METRIZABILITY OF CERTAIN COUNTABLE UNIONS

BY

H. H. Corson and E. Michael[1]

1. Introduction

Suppose the regular[2] space X is the union of a collection $\mathfrak{M}$ of metrizable subsets. A number of conditions for X to be metrizable under these circumstances are known, but in all of them the elements of $\mathfrak{M}$ are either open [8], [10], or closed [7], [10], or separable with X compact [9]. In this paper we consider the case where $\mathfrak{M}$ is countable, and where each $M \in \mathfrak{M}$ is a dense subset of an open set; such sets M will be called *locally dense*.[3]

Now let the regular space X be the union of a countable collection $\mathfrak{M}$ of metrizable, locally dense subsets M_n. As Example 6.5 shows, these assumptions alone do not imply that X is metrizable, even if $\mathfrak{M}$ has only two elements. Further conditions are needed to insure metrizability, and they fall into two classes: On the one hand, our assumptions imply that X has a point-countable base, and this has two immediate consequences. First, if X is separable,[4] it has a countable base and is therefore metrizable; second, if X is compact, it is metrizable by a theorem of A. Mishchenko [6]. On the other hand, it will be shown that X is metrizable if it is normal and there are generalized[5] F_σ (in X) sets $A_n \subset M_n$ which cover X. Some of the principal consequences of these facts are summarized in the following theorem.

Theorem 1.1. *If the normal[6] space X is the union of a countable collection $\mathfrak{M}$ of locally dense, metrizable subsets M_n, then X is metrizable if it satisfies any of the following conditions*:

(a) *X is separable (in particular, each M_n is separable).*

(b) *X is locally compact.*

(c) *X is σ-compact.*[7]

(d) *Every open set in X is an F_σ.*

(e) *There exist F_σ-sets $A_n \subset M_n$ which cover X.*

Received January 20, 1963.

[1] Supported by an N. S. F. contract.

[2] In our terminology, regular, completely regular, and normal spaces are assumed to be T_1.

[3] The justification for the terminology is that M is locally dense in X if and only if each $x \in M$ has a neighborhood U in X such that $M \cap U$ is dense in U. (Another characterization: $M \subset \bar{M}^0$.) Note that all open (but not all closed) sets are locally dense.

[4] A space is *separable* if it has a countable, dense subset.

[5] A_n is a *generalized* F_σ in X if, whenever U is an open set in X containing A_n, then there exists an F_σ-set F in X such that $A_n \subset F \subset U$.

[6] For (a)–(c), it suffices if X is regular. For (d)–(h), Example 6.6 shows that normality is needed.

[7] A space is *σ-compact* if it is the union of countably many compact subsets.

(f) $\mathfrak{M}$ *is finite, and there exist* G_δ*-sets* $A_n \subset M_n$ *which cover* X.
(g) $\mathfrak{M}$ *is finite, and there exist completely metrizable* $A_n \subset M_n$ *which cover* X.
(h) $\mathfrak{M}$ *has only two elements, and* X *is an absolute* G_δ.[8]

It is not known whether "σ-compact" can be weakened to "Lindelöf" in (c), or whether the first requirement is really needed in (h).

As a special case of Theorem 1.1(e), we obtain the following corollary, which the second author needs in the proof of [5; Proposition 11.1].

COROLLARY 1.2. *If the normal space* X *is the union of two metrizable subsets, of which one is dense and the other an open* F_σ, *then* X *is metrizable.*

The proofs of Theorem 1.1(g)–(h) use the following result, which was known (see, for instance, [3; Problem K, p. 207]) if $A = M$.

PROPOSITION 1.3. *Let* X *be a Hausdorff space, and* M *a dense, metrizable subset of* X. *Then any completely metrizable subset* A *of* M *is a* G_δ *in* X.

Section 2 deals with point-countable bases, and offers an alternative proof of Mishchenko's theorem. Section 3 proves Proposition 1.3. Section 4 outlines the proof of Theorem 1.1 and related results, and Section 5 deals with analogous results on paracompactness. Section 6 is devoted to assorted examples, some of which are known.

We are grateful to the referee for numerous helpful comments.

2. Point-countable bases

In [6], A. Mishchenko proved the following result. (The proof given here, which is based on ideas of M. E. Rudin, is different from Mishchenko's, and shows, incidentally, that the result remains true for *countably* compact spaces.)

PROPOSITION 2.1 (Mishchenko). *A point-countable base* $\mathfrak{B}$ *of a compact* T_1*-space* X *is countable.*

Proof. It suffices to show that X is separable. By induction, define a sequence of countable subsets C_n of X, with C_1 empty and $C_{n+1} \supset C_n$, with the following property: If $\mathfrak{B}_n$ consists of those elements of $\mathfrak{B}$ which intersect C_n, then C_{n+1} contains a point of $X - \bigcup \mathfrak{F}$ for every finite $\mathfrak{F} \subset \mathfrak{B}_n$ such that $X - \bigcup \mathfrak{F} \neq \emptyset$. (Since C_n is countable, $\mathfrak{B}_n$ must be countable, and hence has only countably many finite subcollections.) Let $C = \bigcup_{n=1}^{\infty} C_n$. It suffices to show that $X = \bar{C}$.

Suppose $X \neq \bar{C}$. Then there exists an $x_0 \in X - \bar{C}$. Let $\mathfrak{U}$ be the collection of elements of $\mathfrak{B}$ which intersect $\bar{C}$ and miss x_0. Then $\mathfrak{U}$ covers $\bar{C}$ because X is T_1, and $\mathfrak{U}$ is countable because C is countable and dense in $\bar{C}$. Since

[8] A completely regular X is an *absolute* G_δ (= topologically complete in Čech's terminology [2]) if it is a G_δ in every completely regular space containing it as a dense subset. If X is metrizable, Čech [2] showed that this is equivalent to being completely metrizable. See also Proposition 1.3.

$\bar{C}$ is countably compact, $\mathfrak{U}$ has a finite subcollection $\mathfrak{F}$ which covers C. Now $\mathfrak{F} \subset \mathfrak{B}_n$ for n large enough. But $X - \bigcup \mathfrak{F}$ contains x_0, and thus is not empty, so that C_{n+1} contains an element of $X - \bigcup \mathfrak{F}$. This contradicts the fact that $\mathfrak{F}$ covers C_{n+1}, and that proves the theorem.

It follows immediately from Proposition 2.1 that *a compact Hausdorff space with point-countable base is metrizable.* Somewhat more generally, we obtain the following corollaries.

COROLLARY 2.2. *A regular space X, with a dense σ-compact subset D and a point-countable base $\mathfrak{B}$, is metrizable.*

Proof. Let $D = \bigcup_{n=1}^{\infty} C_n$, with each C_n compact. Since C_n has a point-countable base, it is—as observed above—compact metric. Hence each C_n, and therefore X, is separable. But this implies that $\mathfrak{B}$ is countable, so that X is metrizable. That completes the proof.

COROLLARY 2.3. *A locally compact Hausdorff space X with a point-countable base $\mathfrak{B}$ is metrizable.*

Proof. Since every point of X has a compact metric neighborhood, X is covered by a subcollection $\mathfrak{B}'$ of $\mathfrak{B}$ whose elements are separable. Call x, y in X equivalent if there exist $B_0, \cdots, B_n \in \mathfrak{B}'$ with $x \in B_0$, $y \in B_n$, and $B_i \cap B_{i+1} \neq \emptyset$ for $i = 0, \cdots, n-1$. Let C be an equivalence class. Surely C is open. Since each element of $\mathfrak{B}'$ intersects only countably many others, C is the union of a countable subcollection of $\mathfrak{B}'$, hence is separable, and thus (since it has a point-countable base) has a countable base. Since X is regular, C is metrizable. Hence X is the disjoint union of open, metrizable subsets, and is thus itself metrizable. That completes the proof.

As the proof shows, Corollary 2.3 remains true if it is assumed that X is regular and each point of X has a neighborhood with a dense σ-compact subset.

Mishchenko gave an example of a nonmetrizable space with point-countable base which is paracompact, and of another which is Lindelöf but not regular. In Example 6.4 we construct such a space which is *regular* Lindelöf and hereditarily paracompact. An unsolved problem, suggested by Corollary 2.3, is whether a normal (or perhaps paracompact) space X with a point-countable base, which is an absolute G_δ, must be metrizable. (Complete regularity would not suffice, however, as shown by Proposition 4.2 and Example 6.6.)

3. Proof of Proposition 1.3

For $A = M$, an outline of the proof can be found in Kelley [3; Problem K, p. 207], and we follow the same outline in the general case. Let d be some metric on M compatible with the topology, and let M^* be the completion of M with this metric. For each n, let U_n be the set of points in X which have a neighborhood whose intersection with M has d-diameter $< 1/n$. Let $G = \bigcap_{n=1}^{\infty} U_n$. Then there exists a continuous $f : G \to M^*$ which extends the identity map $i : M \to M$. Now if $x \in M$ and $y \in G - M$, then $f(x) \neq f(y)$, because y has a closed neighborhood N which misses x, and $f(y)$ is in the M^*-closure of $f(N \cap M)$ while $f(x)$ is not. Hence $f^{-1}(A) = A$ for any $A \subset M$; if A is completely metrizable, then A is a G_δ in M^*, so $f^{-1}(A)$ is G_δ in $f^{-1}(M^*) = G$, and hence—since G is a G_δ in X—it follows that $A = f^{-1}(A)$ is a G_δ in X. That completes the proof.

We take this opportunity to record the following consequence of Proposition 1.3, which is needed in Section 6.

COROLLARY 3.1. *If a metrizable space M is the union of finitely many completely metrizable subsets, then M is completely metrizable.*

Proof. Let $A_1, \cdots, A_n$ be the completely metrizable subsets. By Proposition 1.3, each A_n is a G_δ in βM. Hence M is a G_δ in βM, and is therefore completely metrizable [2]. (The referee has pointed out that in this proof βM could be replaced by a metric completion of M, while replacing Proposition 1.3 and [2] by the classical result that a subset of a completely metrizable space is completely metrizable if and only if it is a G_δ.)

4. Proof of Theorem 1.1 and related results

LEMMA 4.1. *Let M be a metrizable, locally dense subset of a regular space X. Then there exist collections $\mathcal{V}_i$ $(i = 1, 2, \cdots)$ of open subsets of X such that*

(a) *$M \subset \bigcup \mathcal{V}_i$ for all i.*

(b) *$\mathcal{V}_i$ is locally finite at every point of $\bigcup \mathcal{V}_i$ for all i.*

(c) *If $x \in M$, and U is a neighborhood of x in X, then there exists a V in some $\mathcal{V}_i$ such that $x \in V \subset U$.*

Proof. By the Nagata-Smirnov theorem, M has a base $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$, where each $\mathcal{B}_i$ is a locally finite open covering of M. Let G be an open set containing M as a dense subset. For each $B \in \mathcal{B}$, let B' be an open subset of G such that $B' \cap M = B$, and note that $B' \subset \overline{B}$ since M is dense in G. Let $\mathcal{B}_i' = \{B' \mid B \in \mathcal{B}_i\}$, and let

$$L_i = \{x \in X \mid \mathcal{B}_i' \text{ is locally finite at } x\}.$$

Then L_i is open, and $L_i \supset M$ since M is dense in G (so that, if U and V are open subsets of G which intersect, then $U \cap M$ and $V \cap M$ intersect). Now let[9] $\mathcal{V}_i = \mathcal{B}_i' \mid L_i$. Clearly $\mathcal{V}_i$ satisfies (a) and (b). To check (c), let $x \in M$, and let U be a neighborhood of x in X. Since X is regular, there is an open W in X such that $x \in W \subset \overline{W} \subset U$. Pick an i and a $B \in \mathcal{B}_i$ such that $x \in B \subset (W \cap M)$. Then

$$x \in (B' \cap L_i) \subset \overline{B} \subset \overline{W} \subset U,$$

and $(B' \cap L_i) \in \mathcal{V}_i$. That completes the proof.

---

[9] If $\mathcal{B}$ is a collection of sets, and $L \subset \bigcup \mathcal{B}$, then $\mathcal{B} \mid L$ denotes $\{B \cap L \mid B \in \mathcal{B}\}$.

From Lemma 4.1, we immediately obtain

PROPOSITION 4.2. *If a regular space X is the union of countably many locally dense, metrizable subsets, then X has a point-countable base.*

Call a subset M of a topological space X *metrically embedded* in X if there exist locally finite collections $\mathcal{V}_i$ $(i = 1, 2, \cdots)$ of open subsets of X satisfying condition (c) of Lemma 4.1. From the Nagata-Smirnov metrization theorem, we conclude

LEMMA 4.3. *If a regular space X is the union of countably many metrically embedded subsets, then X is metrizable.*

We now prove

LEMMA 4.4. *If M is a metrizable, locally dense subset of a normal space X, and if $A \subset M$ is a generalized*[5] *F_σ in X, then A is metrically embedded in X.*

Proof. Let $\mathcal{V}_i$ $(i = 1, 2, \cdots)$ be as in Lemma 4.1, and let $V_i = \bigcup \mathcal{V}_i$. Since A is a generalized F_σ, there exist closed subsets F_{ij} $(j = 1, 2, \cdots)$ of X such that

$$A \subset \textstyle\bigcup_{j=1}^{\infty} F_{ij} \subset V_i .$$

Since X is normal, there exist open subsets U_{ij} of X such that $F_{ij} \subset U_{ij}$ and $\overline{U}_{ij} \subset V_i$. Let $\mathcal{V}_{ij} = \mathcal{V}_i \mid U_{ij}$. Then $\mathcal{V}_{ij}$ is a collection of open subsets of X which is locally finite at every x in X. To complete the proof, let $x \in A$, and let U be a neighborhood of x in X. Pick i and $V \in \mathcal{V}_i$ so that $x \in V \subset U$, and then pick j so that $x \in F_{ij}$. Then $x \in (V \cap U_{ij}) \subset U$, and $(V \cap U_{ij}) \in \mathcal{V}_{ij}$, which completes the proof.

From Lemmas 4.3 and 4.4 we conclude

PROPOSITION 4.5. *If the normal space X is the union of countably many metrizable, locally dense subsets M_n, and if there exist generalized F_σ (in X) sets $A_n \subset M_n$ which cover X, then X is metrizable.*

Proof of Theorem 1.1. Parts (a)–(c) follow immediately from Proposition 4.2 and the results of Section 2.[10] Parts (d)–(h) follow from Proposition 4.5, as will now be verified.

(d) This implies that every subset of X is a generalized F_σ.

(e) An F_σ is surely a generalized F_σ.

(f) This implies (d): In fact, let $E \subset X$ be closed. Then, for each n, we have $E \cap A_n$ closed in the metric space A_n, hence a G_δ in A_n, and thus a G_δ in X. But there are only finitely many A_n, so E is the union of finitely many G_δ's and is therefore itself a G_δ.

(g) By Proposition 1.3, and the fact that M_n is locally dense, this implies (f).

[10] As the remark following Corollary 2.3 shows, (a)–(c) may actually be simultaneously weakened to: each point of X has a neighborhood with a dense σ-compact subset.

(h) Here $X = M_1 \cup M_2$. Let us show that any open $U \supset M_1$ is an F_σ. Since $X - U$ is a closed subset of X, it is, just like X, an absolute G_δ [2]. Since $X - U$ is a subset of M_2, and hence metrizable, this implies that $X - U$ is completely metrizable [2]. From Proposition 1.3, and the fact that M_2 is locally dense, it follows that $X - U$ is a G_δ in X, and hence U is an F_σ in X. It follows that M_1 is a generalized F_σ in X, and, similarly, so is M_2. Hence X is metrizable by Proposition 4.5.

5. Paracompactness

Proposition 4.5 remains valid, with much the same proof, with metrizable replaced by paracompact. One thus obtains

PROPOSITION 5.1. *If the normal space X is the union of countably many paracompact, locally dense subsets P_n, and if there exist generalized F_σ (in X) sets $A_n \subset P_n$ which cover X, then X is paracompact.*

If X is actually collectionwise normal, then one can dispense with local denseness.

PROPOSITION 5.2. *If the collectionwise normal space X is the union of countably many paracompact, generalized F_σ-subsets P_n, then X is paracompact.*

Proof. Let $\mathcal{U}$ be an open covering of X. By [3; Theorem 28, p. 156], it suffices to find a σ-discrete open refinement of $\mathcal{U}$. For each n, let $\mathcal{V}_n = \bigcup_{i=1}^{\infty} \mathcal{V}_{n,i}$ be a σ-discrete (with respect to P_n) refinement of $\mathcal{U} \mid P_n$ by sets open in P_n. Let $D_{n,i}$ be the set of points in X having a neighborhood which intersects at most one element of $\mathcal{V}_{n,i}$. Then $D_{n,i}$ is open, and $D_{n,i} \supset P_n$. Since P_n is a generalized F_σ, there exist closed sets $F_{n,i,j}$ ($j = 1, 2, \cdots$) such that

$$P_n \subset \bigcup_{j=1}^{\infty} F_{n,i,j} \subset D_{n,i}.$$

Let

$$\mathcal{A}_{n,i,j} = \{V \cap F_{n,i,j} \mid V \in \mathcal{V}_{n,i}\}.$$

Then $\mathcal{A}_{n,i,j}$ is discrete with respect to X, so, since X is collectionwise normal, there exists a discrete collection $\{W_A \mid A \in \mathcal{A}_{n,i,j}\}$ of open subsets of X such that always $W_A \supset A$. For each $A \in \mathcal{A}_{n,i,j}$ pick a U_A in $\mathcal{U}$ which contains A, and let

$$\mathcal{R}_{n,i,j} = \{W_A \cap U_A \mid A \in \mathcal{A}_{n,i,j}\}.$$

Then $\bigcup_{n,i,j=1}^{\infty} \mathcal{R}_{n,i,j}$ is a σ-discrete open refinement of $\mathcal{U}$, and that completes the proof.

As Example 6.7 shows, "collectionwise normal" cannot be replaced by "normal" in Proposition 5.2.

We conclude this section with an easy result which is needed in the next section.

PROPOSITION 5.3. *Let E be a regular space, and A a subset which is Lindelöf, whose complement is paracompact, and such that every neighborhood of A contains a closed neighborhood of A. Then E is paracompact.*

Proof. Let $\mathcal{U}$ be an open covering of E. Since A is Lindelöf, $\mathcal{U}$ has a countable subcollection $\mathcal{V}$ which covers A. Let $V = \bigcup \mathcal{V}$; by assumption, there exists an open W such that $A \subset W$ and $\bar{W} \subset V$. Now $E - W$ is closed in $E - A$, and hence paracompact, so $\mathcal{U} \mid (E - W)$ has a locally finite, open (with respect to $E - W$) refinement $\mathcal{R}$. Let $\mathcal{S} = \mathcal{R} \mid (E - \bar{W})$, and let $\mathcal{J} = \mathcal{V} \cup \mathcal{S}$; then $\mathcal{J}$ is a σ-locally finite open refinement of $\mathcal{U}$. Since E is regular, this implies that E is paracompact [3; Theorem 28, p. 156].

6. Examples

EXAMPLE 6.1. *A nonmetrizable, compact Hausdorff space, which is the union of two metrizable subsets*:

The one-point compactification of an uncountable discrete space.

EXAMPLE 6.2. *A nonmetrizable, σ-compact, paracompact space, which is the union of two separable, metrizable, F_σ-subsets*:

The integers N, together with one point of $\beta N - N$.

EXAMPLE 6.3. *A nonmetrizable, compact Hausdorff space which is the union of uncountably many dense, separable, metrizable subsets*:

Let X be a nonmetrizable, compact Hausdorff space which is separable and has a countable base at each point (such as the union of the top and bottom edges of the unit square topologized by dictionary ordering). Then every countable subset of X has a countable base. Hence if D is a countable dense subset, then X is the union of all the separable metrizable subsets $D \cup \{x\}$, with $x \in X$.

EXAMPLE 6.4. *A nonmetrizable, hereditarily paracompact, Lindelöf space with a point-countable (in fact σ-disjoint) base*:

Let Y be a subset of the unit interval I which is uncountable, but all of whose compact subsets are countable; such subsets exist by [4; Theorem 1, p. 422]. Let X be the closure of Y in I, and let $M = X - Y$. Denote the usual topology on X by σ. Let τ be the topology on X for which open sets are of the form $U \cup S$, where U is a σ-open subset of X and $S \subset Y$. Then (X, τ) is the required space: Regularity is easily checked. If $E \subset (X, \tau)$, then E is paracompact by Proposition 5.3 (with $A = E \cap M$); hence (X, τ) is hereditarily paracompact. That (X, τ) is Lindelöf follows from the fact that it has a separable metric subset, namely $(M, \tau) = (M, \sigma)$, with the property that every open set N containing it has a countable complement. (This is so because $N = U \cup S$, where U is σ-open and $S \subset Y$; thus $U \supset M$, and $X - U$ is a compact—and thus countable—subset of (Y, σ).) Finally, (X, τ) is not metrizable because every G_δ containing M has countable complement, while M has uncountable complement, so that the closed set M is not a G_δ. (Another argument is that, if (X, τ) were metrizable, then all its subsets would be separable, whereas (Y, τ) is an uncountable discrete space.)

EXAMPLE 6.5. *A nonmetrizable, hereditarily paracompact space, which is the union of two dense, metrizable subsets (one of which is open), and which is also the union of countably many dense, completely metrizable, G_δ-subsets S_n:*

Let R denote the reals, Q the rationals, and Y the irrationals. Let X be the subset of $R \times R$ defined by

$$X = (Q \times \{0\}) \cup (Y \times R).$$

Topologize X by taking as a base all ordinary open sets and all sets of the form $\{y\} \times V$, with $y \in Y$ and V open in R.

Regularity of X is easily checked. If $E \subset X$, then E is paracompact by Proposition 5.3 (with $A = E \cap (Q \times \{0\})$), so X is hereditarily paracompact. Since Q is not a G_δ in R, the closed set $Q \times \{0\}$ is not a G_δ in X, so X is not metrizable.

The two dense metrizable subsets are $Y \times R$ (which is open) and $(Q \times \{0\}) \cup (Y \times (R - \{0\}))$. That the latter set is metrizable follows from Lemmas 4.3 and 4.4, since $Q \times \{0\}$ is clearly metrically embedded in X, while $Y \times (R - \{0\})$ is a metrizable open F_σ in X.

To define the S_n, write Q as a sequence $\{x_n\}_{n=1}^{\infty}$, and let S_n be $Y \times R$ together with the point $x_n \times \{0\}$. Then S_n is metrizable by Lemmas 4.3 and 4.4 (since $x_n \times \{0\}$ has a countable base of neighborhoods in X, and $Y \times R$ is a metrizable open F_σ in X), so S_n is completely metrizable by Corollary 3.1. (It is not hard, by the way, to describe a complete metric on S_n explicitly.) Since S_n is the union of two G_δ-subsets of X, it is itself a G_δ in X, and that completes the proof.

EXAMPLE 6.6. *A nonnormal, completely regular, absolute G_δ space X, all of whose open subsets are F_σ, which is the union of two dense, open, completely metrizable subsets:*

Let X consist of all points (x, y) in the plane with either x irrational and $y \geqq 0$, or with $x = r_n$ (where $r_1, r_2, \cdots$ is an enumeration of the rationals) and $0 < y \leqq 1/n$. A basic neighborhood of (x, y) in X is a vertical interval about (x, y) if x is irrational, and is an ordinary plane neighborhood of (x, y) in X if x is rational. It is easy to check that this defines a completely regular topology on X.

If $A = \{(x, y) \in X \mid x \text{ rational}\}$ and $B = \{(x, y) \in X \mid y = 0\}$, then A and B are disjoint and closed in X. To show that X is not normal, let U be an open set about A, and let us show that $\bar{U}$ intersects B. Let Y be the x-axis in the plane, and let $Q = \{(x, 0) \in Y \mid x \text{ rational}\}$. If

$$U_n = \{(x, 0) \in Y \mid (x, y) \in U \quad \text{for some} \quad y < 1/n\},$$

then each U_n is a neighborhood of Q in Y for the ordinary topology on Y. Since Q is not a G_δ in Y for this topology, there is a point $(x_0, 0)$ of $Y - Q = B$ which lies in every U_n, so that $(x_0, 0) \in \bar{U} \cap B$.

The open, dense, completely metrizable subsets of X which cover X are $G = X - A$ and $H = X - B$. The only nontrivial assertion here is that H is completely metrizable. Let us verify that.

To prove H metrizable, it suffices to find a σ-locally finite base. Now $H = (H \cap A) \cup (H - A)$. Let $\mathcal{C}$ be a countable base for H *with the plane topology*; this will provide a base in H (given topology) at every point of $H \cap A$. As for $H - A$, note that A is closed in the upper half plane (excluding the x-axis) in the plane topology, and hence $H - A$ is the union of countably many open (in H) subsets H_n whose closures (in H) miss A. Now $H - A$ is metrizable, so each $\bar{H}_n$ has a σ-locally finite base $\mathcal{B}_n$. If now $\mathcal{B}_n' = \mathcal{B}_n \mid H_n$ for all n, then $\mathcal{C} \cup (\bigcup_{n=1}^{\infty} \mathcal{B}_n')$ is a σ-locally finite base for H.

Since A is closed in the upper half plane, it is completely metrizable in the plane topology, which is also the topology it inherits as a subset of X. Furthermore $H - A$, as a disjoint union of lines, is also completely metrizable. Hence H is completely metrizable by Corollary 3.1.

That X is an absolute G_δ follows from the fact that G and H—and hence their union X—are dense G_δ's in βX.

Suppose, finally, that V is open in X. Since B and H are metrizable F_σ-subsets of X, the relatively open subsets $V \cap B$ and $V \cap H$ are also F_σ in X, and hence so is their union V.

EXAMPLE 6.7. *A nonparacompact perfectly normal space, which is the union of two metrizable F_σ-subsets*:[11]

The space F described by R. H. Bing in [1; Example H] has the required properties. It is perfectly normal but not collectionwise normal, and hence not paracompact. The subset F_p is closed and metrizable, while $F - F_p$ is a metrizable, open F_σ.

REFERENCES

1. R. H. BING, *Metrization of topological spaces*, Canadian J. Math., vol. 3 (1951), pp. 175–186.
2. E. ČECH, *On bicompact spaces*, Ann. of Math. (2), vol. 38 (1937), pp. 823–844.
3. J. L. KELLEY, *General topology*, New York, Van Nostrand, 1955.
4. C. KURATOWSKI, *Topologie I*, 4th ed., Monografie Matematyczne, vol. 20, Warsaw, Polska Akademia Nauk, 1958.
5. E. MICHAEL, *Cuts*, Acta Math., vol. 111 (1964), to appear.
6. A. MISHCHENKO, *Spaces with point-countable base*, Doklady Akad. Nauk SSSR (N.S.), vol. 144 (1962), pp. 985–988 (= Soviet Math. Dokl., vol. 3 (1962), pp. 855–858).
7. J. NAGATA, *On a necessary and sufficient condition of metrizability*, J. Inst. Polytech. Osaka City Univ., Ser. A. Math., vol. 1 (1950), pp. 93–100.

[11] By Proposition 5.2, such a space cannot be collectionwise normal.

8. Yu. M. Smirnov, *A necessary and sufficient condition for metrizability of a topological space*, Doklady Akad. Nauk SSSR (N.S.), vol. 77 (1951), pp. 197–200 (in Russian).
9. ———, *On the metrizability of bicompacts decomposable into a sum of sets with countable basis*, Fund. Math., vol. 43 (1956), pp. 387–393 (in Russian).
10. A. H. Stone, *Metrisability of unions of spaces*, Proc. Amer. Math. Soc., vol. 10 (1959), pp. 361–366.

University of Washington
Seattle, Washington