MODULES OVER REGULAR LOCAL RINGS

BY
M. PAVAMAN MURTHY

Introduction

In [1] M. Auslander has proved the following:

THEOREM. *Let R be an unramified regular local ring, and M a torsion-free R-module of finite type. If* $\mathrm{Tor}_i^R(M, N) = 0$ *for some R-module N of finite type, then* $\mathrm{Tor}_j^R(M, N) = 0$ *for all* $j \geqq i$.

In this paper we shall prove that for an arbitrary regular ring R and for two R-modules of finite type M and N, if $\mathrm{Tor}_i^R(M, N) = \mathrm{Tor}_{i+1}^R(M, N) = 0$, then $\mathrm{Tor}_j^R(M, N) = 0$ for all $j \geqq i$. In fact we shall prove a general result for complete intersections.[1] In Section 2 using this result and following methods of M. Auslander we shall show that for a regular local ring R and for any two R-modules of finite type M and N if M and $M \otimes N$ are torsion-free and $\mathrm{Tor}_1^R(M, N) = 0$, then

(i) N is torsion-free.
(ii) $\mathrm{Tor}_i^R(M, N) = 0$ for $i > 0$.
(iii) $\mathrm{hd}\, M + \mathrm{hd}\, N = \mathrm{hd}\,(M \otimes N) < \dim R$.

Here $\mathrm{hd}\, M$ denotes the homological dimension of M, and $\dim R$ denotes the Krull dimension of R. We shall also give some sufficient conditions for an R-module to be reflexive.

Let R be a local ring (not necessarily regular). Let M and N be R-modules of finite type such that $\mathrm{hd}\, M < \infty$. Let q be the largest integer such that $\mathrm{Tor}_q^R(M, N) \neq 0$. In [1, Theorem 1.2] the formula

$$\mathrm{codim}\, N = \mathrm{codim}\, \mathrm{Tor}_q^R(M, N) + \mathrm{hd}\, M - q,$$

was established under the hypothesis $\mathrm{codim}\, \mathrm{Tor}_q^R(M, N) \leqq 1$ or $q = 0$, where for an arbitrary R-module E, $\mathrm{codim}\, E$ denotes codimension of E (for notation and basic concepts of homology theory of local rings see for example [3], [4], or [5]). In Section 3 we give an example to show that the above formula is not universally valid. This answers in the negative a question raised by M. Auslander in [1].

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1. A property of Tor over regular rings

Throughout this paper we shall consider only commutative noetherian rings with identity and modules which are unitary and of finite type. We

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[1] In the first version of this note the author proved this result for regular local rings. He is thankful to the referee for the remark that a similar proof yields a generalisation to complete intersections.

recall (see [1]) that a chain complex X over a ring R is *rigid* with respect to an R-module M if $H_i(X \otimes M) = 0$ for some i implies $H_j(X \otimes M) = 0$ for all $j \geqq i$ (where for a chain complex Y, $H_j(Y)$ denotes its j^{th} homology module). Thus to say that a projective resolution (and therefore any projective resolution) of an R-module M is rigid with respect to an R-module N is to say that for any i, $\operatorname{Tor}_i^R(M, N) = 0$ implies $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geqq i$. Let $R = k[[X_1, \cdots, X_n]]$ be the ring of formal power series over a discrete valuation ring k. Let M be an R-module such that the prime element π of k is not a zero divisor for M. For the sake of completeness we include here the proof of the fact that any projective resolution of M is rigid with respect to all modules (see [1, proof of Corollary 2.2]).

Let N be any R-module, and let M be as above. Then we have (see [5, Chapter V]),

$$\operatorname{Tor}_i^{k[[X,Y]]}(M \mathbin{\widehat{\otimes}_k} N, R) \approx \operatorname{Tor}_i^R(M, N) \qquad \text{for } i \geqq 0,$$

where $k[[X, Y]] = k[[X_1, \cdots, X_n, Y_1, \cdots, Y_n]]$ is the ring of formal power series in $2n$ variables and $M \mathbin{\widehat{\otimes}_k} N$ is the complete tensor product of M and N over k (for definition see [5, Chapter V]) and R is considered as a $k[[X, Y]]$-module by identifying it with $k[[X, Y]]/(X_1 - Y_1, \cdots, X_n - Y_n)$. As $X_1 - Y_1, \cdots, X_n - Y_n$ form a $k[[X, Y]]$-sequence, the Koszul complex of $X_1 - Y_1, \cdots, X_n - Y_n$ provides a projective resolution of R as a $k[[X, Y]]$-module (for definition see [4] or [5]). As the Koszul complex is rigid with respect to all modules (see [4, 2.6]), we have

PROPOSITION 1.1. *Let R be a ring of formal power series over a discrete valuation ring k, and let M be a torsion-free R-module. Then any projective resolution of M is rigid with respect to all R-modules.*

LEMMA 1.2. *Let R be an integral domain which has the following property: for any torsion-free module M, every projective resolution of M is rigid with respect to all modules. Then, for any $i \geqq 2$ and for any two modules M, N, $\operatorname{Tor}_i^R(M, N) = 0$ implies $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geqq i$.*

Proof. There exist R-modules L and F such that F is free and the sequence

$$0 \to L \to F \to M \to 0$$

is exact. Then by the exact sequence of Tor, we have

$$\operatorname{Tor}_{j+1}^R(M, N) \approx \operatorname{Tor}_j^R(L, N) \qquad \text{for all } j > 0.$$

Since $i \geqq 2$, we have

$$\operatorname{Tor}_i^R(M, N) \approx \operatorname{Tor}_{i-1}^R(L, N) = 0.$$

Now L is torsion-free, and therefore

$$\operatorname{Tor}_j^R(L, N) = 0 \qquad \text{for all } j \geqq i - 1,$$

i.e., $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geqq i$.

COROLLARY 1.3. *Let R be as in Proposition 1.1; then Lemma 1.2 holds.*

Let R be a local ring, and let M and N be R-modules. Then we have

$$[\mathrm{Tor}_i^R (M, N)]^\wedge \approx \mathrm{Tor}_i^{\hat{R}} (\hat{M}, \hat{N}),$$

where for an R-module E, $\hat{E}$ denotes the completion of E. Further $E = 0$ if and only if $\hat{E} = 0$ (these are consequences of the fact that $\hat{R}$ is R-flat (see for example [5])). We here recall that a regular local ring R is said to be unramified if it is of equal characteristic, or of unequal characteristic with $p \notin m^2$ (m is the maximal ideal of R, and p is the characteristic of R/m).

COROLLARY 1.4. *Let R be an unramified regular local ring. Let M, N be R-modules such that $\mathrm{Tor}_i^R (M, N) = 0$ for some $i \geqq 2$. Then $\mathrm{Tor}_j^R (M, N) = 0$ for all $j \geqq i$.*

Proof. By the above remark we may assume R is complete. By a well-known structure theorem of Cohen [7], R is then a ring of formal power series over a complete discrete valuation ring. Now the result follows from Corollary 1.3.

LEMMA 1.5. *Let Λ be a local ring, and let $R = \Lambda/(x)$ where x is not a zero divisor in Λ. Suppose Λ has the property: for any two Λ-modules A, B and for any $i \geqq 2$*

$$\mathrm{Tor}_{i+k}^\Lambda (A, B) = 0, \qquad 0 \leqq k \leqq d,$$

implies $\mathrm{Tor}_j^\Lambda (A, B) = 0$ for $j \geqq i$. Then for any two R-modules M, N of finite type and for any l,

(I) $$\mathrm{Tor}_{l+k}^R (M, N) = 0, \qquad 0 \leqq k \leqq d + 1,$$

implies $\mathrm{Tor}_j^R (M, N) = 0$ for all $j \geqq l$.

Proof. Let (I) hold. If $l = 0$, then $M \otimes N = 0$, and therefore $M = 0$ or $N = 0$. Hence we may assume $l > 0$. Choose Λ-modules L, F such that the sequence

(i) $$0 \to L \to F \to M \to 0$$

is Λ-exact. Since x annihilates M, we have $M \otimes_\Lambda R \approx M$. Since x is not a zero divisor in Λ, the sequence

$$0 \to \Lambda \xrightarrow{x} \Lambda \to R \to 0$$

is exact. By tensoring this sequence with M we get $\mathrm{Tor}_1^\Lambda (M, R) \approx M$. Now by tensoring the exact sequence (i) with R over Λ we have the exact sequence

(ii) $$0 \to M \to L/xL \xrightarrow{\alpha} F/xF \to M \to 0.$$

Set $\alpha(L/xL) = L'$. Since F/xF is R-free and the sequence

$$0 \to L' \to F/xF \to M \to 0$$

is R-exact, we have

(a) $$\operatorname{Tor}_j^R (L', N) \approx \operatorname{Tor}_{j+1}^R (M, N), \qquad \text{for } j > 0.$$

As x is not a zero divisor for L, we get $\operatorname{Tor}_i^\Lambda (L, R) = 0$, for $i > 0$. Hence

(b) $$\operatorname{Tor}_j^\Lambda (L, N) \approx \operatorname{Tor}_j^R (L/xL, N), \qquad \text{for } j \geqq 0$$

(see [2, VI, 4.11]). Again by (i) we have

(c) $$\operatorname{Tor}_{j+1}^\Lambda (M, N) \approx \operatorname{Tor}_j^\Lambda (L, N), \qquad \text{for all } j > 0.$$

By tensoring the exact sequence

$$0 \to M \to L/xL \to L' \to 0$$

with N over R, and because of the isomorphisms (a), (b), and (c), we get the exact sequence:

$$(*) \quad \cdots \to \operatorname{Tor}_j^R (M, N) \to \operatorname{Tor}_{j+1}^\Lambda (M, N) \to \operatorname{Tor}_{j+1}^R (M, N)$$
$$\to \operatorname{Tor}_{j-1}^R (M, N) \to \operatorname{Tor}_j^\Lambda (M, N) \to \operatorname{Tor}_j^R (M, N) \to \cdots \quad (\text{for } j \geqq 2).$$

Since $\operatorname{Tor}_{l+k}^R (M, N) = 0$, $0 \leqq k \leqq d + 1$, using $(*)$ we have

$$\operatorname{Tor}_{l+k}^\Lambda (M, N) = 0, \qquad 1 \leqq k \leqq d + 1.$$

Hence $\operatorname{Tor}_j^\Lambda (M, N) = 0$, for all $j \geqq l + 1$. Therefore

$$\operatorname{Tor}_{j+1}^R (M, N) \approx \operatorname{Tor}_{j-1}^R (M, N) \qquad \text{for all } j \geqq l + 1.$$

Hence

$$\operatorname{Tor}_l^R (M, N) \approx \operatorname{Tor}_{l+2r}^R (M, N) \quad \text{and} \quad \operatorname{Tor}_{l+1}^R (M, N) \approx \operatorname{Tor}_{l+2r+1}^R (M, N).$$

Therefore $\operatorname{Tor}_j^R (M, N) = 0$ for all $j \geqq l$.

Using Lemma 1.2 and Lemma 1.5, by an easy induction we have

THEOREM 1.6. *Let Λ be a local domain which has the following property: any projective resolution of a torsion-free Λ-module is rigid with respect to all Λ-modules. Let $R = \Lambda/(x_1, \cdots, x_d)$, where $x_1, \cdots, x_d$, $d > 0$ is a Λ-sequence. Then for any two R-modules M, N and for any i*

$$\operatorname{Tor}_{i+k}^R (M, N) = 0, \qquad 0 \leqq k \leqq d,$$

implies $\operatorname{Tor}_j^R (M, N) = 0$, *for all* $j \geqq i$.

We recall that a local ring R is said to be a *complete intersection* if $R = \Lambda/(x_1, \cdots, x_r)$, where Λ is a regular local ring and $x_1, \cdots, x_r$ is a Λ-sequence.

COROLLARY 1.7. *Let $R = \Lambda/(x_1, \cdots, x_d)$, $d > 0$ be a complete intersection with Λ, an unramified regular local ring, and $x_1, \cdots, x_d$ a Λ-sequence. Then the conclusions of Theorem* 1.6 *hold.*

We say that a ring R is *regular* if R_m is a regular local ring for every maximal ideal m of R.

COROLLARY 1.8. *Let R be a regular ring. Then for any two R-modules M and N*

$$\mathrm{Tor}_i^R(M,N) = \mathrm{Tor}_{i+1}^R(M,N) = 0$$

implies $\mathrm{Tor}_j^R(M,N) = 0$ *for all* $j \geqq i$.

Proof. Since for any maximal ideal m of R, R_m is R-flat, we have

$$R_m \otimes \mathrm{Tor}_j^R(M,N) \approx \mathrm{Tor}_j^{R_m}(M_m, N_m).$$

Further for any R-module E, if $E_m = 0$ for all maximal ideals m of R, then $E = 0$. Hence we may assume R is a regular local ring. By the remark following Corollary 1.3 we may assume R is complete. Then by a structure theorem of Cohen (see [7]), $R \approx \Lambda/(x)$, where Λ is a ring of formal power series over a discrete valuation ring k. Now the corollary follows from Proposition 1.1 and Theorem 1.6.

Using Lemma 1.5 and Corollary 1.8, we have by induction

COROLLARY 1.9. *Let $R = \Lambda/(x_1, \cdots, x_d)$ be a complete intersection with R, an arbitrary regular local ring, and $x_1, \cdots, x_d$ a Λ-sequence. Then for any two R-modules M and N,*

$$\mathrm{Tor}_{i+k}^R(M,N) = 0, \qquad 0 \leqq k \leqq d+1,$$

implies $\mathrm{Tor}_j^R(M,N) = 0$ *for all* $j \geqq i$.

2. Some applications of Theorem 1.6

We give here some applications of Theorem 1.6 on the lines of [1].

PROPOSITION 2.1. *Let Λ be a local domain which has the property that any projective resolution of a torsion-free module is rigid with respect to all Λ-modules. Let $R = \Lambda/(x_1, \cdots, x_d)$ be an integral domain with $x_1, \cdots, x_d$ a Λ-sequence. Let M and N be R-modules such that M and $M \otimes N$ are torsion-free and* $\mathrm{Tor}_i^R(M,N) = 0$, $1 \leqq i \leqq d$. *Then*

(i) $\mathrm{Tor}_i^R(M,N) = 0$ *for* $i > 0$.

If further M is of finite homological dimension, then

(ii) *N is torsion-free.*

If N is also of finite homological dimension, then

(iii) $\mathrm{hd}\, M + \mathrm{hd}\, N = \mathrm{hd}\,(M \otimes N) < \mathrm{codim}\, R$.

Proof. (i) As M is torsion-free, we have an exact sequence

$$0 \to M \to F \to F/M \to 0$$

with F free and F/M a torsion-module. Hence the sequence

$$0 \rightarrow \operatorname{Tor}_1^R (F/M, N) \rightarrow M \otimes N \rightarrow F \otimes N \rightarrow (F/M) \otimes N \rightarrow 0$$

is exact. As $M \otimes N$ is torsion-free and $\operatorname{Tor}_1^R (F/M, N)$ is a torsion-module, we have $\operatorname{Tor}_1^R (F/M, N) = 0$. Further $\operatorname{Tor}_i^R (M, N) \approx \operatorname{Tor}_{i+1}^R (F/M, N)$ for all $i \geqq 1$. By hypothesis $\operatorname{Tor}_i^R (M, N) = 0$, $1 \leqq i \leqq d$. Hence $\operatorname{Tor}_j^R (F/M, N) = 0$, $1 \leqq j \leqq d + 1$. Therefore by Theorem 1.6 we have $\operatorname{Tor}_j^R (F/M, N) = 0$ for all $j > 0$, i.e., $\operatorname{Tor}_j^R (M, N) = 0$ for all $j > 0$.

(ii) is now a consequence of the following lemma.

LEMMA 2.2. *Let R be a local domain. Let M and N be R-modules such that*

(a) $\operatorname{hd} M < \infty$,

(b) *$M \otimes N$ is torsion-free,*

(c) $\operatorname{Tor}_i^R (M, N) = 0$ *for $i > 0$.*

Then M and N are torsion-free.

Proof. We prove the lemma by induction on $\dim R$. If $\dim R = 0$, then R is a field, and there is nothing to prove. Let $\dim R = k > 0$. Assume that the lemma is valid for all local domains of dimension $< k$. Then by induction hypothesis M_p and N_p are torsion-free for every nonmaximal prime ideal p of R. Hence no nonzero nonmaximal prime ideal is associated with M or N. Now since $\operatorname{Tor}_i^R (M, N) = 0$ for all $i > 0$, we have [1, Theorem 1.2]

$$\operatorname{codim} N = \operatorname{codim} (M \otimes N) + \operatorname{hd} M.$$

As $M \otimes N$ is torsion-free, $\operatorname{codim} (M \otimes N) > 0$. Since for any R-module X of finite homological dimension we have [3, Theorem 3.7]

$$\operatorname{hd} X + \operatorname{codim} X = \operatorname{codim} R > 0,$$

we have

$$\operatorname{codim} M + \operatorname{codim} N = \operatorname{codim} (M \otimes N) + \operatorname{codim} R.$$

As $\operatorname{codim} M \leqq \operatorname{codim} R$, $\operatorname{codim} N \leqq \operatorname{codim} R$, we have

$$\operatorname{codim} M > 0 \quad \text{and} \quad \operatorname{codim} N > 0.$$

Thus (0) is the only prime ideal associated with M and N, i.e., M and N are torsion-free.

(iii) As $\operatorname{Tor}_i^R (M, N) = 0$ for all $i > 0$, we have (see [1, Corollary 1.3])

$$\operatorname{hd} M + \operatorname{hd} N = \operatorname{hd} (M \otimes N).$$

As $M \otimes N$ is torsion-free, we have $\operatorname{codim} (M \otimes N) > 0$. Hence

$$\operatorname{hd} (M \otimes N) < \operatorname{codim} R.$$

COROLLARY 2.3. *Let R be a regular local ring. Let M and N be R-modules such that M and $M \otimes N$ are torsion-free and* $\operatorname{Tor}_1^R (M, N) = 0$. *Then*

(i) *N is torsion-free,*

(ii) $\operatorname{Tor}_i^R (M, N) = 0$ *for all $i > 0$,*

(iii) $\operatorname{hd} M + \operatorname{hd} N = \operatorname{hd} (M \otimes N) < \dim R$.

Proof. Using the remark following Corollary 1.3, we may assume that R is complete. Then $R \approx \Lambda/(x)$ where Λ is a ring of formal power series over a discrete valuation ring. Now the corollary follows from Proposition 1.1 and Proposition 2.1.

Remark 1. The hypothesis (in Corollary 2.3) that $\operatorname{Tor}_1^R(M, N) = 0$ and that M is torsion-free is not necessary in the case of unramified regular local rings. We do not yet know if it is true for arbitrary regular local rings.

Remark 2. A result similar to Corollary 2.3 can be stated for complete intersections.

THEOREM 2.4. *Let R be a regular domain. Let M be an R-module such that M and $M \otimes M$ are torsion-free and* $\operatorname{Tor}_1^R(M, M) = 0$. *Then M is reflexive.*

Proof. Since M is reflexive if and only if M_m is reflexive for every maximal ideal m of R, and since our hypothesis is preserved under localizations, we may assume R is a regular local ring. We now prove the theorem by induction on R. If $\dim R \leqq 2$, then by Corollary 2.3, $2 \operatorname{hd} M < 2$. Hence M is free and therefore reflexive. Suppose $\dim R = n > 2$. If M is free, there is nothing to prove. Assume M is not free. Since M is torsion-free, we have the exact sequence

$$0 \to M \xrightarrow{\alpha} M^{**} \to M^{**}/M \to 0,$$

where M^{**} denotes the bidual of M, and α is the canonical mapping of M into M^{**}. By induction hypothesis $(M^{**}/M)_p = 0$ for all nonmaximal prime ideals p of R. Therefore the maximal ideal m of R is the only prime ideal associated with M^{**}/M. Therefore if $M^{**}/M \neq 0$, then

$$\operatorname{codim} M^{**}/M = 0 \quad \text{and} \quad \operatorname{hd} M^{**}/M = n - \operatorname{codim} M^{**}/M = n.$$

Now $\operatorname{codim} M^{**} \geqq 2$ (see [6, 4.7]). Hence $\operatorname{hd} M^{**} \leqq n - 2$. Because of the exact sequence

$$0 \to M \to M^{**} \to M^{**}/M \to 0,$$

we have $\operatorname{hd} M = n - 1$. But by Corollary 2.3 (iii) we have $\operatorname{hd} M < n/2$, a contradiction. Hence $M^{**}/M = 0$, i.e., M is reflexive.

3. Formula for codimension

Let R be a local ring, and let M be an R-module. Let $x_1, \cdots, x_r$ be a minimal set of generators for M. Let $F = \sum_{i=1}^{r} Ry_i$ be a free R-module of rank r with the y_i linearly independent. The sequence

$$0 \to L \to F \xrightarrow{\varphi} M \to 0$$

is exact where $\varphi(y_i) = x_i$. The submodule L is uniquely determined up to an isomorphism and does not depend upon the minimal set of generators chosen (see [8, Chapter IV]). We call L the 1st syzygy of M and denote it by $\operatorname{syz}^1 M$. We define $\operatorname{syz}^0 M = M$, and $\operatorname{syz}^{i+1} M = \operatorname{syz}^1(\operatorname{syz}^i M)$, by induction. Thus all the $\operatorname{syz}^i M$ are uniquely determined up to an isomorphism.

In [1, Theorem 1.2] the following was proved:

(*) Let R be a local ring, M an R-module of finite homological dimension, and q the largest integer such that $\operatorname{Tor}_q^R(M, N) \neq 0$. If

$$\text{(i)} \quad \operatorname{codim} \operatorname{Tor}_q^R(M, N) \leqq 1 \qquad \text{or} \qquad \text{(ii)} \quad q = 0,$$

then

$$\operatorname{codim} N = \operatorname{codim} \operatorname{Tor}_q^R(M, N) + \operatorname{hd} M - q.$$

Now by writing the exact sequence of Tor we immediately see that

$$\operatorname{Tor}_i^R(\operatorname{syz}^q M, N) \approx \operatorname{Tor}_{q+i}^R(M, N) \qquad \text{for } i > 0.$$

Hence by (*)(ii) we have

PROPOSITION 3.1. *Let R be a local ring, and M a module of finite homological dimension. Let N be any module, and q the largest integer such that* $\operatorname{Tor}_q^R(M, N) \neq 0$. *Then* $\operatorname{codim} N = \operatorname{codim} \operatorname{syz}^q(M \otimes N) + \operatorname{hd} M - q$.

Finally we give an example to show that (*) is not true in general. Let R be a regular local ring of dimension $\geqq 3$. Let x be a prime element in R, and p a nonmaximal prime ideal of height $\geqq 2$, such that $x \notin p$. Set

$$M = R/(x) \quad \text{and} \quad N = R/(x) \oplus R/p.$$

Then $\operatorname{hd} M = 1$, and $\operatorname{Tor}_1^R(M, N) = R/(x) \neq 0$. Hence $q = 1$. Now

$$\operatorname{codim} N = \min(\operatorname{codim} R/(x), \operatorname{codim} R/p) = \operatorname{codim} R/p \leqq n - 1,$$

whereas $\operatorname{codim} \operatorname{Tor}_1^R(M, N) = n - 1$.

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TATA INSTITUTE OF FUNDAMENTAL RESEARCH
BOMBAY, INDIA