ERGODIC TRANSFORMATION GROUPS WITH A PURE POINT SPECTRUM

BY
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1. Introduction

Let the real line act ergodically on the finite measure space S, μ so as to preserve the measure. Form the Hilbert space $\mathfrak{L}^2(S, \mu)$ and for each real x let U_x be the unitary transformation which takes $f \in \mathfrak{L}^2(S, \mu)$ into its translate f_x ; $f_x(s) = f(sx)$. Then $x \to U_x$ is a unitary representation of the real line whose decomposition into irreducible representations may be studied and correlated with the properties of the given action of the real line on S (Koopman's program [3]). One says that the action has a *pure point spectrum* if there exists an orthonormal basis $\Phi_1, \Phi_2, \cdots$ for $\mathfrak{L}^2(S, \mu)$ such that $U_x(\Phi_j) = e^{i\lambda_j x}\Phi_j$. The numbers λ_j then constitute the *spectrum* of the action. This case has been thoroughly studied by von Neumann in [7] and the following results have been obtained (modulo appropriate regularity assumptions about S, μ and the action): (a) The λ_j which occur form a subgroup Γ of the additive group of the real line and each λ_j occurs with multiplicity one. (b) Given any countable subgroup Γ of the additive group of the real line there exists an action of this group on a finite measure space S, μ whose spectrum is Γ. (c) The action having Γ as its spectrum is uniquely determined (up to the obvious equivalence) and may be constructed from Γ as follows. Give Γ the discrete topology and let $\hat{\Gamma}$ be its (compact) character group. For each real x let $\theta(x)$ be the character $\lambda \to e^{i\lambda x}$. Take S to be $\hat{\Gamma}$, take μ to be Haar measure and define an action of the real line on $S = \hat{\Gamma}$ by setting $\mathfrak{X}x = \mathfrak{X}\theta(x)$.

It is obvious how to formulate these results so that they make sense for any separable locally compact commutative group G. One simply replaces Γ by a countable subgroup of the dual $\hat{G}$ of G, the equation $U_x(\Phi_j) = e^{i\lambda_j x}\Phi_j$ by $U_x(\Phi_j) = \mathfrak{X}_j(x)\Phi_j$ and the character $\lambda \to e^{i\lambda x}$ by the character $\mathfrak{X} \to \mathfrak{X}(x)$. Although no one seems to have bothered to work out the details many mathematicians have realized that it would probably be easy to prove the indicated generalization of von Neumann's result. The special case in which G is the infinite cyclic group has been treated in detail in [2].

In this note we shall go a step further and show that a similarly complete analysis is possible when the real line is replaced by any separable locally compact group G provided that having a pure point spectrum is interpreted to mean that U is a discrete direct sum of finite-dimensional irreducible representations of G. Our method is not a direct generalization of that of von Neumann and yields a different proof when applied to his case. Moreover, as we shall show, it can be used to analyze the still more general situation

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which we obtain when we replace U by any discrete direct sum of finite-dimensional representations of G for which there exists an ergodic system of imprimitivity [5, page 278].

2. Preliminaries and formulation of Theorem 1

We shall use the measure theoretic notions developed in our papers [4] and [6] and begin by recalling some of these. (See also Blackwell [1] and footnote (1) in [6].) Let G be any separable locally compact group and let S be a standard Borel space. If sx is defined and in S for all s in S and all x in G we shall say that S is a Borel G-space if the following conditions are satisfied:

(i) $s, x \to sx$ is a Borel function.
(ii) $(sx_1)x_2 \equiv s(x_1 x_2)$.
(iii) $se = s$.

Let μ be a σ-finite measure defined on all Borel subsets of S which is *invariant* in the sense that $\mu(Ex) = \mu(E)$ for all x in G and all Borel subsets E of S. One says that the system S, G, μ is *ergodic* if whenever E is a Borel set in S with $\mu(E) \neq 0$ and $\mu(S - E) \neq 0$ then for some $x \in G$ we have

$$\mu((E - Ex) \cup (Ex - E)) > 0.$$

It is known [6, Theorem 3] that the system S, G, μ fails to be ergodic if and only if there exists a Borel set E in S with $\mu(E) \neq 0$, $\mu(S - E) \neq 0$, and $Ex = E$ for all x in G.

If S is a standard Borel G-space and μ is an invariant measure in S we define a unitary representation U of G whose space is $\mathfrak{L}^2(S, \mu)$ by setting $U_x(f)(s) = f(sx)$. We shall call U the *representation associated with* the system S, G, μ. When U is a discrete direct sum of finite-dimensional irreducible representations we shall say that the system has a *pure point spectrum*. Let G be any separable locally compact group, let φ be a continuous homomorphism of G onto a dense subgroup of a compact group K and let H be any closed subgroup of K. Using the pair φ, H we may construct an ergodic system S_H^φ, G, μ_H as follows. Let $S_H^\varphi = K/H$, the space of all right H cosets in K, and let μ_H denote the unique invariant measure in K/H such that $\mu_H(K/H) = 1$. Setting $\bar{s}x = \overline{s\varphi(x)}$ where $\bar{s}$ is the coset to which s belongs, we make S_H^φ into a standard G-space. It is easy to see that the system S_H^φ, G, μ_H is ergodic. Indeed if it were not there would exist a nonconstant member of $\mathfrak{L}^2(S_H^\varphi, \mu_H)$ which is invariant under the G action and hence (by continuity) under the K action. Since K acts transitively this is impossible. Now the unitary representation U associated with the system S_H^φ, G, μ_H is the representation $x \to V_{\varphi(x)}$ where V is the unitary representation associated with the system S_H^φ, K, μ. But V is a subrepresentation of the regular representation of K and hence is a discrete direct sum of finite-dimensional irreducible representations. Thus S_H^φ, G, μ_H has a pure point spectrum. Theorem 1 is a converse of this result.

THEOREM 1. *Let S, G, μ be an ergodic system with a pure point spectrum where G is a separable locally compact group, S is a standard Borel G-space and μ is an invariant Borel measure in S. Then there exist a continuous homomorphism φ of G onto a dense subgroup of a compact group K and a closed subgroup H of K such that S, G, μ is equivalent to the system S_H^φ, G, μ_H in the following sense: There exist invariant Borel subsets N_1 and N_2 of S and S_H^φ respectively and a Borel isomorphism θ of $S - N_1$ on $S_H^\varphi - N_2$ such that*

$$\theta(sx) \equiv \theta(s)x, \qquad \mu(N_1) = \mu_H(N_2) = 0.$$

3. Proof of Theorem 1

We begin with two lemmas.

LEMMA 1. *Let $x \to U_x$ be a (continuous unitary) representation of the separable locally compact group G. Let B be any complete Boolean algebra of projections in the space $\mathcal{H}(U)$ of U such that $U_x^{-1}BU_x = B$ for all x in a dense subgroup G' of G. Then $U_x^{-1}BU_x = B$ for all x in G.*

Proof. Let x be any element in G and let P be any element in B. Our task is to show that $U_x^{-1}PU_x \in B$. Let φ and ψ be elements of $\mathcal{H}(U)$ and let y be any element of G'. Then

$$\begin{aligned}((U_x^{-1}PU_x - U_y^{-1}PU_y)(\varphi)\cdot\psi) &= (U_x^{-1}PU_x(\varphi)\cdot\psi) - (U_y^{-1}PU_y)(\varphi)\cdot\psi) \\ &= (PU_x(\varphi)\cdot U_x(\psi)) - (PU_y(\varphi)\cdot U_y(\psi)).\end{aligned}$$

Moreover by taking y sufficiently close to x we may make $U_y(\varphi)$ and $U_y(\psi)$ as close respectively to $U_x(\varphi)$ and $U_x(\psi)$ as we please. Hence $U_x^{-1}PU_x$ is a weak limit of projections of the form $U_y^{-1}PU_y$ and hence of members of B. Since B is weakly closed the lemma is proved.

LEMMA 2. *If the system S, G, μ is ergodic and G is compact then there exists a point s in S such that $\mu(S - sG) = 0$.*

Proof. By Lemma 2 of [6] we may suppose that S may be metrized as a separable metric space in such a manner that $s, x \to sx$ is continuous and so that the Borel sets defined by the topology are the same as the given ones. It follows that the orbits in S are all compact and hence that the space of all orbits is metrizable. Since a continuous image of a separable metric space is again such, the orbit space is countably separated. We may now apply the argument used in proving Theorem 6.3 of [5] to show that there must be an orbit of positive measure. Because of ergodicity the complement of the orbit must be of measure zero.

Proof of Theorem 1. Let $U = L^1 \oplus L^2 \oplus \cdots$ where the L^j are finite-dimensional irreducible representations of G. Let K_j be the image of G in $\mathcal{H}(L^j)$ and let N^j be the set of all x for which $L_x^j = I$. (Here and elsewhere in the paper $\mathcal{H}(W)$ denotes the space in which the representation W acts.) Then the closure $\bar{K}_j$ of K_j is compact, $N = \bigcap_{j=1}^\infty N_j$ is a closed normal subgroup of

G and $x \to L_x^1, L_x^2, \cdots = \Phi(x)$ is a continuous homomorphism of G into the compact group $\tilde{K} = \bar{K}_1 \times \bar{K}_2 \times \cdots$. Let K be the closure of $\Phi(G)$ in $\tilde{K}$. Then K is compact and Φ is a continuous homomorphism of G onto a dense subgroup of K. The kernel of Φ is N so Φ defines a continuous (but not in general bicontinuous) isomorphism of G/N into K. It is now easy to define a continuous unitary representation V of K in $\mathcal{H}(U)$ such that $V_{\Phi(x)} = U_x$ for all x in G. For each y in K of the form $\Phi(x)$ let $V_y = U_x$. If $\Phi(x_1) = \Phi(x_2)$ then $x_1 x_2^{-1} \in N$ so $L_{x_1}^j = L_{x_2}^j$ for all j so $U_{x_1} = U_{x_2}$. Thus V_y is well defined for all $y \in \Phi(G)$. Moreover for each j and each pair θ, θ' of elements of $\mathcal{H}(L^j)$ the mapping $y \to (V_y(\theta)\cdot\theta')$ is continuous in the $\tilde{K}$ topology. Hence $y \to (V_y(\psi_1)\cdot\psi_2)$ is continuous for all ψ_1 and ψ_2 in $\mathcal{H}(U) = \mathcal{H}(V)$. Hence V has a unique continuous extension to all of K.

For each Borel subset E of S let P_E be the projection on the space of all functions which vanish outside of E. Then $U_x P_E U_x^{-1} = P_{[E]x^{-1}}$ for all x and E. Hence if B denotes the complete Boolean algebra of all P_E then $U_x B U_x^{-1} = B$ for all x in G. Hence $V_y B V_y^{-1} = B$ for all y in $\Phi(G)$. Hence by Lemma 1, $V_y B V_y^{-1} = B$ for all y in K. Applying Theorem 4 of [6] we deduce the existence of a standard Borel K-space S' and a projection-valued measure $E \to P'_E$ defined on the Borel subsets of S' such that $V_y P'_E V_y^{-1} = P'_{[E]y^{-1}}$ for all E and y and such that B is the set of all P'_E. We define a Borel measure μ' in S' by setting $\mu'(E) = \mu(F)$ whenever $P'_E = P_F$. It is clear that μ' is an ergodic invariant measure in S' and hence by Lemma 2 that S' has an orbit whose complement has measure zero. It follows at once that we may suppose S' chosen so as to be a transitive K-space and hence (Cf. [5, Theorem 6.1]) that we may choose S' so as to be a coset space K/H where H is a closed subgroup of K. We make $S' = K/H$ into a G-space by setting $sx = s\Phi(x)$. Comparing with S we see that the correspondence defined by the equation $P_E = P'_F$ is an equivalence between the associated Boolean G-spaces ([6] page 328). By Theorem 2 of [6] S and S' must be equivalent as indicated and the proof of our theorem is complete.

COROLLARY. *The measure μ is necessarily finite and the representation U has the following properties.*

(a) *Each irreducible constituent occurs with a finite multiplicity which is less than or equal to its dimension.*

(b) *For each irreducible constituent L of U, $\bar{L}$ is also an irreducible constituent of U.*

(c) *If M_1 and M_2 are irreducible constituents of U then the tensor product $M_1 \otimes M_2$ has an irreducible constituent which appears as an irreducible constituent of U.*

Remark 1. Consider the special case in which G is commutative. Then K must be commutative and K/H is a group. If there exists $x \in G$ with $\Phi(x) \in H$ then $sx = s$ for almost all s. Hence if we normalize, as we may, by factoring out the subgroup of all x with $sx = s$ for almost all s we reach the conclusion that $\Phi(G) \cap H = 0$. For each $x \in G$ let $\theta(x) = h(\Phi(x))$ where h is the canonical mapping of K on K/H. Then θ is a Borel homomorphism of G into K/H and hence is continuous. Thus we may conclude that $H = e$ and that Φ is one-to-one; in other words the given action of G on S is equivalent to the action obtained by restricting the right multiplication on a compact commutative group to a dense subgroup isomorphic to G. The generalization of von Neumann's result to arbitrary separable locally compact commutative groups is immediate.

Remark 2. In the commutative case the action of G on S is determined to within equivalence by the irreducible constituents of the representation U. Whether or not this uniqueness holds in the noncommutative case is clearly equivalent to the following question. Let H_1 and H_2 be closed subgroups of the compact group K. Let U^1 and U^2 denote the unitary representations of K associated with the natural actions of K on K/H_1 and K/H_2 respectively. Does unitary equivalence of U^1 and U^2 imply the existence of an x in K such that $xH_1x^{-1} = H_2$? That this need not be so even if the group is finite has been shown by Todd [8].

4. A further generalization

Let S, G, μ satisfy the hypothesis of Theorem 1. Let a be a complex-valued Borel function defined on $S \times G$ with $|a| = 1$. For each $x \in G$ let

$$(U_x^a f)(s) = a(s, x)f(sx).$$

Then $x \to U_x^a$ will define a unitary representation of G if and only if $a(s, e) = 1$ for almost all s and for each pair $x_1, x_2 \in G$,

$$* \qquad a(s, x_1 x_2) = a(s, x_1)a(sx_1, x_2) \qquad \text{for almost all } s.$$

Suppose that we have such an a and that U^a is a discrete direct sum of finite-dimensional irreducible representations. Can we draw a conclusion about the structure of S, G, μ analogous to and generalizing that in Theorem 1? Theorem 2 provides a positive answer to this and related questions. Before stating this theorem it will be convenient to make a few definitions. Let H be a closed subgroup of the separable compact group K and let L be a unitary representation of H. Let $\mathcal{H}^L$ denote the set of all Borel functions f from K to $\mathcal{H}(L)$ (the space of L) such that $f(\xi x) = L_\xi f(x)$ for all $\xi, x \in H \times K$ and such that $\int (f(x) \cdot f(x))\, dx < \infty$. We make $\mathcal{H}^L$ into a Hilbert space by introducing the obvious norm and define a unitary representation U^L of K whose space is $\mathcal{H}^L$ by setting $U_x^L f(y) = f(yx)$. For each Borel subset E of the coset space K/H let P_E^L be the projection which maps f into $\varphi_{E'} f$ where $\varphi_{E'}$ is the characteristic function of the set E' of all x lying in some coset of E. Let B^L denote the set of all projections of the form P_E^L. Then B^L is a complete Boolean algebra of projections and $U_x^L B^L U_{x^{-1}}^L = B^L$ for all x in K. Now let Φ be a continuous homomorphism of the separable locally compact group G onto a dense subgroup of K. Let $U_x = U_{\Phi(x)}^L$. Then U

is a unitary representation of G which is a discrete direct sum of finite-dimensional irreducible representations and B^L is a complete Boolean algebra of projections which is *invariant* in the sense that $U_x B^L U_x^{-1} = B^L$ for all x in G and *ergodic* in the sense that $U_x P U_x^{-1} = P$ for all x in G implies $P = 0 \sim P = 1$. Theorem 2 is a converse of this result.

THEOREM 2. *Let U be a unitary representation of the separable locally compact group G and let B be a complete Boolean algebra of projections in $\mathcal{H}(U)$ which is invariant and ergodic with respect to U. Suppose that U is a discrete direct sum of finite-dimensional irreducible representations. Then there exist a continuous homomorphism Φ of G onto a dense subgroup of a compact group K, a closed subgroup H of K, a unitary representation L of H and a unitary map W of $\mathcal{H}(U)$ on $\mathcal{H}(U^L)$ such that*

(i) $WBW^{-1} = B^L$.

(ii) $WU_x W^{-1} = U^L_{\Phi(x)}$ *for all x in G.*

Proof. We construct K, Φ and a unitary representation V of K such that $V_{\Phi(x)} \equiv U_x$ exactly as in the proof of Theorem 1. Applying Lemma 1 we conclude that B is invariant under V. Since B is ergodic with respect to U it is a fortiori ergodic with respect to V. Next we apply Theorem 4 of [6] and conclude that there exist a standard Borel K-space and a projection-valued measure P defined on S such that:

(a) The set of all P_E is just B and

(b) $U_x P_E U_x^{-1} = P_{[E]x^{-1}}$ for all x in K.

Choose an element ψ in $\mathcal{H}(U)$ such that $(P_E(\psi)\cdot\psi) = 0$ if and only if $P_E = 0$. Then $E \rightarrow (P_E(\psi)\cdot\psi)$ is a measure on S whose null sets are invariant—a quasi invariant measure. The definition of ergodicity and the proof of Lemma 2 are still valid for measures which are only quasi invariant. Thus we may conclude, as in the proof of Theorem 1, that we may choose S as the right coset space K/H for some closed subgroup H of K. Hence we may apply Theorem 6.6 of [5] and deduce the unitary equivalence of the pair U, B with the pair U^L, B^L for some unitary representation L of H. The truth of the theorem follows at once.

5. An application of Theorem 2

Let G be a separable locally compact group and let S be a standard Borel G-space. Let μ be a σ-finite Borel measure in S whose null sets are invariant under G. Let the action of G on S be ergodic with respect to μ. Let ρ be a positive real-valued Borel function on $S \times G$ such that $f(s) \rightarrow \rho(s, x)f(sx)$ is a unitary operator U_x in $\mathcal{L}^2(S, \mu)$ for each x in G. Let H_0 be a separable Hilbert space. Then $f(s) \rightarrow \rho(s, x)f(sx)$ is also a unitary operator U^0 in $\mathcal{L}^2(S, H_0, \mu)$, that is in the Hilbert space of all square summable functions from S to H_0. Let A be any Borel function from $S \times G$ to the group $\mathfrak{U}_0$ of all unitary operators in H_0 which satisfies the following.

(a) $A(s, e) = I$ for almost all s.
(b) For each pair x_1, x_2 in G,

$$A(s, x_1 x_2) = A(s, x_1)A(sx_1, x_2) \qquad \text{for almost all } s.$$

Then for each x in G, $f(s) \to A(s, x)\rho(s, x)f(sx)$ is a unitary operator ${}^{A}U_x$ in $\mathfrak{L}^2(S, H_0, \mu)$ and $x \to {}^{A}U_x$ is a unitary representation of G.

THEOREM 3. *If the representation ${}^{A}U$ is a discrete direct sum of finite-dimensional irreducible representations then there exists a finite invariant measure μ^1 having the same null sets as μ and the conclusion of Theorem* 1 *holds.*

Proof. For each Borel set E in S let P_E be multiplication by the characteristic function of E and let B be the set of all P_E. Then B is ergodic and invariant under the ${}^{A}U_x$ and we may apply Theorem 2. The unitary mapping W of Theorem 2 sets up an isomorphism between B and the B^L of Theorem 2 to which we may apply Theorem 2 of [6]. The truth of Theorem 3 follows at once.

Remark. Application of Theorem 2 to the representation U^A actually yields more information than the structure of S. We can also conclude that the representation L of H is uniquely determined up to equivalence and that A_1 and A_2 yield equivalent L's if and only if there exists a Borel function $s \to V(s)$ from S to the unitary operators from the space of A_1 to the space of A_2 such that for all x,

$$A_1(s, x) = V(s)^{-1}A_2(s, x)V(sx) \qquad \text{for almost all } s.$$

(Cf [5, §5]). We leave details to the reader. It is important to notice that this analysis applies only to those A's which yield a U^A which is a discrete direct sum of finite-dimensional irreducible representations. Even if S is an S_H^{φ} as defined in §2 there may exist A's for which U^A does not decompose in this manner.

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