1}(l_i')$.

The set $S^2 \cup B_3$ is tame and so there is a homeomorphism f of E^3 onto E^3, which throws S^2 onto the boundary B of a tetrahedron while $f(B_3)$ is a segment meeting B orthogonally in the interior of a face F of the tetrahedron. If U_1 is any open set in E^3 containing $f(b)$, there is a value j such that $f(l_j) \subset U_1$ and the component C_b^2 of $f(D) - f(l_j)$ which contains $f(b)$ lies in U_1. The set $f(D) - C_b^2$ is a disk L^2 and by construction $f(S^2 \cup B_3) \cup L^2$ is locally tame and so tame [2], [14]. It follows that in U_1 there is a tame 3-cell C_u which meets F in a disk on Bd C_u, $C_u \subset \overline{\text{ext } B}$, Bd $C_u \cap L^2$ is a spanning arc of Bd $f(D)$ between $f(l_{j-1})$ and $f(l_j)$ with its end points on $f(l_j)$,

$$\text{Bd } C_u \cup f(S^2) \cup L^2 \cup f(B_3)$$

is tame and Bd $C_u \cap f(B_3)$ is a pair of points one of which is $f(b)$. We assert that C_u may be chosen so that Bd $C_u \cap C_b^2 = f(b)$. For if $\bar{C}_b^2 \cap \text{Bd } C_u$ is not $f(b)$ the tameness of $\bar{C}_b^2$ permits the removal of other intersections with Bd C_u by a homeomorphism of E^3 onto E^3 which is fixed outside of U and leaves L^2 and $f(S^2)$ fixed.

It is now possible to select a sequence of 3-cells $\{C_i\}$ with the following properties:

(i) $C_i \cup f(S^2) \cup f(B_3)$ is tame, $\bigcap C_i = f(b)$;
(ii) Bd $C_i \cap f(D)$ is an arc spanning Bd $f(D)$ plus $f(b)$;
(iii) Bd $C_i \cap f(B_3)$ is a pair of points;
(iv) $C_i \cap f(S^2)$ is a disk on Bd C_i while $C_{i+1} \subset C_i$ and

$$\text{Bd } C_{i+1} \cap \text{Bd } C_i \supset \text{int}(C_i \cap S^2).$$

Imagine a standard model M consisting of the boundary T_1 of a tetrahedron in E^3 which is met by a triangle T_2 at a point b_1 which is a vertex of T_2, $T_2 - b_1 \subset \text{ext } T_1$. One can clearly find a sequence of polyhedral 3-cells $\{C_i'\}$ meeting all conditions (i)–(iv) for the standard model. There is a homeomorphism of E^3 onto E^3 which carries $f(S^2 \cup D)$ onto M. This can be seen by noting that in the 3-cell $L_i = \overline{C_i - C_{i+1}}$, the segment $f(B_3) \cap L_i$ is unknotted and so the disk $f(D) \cap L_i$ is also unknotted in L_i. Thus we can define a homeomorphism f_1 from $\overline{E^3 - C_1}$ onto $\overline{E^3 - C_1'}$ such that $f_1(f(S^2)) = T_1$, $f_1(f(D) - C_1) = T_2 - C_1'$ and f_1 can be extended so that f_1 carries $f(D \cup S^2)$ onto M, by successive extensions to the L_i.

We write some corollaries to the proof of (1.1).

(1.2) COROLLARY. *Let D_1 and D_2 be tame disks in E^3 such that $D_1 \cap D_2 = p$, a point of both* Bd D_1 *and* Bd D_2. *If* Bd $D_1 \cup$ Bd D_2 is tame, *then $D_1 \cup D_2$ is tame.*

(1.3) COROLLARY. *Let S_1 and S_2 be tame 2-spheres in E^3 which meet in a point p. Then $S_1 \cup S_2$ is tame if and only if there is a tame arc J from a point of $S_1 - p$ to a point of $S_2 - p$, $J \subset S_1 \cup S_2$.*

Though this lemma and its corollaries have particular interest where Fox-Artin examples are concerned [9], their main use here will be in the characterization of tame complexes in general.

We extend Theorem 3 of [7].

(1.4) LEMMA. *Let $\{D_i\}$, where $i = 1, 2, \cdots, n$, be a finite collection of tame disks in E^3. If J is an arc on the boundary of each D_i, and if each pair of these disks meets in J only, then $Q = \bigcup_{i=1}^{n} D_i$ is tame.*

Proof. The case $n = 2$ is established in [7]. It will, therefore, be assumed that the theorem has been proved for $n < k$. Proceeding inductively let $n = k$. There is then no loss of generality in assuming that $B = \bigcup_{i=1}^{k-1} D_i$ is a polyhedron and that each D_i, for $i \leq k - 1$, is a polyhedral disk.

Since B is a polyhedron, B lies in a tame 3-cell C and all but at most two of the disks in B span the boundary of C. Further Bd $C \cup B$ is tame. Evidently C may be selected so that $C \cap D_k = J$. It follows from [14] that Bd $C \cup$ Bd D_k is tame and thus the argument in Theorem 3 of [7] can be applied to obtain a homeomorphism g of E^3 onto E^3 such that $g(C \cup D_k)$ is a polyhedron and $g(J)$ is a polygonal path. The disks $g(D_i)$ for $i \leq k - 1$ can now be made polyhedral without disturbing $g(D_k)$ by [14].

(1.5) COROLLARY. *Let K be a finite simplicial 2-dimensional geometric complex and h a homeomorphism from K in E^3. If h carries each 1-simplex and each 2-simplex of K onto a tame set in E^3, then $h(K)$ is locally tame except perhaps at its vertices.*

Proof. If σ^2 is a 2-simplex let $\text{int}(\sigma^2)$ denote its interior. By hypothesis if σ^2 is a 2-simplex of K, then $h(K)$ is locally tame at each point of $h(\mathrm{int}(\sigma^2))$. If σ' is a 1-simplex of K, then $h(K)$ is locally tame at each point of $h(\mathrm{int}(\sigma'))$ by (1.4). Thus, $h(K)$ is locally tame except perhaps at points corresponding to vertices of K.

We note that the converse of (1.2) is certainly false as shown by Example 1.1 of [9]. This example can be rendered 2-dimensional by the traditional "swelling of an arc".

2. Tame stars

In this paragraph we show that the general characterization of tame complexes hold for a 2-dimensional star-complex.

(2.1) THEOREM. *Let K be a* 1- *or* 2-*dimensional complex, v a vertex of K such that* $\mathrm{St}\, v = K$, *and let h be a homeomorphism of K into E^3. If $h(K)$ has a tame* 1-*skeleton and if each* 2-*simplex in $h(K)$ is tame, $h(K)$ is tame.*

Proof. We will establish this result by induction on k, the number of 2-simplices in K. If $k = 0$, then K is an n-frame and tame by hypothesis. Further let B_1 and B_2 be branches of K and suppose that S^2 is a tame 2-sphere such that

$$h(B_1) \cup h(B_2) \subset S^2$$

while

$$h(K) - h(B_1 \cup B_2) \subset \mathrm{ext}(S^2).$$

Then by (1.1), $S^2 \cup h(K)$ is tame.

Suppose we have proved (2.1) for all $k < j$ and that for all K having fewer than j 2-simplices it is true that for each tame 2-sphere S^2 meeting $h(K)$ in just two 1-simplices, while $h(K) \subset \overline{\mathrm{ext}\, S^2}$, $h(K) \cup S^2$ is tame. We suppose that K has j 2-simplices and that $h(K)$ meets the hypothesis of (2.1). Let σ^2 be a 2-simplex of K. Then σ^2 has a 1-simplex σ' which is opposite v, the center of the star. Let σ_1' and σ_2' be the other 1-simplices of K. Since $h(\sigma^2)$ is tame there is by Lemma 5.1 of [10] and the approximation theorem of Bing [3] a 2-sphere S^2 such that $S^2 \cap h(K) = h(\sigma_1' \cup \sigma_2')$, S^2 is locally polyhedral except at points of $h(\sigma_1' \cup \sigma_2')$,

$$h(K) - h(\sigma^2) \subset \mathrm{ext}(S^2), \quad \text{and} \quad h(\sigma^2) - h(\sigma_1' \cup \sigma_2') \subset \mathrm{int}(S^2).$$

Then S^2 is tame by [8]. Let K_1 be the complex obtained from K by deleting the interior of σ^2 and σ'. Then $h(K_1)$ has $(j - 1)$ 2-simplices and so is tame. Further $h(K_1) \cup S^2$ is tame by the inductive hypothesis. So there is a homeomorphism g_1 of E^3 onto E^3 and $g_1(h(K_1) \cup S^2)$ is a polyhedron. Note that $g_1(S^2)$ and $g_1 h(\sigma_1' \cup \sigma_2')$ are polyhedra, while $g_1(h(\sigma^2))$ lies in the interior of the polyhedral 2-sphere $g_1(S^2)$ except for the polygonal path $h(\sigma_1' \cup \sigma_2')$. But now by an application of Moise's theorem on smoothing an annulus [14] as in [7] one can find another homeomorphism of E^3 onto E^3 which is fixed in $\mathrm{ext}(g_1 h(S^2))$ and $g_2 g_1(h(\sigma^2))$ is a polyhedral disk. Thus $g_2 g_1(h(K))$ is a polyhedron and $h(K)$ is tame. It follows by mathematical induction that (2.1) is true.

3. Tame 2- and 3-complexes

The case of the 2-complex will first be considered.

(3.1) THEOREM. *Let K be a finite 2-complex and h a homeomorphism from K into E^3. Then $h(K)$ is tame if and only if each 2-simplex in $h(K)$ is tame and the 1-skeleton of $h(K)$ is tame.*

Proof. The sufficiency of the condition follows from (2.1). For by (2.1), $h(K)$ is locally tame and then by [2] or [14], $h(K)$ is tame. We show the necessity by noting that this follows immediately from (3.2).

(3.2) LEMMA. *If K is a 2-complex and h a homeomorphism of K into E^3 such that $h(K)$ is tame, then there is a homeomorphism g of E^3 onto E^3 which carries $h(K)$ and its 1-skeleton onto polyhedra.*

Proof. If g is a homeomorphism of E^3 onto E^3 which carries $h(K)$ onto a polyhedron and if σ' is a 1-simplex of K such that $gh(\sigma')$ is not a polygonal path, evidently σ' lies on precisely two 2-simplices of K. So by repeated application of the Schoenflies Theorem in the plane we may assume that for each σ', $gh(\sigma')$ is locally polyhedral except perhaps at its end points.

Let v be a vertex of K and suppose that in $\mathrm{St}(v)$ there is a 1-simplex σ' and $gh(\sigma')$ is not locally polyhedral at $gh(v)$. We select in $gh(\mathrm{St}(v))$ a disk D containing $gh(\sigma')$ in its interior except for its end points; D is a subcomplex of $gh(K)$ and D is maximal with respect to the property that $gh(K)$ is locally euclidean at all interior points of D except perhaps at $gh(v)$ or on a single 1-simplex having $gh(v)$ as end point. It is not difficult to see that $gh(\sigma')$ may be thrown onto a path on D which is locally polygonal at $gh(v)$. This procedure can then be applied to each 1-simplex and each of its vertices. This proves (3.2).

(3.3) THEOREM. *Let K be a finite geometric simplicial complex and h a homeomorphism of K into E^3. Then $h(K)$ is tame if and only if h carries the 1-skeleton of K and each 2-simplex of K onto a tame set.*

Proof. The sufficiency of the condition follows from (3.1) and J. W. Alexander's polyhedral Schoenflies Theorem [1].

Following Moise we denote by BK the subcomplex of K consisting of all points at which K is not 3-dimensional along with the 2-simplices of K which are faces of just one 3-simplex. Then if $h(K)$ is tame, $h(BK)$ is tame. By (3.2) we may suppose that each 1- and 2-simplex of $h(BK)$ is a polyhedron. If k is the number of 3-simplices in K, then (3.3) is true for $k = 0$. If (3.3) has been shown for $k < j$, let K have j 3-simplices. Since the subcomplex of K consisting of the closure of those points at which K is not 3-dimensional is carried by h to a polyhedron, we assume without loss of generality that K

is homogeneous and that further K is connected and is separated by no 0- or 1-simplex. Evidently the 1-skeleton of $h(K)$ is locally tame at each vertex of $h(K - BK)$. So let $h(v)$ be a vertex of $h(BK)$ and $h(G)$ the 1-skeleton of $h(BK)$. If σ^2 is a 2-simplex in BK with v as a vertex, let σ^3 be the 3-simplex containing σ^2. Then by (1.1), $\sigma^3 \cup h(G)$ is locally tame at $h(v)$. Evidently by applying (1.1) and [8] to the 3-simplices having v as vertex repeatedly, we can show that the 1-skeleton of $h(K)$ is locally tame at $h(v)$.

If v_1 is a vertex of K in $K - BK$, then $\mathrm{St}(v_1)$ is a closed 3-cell and the 1-skeleton of $h(\mathrm{St}\, v_1)$ is locally tame at $h(v_1)$. Thus $h(K)$ has a tame 1-skeleton. That $h(K)$ has tame 2-simplices follows from [8].

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Virginia Polytechnic Institute
Blacksburg, Virginia
University of Tennessee
Knoxville, Tennessee