GEODESIC SPHERES IN GRASSMANN MANIFOLDS

BY

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1. Introduction

Let $\mathbf{G}_{n,k}(\mathbf{F})$ denote the Grassmann manifold consisting of all n-dimensional subspaces of a left k-dimensional hermitian vectorspace $\mathbf{F}^k$, where $\mathbf{F}$ is the real number field, the complex number field, or the algebra of real quaternions. We view $\mathbf{G}_{n,k}(\mathbf{F})$ as a Riemannian symmetric space in the usual way, and study the connected totally geodesic submanifolds $\mathbf{B}$ in which any two distinct elements have zero intersection as subspaces of $\mathbf{F}^k$. Our main result (Theorem 4 in §8) states that the submanifold $\mathbf{B}$ is a compact Riemannian symmetric space of rank one, and gives the conditions under which it is a sphere. The rest of the paper is devoted to the classification (up to a global isometry of $\mathbf{G}_{n,k}(\mathbf{F})$) of those submanifolds $\mathbf{B}$ which are isometric to spheres (Theorem 8 in §13). If $\mathbf{B}$ is not a sphere, then it is a real, complex, or quaternionic projective space, or the Cayley projective plane; these submanifolds will be studied in a later paper [11].

The key to this study is the observation that any two elements of $\mathbf{B}$, viewed as subspaces of $\mathbf{F}^k$, are at a constant angle (*isoclinic* in the sense of Y.-C. Wong [12]). Chapter I is concerned with sets of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^{2n}$, and we are able to extend Wong's structure theorem for such sets [12, Theorem 3.2, p. 25] to the complex numbers and the quaternions, giving a unified and basis-free treatment (Theorem 1 in §4). Essentially, we introduce a "closure" operation on the collection of all such sets, and characterize the "closed" sets by means of linear transformations which satisfy some equations studied by A. Hurwitz [6] in connection with quadratic forms permitting composition. We give the closed sets the name *isoclinic sphere*; the first result of Chapter II is that an isoclinic sphere on $\mathbf{F}^{2n}$ is a totally geodesic submanifold of $\mathbf{G}_{n,2n}(\mathbf{F})$ which is isometric to a sphere (Theorem 2 in §6). We then prove a strong converse (Theorem 3 in §7) which allows us to prove our main result (Theorem 4 in §8) by reducing it to the case where $\mathbf{B}$ is an isoclinic sphere on a $2n$-dimensional subspace of $\mathbf{F}^k$.

Chapter III is devoted to the classification of isoclinic spheres on $2n$-dimensional subspaces of $\mathbf{F}^k$, up to equivalence under the full group of isometries of $\mathbf{G}_{n,k}(\mathbf{F})$. We first consider the case $k = 2n$. Our structure theorem for isoclinic spheres (Theorem 1) shows that isoclinic spheres can all be obtained from certain representations of Clifford algebras. Sections 10 and 11 are devoted to the study of these representations, and yield (Theorem 6 in §11)

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a complete description of the conditions for their existence and equivalence. From this, we obtain (Theorem 7 in §12) the classification of isoclinic spheres on $\mathbf{F}^{2n}$ under the unitary group, which in turn gives us our final classification theorem (Theorem 8 in §13).

Sections 10 and 11 are based on well-known techniques with semisimple associative algebras. In the course of §10, we give a unified treatment over the real numbers, the complex numbers, and the quaternions, of the Hurwitz-Radon problem on quadratic forms permitting composition. The result (Theorem 5) seems to be new for the quaternions.

I am deeply indebted to Y.-C. Wong's memoir [12]. While our Sections 3, 4, 5, 10, 11, and 12 extend all the results of [12] and often use different techniques, many of the ideas are due to Wong.

Chapter I. Isoclinic Subspaces

2. Notation and terminology

$\mathbf{F}$ will denote one of the real division algebras $\mathbf{R}$ (real), $\mathbf{C}$ (complex), or $\mathbf{K}$ (quaternion), with conjugation $\alpha \to \bar{\alpha}$ over $\mathbf{R}$ and norm $|\alpha| = (\alpha\bar{\alpha})^{1/2}$. For every integer $m > 0$, $\mathbf{F}^m$ will denote a left vectorspace of dimension m over $\mathbf{F}$ endowed with a positive-definite hermitian inner product $u \cdot v$, and $\mathbf{U}(m, \mathbf{F})$ will denote the group of linear transformations of $\mathbf{F}^m$ which preserve the inner product (the *unitary group*). Scalar multiplication is an action on the left, and linear transformations act on the right. With respect to any given basis of $\mathbf{F}^m$, every linear transformation is represented by a matrix. If the basis is orthonormal, then it is well known in case $\mathbf{F} \neq \mathbf{K}$, and easy to verify in case $\mathbf{F} = \mathbf{K}$, that a linear transformation with matrix A is unitary (in $\mathbf{U}(m, \mathbf{F})$) if and only if $A^{-1} = {}^t\bar{A}$, where t denotes transpose and the bar denotes conjugation of each matrix entry.

Let X and Y be subspaces of $\mathbf{F}^m$, and let $\pi_X : \mathbf{F}^m \to X$ and $\pi_Y : \mathbf{F}^m \to Y$ be the orthogonal projections. We will say that X and Y are *isoclinic* if $\mathbf{F}$ has elements α and β such that

$$\pi_X(y_1) \cdot \pi_X(y_2) = \alpha y_1 \cdot y_2 \quad \text{and} \quad \pi_Y(x_1) \cdot \pi_Y(x_2) = \beta x_1 \cdot x_2$$

for any $x_i \in X$ and $y_i \in Y$. The possibilities $x_1 = x_2$ and $y_1 = y_2$ then show that α and β are real. It is easily verified that X and Y are isoclinic if and only if either they are orthogonal ($X \perp Y$), or dimensions satisfy $\dim X = \dim Y$ and one of the π's satisfies the condition above. As the word suggests, this means that X and Y are at a constant angle.

3. Subspaces isoclinic to a given subspace

Let $\mathbf{B}$ be a collection of n-dimensional subspaces of $V = \mathbf{F}^{2n}$ whose every element is isoclinic to some given element $B \in \mathbf{B}$, and let $\mathbf{b} = \{b_1, \cdots, b_n\}$ be an orthonormal basis of B. We will construct a certain real vectorspace $\mathfrak{S}$ of linear transformations of V such that $\mathbf{B} \subset \mathbf{B}_* = \mathfrak{S}(B)$.

If $\mathbf{B} = \{B, B^\perp\}$, then choose $J \in \mathbf{U}(2n, \mathbf{F})$ with square $-I$ (I = identity) and $J(B) = B^\perp$, and define $\mathfrak{S}$ to be the $\mathbf{R}$-vectorspace with basis I and J, so $\mathbf{B}_* = \mathfrak{S}(B) = \{S(B) : S \in \mathfrak{S}\}$.

If $\mathbf{B} \neq \{B, B^\perp\}$, then we define $\mathfrak{S}_0$ to be the $\mathbf{R}$-vectorspace with basis I and $\mathbf{B}_0 = \{B\}$. Now suppose that $\mathfrak{S}_{k-1}$ and $\mathbf{B}_{k-1}$ are defined. If $\mathbf{B} \subset \mathbf{B}_{k-1}$, then we define $\mathbf{B}_* = \mathbf{B}_{k-1}$ and $\mathfrak{S} = \mathfrak{S}_{k-1}$. If $\mathbf{B}$ has an element $B_k \notin \mathbf{B}_{k-1}$, $B_k \neq B^\perp$, then the fact that B_k is isoclinic to B gives us an orthonormal basis $\{u_i\}$ of B_k with $u_i = v_i + w_i$, $v_i \in B$, and $w_i \in B^\perp$, such that $\{\gamma v_i\}$ is an orthonormal basis of B for some real $\gamma > 1$. It follows that B_k has an orthonormal basis $\{u_i^{(k)}\}$ with $u_i^{(k)} = \beta_k b_i + \alpha_k b_i^{(k)}$ for some nonzero real β_k and α_k, and some orthonormal basis $\{b_i^{(k)}\}$ of $B^\perp$. $J_k(b_i) = b_i^{(k)}$ and $J_k(b_i^{(k)}) = -b_i$ defines a unitary automorphism of V with $J_k^2 = -I$, $J_k(B) = B^\perp$, and $B_k = (\beta_k I + \alpha_k J_k)(B)$. Define $\mathbf{B}_k = \mathfrak{S}_k(B)$ where $\mathfrak{S}_k$ is the $\mathbf{R}$-vectorspace generated by J_k and $\mathfrak{S}_{k-1}$. By finite-dimensionality of V, this recursive procedure will eventually give us a smallest $\mathbf{B}_q$ containing $\mathbf{B}$; we then set $\mathbf{B}_* = \mathbf{B}_q$ and $\mathfrak{S} = \mathfrak{S}_q$.

In general $\mathbf{B}_*$ has elements which are not isoclinic to B.

4. Mutually isoclinic subspaces

Now assume that the elements of $\mathbf{B}$ are mutually (pairwise) isoclinic. We will look into the structure of $\mathfrak{S}$ and see that the elements of $\mathbf{B}_*$ are mutually isoclinic.

Suppose that $\mathbf{b}' = \{b_{n+1}, \cdots, b_{2n}\}$ is an orthonormal basis of $B^\perp$, and that J_i has matrix $\begin{pmatrix} 0 & A_i \\ -{}^t\bar{A}_i & 0 \end{pmatrix}$ in the basis $\{\mathbf{b}, \mathbf{b}'\}$ of V (where t denotes transpose and the bar means that we conjugate every entry). We want to show that each $A_i\,{}^t\bar{A}_j + A_j\,{}^t\bar{A}_i$ is a *real* scalar matrix. Observing this to be invariant under conjugation by $\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$, we see that it is independent of choice of $\mathbf{b}'$. Thus we may assume $A_j = I$ and must prove that $A_i + {}^t\bar{A}_i$ is a real scalar. It suffices to prove $A_i + {}^t\bar{A}_i$ scalar; it will clearly then be real.

Choose nonzero real σ and τ with $B_j = (I + \sigma J_j)B$ and $B_i = (I + \tau J_i)B$. In our basis, J_j has matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, so B_j has basis $\{b_q + \sigma b_{n+q}\}_{q \leq n}$ and $B_j^\perp$ has basis $\{-\sigma b_q + b_{n+q}\}_{q \leq n}$. If $A_i = (\alpha_{qk})$, then B_i has a basis $\{x_q\}_{q \leq n}$ which has expressions

$$\begin{aligned} b_q + \tau \textstyle\sum_{k=1}^n \alpha_{qk}\, b_{n+k} &= x_q \\ &= \textstyle\sum_{k=1}^n \sigma_{qk}(b_k + \sigma b_{n+k}) + \sum_{k=1}^n \tau_{qk}(-\sigma b_k + b_{n+k}) \end{aligned}$$

for some matrices $S = (\sigma_{qk})$ and $T = (\tau_{qk})$. S and T are proportional to $\mathbf{F}$-unitary matrices because B_i and B_j are isoclinic. Equating coefficients of the b_k, we see $S = \sigma T + I$, whence $S + {}^t\bar{S}$ and $T + {}^t\bar{T}$ are scalar. Equating coefficients of the b_{n+k} we see that A_i is a linear combination of S and T; thus $A_i + {}^t\bar{A}_i$ is scalar.

Let $\mathfrak{U}$ be the real-linear set of all matrices A for which, in the fixed orthonormal basis $\{\mathbf{b}, \mathbf{b}'\}$ of V, $\mathfrak{S}$ has an element with matrix $\begin{pmatrix} 0 & A \\ -{}^t\bar{A} & 0 \end{pmatrix}$. The A_i above form a basis of $\mathfrak{U}$ over $\mathbf{R}$, and the preceding paragraphs show that $\mathfrak{U}$ has a positive-definite inner product $\langle U_1, U_2 \rangle$ given by

$$U_1 {}^t\bar{U}_2 + U_2 {}^t\bar{U}_1 = 2 \langle U_1, U_2 \rangle I$$

in which $\| A_i \| = 1$. The idea of finding such an inner product, due to Wong [12], is crucial: Let $\{U_1, \cdots, U_r\}$ be an orthonormal basis of $\mathfrak{U}$, and let S_i be the element of $\mathfrak{S}$ with matrix $\begin{pmatrix} 0 & U_i \\ -{}^t\bar{U}_i & 0 \end{pmatrix}$ in the basis $\{\mathbf{b}, \mathbf{b}'\}$ of V; then the S_i satisfy the Hurwitz equations, giving us a basis-free version of an extension of Wong's main result [12, Theorem 3.2]:

THEOREM 1. *Let* $\mathbf{B}$ *be a collection of mutually isoclinic* n*-dimensional subspaces of* $V = \mathbf{F}^{2n}$ *where* $\mathbf{F}$ *is a real division algebra. Given* $B \in \mathbf{B}$, *there is a real vectorspace* $\mathfrak{S}^r$ *of* $\mathbf{F}$*-linear transformations of* V *such that*

(1) $\mathbf{B}_* = \mathfrak{S}^r(B)$ *is a collection of mutually isoclinic* n*-dimensional subspaces of* V *which contains* $\mathbf{B}$.

(2) $\mathfrak{S}^r$ *has a basis* $\{I, S_1, \cdots, S_r\}$ *over* $\mathbf{R}$ *such that each* $S_i \in \mathbf{U}(2n, \mathbf{F})$, *each* $S_i(B) = B^{\perp}$, *and the* S_i *satisfy the Hurwitz equations*

$$(*) \qquad S_i S_j + S_j S_i = -2\delta_{ij} I.$$

Conversely, if B' *is an* n*-dimensional subspace of* V *and* $\mathfrak{S}'^r$ *is a real vectorspace of* $\mathbf{F}$*-linear transformations of* V *satisfying* (2), *then* $\mathbf{B}' = \mathfrak{S}'^r(B')$ *is a collection of mutually isoclinic* n*-dimensional subspaces of* V.

Proof. We have found $\mathfrak{S}^r$ and proved (2), and $\mathbf{B} \subset \mathbf{B}_*$ by construction. We will prove that the elements of $\mathbf{B}_*$ are mutually isoclinic as a consequence of (2); this will complete the proof of (1) and demonstrate the last assertion of the theorem.

Let P and Q be elements of $\mathbf{B}_*$,

$$P = (p_0 I + \textstyle\sum_{i=1}^r p_i S_i)B \quad \text{and} \quad Q = (q_0 I + \textstyle\sum_{i=1}^r q_i S_i)B.$$

If $p_0 = 0$ or if $q_0 = 0$, then $P = B^{\perp}$ or $Q = B^{\perp}$ and we are done, for every element of $\mathbf{B}_*$ is easily seen to be isoclinic to B by $(*)$ and is thus isoclinic to $B^{\perp}$. Again using $(*)$, we write $P = (I + S)B$ and $Q = (I + T)B$ where $T = q_0^{-1} \sum q_i S_i$ and $S = p_0^{-1} \sum p_i S_i$ are proportional to unitary transformations. In an appropriate orthonormal basis $\mathbf{b}'$ of $B^{\perp}$, T has matrix $\begin{pmatrix} 0 & \alpha I \\ -\alpha I & 0 \end{pmatrix}$, and S has matrix $\begin{pmatrix} 0 & U \\ -{}^t\bar{U} & 0 \end{pmatrix}$ in the basis $\{\mathbf{b}, \mathbf{b}'\}$ of V for some real $\alpha > 0$. Now $U + {}^t\bar{U}$ is a real scalar matrix βI, $2\beta = \langle U, I \rangle$. We have matrices $S' = (\sigma_{qk})$ and $T' = (\tau_{qk})$ such that the basis $\{x_q\} = (I + S)\mathbf{b}$ of P has expressions, where $U = (\mu_{qk})$,

$$b_q + \textstyle\sum_{k=1}^n \mu_{qk} b_{n+k} = x_q = \sum_{k=1}^n \sigma_{qk}(b_k + \alpha b_{n+k}) + \sum_{k=1}^n \tau_{qk}(-\alpha b_k + b_{n+k}).$$

Equating coefficients we have $S' = I + \alpha T'$ and $U = \alpha S' + T'$. Thus $U = \alpha I + \gamma T'$ where $\gamma = \alpha^2 + 1 > 0$, and $U = -\alpha^{-1} I + \delta S'$ where $\delta = \alpha + \alpha^{-1} > 0$. From $U + {}^t\bar{U}$ scalar, it follows that $T' + {}^t\bar{T}'$ and $S' + {}^t\bar{S}'$ are scalar. Thus $S' + {}^t\bar{S}' = I + \alpha(T' + {}^t\bar{T}') + \alpha^2 T' \cdot {}^tT'$ shows that T' is proportional to an $\mathbf{F}$-unitary matrix. It follows that P is isoclinic to $Q^{\perp}$, and is thus isoclinic to Q, Q.E.D.

Remark 1. To view Theorem 1 from the viewpoint of Wong's memoir, one considers the space $\mathfrak{U}$ rather than the space $\mathfrak{S}^r$. $\mathfrak{U}$ is a real vectorspace of $\mathbf{F}$-unitary $n \times n$ matrices which depends on the choice of an orthonormal basis $\mathbf{b}'$ of $B^{\perp}$. If $\mathbf{b}'$ is chosen such that $U_r = I$ in an orthonormal basis $\{U_1, \cdots, U_r\}$ of $\mathfrak{U}$, then $i < r$ implies

$$0 = U_i \cdot {}^t\bar{U}_r + U_r \cdot {}^t\bar{U}_i = U_i + {}^t\bar{U}_i = U_i + U_i^{-1}$$

(so $U_i^2 = -I$), whence $j < r$ implies $U_i U_j + U_j U_i = -2\delta_{ij} I$. This is equivalent to (2) of Theorem 1.

DEFINITION. *The sets* $\mathbf{B}_*$ *and* $\mathbf{B}'$ *of Theorem* 1 *will be called isoclinic r-spheres on* V.

Thus Theorem 1 can be rewritten as

THEOREM 1′. *Every collection of mutually isoclinic n-dimensional subspaces of* $V = \mathbf{F}^{2n}$ *is contained in an isoclinic sphere on* V, *and every isoclinic sphere on* V *is a collection of mutually isoclinic n-dimensional subspaces.*

CHAPTER II. GRASSMANN MANIFOLDS

5. Definitions and preliminaries

Given integers $0 < n < k$ and a real division algebra $\mathbf{F}$, $\mathbf{G}_{n,k}(\mathbf{F})$ will denote the *Grassmann manifold* consisting of all n-dimensional subspaces of $\mathbf{F}^k$. The action of $\mathbf{U}(k, \mathbf{F})$ on $\mathbf{F}^k$ induces an action on $\mathbf{G}_{n,k}(\mathbf{F})$; $\mathbf{U}(k, \mathbf{F})$ is transitive on the elements of $\mathbf{G}_{n,k}(\mathbf{F})$. Given $B \in \mathbf{G}_{n,k}(\mathbf{F})$, this allows us to identify $\mathbf{G}_{n,k}(\mathbf{F})$ with the coset space of $\mathbf{U}(k, \mathbf{F})$ by its isotropy subgroup

$$\mathbf{K}_B = \{T \in \mathbf{U}(k, \mathbf{F}) : T(B) = B\}$$

at B, which gives $\mathbf{G}_{n,k}(\mathbf{F})$ the structure of a real analytic manifold.

$\mathbf{G}_{n,k}(\mathbf{F})$ carries a unique (up to real scalar multiplication) $\mathbf{U}(n, \mathbf{F})$-invariant Riemannian structure, described as follows: There is a vectorspace direct-sum decomposition $\mathfrak{U}(n, \mathbf{F}) = \mathfrak{K}_B + \mathfrak{P}_B$ (German letters denote Lie algebras) where $\mathfrak{P}_B = \mathfrak{K}_B^{\perp}$ under the Killing form f on $\mathfrak{U}(n, \mathbf{F})$, and a natural identification of $\mathfrak{P}_B$ with the tangentspace to $\mathbf{G}_{n,k}(\mathbf{F})$ at B under the differential of the projection $T \to T(B)$ of $\mathbf{U}(n, \mathbf{F})$ onto $\mathbf{G}_{n,k}(\mathbf{F})$; thus $-f$ determines an invariant Riemannian metric on $\mathbf{G}_{n,k}(\mathbf{F})$. $\mathbf{G}_{n,k}(\mathbf{F})$ will always be understood to carry this structure, and is thus a Riemannian symmetric space.

If V is a particular left hermitian vectorspace of dimension k over $\mathbf{F}$, then $\mathbf{G}_n(V)$ will denote $\mathbf{G}_{n,k}(\mathbf{F})$ where the elements are viewed as subspaces of V.

If N is a submanifold of a Riemannian manifold M, then N inherits a Riemannian structure from M. We say that N is *totally geodesic* if every geodesic of N is also a geodesic of M. This implies that, for every 2-dimensional subspace S of a tangentspace of N, the sectional curvature of N along S is the same as that of M along S.

6. Isoclinic spheres as spheres

Let $\mathbf{B}$ be an isoclinic sphere on $V = \mathbf{F}^{2n}$. We will examine $\mathbf{B}$ as a submanifold of the Grassmann manifold $\mathbf{G}_{n,2n}(\mathbf{F})$.

Choose $B \in \mathbf{B}$. Theorem 1 says that there is a real vectorspace $\mathfrak{S}'$ of linear transformations of V with basis $\{I, S_1, \cdots, S_r\}$ such that $S_i \in \mathbf{U}(2n, \mathbf{F})$ and $S_i(B) = B^{\perp}$ for each i; $S_i S_j + S_j S_i = -2\delta_{ij} I$, and $\mathbf{B} = \mathfrak{S}'(B)$. $S_i^2 = -I$ shows that each S_i is skew-hermitian, and may thus be viewed as an element of $\mathfrak{U}(2n, \mathbf{F})$. It is easily checked that each $S_i \in \mathfrak{P}_B$ where $\mathfrak{U}(2n, \mathbf{F}) = \mathfrak{K}_B + \mathfrak{P}_B$, as in §5. Let $\mathfrak{S}$ be the subspace of $\mathfrak{P}_B$ spanned by the S_i, and let $B' \in \mathbf{B}$. Then $B' = (a_0 I + \sum a_i S_i)(B)$, and we may assume $\sum_0^r a_i^2 = 1$. We define

$$a = \left(\textstyle\sum_1^r a_i^2\right)^{1/2} \quad \text{and} \quad S = a^{-1} \textstyle\sum a_i S_i,$$

observe that $B' = (a_0 I + aS)(B)$ and $\exp(tS) = \cos(t)I + \sin(t)S$, and conclude that $B' \in \exp(\mathfrak{S})(B)$. Thus $\mathbf{B} = \exp(\mathfrak{S})(B)$.

Let $\mathfrak{R}$ be the curvature tensor on $\mathbf{G}_{n,2n}(\mathbf{F})$. If X, Y, and Z are tangentvectors to $\mathbf{G}_{n,2n}(\mathbf{F})$ at B, then we view them as elements of $\mathfrak{P}_B$ and have [8]

$$\mathfrak{R}(X, Y) \cdot Z = -[[X, Y], Z].$$

In particular, $\mathfrak{R}(S_i, S_j) \cdot S_i = -4S_j$ if $i \neq j$. A short calculation shows the existence of a real number $p > 0$ such that $f(S_i, S_j) = -\delta_{ij} p^{-2}$ where f is the Killing form. As we have chosen $-f$ for metric, the sectional curvature of $\mathbf{G}_{n,2n}(\mathbf{F})$ along a 2-dimensional subspace of the tangentspace at B with $(-f)$-orthonormal basis $\{X, Y\}$ is given by $-\{-f(\mathfrak{R}(X, Y) \cdot X, Y)\}$. It follows that $\mathbf{G}_{n,2n}(\mathbf{F})$ has sectional curvature $4p^4$ along every 2-dimensional subspace of $\mathfrak{S}$. But $[\mathfrak{S}, [\mathfrak{S}, \mathfrak{S}]] \subset \mathfrak{S}$ (i.e., $\mathfrak{S}$ is a Lie triple system), which implies that $\exp(\mathfrak{S})(B)$ is a totally geodesic submanifold of $\mathbf{G}_{n,2n}(\mathbf{F})$, and the preceding paragraph showed that $\mathbf{B} = \exp(\mathfrak{S})(B)$. We conclude that $\mathbf{B}$ is a totally geodesic submanifold of constant positive curvature $4p^4$ in $\mathbf{G}_{n,2n}(\mathbf{F})$. The number p depends only on n and $\mathbf{F}$.

If $\mathbf{F} = \mathbf{R}$, then Y.-C. Wong has shown [12, p. 62] that a maximal isoclinic sphere on $\mathbf{F}^{2n}$, regarded as a submanifold of $\mathbf{G}_{n,2n}(\mathbf{F})$, is homeomorphic to a sphere. More generally, we may see that $\mathbf{B}$ is homeomorphic to an r-sphere as follows. Let $\mathbf{G}$ be the subgroup of $\mathbf{U}(n, \mathbf{F})$ with Lie algebra $\mathfrak{G} = \mathfrak{T} + \mathfrak{S}$, $\mathfrak{T} = [\mathfrak{S}, \mathfrak{S}]$. $\mathfrak{S}$ is spanned by the S_i, and thus has dimension r. $\mathfrak{T}$ is spanned by the $S_i S_j$ $(i < j)$, and thus has dimension $r(r-1)/2$. As $\mathbf{G}$ acts transitively and almost-effectively by isometries on $\mathbf{B}$, and as $\mathbf{B}$ has constant positive curvature, it follows that $\mathbf{G}$ is locally isomorphic to the orthogonal group $\mathbf{O}(r+1) = \mathbf{U}(r+1, \mathbf{R})$. Thus (see [10], for example) $\mathbf{B}$ is the sphere or the projective space. Every geodesic of $\mathbf{B}$ through B passes through $B^{\perp}$; this shows that $\mathbf{B}$ is not the projective space.

We have now proved

THEOREM 2. *Let $\mathbf{B}$ be an isoclinic r-sphere on $V = \mathbf{F}^{2n}$ for some real division algebra $\mathbf{F}$, and view $\mathbf{B}$ as a subset of the Grassmann manifold $\mathbf{G}_{n,2n}(\mathbf{F})$. Then $\mathbf{B}$ is a totally geodesic submanifold of $\mathbf{G}_{n,2n}(\mathbf{F})$, and there is a real number $q > 0$, depending only on $\mathbf{G}_{n,2n}(\mathbf{F})$, such that $\mathbf{B}$ is isometric to the sphere of radius q in $\mathbf{R}^{r+1}$.*

7. A characterization of isoclinic spheres

Let V be a subspace of dimension $2n$ in $\mathbf{F}^k$, and let $\mathbf{B}$ be an isoclinic sphere on V. As $\mathbf{G}_n(V)$ is a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$, Theorem 2 shows that $\mathbf{B}$ is a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ which is isometric to a sphere. $\mathbf{B}$ also has the property that any two distinct elements, viewed as subspaces of $\mathbf{F}^k$, have intersection 0. We will see that these properties characterize isoclinic spheres on $2n$-dimensional subspaces of $\mathbf{F}^k$.

THEOREM 3. *Let $\mathbf{B}$ be a submanifold of the Grassmann manifold $\mathbf{G}_{n,k}(\mathbf{F})$ where $0 < 2n \leqq k$ and $\mathbf{F}$ is a real division algebra. Then these are equivalent:*

(1) *$\mathbf{B}$ is an isoclinic sphere on a 2n-dimensional subspace of $\mathbf{F}^k$.*

(2) *$\mathbf{B} \subset \mathbf{G}_n(V)$ for some 2n-dimensional subspace V of $\mathbf{F}^k$, any two distinct elements of $\mathbf{B}$, viewed as subspaces of $\mathbf{F}^k$, have intersection 0, and $\mathbf{B}$ is a connected totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$.*

(3) *Any two distinct elements of $\mathbf{B}$ have intersection 0, the fundamental group $\pi_1(\mathbf{B})$ has odd finite order, and $\mathbf{B}$ is a connected totally geodesic submanifold of constant positive curvature in $\mathbf{G}_{n,k}(\mathbf{F})$.*

(4) *Any two distinct elements of $\mathbf{B}$ have intersection 0, and $\mathbf{B}$ is a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ which is isometric to an (ordinary) sphere.*

Proof. We have just seen that (1) implies (4), and (4) clearly implies (3). We must prove that (3) implies (2) and (2) implies (1).

Choose $B \in \mathbf{B}$, let $\mathfrak{U}(k, \mathbf{F}) = \mathfrak{K}_B + \mathfrak{P}_B$ be the decomposition of §5, and let $\mathfrak{S}$ be the subspace of $\mathfrak{P}_B$ which is the tangentspace to $\mathbf{B}$ at B. As $\mathbf{B}$ is a connected totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$, we have $\mathbf{B} = \exp(\mathfrak{S})(B)$ and $[\mathfrak{S}, [\mathfrak{S}, \mathfrak{S}]] \subset \mathfrak{S}$. Let E_{ij} denote the $k \times k$ matrix whose only nonzero entry is a 1 in the (i, j)-place. From the fact that $\mathbf{G}_{n,k}(\mathbf{F})$ is a Riemannian symmetric space of rank n, it follows that, given $X \in \mathfrak{P}_B$, we have an orthonormal basis $\mathbf{x}$ of $\mathbf{F}^k$ such that (a) the first n elements of $\mathbf{x}$ span B and the last $k - n$ elements span $B^{\perp}$, and (b) the matrix of X with respect to $\mathbf{x}$ is of the form $\sum_{i=1}^{n} \alpha_i(E_{i,i+n} - E_{i+n,i})$ with $\alpha_i \in \mathbf{R}$. Suppose further that $X \in \mathfrak{S}$. As $\mathbf{B} = \exp(\mathfrak{S})(B)$ cannot have two elements in a $(2n-1)$-dimensional subspace of $\mathbf{F}^k$, the matrix of $\exp(tX)$ $(t \in \mathbf{R})$ relative to $\mathbf{x}$ cannot have a nonzero entry in the (i, j)-place with $i \leqq n < j$ whenever some

$$\exp(t\alpha_m(E_{m,m+n} - E_{m+n,m})) = \pm I.$$

Replacing x_{i+n} by $-x_{i+n}$ if necessary, we may now assume that each $\alpha_i > 0$. By normalizing X so that the largest α_i is π, each α_i must be π because $\exp(X)$ is diagonal in $\mathbf{x}$. Thus we may assume that all the α_i are equal. In particular, every element of $\mathbf{B} = \exp(\mathfrak{S})(B)$ is isoclinic to B. It follows from the proof of the last part of Theorem 1 that (2) implies (1).

Let $\{X_1, \cdots, X_r\}$ be a basis of $\mathfrak{S}$ which is orthonormal relative to the negative of the Killing form of $\mathfrak{U}(n, \mathbf{F})$. As above, we choose an orthonormal basis $\mathbf{x}$ of $\mathbf{F}^k$ whose first n elements span B, whose last $k - n$ elements span $B^\perp$, and with respect to which X_1 has matrix $\alpha\sum_{i=1}^n(E_{i,i+n} - E_{i+n,i})$ for some real $\alpha > 0$. $\mathfrak{P}_B$ is easily seen to consist of all linear transformations of $\mathbf{F}^k$ with matrix of the form $\begin{pmatrix} 0 & A \\ -{}^t\bar{A} & 0 \end{pmatrix}$, for some $n \times (k - n)$ matrix A over $\mathbf{F}$, with respect to $\mathbf{x}$. Let $\{\alpha_0 = 1, \alpha_1, \cdots, \alpha_q\}$ be the usual basis of $\mathbf{F}$ over $\mathbf{R}$: $\alpha_i^2 = -1$, and distinct α_i anticommute for $i > 0$. Then $\mathfrak{P}_B$ has basis (over $\mathbf{R}$) consisting of all linear transformations with matrix (relative to $\mathbf{x}$) $E_{u,v} - E_{v,u}$ or $\alpha_i(E_{u,v} + E_{v,u})$ for $1 \leqq u \leqq n < v \leqq k$ and $1 \leqq i \leqq q$. Let $\mathfrak{P}_B'$ be the subspace of $\mathfrak{P}_B$ spanned by those basis elements with $v \leqq 2n$. If every $X_i \in \mathfrak{P}_B'$, then $\mathbf{B}$ will consist of subspaces of the span of $\{x_1, \cdots, x_{2n}\}$ because $\mathbf{B} = \exp(\mathfrak{S})(B)$, and (2) will follow. Thus we need only prove that each $X_i \in \mathfrak{P}_B'$ provided that $\mathbf{B}$ has constant positive curvature and that its (necessarily finite) fundamental group is of odd order, and we will have proved that (3) implies (2).

Suppose that $\mathbf{B}$ is of constant positive curvature and with fundamental group of odd order. Then let B' denote the element of $\mathbf{B}$ such that, under the universal Riemannian covering of $\mathbf{B}$ by a sphere, B' is the image of the point antipodal to some point in the inverse image of B. Let

$$X = \alpha^{-1}X_1 = \textstyle\sum_{i=1}^n(E_{i,i+n} - E_{i+n,i});$$

then $B' = \exp(\frac{1}{2}\pi X)(B)$, has basis $\{x_{n+1}, \cdots, x_{2n}\}$, and is preserved by $[\mathfrak{S}, \mathfrak{S}]$. If $u \leqq n < v$, then $[X, E_{u,n+v} \pm E_{n+v,u}] = -(E_{n+u,n+v} \pm E_{n+v,n+u})$; thus $[X_1, X_i]$ cannot preserve B' if $X_i \notin \mathfrak{P}_B'$, proving that (3) implies (2), Q.E.D.

Remark 2. Given $0 < n < k$, there is an isometry $\perp : \mathbf{G}_{n,k}(\mathbf{F}) \to \mathbf{G}_{k-n,k}(\mathbf{F})$ given by $B \to B^\perp$. Thus, if $2n \geqq k$ (i.e., if $2(k - n) \leqq k$), and if $\mathbf{A}$ is a submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$, then Theorem 3 shows that these are equivalent:

1. $\mathbf{A}^\perp$ is an isoclinic sphere on a $2n$-dimensional subspace of $\mathbf{F}^k$.
2. $\mathbf{F}^k$ has a subspace $V^\perp$ of dimension $k - 2n$ such that $A_1 \cap A_2 = V^\perp$ for any two distinct $A_i \in \mathbf{A}$, and $\mathbf{A}$ is a connected totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$.
3. $\mathbf{F}^k = A_1 + A_2$ for any two distinct $A_i \in \mathbf{A}$, and $\mathbf{A}$ is a connected totally

geodesic submanifold of constant positive curvature and fundamental group of odd order.

4. **A** is a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ which is isometric to a sphere any two of whose elements span $\mathbf{F}^k$.

Remark 3. If N is a connected totally geodesic submanifold of a Riemannian (or even affine) symmetric space M, then N is preserved by the symmetry of M at any point of N. If N is of constant positive curvature, it follows that N is isometric either to a sphere or to a real projective space. Thus Theorem 3 tells us the following:

Let $\mathbf{N}$ *be a connected totally geodesic submanifold of constant positive curvature in the Grassmann manifold* $\mathbf{G}_{n,k}(\mathbf{F})$ *where* $0 < 2n \leqq k$ *and* $\mathbf{F}$ *is a real division algebra, and suppose that any two distinct elements of* $\mathbf{N}$ *(viewed as subspaces of* $\mathbf{F}^k$*) have zero intersection. If* $\mathbf{N} \subset \mathbf{G}_n(V)$ *for some 2n-dimensional subspace* V *of* $\mathbf{F}^k$*, then* $\mathbf{N}$ *is isometric to a sphere (and is an isoclinic sphere on* V*). Otherwise,* $\mathbf{N}$ *is isometric to a real projective space.*

This will also follow from Theorem 4.

We give an example to show that the latter case occurs. Suppose $k \geqq 3n$, define (relative to an orthonormal basis $\mathbf{x}$ of $\mathbf{F}^k$)

$$X = \textstyle\sum_{i=1}^n (E_{i,i+n} - E_{i+n,i}),$$
$$Y = \textstyle\sum_{i=1}^n (E_{i,i+2n} - E_{i+2n,i}),$$
$$Z = \textstyle\sum_{i=1}^n (E_{i+2n,i+n} - E_{i+n,i+2n}),$$

and observe that $[X, Y] = Z, [Z, X] = Y$ and $[Z, Y] = -X$. Thus the subgroup $\mathbf{G}$ of $\mathbf{U}(k, \mathbf{F})$, with Lie algebra $\mathfrak{G}$ spanned by X, Y, and Z, is locally isomorphic to $\mathbf{SO}(3)$. Let B be the subspace of $\mathbf{F}^k$ with basis $\{x_1, \cdots, x_n\}$, and define $\mathbf{N} = \mathbf{G}(B)$. $\mathbf{N}$ is of constant positive curvature because it carries an $\mathbf{SO}(3)$-invariant Riemannian metric, and is a totally geodesic submanifold. It follows that $\mathbf{N}$ is isometric to a real projective space $\mathbf{G}_{1,3}(\mathbf{R})$ of dimension 2.

Remark 4. Let $\mathbf{B}$ be a connected totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ where $0 < 2n \leqq k$. The proof that (2) implies (1) in Theorem 3 shows that any two elements of $\mathbf{B}$ are isoclinic if and only if any two distinct elements of $\mathbf{B}$ have zero intersection. If that is the case, and if $k = 2n$, then $\mathbf{B}$ is isometric to a sphere.

8. Symmetric spaces of rank one

Recall that the compact Riemannian symmetric spaces of rank one are the spheres, the real projective spaces, the complex projective space, the quaternionic projective spaces, and the Cayley projective plane. They are characterized among Riemannian symmetric spaces by the fact that they have every sectional curvature positive. Theorem 3 shows that a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ $(2n \leqq k)$, which has every two elements isoclinic and is contained in $\mathbf{G}_n(V)$ for some $2n$-dimensional subspace V of $\mathbf{F}^k$, is a symmetric space of rank one.

We can now state and prove the basic result of this paper:

THEOREM 4. *Let* $\mathbf{B}$ *be a connected totally geodesic submanifold of the Grassmann manifold* $\mathbf{G}_{n,k}(\mathbf{F})$, *where* $0 < 2n \leqq k$ *and* $\mathbf{F}$ *is a real division algebra, and suppose that any two distinct elements of* $\mathbf{B}$ *have zero intersection as subspaces of* $\mathbf{F}^k$. *Then* $\mathbf{B}$ *is a compact Riemannian symmetric space of rank one, and these are equivalent:*

(1) $\mathbf{B}$ *is isometric to a sphere.*

(2) $\mathbf{F}^k$ *has a* $2n$*-dimensional subspace which contains every element of* $\mathbf{B}$.

(3) $\mathbf{B}$ *is an isoclinic sphere on a* $2n$*-dimensional subspace of* $\mathbf{F}^k$.

Proof. The second assertion is contained in Theorem 3, and $\mathbf{B}$ is Riemannian symmetric because it is totally geodesic in $\mathbf{G}_{n,k}(\mathbf{F})$. It suffices to prove that every sectional curvature of $\mathbf{B}$ is nonzero; such a curvature will then be positive because it is a curvature of $\mathbf{G}_{n,k}(\mathbf{F})$, and it will follow that $\mathbf{B}$ is compact and of rank one.

Let $B \in \mathbf{B}$; we will prove that every sectional curvature of $\mathbf{B}$ at B is nonzero. Let $\mathfrak{U}(k, \mathbf{F}) = \mathfrak{K}_B + \mathfrak{P}_B$ be the decomposition mentioned in §5, let $\mathfrak{S} \subset \mathfrak{P}_B$ be the tangentspace to $\mathbf{B}$ at B, and let $\{X, Y\} \subset \mathfrak{S}$ be linearly independent; it suffices to prove that $[X, Y] \neq 0$. Let $\mathbf{x} = \{x_1, \cdots, x_k\}$ be an orthonormal basis of $\mathbf{F}^k$ whose first n elements span B, chosen such that X is a real multiple of the linear transformation with matrix

$$\sum_{i=1}^{n}(E_{i,i+n} - E_{i+n,i})$$

relative to $\mathbf{x}$. This was seen to be possible during the proof of Theorem 3. We may replace X by that multiple. Retaining the notation of the proof of Theorem 3, $\mathfrak{P}_B$ is spanned by the linear transformations with matrix

$$\alpha_i(E_{u,v} - \alpha_i^2 E_{v,u})$$

relative to x, where $1 \leqq u \leqq n < v \leqq k$ and $\{\alpha_i\}$ is our basis of $\mathbf{F}$ over $\mathbf{R}$. $\mathfrak{P}_B'$ was the subspace for which $v \leqq 2n$; let $\mathfrak{P}_B''$ be the subspace for which $v > 2n$. Finally, let $T : \mathfrak{P}_B \to \mathfrak{P}_B$ be the transformation $Z \to [X, [X, Z]]$.

Suppose that $[X, Y] = 0$. In particular, $T(Y) = 0$. Now $Y = Y' + Y''$ with $Y^i \in \mathfrak{P}_B^i$, $T(Z) = -Z$ for $Z \in \mathfrak{P}_B''$, and $T(\mathfrak{P}_B') \subset \mathfrak{P}_B'$. It follows that $Y \in \mathfrak{P}_B'$. Let $\mathfrak{S}' = \mathfrak{S} \cap \mathfrak{P}_B'$. $[\mathfrak{P}_B', [\mathfrak{P}_B', \mathfrak{P}_B']] \subset \mathfrak{P}_B'$ and $[\mathfrak{S}, [\mathfrak{S}, \mathfrak{S}]] \subset \mathfrak{S}$; it follows that

$$[\mathfrak{S}', [\mathfrak{S}', \mathfrak{S}']] \subset \mathfrak{S} \cap \mathfrak{P}_B' = \mathfrak{S}',$$

i.e., $\mathfrak{S}'$ is a Lie triple system. Thus $\mathbf{B}' = \exp(\mathfrak{S}')(B)$ is a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$. Two distinct elements of $\mathbf{B}'$ have zero intersection in $\mathbf{F}^k$ because they are elements of $\mathbf{B}$. By construction of $\mathfrak{S}'$, every element of $\mathbf{B}'$ lies in the subspace of $\mathbf{F}^k$ with basis $\{x_1, \cdots, x_{2n}\}$; it follows from Theorem 3 that $\mathbf{B}'$ is isometric to a sphere. In particular, two independent elements of $\mathfrak{S}'$ must have nonzero bracket. As X and Y lie in $\mathfrak{S}'$, this contradicts $[X, Y] = 0$, Q.E.D.

Remark 5. Suppose, in Theorem 4, that $\mathbf{B}$ is not a sphere. Then $\mathbf{B}$ is a real, complex, or quaternionic projective space, or the Cayley projective plane. Each of its projective lines is isometric to an s-sphere, where $s = 1$, 2, 4, or 8, respectively. It can be proved that $\mathbf{B}$ has no totally geodesic submanifold isometric to an $(s + 1)$-sphere, for two projective lines have a unique point of intersection. Thus we may choose $\mathbf{x}$ so that $\mathbf{B}'$ is an s-sphere, but we cannot choose $\mathbf{x}$ such that $\mathbf{B}'$ is an $(s + 1)$-sphere. This will be the basis for our classification when $\mathbf{B}$ is a projective space.

Chapter III. The Classification of Isoclinic Spheres

9. The connection with Clifford algebras

Recall that the *abstract Clifford algebra* $\mathfrak{C}_r$ is the real associative algebra with identity 1, generators $\{e_1, \cdots, e_r\}$, and relations $e_i e_j + e_j e_i = -2\delta_{ij}$ where δ_{ij} is the Kronecker symbol. $\mathfrak{C}_r'$ denotes the real subspace of $\mathfrak{C}_r$ with basis $\{1, e_1, \cdots, e_r\}$. A *representation* of $\mathfrak{C}_r$ on $\mathbf{F}^m$ is a homomorphism $\phi : \mathfrak{C}_r \to \mathfrak{E}(\mathbf{F}^m)_R$ of $\mathbf{R}$-algebras where $\phi(1) = I$; here $\mathfrak{E}(\mathbf{F}^m)$ denotes the $\mathbf{F}$-algebra of all linear transformations of $\mathbf{F}^m$, the subscript R means that we view it as an algebra over $\mathbf{R}$, and I denotes the identity transformation. ϕ is *equivalent* (resp. *unitarily equivalent*) to another representation ψ if $\mathfrak{E}(\mathbf{F}^m)$ has an invertible (resp. unitary) element T such that each $\psi(x) = T\cdot\phi(x)\cdot T^{-1}$.

We define ϕ to be *unitary* if each $\phi(e_i) \in \mathbf{U}(m, \mathbf{F})$, and to be *translational* (with *basepoint* U) if U is a subspace of $\mathbf{F}^m$ such that each $\phi(e_i)(U) = U^\perp$. If (ϕ, U) and (ψ, P) are translational representations with basepoint of $\mathfrak{C}_r$ on $\mathbf{F}^m$, then we say that they are *strictly equivalent* if there exists $T \in \mathbf{U}(m, \mathbf{F})$ such that $T(U) = P$ and each $\psi(x) = T\cdot\phi(x)\cdot T^{-1}$.

It is clear from Theorem 1 that translational representations with basepoint correspond to isoclinic spheres under

$$(\phi, U) \to \phi(\mathfrak{C}_r')(U)$$

and that strict equivalence of (ϕ, U) results in unitary equivalence of $\phi(\mathfrak{C}_r')(U)$. On the other hand, $(\phi, U^\perp)$ is a translational representation with basepoint not necessarily strictly equivalent to (ϕ, U), while $\phi(\mathfrak{C}_r')(U) = \phi(\mathfrak{C}_r')(U^\perp)$. But we will see that the unitary equivalence class of $\phi(\mathfrak{C}_r')(U)$ determines the pair consisting of the strict equivalence class of (ϕ, U) and that of $(\phi, U^\perp)$. This will allow us to classify the isoclinic spheres up to unitary equivalence. We will then see that unitary equivalence of isoclinic spheres is the same as equivalence under the full group of isometries of the Grassmann manifold.

10. Translational representations

We will collect some information on translational representations which will be useful in §11. Let W denote a vectorspace $\mathbf{F}^m$.

LEMMA 1. *Every representation of $\mathfrak{C}_r$ on W is equivalent to a unitary representation. If r is even, then every representation of $\mathfrak{C}_r$ is faithful, and any two unitary representations are unitarily equivalent.*

Proof. $\{e_1, \cdots, e_r\}$ generates a finite multiplicative group G_r in $\mathfrak{C}_r$, and every $\mathbf{F}$-representation of a finite group is equivalent to an $\mathbf{F}$-unitary representation; thus every representation of $\mathfrak{C}_r$ on W is equivalent to a unitary representation. Now suppose that r is even, whence [4, §2.2] $\mathfrak{C}_r$ is a central simple (= normal simple) algebra over $\mathbf{R}$. $1 \to I \neq 0$ shows every representation nonzero, and thus faithful by simplicity of $\mathfrak{C}_r$.

Let ϕ and ψ be unitary representations of $\mathfrak{C}_r$ (r even) on W. They are faithful, so we have an isomorphism $h : \phi(\mathfrak{C}_r) \cong \psi(\mathfrak{C}_r)$. If $\mathbf{F} \neq \mathbf{C}$, then $\mathfrak{E}(W)_R$ is central simple over $\mathbf{R}$ and [1, Theorem 4.14] h extends to an automorphism h' of $\mathfrak{E}(W)_R$; h' is conjugation by some invertible element of $\mathfrak{E}(W)$ because $\mathfrak{E}(W)_R$ is central simple. If $\mathbf{F} = \mathbf{C}$, then h extends to an isomorphism h'' between the complexifications of $\phi(\mathfrak{C}_r)$ and $\psi(\mathfrak{C}_r)$ because $\mathfrak{C}_r$ is central simple over $\mathbf{R}$, and $\mathfrak{E}(W)$ is central simple over $\mathbf{C}$; thus [1, Theorem 4.14] h'' extends to an automorphism h' of $\mathfrak{E}(W)$, and h' is conjugation by an invertible element.

We have just seen that ϕ and ψ are equivalent. In particular, they induce equivalent $\mathbf{F}$-representations of G_r. Thus they induce $\mathbf{F}$-unitarily equivalent $\mathbf{F}$-representations of G_r (the usual proof is valid over $\mathbf{K}$). In conclusion, ϕ and ψ are unitarily equivalent, Q.E.D.

LEMMA 2. *$\mathfrak{C}_r$ ($r > 0$) has a translational representation on W if and only if W has some even dimension $2n$, and has an n-dimensional subspace U on which there is a representation of $\mathfrak{C}_{r-1}$. Any two translational representations of $\mathfrak{C}_r$ on W are unitarily equivalent.*

Proof. The first statement is clear from §4, and the second statement follows from Lemma 1 in case r is even. Now let ϕ and ψ be translational representations of $\mathfrak{C}_r$ (r odd) on W. Replacing ψ by a unitary equivalent, we have a subspace B of W such that $\psi(e_i)(B) = B^\perp = \phi(e_i)(B)$ for $1 \leqq i \leqq r$. Let $\mathbf{b}$ be an orthonormal basis of B, and let $\mathbf{c}$ be an orthonormal basis of $B^\perp$. In the orthonormal basis $\{\mathbf{b}, \mathbf{c}\}$ of W, $\phi(e_r)$ has some matrix $\begin{pmatrix} 0 & X \\ -{}^t\bar{X} & 0 \end{pmatrix}$ and $\psi(e_r)$ has matrix $\begin{pmatrix} 0 & Y \\ -{}^t\bar{Y} & 0 \end{pmatrix}$, where X and Y are $\mathbf{F}$-unitary matrices. Let Z be the unitary automorphism of W whose matrix in $\{\mathbf{b}, \mathbf{c}\}$ is $\begin{pmatrix} I & 0 \\ 0 & X^{-1}Y \end{pmatrix}$. Then $Z(B) = B$ and $Z \cdot \psi(e_r) \cdot Z^{-1} = \phi(e_r)$. Thus we may assume $\psi(e_r) = \phi(e_r)$.

Now let $\mathbf{b}' = \phi(e_r)(\mathbf{b})$, orthonormal basis of $B^\perp$ such that $\phi(e_r) = \psi(e_r)$ has matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ in the orthonormal basis $\{\mathbf{b}, \mathbf{b}'\}$ of W. In $\{\mathbf{b}, \mathbf{b}'\}$, $\phi(e_i)$ has matrix $\begin{pmatrix} 0 & U_i \\ -{}^t\bar{U}_i & 0 \end{pmatrix}$, and $\psi(e_i)$ has matrix $\begin{pmatrix} 0 & V_i \\ -{}^t\bar{V}_i & 0 \end{pmatrix}$ for $1 \leqq i \leqq r - 1$, where U_i and V_i are $\mathbf{F}$-unitary matrices. The relation $e_i e_j + e_j e_i = -2\delta_{ij}$ gives $U_i U_j + U_j U_i = -2\delta_{ij} \cdot I = V_i V_j + V_j V_i$, whence we have unitary representations of $\mathfrak{C}_{r-1}$ on B defined by: $\phi'(e_i)$ has matrix U_i, and $\psi'(e_i)$ has matrix V_i in the basis $\mathbf{b}$. By Lemma 1, we have an $\mathbf{F}$-unitary matrix A such that $AU_iA^{-1} = V_i$, for $r - 1$ is even (because r was assumed odd). Now let D be the unitary automorphism of W with matrix $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ in the basis $\{\mathbf{b}, \mathbf{b}'\}$, and we have $D \cdot \phi(e_i) \cdot D^{-1} = \psi(e_i)$ for $1 \leqq i \leqq r$, proving that ϕ and ψ are unitarily equivalent, Q.E.D.

The following result, stated as a theorem because of its historical interest, solves a classical problem of quadratic forms permitting composition over a real division algebra $\mathbf{F}$. For $\mathbf{F} = \mathbf{C}$, the problem was formulated and solved by A. Hurwitz [6]; later, J. Radon solved it for $\mathbf{F} = \mathbf{R}$ [9]. The problem was solved for composition of a form with itself over an arbitrary commutative field by A. A. Albert [2], and representation-theoretic proofs for the case $\mathbf{F} = \mathbf{R}$ have been given by B. Eckmann [5] and H. C. Lee [7]. Our proof, based on a close look at subalgebras of a total matrix algebra, gives a brief and unified treatment of the three possibilities for $\mathbf{F}$. We believe the result to be new for $\mathbf{F} = \mathbf{K}$.

THEOREM 5. *Let U be a left vectorspace of dimension n over a real division algebra $\mathbf{F}$, $n = 2^{4a+b}u = 2^q u$ where u is odd and $0 \leqq b \leqq 3$. Then $\mathfrak{C}_{r-1}$ has a representation on U if and only if*

(1) $\mathbf{F} = \mathbf{R}$: $r \leqq 8a + 2^b$.

(2) $\mathbf{F} = \mathbf{C}$: $r \leqq 8a + 2b + 2 = 2q + 2$.

(3) $\mathbf{F} = \mathbf{K}$: $r \leqq 8a + 2^b + \frac{1}{2}(b + 2)(3 - b)$.

Proof. If $\mathfrak{M}$ is an algebra with identity element 1, then $\mathfrak{A} \subset \mathfrak{M}$ will mean that $\mathfrak{A}$ is a subalgebra of $\mathfrak{M}$ and 1 is the identity element of $\mathfrak{A}$. If $\mathfrak{A}$ is central simple, then [1, Theorem 4.6] $\mathfrak{M}$ has a subalgebra $\mathfrak{A}'$ such that $\mathfrak{M}$ is the Kronecker product (= tensor product) $\mathfrak{A} \otimes \mathfrak{A}'$; this result will be used without reference. $\mathfrak{M}_k(\mathbf{F})$ will denote the algebra of $k \times k$ matrices over $\mathbf{F}$ as an algebra over $\mathbf{F}$; $\mathfrak{M}_k(\mathbf{F})_R$ denotes $\mathfrak{M}_k(\mathbf{F})$ viewed as an algebra over $\mathbf{R}$.

The Clifford algebras have the following structure (see [4, Chapter II]):

$$\begin{aligned}
\mathfrak{C}_{8t} &= \mathfrak{M}_{2^{4t}}(\mathbf{R}), & \mathfrak{C}_{8t+1} &= \mathfrak{M}_{2^{4t}}(\mathbf{C})_R, \\
\mathfrak{C}_{8t+2} &= \mathfrak{M}_{2^{4t}}(\mathbf{K})_R, & \mathfrak{C}_{8t+3} &= \mathfrak{C}_{8t+2} \oplus \mathfrak{C}_{8t+2}, \\
\mathfrak{C}_{8t+4} &= \mathfrak{M}_{2^{4t+1}}(\mathbf{K})_R, & \mathfrak{C}_{8t+5} &= \mathfrak{M}_{2^{4t+2}}(\mathbf{C})_R, \\
\mathfrak{C}_{8t+6} &= \mathfrak{M}_{2^{4t+3}}(\mathbf{R}), & \mathfrak{C}_{8t+7} &= \mathfrak{C}_{8t+6} \oplus \mathfrak{C}_{8t+6}.
\end{aligned}$$

In the cases where $\mathfrak{C}_m$ is not simple, none of the e_i lie in a simple summand. Thus $\mathfrak{C}_m$ has a representation on U if and only if

$$m = 8t \quad \text{and} \quad \mathfrak{M}_{2^{4t}}(\mathbf{R}) \subset \mathfrak{E}(U)_R = \mathfrak{M}_n(\mathbf{F})_R,$$

or $\quad m = 8t + 1$ and $\mathfrak{M}_{2^{4t}}(\mathbf{C})_R \subset \mathfrak{M}_n(\mathbf{F})_R$,

or $\quad m = 8t + 2$ and $\mathfrak{M}_{2^{4t}}(\mathbf{K})_R \subset \mathfrak{M}_n(\mathbf{F})_R$,

or $\quad m = 8t + 3$ and $\mathfrak{M}_{2^{4t}}(\mathbf{K})_R \subset \mathfrak{M}_n(\mathbf{F})_R$,

or $\quad m = 8t + 4$ and $\mathfrak{M}_{2^{4t+1}}(\mathbf{K})_R \subset \mathfrak{M}_n(\mathbf{F})_R$,

or $\quad m = 8t + 5$ and $\mathfrak{M}_{2^{4t+2}}(\mathbf{C})_R \subset \mathfrak{M}_n(\mathbf{F})_R$,

or $\quad m = 8t + 6$ and $\mathfrak{M}_{2^{4t+3}}(\mathbf{R}) \subset \mathfrak{M}_n(\mathbf{F})_R$,

or $\quad m = 8t + 7$ and $\mathfrak{M}_{2^{4t+3}}(\mathbf{R}) \subset \mathfrak{M}_n(\mathbf{F})_R$.

If we use the fact that $\mathfrak{M}_p(\mathbf{F})_R \subset \mathfrak{M}_s(\mathbf{F})_R$, $\mathfrak{M}_p(\mathbf{C})_R \subset \mathfrak{M}_{2s}(\mathbf{R})$, $\mathfrak{M}_p(\mathbf{K})_R \subset \mathfrak{M}_{4s}(\mathbf{R})$, $\mathfrak{M}_p(\mathbf{R}) \subset \mathfrak{M}_s(\mathbf{C})_R$, $\mathfrak{M}_p(\mathbf{K})_R \subset \mathfrak{M}_{2s}(\mathbf{C})_R$, $\mathfrak{M}_p(\mathbf{R}) \subset \mathfrak{M}_s(\mathbf{K})_R$, and $\mathfrak{M}_p(\mathbf{C})_R \subset \mathfrak{M}_s(\mathbf{K})_R$ are each equivalent to s being divisible by p, it follows that $\mathfrak{C}_{r-1}$ has a representation on U if and only if Table I holds.

TABLE I

	$\mathbf{F} = \mathbf{R}$	$\mathbf{F} = \mathbf{C}$	$\mathbf{F} = \mathbf{K}$
$r - 1 = 8t$	$r \leqq 2q + 1$	$r \leqq 2q + 1$	$r \leqq 2q + 1$
$r - 1 = 8t + 1$	$r \leqq 2q$	$r \leqq 2q + 2$	$r \leqq 2q + 2$
$r - 1 = 8t + 2$	$r \leqq 2q - 1$	$r \leqq 2q + 1$	$r \leqq 2q + 3$
$r - 1 = 8t + 3$	$r \leqq 2q$	$r \leqq 2q + 2$	$r \leqq 2q + 4$
$r - 1 = 8t + 4$	$r \leqq 2q - 1$	$r \leqq 2q + 1$	$r \leqq 2q + 3$
$r - 1 = 8t + 5$	$r \leqq 2q$	$r \leqq 2q + 2$	$r \leqq 2q + 2$
$r - 1 = 8t + 6$	$r \leqq 2q + 1$	$r \leqq 2q + 1$	$r \leqq 2q + 1$
$r - 1 = 8t + 7$	$r \leqq 2q + 2$	$r \leqq 2q + 2$	$r \leqq 2q + 2$

Let s be the largest integer such that $\mathfrak{C}_{s-1}$ has a representation on U. Then $\mathfrak{C}_{r-1}$ has a representation on U if and only if $r \leqq s$, and our considerations show that s is given by Table II.

TABLE II

	$\mathbf{F} = \mathbf{R}$	$\mathbf{F} = \mathbf{C}$	$\mathbf{F} = \mathbf{K}$
$q = 4a$	$s = 2q + 1$	$s = 2q + 2$	$s = 2q + 4$
$q = 4a + 1$	$s = 2q$	$s = 2q + 2$	$s = 2q + 3$
$q = 4a + 2$	$s = 2q$	$s = 2q + 2$	$s = 2q + 2$
$q = 4a + 3$	$s = 2q + 2$	$s = 2q + 2$	$s = 2q + 2$

Thus

$$\begin{aligned} s &= 8a + 2^b && \text{if } \mathbf{F} = \mathbf{R}, \\ s &= 8a + 2b + 2 = 2q + 2 && \text{if } \mathbf{F} = \mathbf{C}, \\ s &= 8a + 2^b + \tfrac{1}{2}(b + 2)(3 - b) && \text{if } \mathbf{F} = \mathbf{K}, \end{aligned}$$

which proves the theorem, Q.E.D.

11. Translational representations with basepoint

The study of translational representations with basepoint is facilitated by the algebra

$$\mathfrak{D}_r = \mathfrak{C}_r + s\cdot\mathfrak{C}_r : \quad se_i s^{-1} = -e_i , \quad s^2 = 1$$

and the behavior of its elements

$$z = se_1 e_2 \cdots e_r , \qquad a_1 = \tfrac{1}{2}(1+z), \quad a_2 = \tfrac{1}{2}(1-z).$$

For if (ϕ, U) is a translational representation with basepoint (see §9 for definitions) of $\mathfrak{C}_r$ on W, then we have an induced representation of $\mathfrak{D}_r$ on W by

$$\phi' : x + sy \rightarrow \phi(x) + S_U\cdot\phi(y) \qquad\qquad (x, y \in \mathfrak{C}_r)$$

where S_U is the unitary transformation of W which is I on U and is $-I$ on $U^{\perp}$. On the other hand, if ψ'' is a representation of $\mathfrak{D}_r$ on W $(r > 0)$, then its restriction ψ to $\mathfrak{C}_r$ is a representation of $\mathfrak{C}_r$ on W; if ψ is unitary, then $W = P \oplus P^{\perp}$ where $\psi''(s)$ is I on P and $-I$ on $P^{\perp}$ and where each $\psi(e_i)$ interchanges P and $P^{\perp}$, because each $\psi(e_i)$ anticommutes with $\psi''(s)$ and $\psi''(s)^2 = I$, whence (ψ, P) is a translational representation with basepoint and $\psi' = \psi''$.

We will only use $\mathfrak{D}_r$ when r is even. In that case, the center $\mathfrak{Z}$ of $\mathfrak{D}_r$ is spanned by 1 and z, and $\mathfrak{D}_r \cong \mathfrak{C}_r \otimes_R \mathfrak{Z}$.

$$z^2 = (-1)^r\cdot s^2\cdot(e_1 \cdot\cdots\cdot e_r)^2 = (-1)^r\cdot(-1)^{r(r+1)/2},$$

whence $\mathfrak{Z} \cong \mathbf{C}$ if $r \equiv 2 \pmod 4$ and $\mathfrak{Z} \cong \mathbf{R} \oplus \mathbf{R}$ if $r \equiv 0 \pmod 4$. Thus $\mathfrak{D}_r$ (r even) is given by

$$\begin{aligned}
\mathfrak{D}_{8t} &= \mathfrak{M}_{2^{4t}}(\mathbf{R}) \oplus \mathfrak{M}_{2^{4t}}(\mathbf{R}),\\
\mathfrak{D}_{8t+2} &= \mathfrak{M}_{2^{4t}}(\mathbf{K}) \otimes_R \mathbf{C} = \mathfrak{M}_{2^{4t+1}}(\mathbf{C})_R ,\\
\mathfrak{D}_{8t+4} &= \mathfrak{M}_{2^{4t+1}}(\mathbf{K})_R \oplus \mathfrak{M}_{2^{4t+1}}(\mathbf{K})_R ,\\
\mathfrak{D}_{8t+6} &= \mathfrak{M}_{2^{4t+3}}(\mathbf{R}) \otimes_R \mathbf{C} = \mathfrak{M}_{2^{4t+3}}(\mathbf{C})_R .
\end{aligned}$$

Finally, it is easy to verify that two translational representations with basepoint of $\mathfrak{C}_r$ are strictly equivalent if the associated representations of $\mathfrak{D}_r$ are equivalent.

LEMMA 3. *If r is odd, or if $r \equiv 2 \pmod 4$ and $\mathbf{F} \neq \mathbf{C}$, then any two translational representations with basepoint of $\mathfrak{C}_r$ on W are strictly equivalent.*

Proof. If r is odd, this was seen during the proof of Lemma 2.

Now let $r \equiv 2 \pmod 4$ and $\mathbf{F} \neq \mathbf{C}$. By the preceding discussion, we need only take a complex matrix algebra $\mathfrak{M}_q(\mathbf{C})$ and prove that two $\mathbf{R}$-algebra representations ϕ and ψ of it on W are equivalent. Let $J = \sqrt{-1}\, I \in \mathfrak{M}_q(\mathbf{C})$. As $\mathbf{F} \neq \mathbf{C}, \phi(J)$ is conjugate to $\psi(J)$ because they both have square $-I \in \mathfrak{E}(W)$; thus we may assume $\phi(J) = J' = \psi(J)$. Let $\mathfrak{A}$ be the centralizer of J' in $\mathfrak{E}(W)$; $\mathfrak{A}$ carries the structure of a complex matrix algebra in which $J' = \sqrt{-1}\, I$, and ϕ and ψ are $\mathbf{C}$-algebra homomorphisms of $\mathfrak{M}_q(\mathbf{C})$ into $\mathfrak{A}$. It follows [1, Theorems 4.5 and 4.14] that ϕ and ψ are equivalent by an element of $\mathfrak{A}$, Q.E.D.

The obstacles to extending Lemma 3 are made explicit in the following definition.

DEFINITION. *Let (ϕ, U) be a translational representation with basepoint of $\mathfrak{C}_r$ on W, and let ϕ' be the associated representation of $\mathfrak{D}_r$ on W. If $\mathbf{F} = \mathbf{C}$ and $r \equiv 2 \pmod 4$, then $W = W_1 \oplus W_2$ where $\phi'(z)$ is scalar multiplication by $\sqrt{-1}$ on W_1 and by $-\sqrt{-1}$ on W_2; we define the index $\nu(\phi, U)$ to be the ordered pair $\{\dim W_1, \dim W_2\}$. If $r \equiv 0 \pmod 4$, then $W = W_1 \oplus W_2$ where $\phi'(z)$ is scalar multiplication by $+1$ on W_1 and by -1 on W_2; we define the index $\nu(\phi, U)$ to be the ordered pair $\{\dim W_1, \dim W_2\}$.*

LEMMA 4. *If $r \equiv 0 \pmod 4$, or if $\mathbf{F} = \mathbf{C}$ and $r \equiv 2 \pmod 4$, then two translational representations with basepoint of $\mathfrak{C}_r$ on W are strictly equivalent if and only if their indices are equal; a translational representation with basepoint (ϕ, U) of $\mathfrak{C}_r$ on W extends to a translational representation with basepoint (τ, U) of $\mathfrak{C}_{r+1}$ on W if and only if $u = v$ where $\nu(\phi, U) = \{u, v\}$.*

Proof. Let (ϕ, U) and (ψ, P) be translational representations of $\mathfrak{C}_r$ on W. If they are strictly equivalent, then ϕ' and ψ' are equivalent representations of $\mathfrak{D}_r$ on W, whence $\phi'(z)$ is conjugate to $\psi'(z)$, proving that $\nu(\phi, U) = \nu(\psi, P)$. Now suppose $\nu(\phi, U) = \nu(\psi, P)$; we will prove that (ϕ, U) is strictly equivalent to (ψ, P). Submitting (ψ, P) to a strict equivalence, we may assume that $\psi'(z) = \phi'(z)$. Now let $W = W_1 \oplus W_2$ be the (necessarily orthogonal) direct-sum decomposition of W described in the definition of $\nu(\phi, U)$, let $\mathbf{w}_i$ be an orthonormal basis of W_i, and let $\mathfrak{A}$ be the centralizer of $\phi'(z) = \psi'(z)$ in $\mathfrak{E}(W)$. It is clear that $\mathfrak{A} = \mathfrak{E}(W_1) \oplus \mathfrak{E}(W_2)$.

Suppose first that $r \equiv 2 \pmod 4$, $r = 2r'$, and $\mathbf{F} = \mathbf{C}$. Then $\mathfrak{D}_r = \mathfrak{M}_{2^{r'}}(\mathbf{C})$, and ϕ' and ψ' each defines $\mathbf{C}$-algebra homomorphisms of $\mathfrak{D}_r$ into $\mathfrak{E}(W_1)$ and into $\mathfrak{E}(W_2)$ which carry 1 to I and z to $\pm\sqrt{-1}\, I$. It follows that

$$\phi' = \phi_1 \oplus \phi_2 \quad \text{and} \quad \psi' = \psi_1 \oplus \psi_2,$$

where ϕ_1 and ψ_1 are $\mathbf{C}$-representations of $\mathfrak{M}_{2^{r'}}(\mathbf{C})$ on W_1 with z viewed as $\sqrt{-1}\, I \in \mathfrak{M}_{2^{r'}}(\mathbf{C})$, and ϕ_2 and ψ_2 are $\mathbf{C}$-representations of $\mathfrak{M}_{2^{r'}}(\mathbf{C})$ on W_2 with z viewed as $-\sqrt{-1}\, I \in \mathfrak{M}_{2^{r'}}(\mathbf{C})$. As ϕ_i is equivalent to ψ_i [1, Theorems 4.5 and 4.14], ϕ' is equivalent to ψ', whence (ϕ, U) and (ψ, P) are strictly equivalent.

Now let $r \equiv 0 \pmod 4$, so $\mathfrak{D}_r = a_1 \cdot \mathfrak{D}_r \oplus a_2 \cdot \mathfrak{D}_r$ where a_i is the identity element of $a_i \cdot \mathfrak{D}_r$, and $a_i \cdot \mathfrak{D}_r$ is isomorphic to $\mathfrak{M}_{r'}(\mathbf{R})$ (if $r \equiv 0 \pmod 8$), or to $\mathfrak{M}_{r'}(\mathbf{K})_R$ (if $r \equiv 4 \pmod 8$). As above, $\phi' = \phi_1 \oplus \phi_2$ and $\psi' = \psi_1 \oplus \psi_2$ where ϕ_i and ψ_i are $\mathbf{R}$-representations of $a_i \cdot \mathfrak{D}_r$ on W_i which send a_i to I. If $\mathbf{F} \neq \mathbf{C}$, then [1, Theorems 4.5 and 4.14] ϕ_i is equivalent to ψ_i, and it follows that (ϕ, U) is strictly equivalent to (ψ, P). Now assume $\mathbf{F} = \mathbf{C}$. Then ϕ_i and ψ_i extend to $\mathbf{C}$-representations of $a_i \cdot \mathfrak{D}_r \otimes_R \mathbf{C} = a_i \cdot (\mathfrak{D}_r \otimes_R \mathbf{C})$ on W_i, and the same argument shows that (ϕ, U) is strictly equivalent to (ψ, P).

If (ϕ, U) extends to a translational representation with basepoint (τ, U) of $\mathfrak{C}_{r+1}$ on W, then

$$se_{r+1}s^{-1} = -e_{r+1} \quad \text{and} \quad (e_i \cdot \cdots \cdot e_r)e_{r+1} = e_{r+1}(e_1 \cdot \cdots \cdot e_r)$$

show that $\tau'(z)$ anticommutes with $\tau(e_{r+1})$; it follows that $\tau(e_{r+1})$ interchanges W_1 and W_2, and thus that $\dim W_1 = \dim W_2$, proving that $u = v$ where $\nu(\phi, U) = \{u, v\}$.

Now let $\nu(\phi, U) = \{u, u\}$. The representations ϕ_1 and ϕ_2 of $\mathfrak{D}_r$ on W_1 and W_2 are equivalent under some unitary transformation of W_1 onto W_2. Thus we may assume the orthonormal basis $\mathbf{w}_2$ of W_2 chosen such that $\phi(e_i)$ has matrix $\begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix}$ in the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ of W. We define $\tau(e_i) = \phi(e_i)$ for $1 \leqq i \leqq r$ and define $\tau(e_{r+1})$ to be the unitary transformation with matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ in the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$. Then $\tau(e_{r+1})$ anticommutes with each of the other $\tau(e_i)$, so τ defines a representation of $\mathfrak{C}_{r+1}$ on W which extends ϕ. $\tau(e_{r+1})$ commutes with $\phi(e_1 \cdot \cdots \cdot e_r)$ because r is even, and anticommutes with $\phi'(z)$ because it has matrix $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ in the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$; thus $\tau(e_{r+1})$ anticommutes with $\phi'(s)$. It follows that $\tau(e_{r+1})$ interchanges U and $U^{\perp}$, so (τ, U) is a translational representation with basepoint of $\mathfrak{C}_{r+1}$ on W which extends (ϕ, U), Q.E.D.

We now summarize the last two sections in

THEOREM 6. *Let V be a positive-definite hermitian vectorspace of dimension $2n$ over a real division algebra $\mathbf{F}$, express $n = 2^{4a+b}w = 2^q w$ with w odd and $0 \leqq b \leqq 3$, and define*

$$f(V) = 8a + 2^b \qquad \text{if} \quad \mathbf{F} = \mathbf{R},$$

$$f(V) = 8a + 2b + 2 = 2q + 2 \qquad \text{if} \quad \mathbf{F} = \mathbf{C},$$

$$f(V) = 8a + 2^b + \tfrac{1}{2}(b + 2)(3 - b) \quad \text{if} \quad \mathbf{F} = \mathbf{K}.$$

Then $\mathfrak{C}_r$ has a translational representation on V if and only if $r \leqq f(V)$, and any two translational representations of $\mathfrak{C}_r$ on V are unitarily equivalent. Let (ϕ, U) and (ψ, P) be translational representations with basepoint of $\mathfrak{C}_r$ on V. Then (ϕ, U) is strictly equivalent to (ψ, P) if r is odd or if both $r \equiv 2 \pmod 4$ and $\mathbf{F} \neq \mathbf{C}$; if $r \equiv 0 \pmod 4$, or if $r \equiv 2 \pmod 4$ and $\mathbf{F} = \mathbf{C}$, then (ϕ, U) is strictly equivalent to (ψ, P) if and only if the index $\nu(\phi, U) = \nu(\psi, P)$. If r is odd, or if $r \equiv 2 \pmod 4$ and $\mathbf{F} \neq \mathbf{C}$, or if $\nu(\phi, U) = \{n, n\}$, then (ϕ, U) extends to a translational representation with basepoint (τ, U) of $\mathfrak{C}_{f(V)}$ on V;

TABLE III

F	r	index	conditions[2]
R, C, K	$8t$	$\{2^{4t}c,\ 2n - 2^{4t}c\}$	$1 \leqq t \leqq [(q+1)/4]$ and $0 \leqq c \leqq 2^{q-4t+1}w$ where $c \neq 2^{q-4t}w$.
R, C, K $\dim_R \mathbf{F} = 2^x$	$8t+4$	$\{2^{4t+3-x}c,\ 2n - 2^{4t+3-x}c\}$	$0 \leqq t \leqq [(q-2+x)/4]$ and $0 \leqq c \leqq 2^{q-4t-2+x}w$ where $c \neq 2^{q-4t-3+x}w$.
C	$4s+2$	$\{2^{s+1}d,\ 2n - 2^{s+1}d\}$	$0 \leqq s \leqq [q/2]$ and $0 \leqq d \leqq 2^{q-2s}w$ where $d \neq 2^{q-2s-1}w$.
R	$f(V)$		$f(V) \not\equiv 0 \pmod 4$, i.e., $b = 0$ or $b = 1$
K	$f(V)$		$f(V) \not\equiv 0 \pmod 4$, i.e., $b = 1$ or $b = 2$

otherwise, (ϕ, U) does not extend to a translational representation with basepoint (σ, U) of $\mathfrak{C}_{r+1}$ on V, i.e., (ϕ, U) is maximal. In particular, the strict equivalence classes of maximal translational representations with basepoint of $\mathfrak{C}_r$ on V are enumerated in Table III.

Theorem 6 extends the results of Part II of Y.-C. Wong's memoir [12].

12. Unitary classification of isoclinic spheres

According to Theorem 1, every set of pairwise isoclinic n-dimensional subspaces of V ($\dim V = 2n$) lies in an isoclinic sphere $\mathbf{B}^r = \phi(\mathfrak{C}_r')(U)$ where (ϕ, U) is a translational representation with basepoint of $\mathfrak{C}_r$ on V. We will say that $\mathbf{B}^r$ and (ϕ, U) are *associated*. As remarked earlier, (ϕ, U) determines $\mathbf{B}^r$ but $\mathbf{B}^r$ does not determine (ϕ, U); in fact, $\mathbf{B}^r$ is associated with $(\phi, U^\perp)$, which need not be strictly equivalent to (ϕ, U) (in which case $\nu(\phi, U) = \{u, v\}$ and $\nu(\phi, U^\perp) = \{v, u\}$). This lack of uniqueness is clarified by

LEMMA 5. *Let $\mathbf{B}^r$ be an isoclinic r-sphere on V, and let (ϕ, U) and (ψ, P) be translational representations with basepoint associated with $\mathbf{B}^r$. If these representations are not strictly equivalent, and if $\nu(\phi, U) = \{u, v\}$, then $\nu(\psi, P) = \{v, u\}$.*

Proof. Given $B \in \mathbf{B}^r$, the constructions of §3 and §4 show that the set $\mathfrak{S}^r$ of Theorem 1 is uniquely determined. As the inner product on $\mathfrak{S}^r$ is

[2] Here, of course, $[y]$ denotes the integral part of $y \in \mathbf{R}$.

canonical, the orthonormal basis $\{I, S_1, \cdots, S_r\}$ of $\mathfrak{S}^r$ is determined by B up to an orthogonal transformation of the S_i; it follows that B determines the product $S = S_1 S_2 \cdots S_r$ up to sign, and thus determines $Z_B = S_B S_1 \cdots S_r$ up to sign, where S_B is the unitary transformation of V which is $+1$ on B and is -1 on $B^\perp$. If we move B in $\mathbf{B}^r$, then Z_B moves continuously. The lemma now follows from Lemma 3 and the definition of the indices $\nu(\phi, U)$ and $\nu(\psi, P)$, Q.E.D.

In view of Lemma 5, if $r \equiv 0 \pmod 4$, or if $r \equiv 2 \pmod 4$ and $\mathbf{F} = \mathbf{C}$, we define the *index* $\nu(\mathbf{B}^r)$ of an isoclinic r-sphere $\mathbf{B}^r$ on V to be the *unordered* pair $\{u, v\}$ where $\mathbf{B}^r$ is associated with some (ϕ, U) of index $\{u, v\}$.

Combining Theorem 1, Theorem 6, and Lemma 5, we have the unitary classification of isoclinic spheres:

THEOREM 7. *Let V be a positive-definite hermitian vectorspace of dimension $2n$ over a real division algebra $\mathbf{F}$. Express $n = 2^{4a+b}w = 2^q w$ with w odd and $0 \leqq b \leqq 3$, define $f(V)$ to be $8a + 2^b$ if $\mathbf{F} = \mathbf{R}$, $2q + 2$ if $\mathbf{F} = \mathbf{C}$, and $8a + 2^b + \frac{1}{2}(b+2)(3-b)$ if $\mathbf{F} = \mathbf{K}$. Then every family of pairwise isoclinic n-dimensional subspaces of V is a subset of an isoclinic sphere on V, and every isoclinic sphere on V is a family of pairwise isoclinic n-dimensional subspaces of V. There is an isoclinic r-sphere on V if and only if $r \leqq f(V)$; if r is odd, or if $r \equiv 2 \pmod 4$ and $\mathbf{F} \neq \mathbf{C}$, then any two isoclinic r-spheres on V are unitarily equivalent (under the unitary group of V); if $r \equiv 0 \pmod 4$, or if $r \equiv 2 \pmod 4$ and $\mathbf{F} = \mathbf{C}$, then two isoclinic r-spheres on V are unitarily equivalent if and only if they have the same index. Every isoclinic sphere lies in a maximal isoclinic sphere, any two nonmaximal isoclinic r-spheres on V are unitarily equivalent, and the unitary equivalence classes of maximal isoclinic r-spheres on V are enumerated in Table IV.*

TABLE IV

F	r	index	conditions[2]
R, C, K	$8t$	$\{2^{4t}c, 2n - 2^{4t}c\}$	$1 \leqq t \leqq [(q+1)/4]$ and $0 \leqq c < 2^{q-4t}w$.
R, C, K $\dim_R \mathbf{F} = 2^x$	$8t + 4$	$\{2^{4t+3-x}c, 2n - 2^{4t+3-x}c\}$	$0 \leqq t \leqq [(q-2+x)/4]$ and $0 \leqq c < 2^{q-4t-3+x}w$.
C	$4s + 2$	$\{2^{s+1}d, 2n - 2^{s+1}d\}$	$0 \leqq s \leqq [q/2]$ and $0 \leqq d < 2^{q-2s-1}w$.
R	$f(V)$		$f(V) \not\equiv 0 \pmod 4$, i.e., $b = 0$ or $b = 1$
K	$f(V)$		$f(V) \not\equiv 0 \pmod 4$, i.e., $b = 1$ or $b = 2$

Remark 6. Except for his §I.4, we have extended all the results of Y.-C. Wong's memoir to an arbitrary real division algebra. To extend his §I.4, one need only view V as a real euclidean space and observe that $\mathbf{F}$-isoclinic subspaces are $\mathbf{R}$-isoclinic. And as pointed out by Wong for the real case, Theorem 7 gives the classification of maximal sets of Clifford-parallel linear subspaces of dimension (over $\mathbf{F}$) $n - 1$ in the projective space of dimension $2n - 1$ over $\mathbf{F}$. For the motions of that projective space $\mathbf{G}_{1,2n}(\mathbf{F})$ induced by $\mathbf{U}(2n, \mathbf{F})$ are precisely its isometries.

Remark 7. The following shows how Theorem 7 gives the unitary classification of isoclinic spheres on $2n$-dimensional subspaces of $\mathbf{F}^k$:

Let V be a $2n$-dimensional subspace of $\mathbf{F}^k$. Then two isoclinic spheres on V are unitarily equivalent in V if and only if they are unitarily equivalent in $\mathbf{F}^k$. An isoclinic sphere on a $2n$-dimensional subspace of $\mathbf{F}^k$ is unitarily equivalent in $\mathbf{F}^k$ (by an element of $\mathbf{U}(k, \mathbf{F})$) to an isoclinic sphere on V; thus the unitary equivalence (in $\mathbf{F}^k$) classes of isoclinic spheres on $2n$-dimensional subspaces of $\mathbf{F}^k$ are in one-to-one correspondence with the unitary equivalence (in V) classes of isoclinic spheres on V.

To see this, let $\mathbf{B}_i$ be isoclinic spheres on V. If the $\mathbf{B}_i$ are unitarily equivalent in V, we choose a unitary transformation of V carrying $\mathbf{B}_1$ to $\mathbf{B}_2$, and extend it to an element of $\mathbf{U}(k, \mathbf{F})$ by defining it to be the identity on $V^{\perp}$, thus proving the $\mathbf{B}_i$ unitarily equivalent in $\mathbf{F}^k$. If the $\mathbf{B}_i$ are unitarily equivalent in $\mathbf{F}^k$, we choose $T \in \mathbf{U}(k, \mathbf{F})$ such that $T(\mathbf{B}_1) = \mathbf{B}_2$. $T(V) = V$ automatically if $\dim \mathbf{B}_i > 0$, and can be arranged if $\dim \mathbf{B}_i = 0$. Now the restriction $T|_V$ is a unitary transformation of V carrying $\mathbf{B}_1$ to $\mathbf{B}_2$, so the $\mathbf{B}_i$ are unitarily equivalent in V.

13. The classification of isoclinic spheres under rigid motions of the Grassmann manifold

If M is a Riemannian manifold, then $\mathbf{I}(M)$ will denote the group of all isometries (differentiable homeomorphisms which preserve the Riemannian structure) of M. Recall that $\mathbf{I}(M)$ is a Lie group in the compact-open topology; $\mathbf{I}_0(M)$ will denote the identity component of $\mathbf{I}(M)$.

$\mathbf{U}(k, \mathbf{F})$ acts by isometries on $\mathbf{G}_{n,k}(\mathbf{F})$; let $\mathbf{I}'(\mathbf{G}_{n,k}(\mathbf{F}))$ denote the group of isometries of $\mathbf{G}_{n,k}(\mathbf{F})$ induced by $\mathbf{U}(k, \mathbf{F})$. There is an isometry β of $\mathbf{G}_{n,2n}(\mathbf{F})$ given by $\beta(P) = P^{\perp}$. Also, choice of an orthonormal basis of $\mathbf{C}^k$ allows us to extend the conjugation of $\mathbf{C}$ over $\mathbf{R}$ to a transformation of $\mathbf{C}^k$, and this transformation induces an isometry α of $\mathbf{G}_{n,k}(\mathbf{C})$. Finally, the triality automorphism of $\mathfrak{SO}(8)$ induces an isometry τ of $\mathbf{G}_{4,8}(\mathbf{R})$. It is known [3] that $\mathbf{I}(\mathbf{G}_{n,k}(\mathbf{F}))$ is given as follows:

1. $\mathbf{I}(\mathbf{G}_{n,k}(\mathbf{K})) = \mathbf{I}_0(\mathbf{G}_{n,k}(\mathbf{K}))$ if $k \neq 2n$,

   $\mathbf{I}(\mathbf{G}_{n,2n}(\mathbf{K})) = \{1, \beta\} \cdot \mathbf{I}_0(\mathbf{G}_{n,2n}(\mathbf{K}))$,

   $\mathbf{I}_0(\mathbf{G}_{n,k}(\mathbf{K})) = \mathbf{I}'(\mathbf{G}_{n,k}(\mathbf{K}))$,

2. $\mathbf{I}(\mathbf{G}_{n,k}(\mathbf{C})) = \{1, \alpha\} \cdot \mathbf{I}_0(\mathbf{G}_{n,k}(\mathbf{C}))$ if $k \neq 2n$,

$\mathbf{I}(\mathbf{G}_{n,2n}(\mathbf{C})) = \{1, \alpha, \beta, \alpha\beta\} \cdot \mathbf{I}_0(\mathbf{G}_{n,2n}(\mathbf{C}))$,

$\mathbf{I}_0(\mathbf{G}_{n,k}(\mathbf{C})) = \mathbf{I}'(\mathbf{G}_{n,k}(\mathbf{C}))$,

3. $\mathbf{I}(\mathbf{G}_{n,k}(\mathbf{R})) = \mathbf{I}'(\mathbf{G}_{n,k}(\mathbf{R}))$ if $k \neq 2n$,

$\mathbf{I}(\mathbf{G}_{n,2n}(\mathbf{R})) = \{1, \beta\} \cdot \mathbf{I}'(\mathbf{G}_{n,2n}(\mathbf{R}))$ if $n \neq 4$,

$\mathbf{I}(\mathbf{G}_{4,8}(\mathbf{R})) = \{1, \beta, \tau, \tau\beta, \tau^2, \tau^2\beta\} \cdot \mathbf{I}'(\mathbf{G}_{4,8}(\mathbf{R}))$,

$\mathbf{I}_0(\mathbf{G}_{n,k}(\mathbf{R}))$ has index 1 or 2 in $\mathbf{I}'(\mathbf{G}_{n,k}(\mathbf{R}))$.

LEMMA 6. *Let V_1 and V_2 be $2n$-dimensional subspaces of $\mathbf{F}^k$, let $\mathbf{B}_i$ be an isoclinic sphere on V_i, and view the $\mathbf{B}_i$ as submanifolds of $\mathbf{G}_{n,k}(\mathbf{F})$. Then these are equivalent:*

(1) *$\mathbf{B}_1$ is unitarily equivalent to $\mathbf{B}_2$ in $\mathbf{F}^k$.*
(2) *An element of $\mathbf{I}'(\mathbf{G}_{n,k}(\mathbf{F}))$ maps $\mathbf{B}_1$ onto $\mathbf{B}_2$.*
(3) *An element of $\mathbf{I}(\mathbf{G}_{n,k}(\mathbf{F}))$ maps $\mathbf{B}_1$ onto $\mathbf{B}_2$.*

Proof. (1) and (2) are equivalent by definition of $\mathbf{I}'(\mathbf{G}_{n,k}(\mathbf{F}))$, and it is clear that (2) implies (3). Now assume (3). As $\beta(\mathbf{B}_i) = \mathbf{B}_i$ in case $k = 2n$, we need only prove that α and τ cannot change the unitary equivalence class of $\mathbf{B}_i$, and (1) will follow.

First consider the case of $\mathbf{G}_{4,8}(\mathbf{R})$. According to Theorem 7, any two isoclinic r-spheres on $\mathbf{R}^8$ are unitarily equivalent. Remark 7 now shows that τ cannot change the unitary equivalence class of $\mathbf{B}_i$.

Now consider the case $\mathbf{F} = \mathbf{C}$, and let $r = \dim \mathbf{B}_i$. If r is odd, then Theorem 7 and Remark 7 show $\mathbf{B}_i$ unitarily equivalent to $\alpha(\mathbf{B}_i)$. Now assume r to be even. If $r \equiv 0 \pmod 4$, and if we alter V_1 and V_2 by an element of $\mathbf{U}(k, \mathbf{F})$ such that they are equal and are invariant under α, then Lemma 4 and Remark 7 show $\mathbf{B}_i$ to be unitarily equivalent to $\alpha(\mathbf{B}_i)$. Finally, if $r \equiv 2 \pmod 4$, then Lemma 4, the discussion preceding Lemma 5, and Remark 7 show that $\mathbf{B}_i$ and $\alpha(\mathbf{B}_i)$ are unitarily equivalent, Q.E.D.

Combining Theorem 7, Remark 7, and Lemma 6, we have our final classification theorem:

THEOREM 8. *Consider the Grassmann manifold $\mathbf{G}_{n,k}(\mathbf{F})$ where $2n \leqq k$ and $\mathbf{F}$ is a real division algebra. Express $n = 2^{4a+b}w = 2^q w$ with w odd and $0 \leqq b \leqq 3$, and define $f(n, \mathbf{K})$ to be $8a + 2^b$ if $\mathbf{F} = \mathbf{R}$, $2q + 2$ if $\mathbf{F} = \mathbf{C}$, and $8a + 2^b + \frac{1}{2}(b + 2)(3 - b)$ if $\mathbf{F} = \mathbf{K}$. Then $\mathbf{G}_{n,k}(\mathbf{F})$ contains an isoclinic r-sphere on a $2n$-dimensional subspace of $\mathbf{F}^k$ if and only if $r \leqq f(n, \mathbf{K})$. Every isoclinic sphere on a $2n$-dimensional subspace of $\mathbf{F}^k$ lies in a maximal such isoclinic sphere, any two nonmaximal such isoclinic r-spheres are equivalent under an isometry of $\mathbf{G}_{n,k}(\mathbf{F})$, and the $\mathbf{I}(\mathbf{G}_{n,k}(\mathbf{F}))$-equivalence classes of maximal isoclinic r-spheres are given by Table IV in Theorem 7.*

Theorems 4 and 8 give a complete description of the totally geodesic submanifolds of $\mathbf{G}_{n,k}(\mathbf{F})$ in which any two distinct elements have zero intersection in $\mathbf{F}^k$, and every element is contained in some fixed $2n$-dimensional subspace of $\mathbf{F}^k$. This applies to the case $2n \leqq k$; Remark 2 gives the corresponding result for the case $2n \geqq k$.

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