LOCAL TIMES FOR A CLASS OF MARKOFF PROCESSES

BY
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Let $X(\tau, \omega)$ be a one-dimensional Markoff process defined on a probability space $(\Omega, \mathfrak{F}, P)$. For any positive real t and any $\omega \in \Omega$ we can define a measure, $\mu(\cdot, t, \omega)$, on R, the real numbers, by

$$(0.1) \qquad \mu(A, t, \omega) = \text{LM}\{\tau : X(\tau, \omega) \in A, 0 \leqq \tau < t\},$$

where LM represents Lebesgue measure. Trotter [3] showed that if $X(\tau, \omega)$ is Brownian motion, then, for almost all ω, $\mu(\cdot, t, \omega)$ has a continuous density function, i.e., there exists a function $L(x, t, \omega)$, defined for all $x \in R$ and all positive t, continuous jointly in x and t, such that

$$(0.2) \qquad \mu(A, t, \omega) = \int_A L(x, t, \omega)\, dx$$

for every Borel set A. $L(x, t, \omega)$ is called the "local time" at x up to time t. In this paper we investigate the following problem: For θ a given Borel measure on R, when will $\mu(\cdot, t, \omega)$ have a continuous θ-density, i.e., when will there exist, for almost all ω, a function $L(x, t, \omega)$, defined for all $x \in R$ and all positive t, continuous jointly in x and t, such that

$$(0.3) \qquad \mu(A, t, \omega) = \int_A L(x, t, \omega)\theta\{dx\}$$

for every Borel set A? We shall show that such an $L(x, t, \omega)$ will exist, for almost all ω, whenever the transition probabilities of $X(\tau, \omega)$ satisfy certain conditions (involving θ). In particular, we shall show that if $X(\tau, \omega)$ is a stable process of index α, $1 < \alpha \leqq 2$, then, for almost all ω, there will exist a function $L(x, t, \omega)$ satisfying (0.2). Since Brownian motion is a stable process of index 2, this offers a new proof of Trotter's result.

1. Preliminary material

The purpose of this section is to explain briefly certain concepts arising in the theory of Markoff processes which will be used in later sections.

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Throughout this paper $X(\tau, \omega)$ will be a time-homogeneous Markoff process with transition probabilities; that is, for every $x \in R$ and nonnegative t there is given a measure on R, which we shall denote by $P_t(x, dy)$ or $P_t(x, \cdot)$, such that the following conditions are satisfied:

1. $P_t(x, R) = 1$, i.e., $P_t(x, \cdot)$ is a probability measure on R for every x and t;
2. if A is any fixed Borel set, then $P_t(x, A)$ is Borel measurable in the pair (t, x);
3. the relation

$$\int_R P_t(y, A)P_s(x, dy) = P_{t+s}(x, A) \tag{1.1}$$

holds identically;

4. $P(X(s) \in A \mid X(\tau), 0 \leqq \tau \leqq t) = P_{s-t}(X(t), A)$ for $s \geqq t \geqq 0$.

If $\omega \in \Omega$, the sample path associated with ω, often denoted $X(\omega)$, is the function mapping $[0, \infty)$ into R defined by $t \to X(t, \omega)$. *We shall assume that almost every sample path is right continuous.*

A positive measurable function, S, on Ω will be called a terminal time for $X(\tau, \omega)$ if S is independent of the functions $X(\tau)$ and

$$P(S > t) = e^{-t}.$$

By modifying Ω somewhat we may always assume the existence of a terminal time for any given process. Indeed, let $(R_+, \mathcal{B}_+, P^*)$ be the probability space consisting of R_+, the positive reals, with field of measurable sets $\mathcal{B}_+$, the Borel sets in R_+, and measure P^* where

$$P^*(A) = \int_A e^{-x}\, dx.$$

Let $\bar{\Omega} = \Omega \times R_+$ be the product measure space of Ω and R_+ with product measure $\bar{P} = P \times P^*$. If $(\omega, t) = \bar{\omega} \in \bar{\Omega}$, let $X(\tau, \bar{\omega}) = X(\tau, \omega)$ and $S(\bar{\omega}) = t$. $X(\tau, \bar{\omega})$ has the same transition probabilities as $X(\tau, \omega)$, and S is clearly a terminal time for $X(\tau, \bar{\omega})$.

Suppose S is a terminal time for $X(\tau, \omega)$, and T is a positive (possibly infinite) measurable function on Ω independent of S. Let

$$\Omega' = \{\omega : T(\omega) < S(\omega)\}.$$

If $P(\Omega') > 0$, we may consider Ω' as a probability space $(\Omega', \mathfrak{F}', P')$ where $\mathfrak{F}'$ is the trace of $\mathfrak{F}$ on Ω' (the field of subsets of Ω' belonging to $\mathfrak{F}$) and $P'(A)$ is the conditional probability $P(A)/P(\Omega')$. On Ω' we may define a new process (in general just a stochastic process), $Y(\tau, \omega)$, by

$$Y(\tau, \omega) = X(\tau + T(\omega), \omega). \tag{1.2}$$

The function $S' = S - T$ is a terminal time for $Y(\tau, \omega)$. If $T(\omega) \equiv t$, then

$Y(\tau, \omega)$ is not merely a stochastic process, but a Markoff process having the same transition probabilities as $X(\tau, \omega)$.

Suppose A is a Borél set, and $T(\omega)$ is the infimum of those τ, $0 \leqq \tau < \infty$, such that $X(\tau, \omega) \epsilon A$, or ∞ if there is no such τ. T clearly depends only on the $X(\tau)$ and is hence independent of any terminal time S. $T(\omega)$ is called the time $X(\omega)$ "hits" A. *We shall assume that if $T(\omega)$ is the time $X(\omega)$ hits the interval $[a, b)$, a and b real numbers, $a < b$, then the stochastic process $Y(\tau, \omega)$ defined by (1.2) is a Markoff process with the same transition probabilities as $X(\tau, \omega)$.*

The initial distribution, ρ_X, of a Markoff process $X(\tau, \omega)$ is the probability measure on R defined by

$$\rho_X(A) = P\{\omega : X(0, \omega) \epsilon A\}.$$

If $X(0, \omega) \equiv x$ for some $x \epsilon R$, then $\rho_X = \rho_x$ is the measure giving unit mass to the point x. *We shall assume that for every $x \epsilon R$ there is a Markoff process $X_x(\tau, \omega)$, defined on a probability space $(\Omega_x, \mathfrak{F}_x, P_x)$, having almost all sample paths right continuous, initial distribution ρ_x, and the same transition probabilities as $X(\tau, \omega)$.*

A Markoff process satisfying the above three assumptions will be said to satisfy Hypothesis A. We shall refer to the individual assumptions as A1, A2, and A3, respectively. Although the transition probabilities do not uniquely determine a Markoff process, they play a vital part in determining whether Hypothesis A will be satisfied. Blumenthal [1] gives conditions on the $P_t(x, dy)$ which insure that Hypothesis A will be satisfied.

If t is any nonnegative real number, we can define an operator P_t, mapping the space of bounded, measurable, real-valued functions on R into itself, by

$$P_t f(x) = \int P_t(x, dy) f(y).$$

We define another operator, U, having the same domain and range as the P_t, by

$$Uf(x) = \int_0^\infty e^{-t} P_t f(x)\, dt.$$

U is called the potential operator associated with the transition probabilities $P_t(x, dy)$. An alternate expression for $Uf(x)$ is

$$Uf(x) = E\left(\int_0^{S(\omega)} f(X_x(\tau, \omega))\, d\tau\right), \tag{1.3}$$

where S is a terminal time for $X_x(\tau, \omega)$. Indeed more than (1.3) is true.

LEMMA 1. *Let $Y(\tau, \omega)$ be a Markoff process with almost all sample paths right continuous and having the same transition probabilities as $X(\tau, \omega)$. If f is any bounded, measurable, real-valued function on R, let*

$$F(n, Y) = E\left(\left(\int_0^{S(\omega)} f(Y(\tau, \omega))\, d\tau\right)^n\right)$$

and

$$F(n, x) = E\left(\left(\int_0^{S(\omega)} f(X_x(\tau, \omega))\, d\tau\right)^n\right).$$

Then

$$F(n, Y) = \int F(n, x)\rho_Y\{dx\} \tag{1.4}$$

and

$$F(n, x) = nU(fF(n-1, \cdot))(x) \tag{1.5}$$

where $F(0, x) \equiv 1$ *and* $F(n, \cdot)$ *is the function* $x \to F(n, x)$. *For* $n = 1$, (1.5) *is* (1.3).

Proof of Lemma 1. To prove (1.4) it suffices to prove the stronger statement

$$E\left(\left(\int_0^{S(\omega)} f(Y(\tau, \omega))\, d\tau\right)^n \,\middle|\, Y(0)\right) = F(n, Y(0)) \tag{1.6}$$

(where $E(\ |\ )$ represents conditional expectation). If almost all sample paths of $Y(\tau, \omega)$ are right continuous, then, for almost all ω,

$$\left(\int_0^{S(\omega)} f(Y(\tau, \omega))\, d\tau\right)^n = n\int_0^{S(\omega)} f(Y(t, \omega))\left(\int_t^{S(\omega)} f(Y(\tau, \omega))\, d\tau\right)^{n-1} dt.$$

This formula may be verified by integration by parts.

Let $\Omega_1 = \{(\omega, t) : \omega \in \Omega, 0 \leqq t < S(\omega)\}$. We may consider Ω_1 as a measurable subset of $\Omega \times [0, \infty)$. Let P_1 be the restriction to Ω_1 of the product measure $P \times \text{LM}$ on $\Omega \times [0, \infty)$. If $(\omega, t) = \omega_1 \in \Omega_1$, let

$$F(n, \omega_1) = f(Y(t, \omega))\left(\int_t^{S(\omega)} f(Y(\tau, \omega))\, d\tau\right)^{n-1}.$$

Then, integrating first with respect to t and then ω, we have

$$F(n, Y) = n\int F(n, \omega_1)P_1\{d\omega_1\}.$$

Applying Fubini's theorem and integrating first with respect to ω and then t, we have

$$F(n, Y) = n\int_0^\infty \int_{\Omega_t} f(Y(t, \omega))\left(\int_t^{S(\omega)} f(Y(\tau, \omega))\, d\tau\right)^{n-1} P\{d\omega\}\, dt$$

(where $\Omega_t = \{\omega : S(\omega) > t\}$)

$$= n\int_0^\infty e^{-t} E\left(f(Y_t(0, \omega))\left(\int_0^{S'(\omega)} f(Y_t(\tau, \omega))\, d\tau\right)^{n-1}\right) dt,$$

where the expectation is taken over Ω_t considered as a probability space, $Y_t(\tau, \omega)$ is the Markoff process on Ω_t defined by (1.2) with $T(\omega) \equiv t$, and $S' = S - T$ is a terminal time for $Y_t(\tau, \omega)$. If $n = 1$ and $Y(\tau, \omega)$ is $X_x(\tau, \omega)$, we then have

$$F(1, x) = \int_0^\infty e^{-t} \int f(y) P_t(x, dy)\, dt = \int_0^\infty e^{-t} P_t f(x)\, dt = U(f)(x).$$

In general

$$F(1, Y) = \int_0^\infty e^{-t} \iint f(y) P_t(x, dy) \rho_Y\{dx\}\, dt = \int F(1, x) \rho_Y\{dx\}.$$

Hence (1.4) and (1.5) are true for $n = 1$. The validity of (1.6) for $n = 1$ is also clear.

We now proceed by induction. If $Y(\tau, \omega)$ is $X_x(\tau, \omega)$, then $Y_t(\tau, \omega)$ has almost all sample paths right continuous, the same transition probabilities as $X(\tau, \omega)$, and initial distribution $P_t(x, \cdot)$. Hence

$$E\left(f(Y_t(0, \omega)) \left(\int_0^{S'(\omega)} f(Y_t(\tau, \omega))\, d\tau\right)^{n-1}\right) = \int f(y) F(n - 1, y) P_t(x, dy).$$

Therefore

$$\begin{aligned} F(n, x) &= n \int_0^\infty e^{-t} \int f(y) F(n - 1, y) P_t(x, dy)\, dt \\ &= n \int_0^\infty e^{-t} P_t(fF(n - 1, \cdot))(x)\, dt \qquad (1.7) \\ &= nU(fF(n - 1, \cdot))(x), \end{aligned}$$

which is exactly (1.5). The validity of (1.4) and (1.6) for all n now follows as above.

A nonnegative function, $U(x, y)$, defined for all $x \in R$ and $y \in R$, is said to be a θ-kernel for U, θ a σ-finite measure on R, if

$$Uf(x) = \int U(x, y) f(y) \theta\{dy\}$$

for every bounded, measurable function f. In general such a θ-kernel may not exist. If, for each positive t, $P_t(x, \cdot)$ has a measurable θ-density $p_t(x, y)$ (i.e.,

$$P_t(x, A) = \int_A p_t(x, y) \theta\{dy\}$$

for every Borel set A), then U will have a θ-density, and, moreover,

$$U(x, y) = \int_0^\infty e^{-t} p_t(x, y)\, dt. \qquad (1.8)$$

The validity of (1.8) may be derived by a simple application of Fubini's theorem.

Many of the concepts discussed above, including that of the potential operator, are discussed in Hunt's fundamental paper [2].

We conclude this section by defining some terms which will be useful in formulating Theorem 1 of the next section.

A measure θ on R is called a Borel measure if it gives finite measure to every compact subset of R. It is clear that every Borel measure is σ-finite. A measure θ on R is called a continuous measure if $\theta(\{x\}) = 0$ for every $x \in R$. Every σ-finite measure θ may be written uniquely as $\theta_c + \theta_s$ where θ_c is continuous, and θ_s gives positive measure to at most a countable number of points of R, and measure zero to any set not containing any of these points.

For want of better terminology, a nonnegative, monotonic function g, defined on an interval $[0, \varepsilon_1]$, will be called a good function if

$$\sum_{n \geqq n_1} ng(2^{-n})^{1/2} < \infty,$$

where n_1 is so large that $2^{-n_1} \leqq \varepsilon_1$. A sufficient condition for g to be good is for $|g(x)^{1/2} \log x|$ to decrease to zero as x decreases to zero and

$$\left| \int_0^{\varepsilon_1} g(x)^{1/2} \frac{\log x}{x} \, dx \right| < \infty.$$

The functions $g(x) = x^\beta$, $\beta > 0$, are all good functions.

Finally, for simplicity of notation, we shall let $\langle x, y \rangle$ denote either (x, y) or (y, x) depending on whether $x < y$ or $y < x$.

2. Statement and proof of theorem

THEOREM 1. *Let $X(\tau, \omega)$ be a one-dimensional Markoff process, defined on a probability space $(\Omega, \mathfrak{F}, P)$, satisfying Hypothesis* A, *having transition probabilities $P_t(x, dy)$, and corresponding potential operator U. If U has a θ-kernel $U(x, y)$, $\theta = \theta_c + \theta_s$ a Borel measure on R, and if there exists a good function g, defined on an interval $[0, \varepsilon_1]$, such that*

$$\begin{aligned} &|U(x, x) - U(x, y)| \leqq g(\tfrac{1}{2}\theta_c(\langle x, y \rangle)), \qquad \theta_c(\langle x, y \rangle) \leqq 2\varepsilon_1, \\ &|U(x, x) - U(y, x)| \leqq g(\tfrac{1}{2}\theta_c(\langle x, y \rangle)), \qquad \theta_c(\langle x, y \rangle) \leqq 2\varepsilon_1, \end{aligned} \tag{2.1}$$

then, for almost all ω, there exists a function $L(x, t, \omega)$, defined for all $x \in R$ and $t \in R_+$, continuous jointly in x and t, satisfying (0.3) *for every Borel set A.*

Proof of Theorem 1. We shall assume that θ satisfies the following additional assumption: If E is any nonempty open interval, then $\theta_c(E) > 0$. In reality this is no additional assumption because it can be shown that any measure θ satisfying the hypotheses of Theorem 1 has this property. However, the proof is rather involved and, for this reason, will not be given.

We shall now assume that for every positive integer M there is a positive real, a_M^0, and a negative real, a_{-M}^0, such that $\theta_c([0, a_M^0)) = \theta_c([a_{-M}^0, 0)) = M$.

This assumption is by no means vital and is made only to simplify the presentation of the proof of Theorem 1. We shall define a_0^0 as 0. Hence for every integer i we have defined a real number a_i^0.

We now define, for every integer n, a sequence of numbers a_i^n. If $i \geqq 0$, let a_i^n be that number z such that $\theta_c([0, z)) = i2^{-n}$. If $i < 0$, let a_i^n be that number z such that $\theta_c([z, 0)) = i2^{-n}$. We have, for all i, $a_i^n = a_{2i}^{n+1}$, $a_i^n < a_{i+1}^n$, and $\theta_c([a_i^n, a_{i+1}^n)) = 2^{-n}$. Let $I_i^n = [a_i^n, a_{i+1}^n)$. Then

$$I_i^n = I_{2i}^{n+1} \cup I_{2i+1}^{n+1},$$

and $\mathcal{I}_i^n = \{I_i^n\}$ is a sequence of ever finer partitions of R.

For each n we can define an approximate θ-density, $L_n(x, t, \omega)$, for $\mu(\cdot, t, \omega)$ by

$$L_n(x, t, \omega) = (\theta(I_i^n))^{-1}\mu(I_i^n, t, \omega)$$

for $x \in I_i^n$. We shall show that, for almost all ω, for all x, $a_{-M}^0 \leqq x < a_M^0$, and all $t \leqq N$, M and N arbitrary integers, $L_n(x, t, \omega)$ converges to a limit, which we shall denote by $L(x, t, \omega)$. Moreover, $L(x, t, \omega)$ will be jointly continuous in x and t and satisfy (0.3) for every Borel set A contained in $[a_{-M}^0, a_M^0)$. Letting $M = 1, 2, \cdots$ we see that for almost all ω there is a function $L(x, t, \omega)$, defined for all x and all $t \leqq N$, continuous jointly in x and t, satisfying (0.3) for every Borel set A. Letting $N = 1, 2, \cdots$ we have the result stated in Theorem 1.

$L_n(x, t, \omega)$ is continuous except, possibly, at the points a_i^n. If $z = a_i^n$, let $\Delta(z, n, t, \omega)$ be the discontinuity of $L_n(x, t, \omega)$ at z. Then $x \in I_i^n$ and $y \in I_{i-1}^n \cup I_i^n$ imply

$$|L_n(x, t, \omega) - L_n(y, t, \omega)| \leqq \Delta(a_i^n, n, t, \omega).$$

For $z = a_i^n$, let

$$J(z, n, d, \omega) = \{t : \Delta(z, n, t, \omega) \geqq d, 0 \leqq t \leqq N\}$$

and

$$H(z, n, d) = \{\omega : \mathrm{LM}(J(z, n, d, \omega)) \geqq 2^{-2n}\}.$$

LEMMA 2. *If n is sufficiently large and $a_{-M}^0 \leqq z < a_M^0$, then*

$$P(H(z, n, d)) \leqq K2^{2n} \exp(-kg(2^{-n})^{-1/2} d)$$

where K and k are positive constants depending only on N and M.

The proof of Lemma 2 is lengthy and will be given in the next section.

Let $H(n, d) = \bigcup_z H(z, n, d)$ where the union is over those $z = a_i^n$ where $a_{-M}^0 \leqq a_i^n < a_M^0$. There will be exactly $2M2^n$ such z. (We suppress the dependence of the various sets on M and N. Also, we shall denote all sufficiently large positive constants depending only on M and N by K.) If n is so large that Lemma 2 applies, then

$$\begin{aligned} P(H(n, d)) &\leqq \textstyle\sum_z P(H(z, n, d)) \\ &\leqq 2MK2^{3n} \exp(-kg(2^{-n})^{-1/2} d) = K2^{3n} \exp(-kg(2^{-n})^{-1/2} d). \end{aligned}$$

Let $d_n = 3nk^{-1}g(2^{-n})^{1/2}$ and $G_m = \bigcup_{n\geqq m} H(n, d_n)$. If m is so large that Lemma 2 applies, then

$$P(G_m) \leqq \sum_{n\geqq m} P(H(n, d_n)) \leqq K\sum_{n\geqq m} 2^{3n}e^{-3n} = K\sum_{n\geqq m} (2/e)^{3n},$$

and since this last is a convergent series,

$$P(\lim_m G_m) = \lim_m P(G_m) = 0.$$

For every x and n, $L_n(x, t, \omega)$ is an increasing function of t. Furthermore,

$$| L_n(x, t, \omega) - L_n(x, t', \omega)| \leqq 2^n | t - t' |$$

because $| \mu(A, t, \omega) - \mu(A, t', \omega)| \leqq | t - t' |$ for any Borel set A. Hence if $| t - t' | \leqq 2^{-2n}$, then $| L_n(x, t, \omega) - L_n(x, t', \omega)| \leqq 2^{-n}$. Consequently, unless $\omega \in G_m$, for any $t \leqq N$, $n \geqq m$, and any $z = a_i^n$, $a_{-M}^0 \leqq z < a_M^0$, we can find a t', $0 \leqq t' \leqq N$, $| t - t' | \leqq 2^{-2n}$, such that $\Delta(z, n, t', \omega) \leqq d_n$. Hence

$$\Delta(z, n, t, \omega) \leqq \Delta(z, n, t', \omega) + 2\cdot 2^{-n} \leqq d_n + 2^{-n+1}$$

for $n \geqq m$, $a_{-M}^0 \leqq z < a_M^0$, and $t \leqq N$.

Now, since $I_i^n = I_{2i}^{n+1} \cup I_{2i+1}^{n+1}$, it follows from the inequality

$$\left| \frac{a + b}{c + d} - \frac{a}{c} \right| \leqq \left| \frac{a}{c} - \frac{b}{d} \right|,$$

where a and b are nonnegative, c and d positive real numbers, that

$$\max_{a^0_{-M} \leqq x < a^0_M} | L_{n+1}(x, t, \omega) - L_n(x, t, \omega) | \leqq \max_{a^0_{-M} \leqq z < a^0_M} (\Delta(z, n + 1, t, \omega)).$$

Then, unless $\omega \in G_m$, for any n, p with $n \geqq m$,

$$| L_{n+p}(x, t, \omega) - L_n(x, t, \omega)| \leqq \sum_n^{n+p-1} | L_{i+1}(x, t, \omega) - L_i(x, t, \omega)| < \sum_n^\infty d_i + 2^{-i+1} = q(n).$$

Now, since g is a good function, $\sum_n^\infty d_i + 2^{-i+1} < \infty$. Hence $q(n) \to 0$ as $n \to \infty$. Hence if $\omega \notin G_m$, the sequence $L_n(x, t, \omega)$ converges, uniformly for $a_{-M}^0 \leqq x < a_M^0$ and $t \leqq N$, to a limit which we shall denote by $L(x, t, \omega)$. Furthermore,

$$| L(x, t, \omega) - L_n(x, t, \omega)| \leqq q(n)$$

for $n \geqq m$, $a_{-M}^0 \leqq x < a_M^0$, and $t \leqq N$. It is clear that the measures on $[a_{-M}^0, a_M^0)$ having $L_n(x, t, \omega)$ as θ-densities converge at least weakly to $\mu(\cdot, t, \omega)$ restricted to $[a_{-M}^0, a_M^0)$, and consequently $L(x, t, \omega)$ is a θ-density function for $\mu(\cdot, t, \omega)$ restricted to $[a_{-M}^0, a_M^0)$. The joint continuity of $L(x, t, \omega)$ remains to be established.

Let $\delta(n)$ be a number so small that for every x, $a_{-M}^0 \leqq x < a_M^0$, $x \in I_i^n$ and $| x - y | < \delta(n)$ imply $y \in I_{i-1}^n \cup I_i^n$. Suppose $\omega \notin G_m$. If $| x - y | < \delta(n)$

where $n \geqq m$ and $a^0_{-M} \leqq x, y < a^0_M$, then

$$(2.2)\quad \begin{aligned} |L(x,t,\omega) - L(y,t,\omega)| &\leqq |L(x,t,\omega) - L_n(x,t,\omega)| \\ &\quad + |L_n(x,t,\omega) - L_n(y,t,\omega)| \\ &\quad + |L_n(y,t,\omega) - L(y,t,\omega)| \\ &\leqq 2q(n) + d_n + 2^{-n+1} \leqq 3q(n). \end{aligned}$$

If $n \geqq m$ and $q(n) < \varepsilon/3$ for ε an arbitrary positive real, then $|x - y| < \delta = \delta(n)$ implies $|L(x,t,\omega) - L(y,t,\omega)| < \varepsilon$. Note that the choice of δ for given ε does not depend upon t.

Turning to continuity with respect to t, we have, for $\omega \notin G_m$ and $n \geqq m$,

$$(2.3)\quad \begin{aligned} |L(x,t,\omega) - L(x,t',\omega)| &\leqq |L(x,t,\omega) - L_n(x,t,\omega)| \\ &\quad + |L_n(x,t,\omega) - L_n(x,t',\omega)| \\ &\quad + |L_n(x,t',\omega) - L(x,t',\omega)| \\ &\leqq 2q(n) + 2^n |t - t'|. \end{aligned}$$

If ε is an arbitrary positive number, let $\delta = (\varepsilon/2)2^{-n}$ where $n \geqq m$ and $q(n) < \varepsilon/4$. Then $|t - t'| < \delta$ implies $|L(x,t,\omega) - L(x,t',\omega)| < \varepsilon$. Note that in this case the choice of δ for given ε is independent of x. Hence $L(x,t,\omega)$ is continuous jointly in x and t.

Since $\lim_{m\to\infty} P(G_m) = 0$, we have shown that, for almost all ω, $L_n(x,t,\omega)$ converges, uniformly for $a^0_{-M} \leqq x < a^0_M$ and $t \leqq N$, to a function $L(x,t,\omega)$ which is continuous jointly in x and t and satisfies (0.3) for any Borel set contained in $[a^0_{-M}, a^0_M)$. As we have remarked above, this is sufficient to establish the validity of Theorem 1.

3. Proof of Lemma 2

To avoid interruption of the argument at a later stage we prove a preliminary lemma.

LEMMA 3. *If* $a^0_{-M} \leqq x < a^0_M$ *and* $\theta(\langle x, y\rangle) \leqq \varepsilon_1$, *then* $U(x,y)$ *is bounded, say by* $c_1 < \infty$.

Proof. It suffices to show that $U(x,x)$ is bounded if $a^0_{-M} \leqq x < a^0_M$. Suppose n is so large that $2^{-n} \leqq \varepsilon_1$ and $x \geqq 0 = a^n_0$. For (2.1) to be valid, $U(0,0)$ must be finite. The lemma now follows from the inequality

$$\begin{aligned} U(x,x) \leqq U(0,0) + |U(a^n_1,0) - U(0,0)| + |U(a^n_1,a^n_1) - U(a^n_1,0)| + \\ \cdots + |U(x,x) - U(x,a^n_{m-1})| \leqq U(0,0) + 2mg(\varepsilon_1), \\ a^n_{m-1} < x \leqq a^n_m, \end{aligned}$$

since $x < a^0_M$ implies $m \leqq M2^n$. The proof is similar if $x \leqq 0$.

We now assume that there are terminal times defined for the Markoff processes $X(\tau, \omega)$ and $X_x(\tau, \omega)$. We denote all such terminal times by S. We may now associate with each $\omega \in \Omega$ a measure, $\mu(\cdot, \omega)$, on R by

$$\mu(A, \omega) = \mu(A, S(\omega), \omega) = \text{LM}\{\tau : X(\tau, \omega) \in A, 0 \leqq \tau < S(\omega)\}$$

for every Borel set A. As above, we may define for every $n > 0$ an approximate θ-density, $L_n(x, \omega)$, by

$$L_n(x, \omega) = (\theta(I_i^n))^{-1} \mu(I_i^n, \omega)$$

for $x \in I_i^n$. Let $\Delta(z, n, \omega)$ be the discontinuity of $L_n(x, \omega)$ at $z = a_i^n$.

$\Delta(z, n, \omega)$ has the same distribution as the absolute value of

$$R_n(z, \omega) = \int_0^{S(\omega)} V_z(X(\tau, \omega))\, d\tau$$

where

$$\begin{aligned} V_z(y) &= (\theta(I_{i-1}^n))^{-1} \quad \text{for} \quad y \in I_{i-1}^n = [a_{i-1}^n, a_i^n), \\ &= -(\theta(I_i^n))^{-1} \quad \text{for} \quad y \in I_i^n = [a_i^n, a_{i+1}^n), \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

Let $c = \max(1, c_1)$ and $c_n = g(2^{-n})^{-1/2}$.

LEMMA 4. *There is a positive constant K such that*

$$E(\exp((24c)^{-1} c_n \mid R_n(z, \omega) \mid)) \leqq K$$

for all z, $a_{-M}^0 \leqq z < a_M^0$, and all n so large that $g(2^{-n}) \leqq c^2$.

Proof. Let $Q(m) = E((c_n \mid R_n(z, \omega) \mid)^m)$. We clearly have

$$E(\exp((24c)^{-1} c_n \mid R_n(z, \omega) \mid)) = \sum_{m \geqq 0} \frac{Q(m)}{(24c)^m\, m!}.$$

To prove the lemma it suffices to show that $Q(2m) \leqq (12c)^{2m}(2m!)$ since then, using the inequality $Q(2m-1) \leqq 1 + Q(2m)$, we have

$$\begin{aligned} E(\exp((24c)^{-1} c_n \mid R_n(z, \omega) \mid)) &\leqq \sum_{m \geqq 0} 2^{-2m} + 24c \left(\frac{2m}{2^{2m}}\right) + \frac{1}{(2m-1)!\,(24c)^{2m-1}} \\ &\leqq K < \infty. \end{aligned}$$

Let

$$R(m) = E((c_n R_n(z, \omega))^m)$$

and

$$R(m, x) = E\left(\left(c_n \int_0^{S(\omega)} V_z(X_x(\tau, \omega))\, d\tau\right)^m\right).$$

By the definition of V_z we have

$$E\left(\left(c_n \int_0^{S(\omega)} V_z(X(\tau, \omega))\, d\tau\right)^m\right) = \int_{\Omega'} \left(c_n \int_{T(\omega)}^{S(\omega)} V_z(X(\tau, \omega))\, d\tau\right)^m P\{d\omega\}$$

where $T(\omega)$ is the time $X(\omega)$ hits $[a_{i-1}^n, a_{i+1}^n)$ and $\Omega' = \{\omega : T(\omega) < S(\omega)\}$. If $P(\Omega') = 0$, then $R_n(z, \omega) = 0$ almost everywhere, and hence $E(\exp((24c)^{-1}c_n \mid R_n(z, \omega) \mid)) = 1$ independently of n. If $P(\Omega') > 0$, then

$$R(m) = P(\Omega')E\left(\left(c_n \int_0^{S'(\omega)} V_z(Y(\tau, \omega))\, d\tau\right)^m\right),$$

where the expectation is taken over Ω' considered as a probability space, $Y(\tau, \omega)$ is the stochastic process defined by (1.2) with $T(\omega)$ the time $X(\omega)$ hits $[a_{i-1}^n, a_{i+1}^n)$, and $S' = S - T$ is a terminal time for $Y(\tau, \omega)$. By A2, $Y(\tau, \omega)$ is a Markoff process having the same transition probabilities as $X(\tau, \omega)$. Hence, by (1.4),

$$R(m) = P(\Omega') \int R(m, x)\rho_Y\{dx\}$$

where ρ_Y is the initial distribution of $Y(\tau, \omega)$. By the right-continuity of almost all sample paths of $X(\tau, \omega)$, $\rho_Y([a_{i-1}^n, a_{i+1}^n]) = 1$. Hence to show that $Q(2m) = R(2m) \leqq (12c)^{2m}(2m!)$ it suffices to show that $\mid R(m, x) \mid \leqq (12c)^m\, m!$ for all $x \in [a_{i-1}^n, a_{i+1}^n]$ and all m. We shall show by induction on m that, for all m,

(a) $\quad \mid R(m,x) - R(m, a_i^n) \mid \leqq 4\,g(2^{-n})^{1/2}\, m!\,(12c)^{m-1}, \qquad a_{i-1}^n \leqq x \leqq a_{i+1}^n$

and

(b) $\quad \mid R(m, x) \mid \leqq (12c)^m\, m!, \qquad a_{i-1}^n \leqq x \leqq a_{i+1}^n$

are true. Using (1.5) with $f = c_n V_z$, we obtain

$$\begin{aligned} R(m, x) &= mU(c_n\, V_z\, R(m - 1, \cdot))(x) \\ &= mc_n \int U(x, y)V_z(y)R(m - 1, y)\theta\{dy\} \end{aligned} \tag{3.1}$$

where $R(0, y) \equiv 1$.

By construction, we may write (3.1) as

$$\begin{aligned} R(m, x) = mc_n \Big(&\int_{I_{i-1}^n} U(x, y)R(m - 1, y)\theta_1\{dy\} \\ &- \int_{I_i^n} U(x, y)R(m - 1, y)\theta_2\{dy\}\Big) \end{aligned} \tag{3.2}$$

where θ_1 and θ_2 are probability measures on I_{i-1}^n and I_i^n respectively. Hence

$$\begin{aligned} R(m, x) = mc_n &\int_{I_{i-1}^n} U(x, y)R(m - 1, y) \\ &- U(x, a_i^n)R(m - 1, a_i^n)\theta_1\{dy\} \\ + mc_n &\int_{I_i^n} U(x, a_i^n)R(m - 1, a_i^n) - U(x, y)R(m - 1, y)\theta_2\{dy\}. \end{aligned} \tag{3.3}$$

By definition of the a_i^n, $a_{i-1}^n \leqq x, y \leqq a_{i+1}^n$ implies $\theta_c(\langle x, y\rangle) \leqq 2^{-n+1}$. Hence $| U(x, y_1) - U(x, y_2) | \leqq 2g(2^{-n})$ and $| U(y_1, x) - U(y_2, x) | \leqq 2g(2^{-n})$ for $a_{i-1}^n \leqq x, y_1, y_2 \leqq a_{i+1}^n$. For $m = 1$, (3.3) becomes

$$R(1, x) = c_n \left(\int_{I_{i-1}^n} U(x, y) - U(x, a_i^n)\theta_1\{dy\} + \int_{I_i^n} U(x, a_i^n) - U(x, y)\theta_2\{dy\} \right),$$

and therefore

$$| R(1, x) | \leqq c_n 4g(2^{-n}) = 4g(2^{-n})^{1/2} \leqq 12c, \qquad a_{i-1}^n \leqq x \leqq a_{i+1}^n .$$

From (3.1) we have

$$R(m, x) - R(m, a_i^n) = mc_n \int (U(x, y) - U(a_i^n, y))V_z(y)R(m - 1, y)\theta\{dy\}. \tag{3.4}$$

If $m = 1$, then it follows, as above, that

$$| R(1, x) - R(1, a_i^n) | \leqq 4 c_n g(2^{-n}) = 4 g(2^{-n})^{1/2}, \qquad a_{i-1}^n \leqq x \leqq a_{i+1}^n .$$

Hence (a) and (b) are true for $m = 1$. Suppose (a) and (b) are true for $m - 1$. From (3.4) it then follows that

$$\begin{aligned} | R(m, x) - R(m, a_i^n) | &\leqq mc_n 4g(2^{-n})(m - 1)!\, (12c)^{m-1} \\ &= 4g(2^{-n})^{1/2} m!\, (12c)^{m-1}, \qquad a_{i-1}^n \leqq x \leqq a_{i+1}^n . \end{aligned}$$

Hence (a) is true for m. Now

$$\begin{aligned} mc_n \int_{I_{i-1}^n} & U(x, y)R(m - 1, y) - U(x, a_i^n)R(m - 1, a_i^n)\theta_1\{dy\} \\ &= mc_n \int_{I_{i-1}^n} (U(x, y) - U(x, a_i^n))R(m - 1, y)\theta_1\{dy\} \\ &\quad + mc_n \int_{I_{i-1}^n} U(x, a_i^n)(R(m - 1, y) - R(m - 1, a_i^n))\theta_1\{dy\}. \end{aligned}$$

By the inductive assumptions, we have

$$\begin{aligned} &\left| mc_n \int_{I_{i-1}^n} (U(x, y) - U(x, a_i^n))R(m - 1, y)\theta_1\{dy\} \right| \\ &\qquad \leqq mc_n 2g(2^{-n})(m - 1)!\, (12c)^{m-1} = 2g(2^{-n})^{1/2} m!\, (12c)^{m-1} \\ &\qquad \leqq 2cm!\, (12c)^{m-1} \end{aligned}$$

and

$$\begin{aligned} &\left| mc_n \int_{I_{i-1}^n} U(x, a_i^n)(R(m - 1, y) - R(m - 1, a_i^n))\theta_1\{dy\} \right| \\ &\qquad \leqq mc_n 4cg(2^{-n})^{1/2}(12c)^{m-1}(m - 1)! = 4cm!\, (12c)^{m-1}. \end{aligned}$$

Hence

$$\left| mc_n \int_{I_{i-1}^n} U(x, y)R(m - 1, y) - U(x, a_i^n)R(m - 1, a_i^n)\theta_1\{dy\} \right|$$
$$\leqq 4cm!\,(12c)^{m-1} + 2cm!\,(12c)^{m-1} = 6cm!\,(12c)^{m-1}.$$

Similarly,

$$\left| mc_n \int_{I_i^n} U(x, a_i^n)R(m - 1, a_i^n) - U(x, y)R(m - 1, y)\theta_2\{dy\} \right|$$
$$\leqq 6cm!\,(12c)^{m-1}.$$

Therefore,

$$| R(m, x) | \leqq 6cm!\,(12c)^{m-1} + 6cm!\,(12c)^{m-1}$$
$$= m!\,(12c)^m, \qquad a_{i-1}^n \leqq x \leqq a_{i+1}^n .$$

The proof of Lemma 4 is now complete.

COROLLARY. *$P\{\omega : \Delta(z, n, \omega) \geqq d\} \leqq K \exp(-(12c)^{-1}g(2^{-n+1})^{-1/2} d)$ for $a_{-M}^0 \leqq z < a_M^0$.*

Proof.
$$\begin{aligned} P(\Delta(z, n, \omega) \geqq d) &= P(| R_n(z, \omega) | \geqq d) \\ &= P((12c)^{-1}c_n \,| R_n(z, \omega) | \geqq (12c)^{-1}c_n\, d) \\ &\leqq E(\exp((12c)^{-1}c_n \,| R_n(z, \omega) |)) \\ &\qquad \cdot\exp(-(12c)^{-1}c_n\, d) \\ &\leqq K \exp(-(12c)^{-1}g(2^{-n})^{-1/2} d). \end{aligned}$$

Let $\bar{\Omega} = \Omega \times R_+$ with measure $\bar{P} = P \times P^*$, and let $X(\tau, \bar{\omega})$ and $S(\bar{\omega})$ be as in Section 1. For $\bar{\omega} \in \bar{\Omega}$ we can then define a measure $\mu(\cdot, \bar{\omega})$ and construct approximate θ-density functions $L_n(x, \bar{\omega})$. Let $\Delta(z, n, \bar{\omega})$ be the discontinuity of $L_n(x, \bar{\omega})$ at $z = a_i^n$, and let $\bar{H}(z, n, d) = \{\bar{\omega} : \Delta(z, n, \bar{\omega}) \geqq d\}$. We have shown that if $a_{-M}^0 \leqq z < a_M^0$ and n is sufficiently large, then

$$\bar{P}(\bar{H}(z, n, d)) \leqq K \exp(-(12c)^{-1}\, g(2^{-n})^{1/2}\, d).$$

Let $\gamma(z, n, d) = P(H(z, n, d))$. If $\omega \in H(z, n, d)$ and $t \in J(z, n, d, \omega)$, then $(\omega, t) = \bar{\omega} \in \bar{H}(z, n, d)$. Let

$$\bar{H}_1(z, n, d) = \{(\omega, t) : \omega \in H(z, n, d), t \in J(z, n, d, \omega)\}.$$

By definition of P^*, $\mathrm{LM}(J(z, n, d, \omega)) \geqq 2^{-2n}$ implies $P^*(J(z, n, d, \omega)) \geqq e^{-N}2^{-2n}$ since $J(z, n, d, \omega) \subset [0, N]$. Hence, using Fubini's theorem, we have

$$\begin{aligned} 2^{-2n}e^{-N}\, \gamma(z, n, d) &\leqq \bar{P}(\bar{H}_1(z, n, d) \leqq \bar{P}(\bar{H}(z, n, d)) \\ &\leqq K \exp(-(12c)^{-1}g(2^{-n})^{-1/2}\, d) \end{aligned}$$

or

$$\gamma(z, n, d) \leqq Ke^N2^{2n} \exp(-(12c)^{-1}g(2^{-n})^{-1/2}\, d)$$

if n is sufficiently large and $a_{-M}^0 \leqq z < a_M^0$. This is, however, exactly the assertion of Lemma 2.

4. Examples and applications

We shall now exhibit a large class of Markoff processes satisfying the conditions of Theorem 1 with $\theta = \text{LM}$. A stable process of index α, $0 < \alpha \leqq 2$, is a Markoff process with transition probabilities $P_t(x, dy)$ where $P_t(x, dy)$ is the measure on R having (LM) density $p_t(x, y)$ where

$$(4.1)\quad \begin{aligned} p_t(x, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(x-y)v} \\ &\cdot \exp(-t(id_1 v + d_2 \,|\, v \,|^{\alpha} (1 + id_3(v/|\, v \,|) \tan \tfrac{1}{2}\pi\alpha))) \, dv, \\ &d_1, d_2, d_3 \quad \text{fixed real numbers}, \quad d_2 > 0, \,|\, d_3 \,|\, \leqq 1. \end{aligned}$$

It is well known that there is a Markoff process having these transition probabilities which satisfies Hypothesis A. If $d_1 = d_3 = 0$, $d_2 = 1$, and $\alpha = 2$, then $P_t(x, dy)$ is the familiar normal distribution with mean zero and variance t. Thus Brownian motion is a stable process of index 2. It is clear that (for fixed d_1, d_2, and d_3) $p_t(x, y)$ depends only on t and $x - y$. We therefore write $p_t(x, y)$ as $p_t(z)$ where $z = x - y$. As was noted in Section 1, the potential operator U corresponding to the $P_t(x, dy)$ will have an LM-density $U(x, y)$ where

$$U(x, y) = \int_0^{\infty} e^{-t} p_t(x, y) \, dt.$$

Since $p_t(x, y)$ depends only on $x - y$, so does $U(x, y)$, and we write $U(x, y)$ as $U(z)$ where $z = x - y$. To show that the conditions of Theorem 1 are satisfied, it suffices to show that there exists a good function g such that

$$(4.2)\quad |\, U(x) - U(0) \,|\, \leqq g(\tfrac{1}{2} \,|\, x \,|)$$

for $|\, x \,|$ sufficiently small. We shall show that $g(x) = Kx^{\alpha-1}$ will satisfy (4.2) for $1 < \alpha \leqq 2$. We have, letting $\gamma = 1/\sqrt{(2\pi)}$,

$$\begin{aligned} U(x) &= \int_0^{\infty} e^{-t} p_t(x) \, dt \\ &= \gamma \int_0^{\infty} e^{-t} \int_{-\infty}^{\infty} e^{-ixv} \exp(-t(id_1 v + d_2 \,|v|^{\alpha}(1 + id_3(v/|v|) \tan \tfrac{1}{2}\pi\alpha))) \, dv \, dt \\ &= \gamma \int_{-\infty}^{\infty} \int_0^{\infty} e^{-t} e^{-ixv} \exp(-t(id_1 v + d_2|v|^{\alpha}(1 + id_3(v/|v|) \tan \tfrac{1}{2}\pi\alpha))) \, dt \, dv \\ &= \gamma \int_{-\infty}^{\infty} \frac{e^{-ixv}}{1 + d_2 \,|\, v \,|^{\alpha} (1 + id_3(v/|\, v \,|) \tan \tfrac{1}{2}\pi\alpha) + id_1 v} \, dv. \end{aligned}$$

If $\alpha = 2$, then $\tan \frac{1}{2}\pi\alpha = 0$, and we have

$$U(x) = \gamma \int_{-\infty}^{\infty} \frac{e^{-ixv}}{1 + d_2 v^2 + id_1 v} \, dv.$$

Now $1 + d_2 v^2 + id_1 v$ is a quadratic polynomial whose roots are easily seen to lie on the imaginary axis. Therefore, by using contour integration and the theory of residues, there are real numbers a, b, c, depending on d_1, d_2, and d_3, such that

$$U(x) = ae^{-bx} \quad \text{if} \quad x \geqq 0,$$
$$= ae^{-cx} \quad \text{if} \quad x \leqq 0.$$

Hence $| U(x) - U(0) | \leqq a \max (|b|, |c|) | x | = K | x |$.

Suppose that $1 < \alpha < 2$. Then

$$U(x) - U(0) = \gamma \int_{-\infty}^{\infty} \frac{e^{-ixv} - 1}{1 + d_2 | v |^\alpha (1 + id_3(v/| v |) \tan \tfrac{1}{2}\pi\alpha) + id_1 v} dv$$
$$= \gamma \int_{-\infty}^{\infty} \frac{1}{x} \left(\frac{e^{-iy} - 1}{1 + d_2 | y/x |^\alpha (1 + id_3(y/| y |) \tan \tfrac{1}{2}\pi\alpha) + id_1 yx^{-1}} \right) dy$$

(by letting $y = xv$). Hence, if $x > 0$,

$$| U(x) - U(0) | x^{1-\alpha}$$
$$\leqq \gamma \int_{-\infty}^{\infty} \frac{| e^{-iy} - 1 |}{| x^\alpha + d_2 | y |^\alpha (1 + id_3(y/| y |) \tan \tfrac{1}{2}\pi\alpha) + id_1 yx^{\alpha-1} |} dy$$
$$\leqq \gamma \int_{-\infty}^{\infty} \frac{| e^{-iy} - 1 |}{d_2 | y |^\alpha} dy = K < \infty.$$

The last inequality follows from the inequality

$$d_2 | y |^\alpha \leqq | x^\alpha + d_2 | y |^\alpha (1 + id_3(y/| y |) \tan \tfrac{1}{2}\pi\alpha) + id_1 yx^{\alpha-1} |.$$

Thus we have shown that

$$| U(x) - U(0) | \leqq K x^{\alpha-1}$$

for $x \geqq 0$. For $x \leqq 0$ the same inequality holds by making the substitution $y = -xv$ above.

If $\limsup_{x\to 0} g(x)/g(2x) < 1$, then we are in a position to construct explicit moduli of continuity for $L(x, t, \omega)$ in x and t. For simplicity, we shall perform the construction for $\theta = \text{LM}$. If $\limsup_{x\to 0} g(x)/g(2x) < 1$, then $q(n) \leqq K(ng(2^{-n})^{1/2} + 2^{-n+1})$ for K a sufficiently large positive constant. If $\theta = \text{LM}$, then $a_i^n = i2^{-n}$. Then (2.2) becomes

$$(4.3) \qquad \begin{aligned} | L(x, t, \omega) - L(x', t, \omega) | &\leqq 3q(n) \leqq K(ng(2^{-n})^{1/2} + 2^{-n+1}) \\ &\leqq K(| \log | x - x' | | g(| x - x' |)^{1/2} + 2 | x - x' |) \end{aligned}$$

if $| x | \leqq M$, $| x' | \leqq M$, $t \leqq N$, $\omega \notin G_m$, and $2^{-n-1} < | x - x' | \leqq 2^{-n}$ where $n \geqq m$. For many Markoff processes, including the stable processes, almost all sample paths are bounded in any finite interval. Hence, for almost all ω, there is an $M(\omega)$ such that $L(x, t, \omega) \equiv 0$ for $t \leqq N$ and $| x | \geqq M(\omega)$. When such is the case, then by choosing K sufficiently large (4.3) will hold for all x and x' provided only that $t \leqq N$. If, for x sufficiently small, there are positive constants K_1 and β such that $g(x) \geqq (K_1 x)^\beta$, then

$$| L(x, t, \omega) - L(x', t, \omega) | \leqq K(| \log | x - x' | | g(| x - x' |)^{1/2})$$

for K sufficiently large. For the stable processes we have

$$(4.4) \quad | L(x, t, \omega) - L(x', t, \omega) | \leqq K(| \log | x - x' | | (| x - x' |)^{(\alpha-1)/2}).$$

If we turn to a modulus of continuity for $L(x, t, \omega)$ in t, (2.3) becomes

$$\begin{aligned} | L(x, t, \omega) - L(x, t', \omega) | &\leqq 2q(n) + 2^n | t - t' | \\ &\leqq 2K(ng(2^{-n})^{1/2} + 2^{-n+1}) + 2^n | t - t' |, \end{aligned}$$

for $| x | \leqq M$, $0 < t, t' \leqq N$, $\omega \notin G_m$, and $n \geqq m$. The minimum value of the right-hand side for $| t - t' |$ fixed and n ranging from m to infinity is a function of $| t - t' |$, say $l(| t - t' |)$. Then

$$| L(x, t, \omega) - L(x, t', \omega) | \leqq l(| t - t' |)$$

for $| x | \leqq M$, $0 < t, t' \leqq N$, and $\omega \notin G_m$. In the case of the stable processes we have

$$(4.5) \quad | L(x, t, \omega) - L(x, t', \omega) | \leqq K | t - t' |^{(\alpha-1)/(\alpha+1)} | \log | t - t' | |^{2/(\alpha+1)}$$

for $0 < t, t' \leqq N$, and K sufficiently large. (4.5) may be derived in the following manner: Choose n so that 2^{-n} is approximately equal to $| t - t' |^{2/(\alpha+1)} | \log | t - t' | |^{-2/(\alpha+1)}$. If $| t - t' |$ is sufficiently small, then n will be greater than or equal to m, where $\omega \notin G_m$. It then follows that for $| t - t' |$ so small

$$l(| t - t' |) \leqq K | t - t' |^{(\alpha-1)/(\alpha+1)} | \log | t - t' | |^{2(\alpha+1)}.$$

Choosing K sufficiently large we have the inequality holding for all t and t', $0 < t, t' \leqq N$. Note that (4.5) holds for all x (and not just for $| x | < M$ for some M) since almost all sample paths are bounded in any finite interval.

In the Brownian-motion case the moduli of continuity derived here are exactly the same as those given by Trotter [3].

5. Extensions

One shortcoming of Theorem 1 is that (2.1) involves θ_c and not θ. It is not very clear what is the significance of the decomposition of θ into $\theta_c + \theta_s$. As was noted in Section 2, under the assumptions of Theorem 1, θ_c gives positive measure to every nonempty open set. It would be desirable to know that the conclusions of Theorem 1 hold when, instead of (2.1), only

$$(5.1) \quad \begin{aligned} | U(x, x) - U(x, y) | &\leqq g(\tfrac{1}{2}\varphi(\langle x, y \rangle)), \quad \varphi(\langle x, y \rangle) \leqq 2\varepsilon_1, \\ | U(x, x) - U(y, x) | &\leqq g(\tfrac{1}{2}\varphi(\langle x, y \rangle)), \quad \varphi(\langle x, y \rangle) \leqq 2\varepsilon_1, \end{aligned}$$

is true, where g is a good function defined on an interval $[0, \varepsilon_1]$ and φ is a Borel measure on R which gives positive measure to every nonempty open set. At this time, however, only a slightly weaker result can be shown.

We shall say that a Markoff process satisfies Hypothesis B if it satisfies A1 and A2 and satisfies A3 for intervals of the form $(a, b]$ as well as intervals of the form $[a, b)$.

THEOREM 2. *Let $X(\tau, \omega)$ be a one-dimensional Markoff process defined on a probability space $(\Omega, \mathfrak{F}, P)$ satisfying Hypothesis* B, *having transition probabilities $P_t(x, dy)$ and corresponding potential operator U. Let $\bar{\Omega} = \Omega \times R_+$ with measure $\bar{P} = P \times P^*$, and let $X(\tau, \bar{\omega})$ and $S(\bar{\omega})$ be as in Section* 1. *With $(\omega, t) = \bar{\omega} \in \bar{\Omega}$ we associate a measure, $\mu(\cdot, \bar{\omega})$, on R by*

$$\mu(A, \bar{\omega}) = \mu(A, t, \omega) = \text{LM}\{\tau : X(\tau, \bar{\omega}) \in A, \quad 0 \leqq \tau < S(\bar{\omega})\}$$

for every Borel set A. If U has a θ-kernel $U(x, y)$, θ a Borel measure on R, and if there exist a good function g, defined on an interval $[0, \varepsilon_1]$, and a Borel measure φ on R, giving positive measure to every nonempty open set, such that (5.1) *holds, then, for almost all $\bar{\omega}$, there exists a function $L(x, \bar{\omega})$, defined and continuous for all $x \in R$, such that*

$$\mu(A, \bar{\omega}) = \int_A L(x, \bar{\omega})\theta\{dx\} \tag{5.2}$$

for every Borel set A.

It follows that, for almost all ω, there exists a function $L(x, t, \omega)$ defined for all $x \in R$ and all $t \in R_+$, continuous in x for almost all t (with respect to Lebesgue measure), satisfying (0.2) for every Borel set A.

Proof. We first show that θ gives positive measure to every nonempty open set. Suppose D is an open set and $\theta(D) = 0$. Let 1_D be the indicator of D, i.e., $1_D(x)$ is either one or zero depending on whether $x \in D$ or not. Then, by assumption,

$$U(1_D)(x) = \int 1_D(y)U(x, y)\theta\{dy\} = \int_D U(x, y)\theta\{dy\} = 0$$

for all $x \in R$. But, if $x \in D$, by (1.3),

$$U(1_D)(x) = E\left(\int_0^S 1_D(X_x(\tau, \omega))\, d\tau\right) > 0,$$

since the right continuity of almost all sample paths of $X_x(\tau, \omega)$ implies

$$\int_0^S 1_D(X_x(\tau, \omega))\, d\tau > 0$$

for almost all ω. Thus θ gives positive measure to every nonempty open set.

Since φ is a Borel measure, we can find numbers a_i^0, $i = 0, \pm 1, \pm 2, \cdots$ such that $a_i^0 < a_{i+1}^0$ for all i,

$$\lim_{i\to+\infty} a^0_{-i} = -\infty, \quad \lim_{i\to\infty} a^0_i = \infty, \quad \text{and} \quad \varphi((a^0_i, a^0_{i+1})) \leqq 1.$$

Between a^0_i and a^0_{i+1} there is at least one point a such that $\varphi((a^0_i, a)) \leqq \frac{1}{2} = 2^{-1}$ and $\varphi((a, a^0_{i+1})) \leqq \frac{1}{2} = 2^{-1}$. Choosing such an a (if necessary) we denote it by a^1_{2i+1}, and we denote a^0_i by a^1_{2i}. Thus we have constructed a sequence of points a^1_i such that, for all i, $a^1_i < a^1_{i+1}$, $a^0_i = a^1_{2i}$,

$$\lim_{i\to+\infty} a^1_{-i} = -\infty, \quad \lim_{i\to\infty} a^1_i = \infty, \quad \text{and} \quad \varphi((a^1_i, a^1_{i+1})) \leqq 2^{-1}.$$

In similar fashion we can find, for every n, a sequence of numbers a^n_i such that, for all i, $a^n_i < a^n_{i+1}$, $a^n_i = a^{n+1}_{2i}$,

$$\lim_{i\to\infty} a^n_{-i} = -\infty, \quad \lim_{i\to\infty} a^n_i = \infty, \quad \text{and} \quad \varphi((a^n_i, a^n_{i+1})) \leqq 2^{-n}.$$

Let $I^n_i = [a^n_i, a^n_{i+1})$. Then $I^n_i = I^{n+1}_{2i} \cup I^{n+1}_{2i+1}$, and $\mathcal{I}^n = \{I^n_i\}$ is a sequence of ever finer partitions of R.

We can now associate, for every n, an approximate θ-density, $L_n(x, \bar{\omega})$, for $\mu(\cdot, \bar{\omega})$ by

$$L_n(x, \bar{\omega}) = \mu(I^n, \bar{\omega})/\theta(I^n_i) \tag{5.3}$$

for all $x \in I^n_i$. We now will show, using the methods developed in Sections 2 and 3, that $L_n(x, \bar{\omega})$ converges, for almost all $\bar{\omega}$, to a limit function $L(x, \bar{\omega})$ which satisfies (5.2) for every Borel set A. To be more precise, we show that $L(x, \bar{\omega})$ satisfies (5.2) for all sets A in the σ-ring generated by the intervals I^n_i, $i = 1, 2, \cdots$, $n = 1, 2, \cdots$. However, since φ gives positive measure to every nonempty open set, it follows that the points a^n_i, $i = 1, 2, \cdots$, $n = 1, 2, \cdots$ are dense in R, and therefore the σ-ring generated by all the intervals I^n_i, $i = 1, 2, \cdots$, $n = 1, 2, \cdots$ is the σ-ring of all Borel sets.

We again show this convergence only for x in $[a^0_{-M}, a^0_M)$ and let M approach infinity. Let

$$\Delta(a^n_{2i+1}, n, \bar{\omega}) = L_n(a^n_{2i}, \bar{\omega}) - L_n(a^n_{2i+1}, \omega).$$

If we use once again the inequality $\left| \dfrac{a+b}{c+d} - \dfrac{a}{c} \right| \leqq \left| \dfrac{a}{c} - \dfrac{b}{d} \right|$, it follows that

$$| L_n(x, \bar{\omega}) - L_{n-1}(x, \bar{\omega}) | \leqq | \Delta(a^n_{2i+1}, n, \bar{\omega}) |$$

for all $x \in I^{n-1}_i$. Once again let $c_n = g(2^{-n})^{-1/2}$. By definition,

$$\Delta(a^n_{2i+1}, n, \bar{\omega}) = \int_0^{S(\bar{\omega})} V_{a^n_{2i+1}}(X(\tau, \omega))\, d\tau$$

where

$$\begin{aligned} V_{a^n_{2i+1}}(x) &= (\theta(I^n_{2i}))^{-1} && \text{for } x \in I^n_{2i}, \\ &= -(\theta(I^n_{2i+1}))^{-1} && \text{for } x \in I^n_{2i+1}, \\ &= 0 && \text{elsewhere.} \end{aligned}$$

By construction, if $x \in I^n_{2i}$ and $y \in I^n_{2i+1}$, then $\varphi(\langle x, y\rangle) \leqq 2^{-n+1}$. It now follows, exactly as in the proof of Lemma 4, that there exist positive constants k and K such that

$$(5.4) \qquad E(\exp(k\, c_n \mid \Delta(a_{2i+1}^n, n, \bar{\omega}) \mid)) \leqq K$$

for all n sufficiently large and all a_{2i+1}^n where $a_{-M}^0 \leqq a_{2i+1}^n < a_M^0$.

For $z = a_{2i+1}^n$, let

$$H(z, n, d) = \{\bar{\omega} : \mid \Delta(z, n, \omega) \mid \geqq d\}.$$

It follows from (5.4) that for n sufficiently large and $a_{-M}^0 \leqq z < a_M^0$,

$$\bar{P}(H(z, n, d)) \leqq K \exp(-k\, g(2^{-n})^{-1/2}\, d)$$

where K and k are positive constants depending only on M. Let $H(n, d) = \bigcup_z H(z, n, d)$ where the union is over those $z = a_{2i+1}^n$ where $a_{-M}^0 \leqq z < a_M^0$. There will be exactly $M2^n$ such z. If n is sufficiently large, then

$$\bar{P}(H(n, d)) \leqq \textstyle\sum_z \bar{P}(H(n, z, d)) \leqq MK\, 2^n \exp(-k\, g(2^{-n})^{-1/2}\, d).$$

Let $d_n = nk^{-1} g(2^{-n})^{1/2}$, and $G_m = \bigcup_{n \geqq m} H(n, d_n)$. If m is sufficiently large, then

$$\bar{P}(G_m) \leqq \textstyle\sum_{n \geqq m} \bar{P}(H(n, d_n)) \leqq MK \sum_{n \geqq m} 2^n e^{-n} = MK \sum_{n \geqq m} (2/e)^n,$$

and since this last is a convergent series,

$$P(\lim_m G_m) = \lim_m P(G_m) = 0.$$

As we have shown above,

$$\max_{a_{-M}^0 \leqq x < a_M^0} \mid L_n(x, \bar{\omega}) - L_{n-1}(x, \bar{\omega}) \mid \leqq \max_{a_{-M}^0 \leqq z < a_M^0} \mid (\Delta(z, n, \bar{\omega})) \mid.$$

Then, unless $\bar{\omega} \in G_m$, for any n, p with $n \geqq m$,

$$\mid L_{n+p}(x, \bar{\omega}) - L_n(x, \bar{\omega}) \mid \leqq \textstyle\sum_{n+1}^{n+p} \mid L_i(x, \bar{\omega}) - L_{i-1}(x, \bar{\omega}) \mid < \sum_{i=n}^{i=\infty} d_i = q(n).$$

Now, since g is a good function, $\sum_{i=n}^{i=\infty} d_i < \infty$. Hence $q(n) \to 0$ as $n \to \infty$. Hence if $\bar{\omega} \notin G_m$, the sequence $L_n(x, \bar{\omega})$ converges, uniformly for $a_{-M}^0 \leqq x < a_M^0$, to a limit which we shall denote by $L(x, \bar{\omega})$. Furthermore,

$$\mid L(x, \bar{\omega}) - L_n(x, \bar{\omega}) \mid \leqq q(n)$$

for $n \geqq m$, $a_{-M}^0 \leqq x < a_M^0$. It is clear that the measures on $[a_{-M}^0, a_M^0)$ having $L_n(x, \bar{\omega})$ as θ-densities converge at least weakly to $\mu(\cdot, \bar{\omega})$ restricted to $[a_{-M}^0, a_M^0)$, and consequently $L(x, \bar{\omega})$ is a θ-density function for $\mu(\cdot, \bar{\omega})$ restricted to $[a_{-M}^0, a_M^0)$. The continuity of $L(x, \bar{\omega})$ remains to be established.

We first show that $L(x, \bar{\omega})$ is right continuous for all x. If $x \in R$, let $\delta(n, x)$ be a positive number so small that $x \in I_i^n$ and $0 \leqq y - x < \delta(n, x)$ imply $y \in I_i^n$. Suppose $\bar{\omega} \notin G_m$. If $0 \leqq y - x < \delta(n, x)$ where $n \geqq m$ and $a_{-M}^0 \leqq x < a_M^0$, then

$$
\begin{aligned}
| L(x, \bar{\omega}) - L(y, \bar{\omega}) | \leqq & \; | L(x, \bar{\omega}) - L_n(x, \bar{\omega}) | \\
& + | L_n(x, \bar{\omega}) - L_n(y, \bar{\omega}) | \\
& + | L_n(y, \bar{\omega}) - L(y, \bar{\omega}) | \\
\leqq & \; 2q(n).
\end{aligned} \tag{5.5}
$$

Hence if $n \geqq m$ and $q(n) < \varepsilon/2$, then $0 \leqq y - x < \delta(n, x)$ implies $| L(x, \bar{\omega}) - L(y, \bar{\omega}) | < \varepsilon$. Thus $L(x, \bar{\omega})$ is right continuous in x.

Now let $J_i^n = (a_i^n, a_{i+1}^n]$, and let $Q_n(x, \bar{\omega}) = \mu(J_i^n, \bar{\omega})/\theta(J_i^n)$ for all $x \in J_i^n$. Then, by exactly the same logic as above, for almost all $\bar{\omega}$, $Q_n(x, \bar{\omega})$ converges to a limit function $Q(x, \bar{\omega})$ which is a θ-density for $\mu(\cdot, \bar{\omega})$ restricted to $(a_{-M}^0, a_M^0]$. In this case, however, $Q(x, \bar{\omega})$ will be left-continuous in x. If A is any Borel set in (a_{-M}^0, a_M^0), we have

$$\int_A Q(x, \bar{\omega})\theta\{dx\} = \int_A L(x, \bar{\omega})\theta\{dx\}$$

since both integrals equal $\mu(A, \bar{\omega})$. It therefore follows that $Q(x, \bar{\omega}) = L(x, \bar{\omega})$ for almost all x (with respect to θ) in (a_{-M}^0, a_M^0). Now, since θ gives positive measure to all nonempty open sets, it follows that the set of points where $Q(x, \bar{\omega}) = L(x, \bar{\omega})$ is dense in (a_{-M}^0, a_M^0). Hence $Q(x, \bar{\omega}) = L(x, \bar{\omega})$ everywhere in (a_{-M}^0, a_M^0) and is continuous for all x in (a_{-M}^0, a_M^0). This completes the proof of Theorem 2.

If $\bar{\omega} = (\omega, t_0)$ and $L(x, \bar{\omega})$ exists and is continuous in x, then $L(x, t_0, \omega) = L(x, \bar{\omega})$ is a continuous function in x and satisfies (0.2) (for $t = t_0$) for every Borel set A. Since

$$\bar{P}(\{\bar{\omega} : L(x, \bar{\omega}) \text{ is not well defined}\}) = 0,$$

it follows that

$$P(\{\omega : \text{LM}(\{t : L(x, \bar{\omega}) \text{ is not well defined for } \bar{\omega} = (\omega, t)\}) > 0\}) = 0.$$

Thus we have shown that, for almost all ω, there exists a function $L(x, t, \omega)$, defined for all $x \in R$ and almost all t (with respect to Lebesgue measure), satisfying (0.2) (for all such t) for every Borel set A. If t_0 is a point where it is not defined by this procedure, then define $L(x, t_0, \omega) = \lim_{t \downarrow t_0} L(x, t, \omega)$. By Lesbesgue's bounded convergence theorem it follows that $L(x, t_0, \omega)$ satisfies (0.2) (for $t = t_0$) for every Borel set A. Note that alternatively one could also define $L(x, t_0, \omega) = \lim_{t \uparrow t_0} L(x, t, \omega)$.

The following property which processes having local times possess was pointed out to me by F. Spitzer.

THEOREM 3. *If, for almost all* ω, $\mu(\cdot, t, \omega)$ *has a* θ*-density* $L(x, t, \omega)$, θ *a Borel measure on* R, *then* $J \subset [0, \infty]$, $\text{LM}(J) > 0$ *imply that* $\theta(\{X(\tau, \omega) : \tau \in J\}) > 0$ *for almost all* ω.

Proof. It suffices to consider the case where $J \subset [0, N]$ for some N. Let

$\omega \in \Omega$ be such that $L(x, N, \omega)$ is defined. Let $A = \{X(\tau, \omega) : \tau \in J\}$. Suppose $\theta(A) = 0$. Then

$$0 < \text{LM}(J) \leqq \mu(A, N, \omega) = \int_A L(x, N, \omega)\theta\{dx\} = 0,$$

which is a contradiction.

In particular, applying Theorems 1 and 2 with $\theta = \text{LM}$, we have sufficient conditions that the range of almost all sample paths have nonzero Lebesgue measure whenever the domain has nonzero Lebesgue measure.

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