ELLIPTIC SPACES IN GRASSMANN MANIFOLDS

BY

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1. Introduction

Let $\mathbf{G}_{n,k}(\mathbf{F})$ denote the Grassmann manifold of n-dimensional subspaces of $\mathbf{F}^k$, with its usual structure as a Riemannian symmetric space, where $\mathbf{F}$ denotes the real numbers, the complex numbers, or the quaternions. In an earlier paper [5] we studied the connected totally geodesic submanifolds $\mathbf{B}$ of $\mathbf{G}_{n,k}(\mathbf{F})$ with the property that any two distinct elements of $\mathbf{B}$ have zero intersection as subspaces of $\mathbf{F}^k$. We proved [5, Theorem 4] that $\mathbf{B}$ is isometric to a sphere, to a real, complex, or quaternionic projective space, or to the Cayley projective plane; we then [5, Theorem 8] classified (up to an isometry of $\mathbf{G}_{n,k}(\mathbf{F})$) the manifolds $\mathbf{B}$ which are isometric to spheres. In Chapter I of this paper we show that $\mathbf{B}$ cannot be the Cayley projective plane (Theorem 2), and we classify the manifolds $\mathbf{B}$ which are not isometric to spheres (Theorem 3). The main technique is the application of the results of the preceding paper [5] to the projective lines of $\mathbf{B}$, which are totally geodesic spheres in $\mathbf{G}_{n,k}(\mathbf{F})$, resulting in a structure theorem (Proposition 1) for $\mathbf{B}$.

The key to the study of the manifolds $\mathbf{B}$ is the observation [5, Remark 4] that any two elements of $\mathbf{B}$ are isoclinic (constant angle) in the sense of Y.-C. Wong [6]. Chapter II is devoted to the converse problem. We define a closure operation on sets of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$, and prove (Lemma 10 and Theorem 4) that the closed sets are finite disjoint unions $\mathbf{B}^1 \cup \cdots \cup \mathbf{B}^m$ of manifolds $\mathbf{B}$ where every element of $\mathbf{B}^i$ is orthogonal to every element of $\mathbf{B}^j$ (as subspaces of $\mathbf{F}^k$) whenever $i \neq j$. Thus the notion "set of mutually isoclinic n-dimensional subspaces of $\mathbf{F}^k$" coincides with the notion "subset of a finite union of mutually orthogonal submanifolds $\mathbf{B}$ of $\mathbf{G}_{n,k}(\mathbf{F})$". As our structure and classification theorems completely describe the manifolds $\mathbf{B}$, this gives a thorough analysis of the sets of pairwise isoclinic subspaces of any given dimension in $\mathbf{F}^k$; a similar analysis results for sets of pairwise Clifford-parallel linear subspaces of any given dimension in the projective space $\mathbf{P}^{k-1}(\mathbf{F})$.

CHAPTER I. THE ELLIPTIC SPACES

2. Definitions and notation

$\mathbf{F}$ will always denote one of the real division algebras $\mathbf{R}$ (real numbers), $\mathbf{C}$ (complex numbers), or $\mathbf{K}$ (real quaternions) with conjugation $\alpha \to \bar{\alpha}$ over $\mathbf{R}$. Given an integer $k > 0$, $\mathbf{F}^k$ denotes a hermitian positive-definite left

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vectorspace of dimension k over $\mathbf{F}$, and $\mathbf{U}(k, \mathbf{F})$ denotes the unitary group (all linear transformations which preserve the hermitian structure) of $\mathbf{F}^k$. $\mathbf{U}(k, \mathbf{R})$ is the orthogonal group $\mathbf{O}(k)$; $\mathbf{U}(k, \mathbf{C})$ is the unitary group $\mathbf{U}(k)$; $\mathbf{U}(k, \mathbf{K})$ is the symplectic group (= unitary symplectic group) $\mathbf{Sp}(k)$.

The Grassmann manifold $\mathbf{G}_{n,k}(\mathbf{F})$, defined whenever $0 < n < k$, is the set of all n-dimensional subspaces of $\mathbf{F}^k$ with a structure as Riemannian symmetric space. This structure is defined as follows. $\mathbf{U}(k, \mathbf{F})$ acts transitively on the elements of $\mathbf{G}_{n,k}(\mathbf{F})$; given $B \in \mathbf{G}_{n,k}(\mathbf{F})$, $\mathbf{K}_B$ will denote the isotropy subgroup $\{T \in \mathbf{U}(k, \mathbf{F}) : T(B) = B\}$ of $\mathbf{U}(k, \mathbf{F})$ at B. This allows us to identify $\mathbf{G}_{n,k}(\mathbf{F})$ with the coset space $\mathbf{U}(k, \mathbf{F})/\mathbf{K}_B$ under $T \to T(B)$; as $\mathbf{U}(k, \mathbf{F})$ is a compact Lie group and $\mathbf{K}_B$ is a closed subgroup, this identification gives $\mathbf{G}_{n,k}(\mathbf{F})$ the structure of a compact analytic manifold.

German letters denote Lie algebras, and f is the Killing form on $\mathfrak{U}(k, \mathbf{F})$. Define $\mathfrak{P}_B = \mathfrak{K}_B^{\perp}$ (relative to f); then there is a vectorspace direct-sum decomposition $\mathfrak{U}(k, \mathbf{F}) = \mathfrak{K}_B + \mathfrak{P}_B$. This is a Cartan decomposition; we will call it the decomposition of $\mathfrak{U}(k, \mathbf{F})$ at B. The restriction of $-f$ to $\mathfrak{P}_B$ is positive-definite and $\mathbf{K}_B$-invariant. There is a $\mathbf{K}_B$ -equivariant identification of $\mathfrak{P}_I$ with the tangentspace to $\mathbf{G}_{n,k}(\mathbf{F})$ at B, under the differential of the projection $T \to T(B)$; thus $-f$ induces a $\mathbf{U}(k, \mathbf{F})$-invariant Riemannian metric on $\mathbf{G}_{n,k}(\mathbf{F})$. We will always view $\mathbf{G}_{n,k}(\mathbf{F})$ with this Riemannian structure. It is Riemannian symmetric, the symmetry at B being induced by the element of $\mathbf{U}(k, \mathbf{F})$ which is I (= identity) on B and is $-I$ on $B^{\perp}$ (= orthogonal complement of B in $\mathbf{F}^k$).

If $n = 1$, then $\mathbf{G}_{n,k}(\mathbf{F})$ is just a projective space: $\mathbf{G}_{1,t+1}(\mathbf{F}) = \mathbf{P}^t(\mathbf{F})$, where $\mathbf{P}^t(\mathbf{F})$ carries its usual elliptic metric. The Cayley projective plane $\mathbf{P}^2(\mathbf{Cay})$ cannot be realized this way.

Recall that a submanifold of a Riemannian manifold is *totally geodesic* if every geodesic of the submanifold is a geodesic of the ambient manifold, or, equivalently, if the submanifold contains every geodesic of the ambient manifold which is tangent to the submanifold at some point. Let

$$\exp : \mathfrak{U}(k, \mathbf{F}) \to \mathbf{U}(k, \mathbf{F})$$

denote the exponential map. If $\mathfrak{S}$ is a subspace of $\mathfrak{P}_B$, $B \in \mathbf{G}_{n,k}(\mathbf{F})$, then $\exp(\mathfrak{S})(B)$ is a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ if and only if $\mathfrak{S}$ is a Lie triple system, i.e., if and only if the Lie product $[\mathfrak{S}, [\mathfrak{S}, \mathfrak{S}]] \subset \mathfrak{S}$. For example, it follows that the elements of $\mathbf{G}_{n,k}(\mathbf{F})$ lying in a fixed subspace of $\mathbf{F}^k$ form a connected totally geodesic submanifold. In particular, the projective lines of $\mathbf{P}^t(\mathbf{F})$ are totally geodesic submanifolds which are isometric to spheres; the same is true for $\mathbf{P}^2(\mathbf{Cay})$.

If M is a Riemannian manifold, then $\mathbf{I}(M)$ denotes the full group of isometries (self-diffeomorphisms which preserve the Riemannian structure) of M. For example, $\mathbf{I}(\mathbf{P}^2(\mathbf{Cay}))$ is the compact exceptional group $\mathbf{F}_4$. $\mathbf{I}_0(M)$ denotes the identity component of $\mathbf{I}(M)$.

We will assume familiarity with the first two chapters of the preceding paper [5], and with the geometry of the projective spaces $\mathbf{P}^t(\mathbf{F})$ and $\mathbf{P}^2(\mathbf{Cay})$. A short but sufficient exposition of $\mathbf{P}^2(\mathbf{Cay})$ can be found in [2].

3. Geodesic submanifolds of projective spaces

We need to know the dimensions for which there exist totally geodesic spheres in projective spaces (Lemma 2). As it involves little extra effort, we will also derive the classification of totally geodesic submanifolds in a Riemannian symmetric space of rank one (Theorem 1). $\mathbf{M}$ will denote a projective space $\mathbf{P}^t(\mathbf{F})$ or $\mathbf{P}^2(\mathbf{Cay})$.

LEMMA 1. *Let N be a connected submanifold of* $\mathbf{M}$. *Then N is a totally geodesic submanifold of* $\mathbf{M}$ *which is isometric to a sphere if and only if N is a totally geodesic submanifold of a projective line of* $\mathbf{M}$.

Proof. Sufficiency is clear because the projective lines of $\mathbf{M}$ are totally geodesic submanifolds which are isometric to spheres. Now suppose that N is totally geodesic in $\mathbf{M}$ and is isometric to a sphere. Choose $x \in N$, and let x' be the antipodal point of x on N. Given $y \in N - \{x, x'\}$, there is a unique geodesic γ_y on N which contains x and y. Observe that $x' \in \gamma_y$ and that γ_y is contained in the projective line L_y of $\mathbf{M}$ determined by x and y. Let L be the projective line of $\mathbf{M}$ determined by x and x'; it follows that $L_y = L$. Thus $N \subset L$. Now a geodesic of N is a geodesic of $\mathbf{M}$ which is contained in L, and which is thus a geodesic of L. This shows that N is totally geodesic in L, Q.E.D.

LEMMA 2. $\mathbf{M}$ *has a totally geodesic submanifold isometric to an r-sphere if and only if*

(1) $\mathbf{M} = \mathbf{P}^t(\mathbf{R})$ *and* $r \leqq 1$,
(2) $\mathbf{M} = \mathbf{P}^t(\mathbf{C})$ *and* $r \leqq 2$,
(3) $\mathbf{M} = \mathbf{P}^t(\mathbf{K})$ *and* $r \leqq 4$, *or*
(4) $\mathbf{M} = \mathbf{P}^2(\mathbf{Cay})$ *and* $r \leqq 8$.

If N_1 and N_2 are totally geodesic submanifolds of $\mathbf{M}$ *which are isometric to r-spheres, then* $\mathbf{I}_0(\mathbf{M})$ *has an element which maps N_1 onto N_2.*

Proof. The first statement follows from Lemma 1 because a projective line of $\mathbf{M}$ is a sphere of dimension 1, 2, 4, or 8, respectively. The second statement follows in the first three cases from transitivity of $\mathbf{SO}(t+1)$, $\mathbf{SU}(t+1)$ or $\mathbf{Sp}(t+1)$ on 2-dimensional subspaces of $\mathbf{R}^{t+1}$, $\mathbf{C}^{t+1}$ or $\mathbf{K}^{t+1}$, respectively.

Now let $\mathbf{M} = \mathbf{P}^2(\mathbf{Cay})$. Applying an element of $\mathbf{I}_0(\mathbf{M})$ to N_1, we may assume both N_1 and N_2 to lie in the same projective line L of $\mathbf{M}$, for $\mathbf{I}_0(\mathbf{M})$ acts transitively on the projective lines of $\mathbf{M}$. Let x be the pole of L, i.e., the (unique) focal point of the submanifold L. The isotropy subgroup of $\mathbf{F}_4 = \mathbf{I}_0(\mathbf{M})$ at x is isomorphic to $\mathbf{Spin}(9)$, the universal covering group of the identity component $\mathbf{SO}(9)$ of $\mathbf{O}(9) = \mathbf{U}(9, \mathbf{R})$; it preserves L, and its action on L is that of the usual (linear) action of $\mathbf{SO}(9)$ on $\mathbf{S}^8$, so one of its elements carries N_1 onto N_2, Q.E.D.

LEMMA 3. *Let N be a connected totally geodesic submanifold of* $\mathbf{M}$ *which is not isometric to a sphere. Then*

(1) $\mathbf{M} = \mathbf{P}^t(\mathbf{R})$ *and* $N = \mathbf{P}^r(\mathbf{R})$ $(2 \leqq r \leqq t)$, *or*

(2) $\mathbf{M} = \mathbf{P}^t(\mathbf{C})$ *and* $N = \mathbf{P}^r(\mathbf{R}$ *or* $\mathbf{C})$ $(2 \leqq r \leqq t)$, *or*

(3) $\mathbf{M} = \mathbf{P}^t(\mathbf{K})$ *and* $N = \mathbf{P}^t(\mathbf{R}, \mathbf{C}$ *or* $\mathbf{K})$ $(2 \leqq r \leqq t)$, *or*

(4) $\mathbf{M} = \mathbf{P}^2(\mathbf{Cay})$ *and* $N = \mathbf{P}^2(\mathbf{R}, \mathbf{C}, \mathbf{K}$ *or* $\mathbf{Cay})$.

Proof. We first observe that $\mathbf{P}^{t+1}(\mathbf{F})$ cannot be a totally geodesic submanifold of $\mathbf{P}^t(\mathbf{F}')$. For suppose it is. Choose $x \in \mathbf{P}^{t+1}(\mathbf{F})$, let L and L' be the respective polars (focal sets) of x in $\mathbf{P}^{t+1}(\mathbf{F})$ and $\mathbf{P}^t(\mathbf{F}')$, and observe that $L \subset L'$ because $\mathbf{P}^{t+1}(\mathbf{F})$ is totally geodesic in $\mathbf{P}^t(\mathbf{F}')$. L is totally geodesic in $\mathbf{P}^{t+1}(\mathbf{F})$, thus also in $\mathbf{P}^t(\mathbf{F}')$, thus also in L'. Now $L = \mathbf{P}^t(\mathbf{F})$ and $L' = \mathbf{P}^{t-1}(\mathbf{F}')$, so we have reduced t. Iterating this procedure, we obtain $\mathbf{P}^2(\mathbf{F})$ as a totally geodesic submanifold of a sphere $\mathbf{P}^1(\mathbf{F}')$, which is impossible because $\mathbf{P}^2(\mathbf{F})$ is not isometric to a sphere. This proves $r \leqq t$ in (1), (2), and (3); the same argument proves (4) if N is a projective space.

N is a projective space because it is a Riemannian symmetric space of rank one which is not isometric to a sphere; thus Lemma 2 gives the dimensions of the totally geodesic spheres in N. Such a sphere is a totally geodesic sphere in $\mathbf{M}$. Our lemma now follows from Lemma 2, Q.E.D.

LEMMA 4. *The inclusions of Lemma* 3 *all exist.*

Proof. The inclusions of (1), (2), and (3) obviously exist; thus we need only find a totally geodesic submanifold of $\mathbf{P}^2(\mathbf{Cay})$ which is isometric to $\mathbf{P}^2(\mathbf{K})$.

We choose [3, p. 219] a maximal subgroup G of $\mathbf{F}_4 = \mathbf{I}(\mathbf{P}^2(\mathbf{Cay}))$ which is locally isomorphic to $\mathbf{Sp}(3) \times \mathbf{Sp}(1)$, and let H be the subgroup of G for the local factor $\mathbf{Sp}(3)$. G is normalized by a symmetry of $\mathbf{P}^2(\mathbf{Cay})$, and this symmetry normalizes H; this gives $x \in \mathbf{P}^2(\mathbf{Cay})$ such that $G(x)$ and $H(x)$ are totally geodesic submanifolds.

$G(x)$ is not a sphere. For if it were a sphere of dimension > 0, it would be contained in a projective line L by Lemma 1, and G would preserve L. Then G would leave fixed the pole of L, and would be contained in an isotropy subgroup $\mathbf{Spin}(9)$ of $\mathbf{F}_4$, contradicting maximality of G in $\mathbf{F}_4$.

$H(x)$ is not a sphere. For $H(x) = x$ implies that H preserves every element of $G(x)$, and thus preserves every projective line with two points in $G(x)$. As $G(x)$ is not a sphere, H would preserve many projective lines, and would thus act trivially on $\mathbf{P}^2(\mathbf{Cay})$; this is impossible. If $H(x)$ is a sphere of positive dimension, then H preserves the projective line L containing $H(x)$, whence H preserves the pole y of L. $G(y)$ is totally geodesic, so the preceding argument shows $H(y) \neq y$.

$H(x) \neq \mathbf{P}^2(\mathbf{R}, \mathbf{C}, \text{or } \mathbf{Cay})$. For equality would give $\mathbf{P}^2(\mathbf{R}, \mathbf{C}, \text{or } \mathbf{Cay})$

as a coset space H/K of H. Nonvanishing of the Euler characteristic

$$\chi(\mathbf{P}^2(\mathbf{R}, \mathbf{C}, \text{or } \mathbf{Cay})$$

implies [4, p. 15] that rank K = rank $H = 3$. The homotopy sequence

$$\{1\} = \pi_2(H) \to \pi_2(\mathbf{P}^2(-)) \to \pi_1(K) \to \pi_1(H) = \{1\}$$

shows (see [2] for $\mathbf{P}^2(\mathbf{Cay})$) that K has center of dimension 1 for $\mathbf{R}$ or $\mathbf{C}$, and K is semisimple for $\mathbf{Cay}$. Now dim $H = 21$, whence dim K is 19 for $\mathbf{R}$, 17 for $\mathbf{C}$, and 5 for $\mathbf{Cay}$. But there is no semisimple Lie group of rank 3 and dimension 5, nor of rank 2 and dimension 16, nor of rank 2 and dimension 18.

As $H(x)$ is not isometric to a sphere, Lemma 2 shows that it is isometric to $\mathbf{P}^2(\mathbf{R}, \mathbf{C}, \mathbf{K}, \text{or } \mathbf{Cay})$. We have just eliminated all except $\mathbf{P}^2(\mathbf{K})$, Q.E.D.

Let $\mathbf{S}^m$ denote the m-sphere in a Riemannian metric of constant positive curvature.

We have arrived at the goal of §3:

THEOREM 1. *Let M be a connected compact Riemannian symmetric space of rank one, and let N be a connected totally geodesic submanifold of M. Then*

(1) $M = \mathbf{S}^t$ *and* $N = \mathbf{S}^r$ $(1 \leqq r \leqq t)$; *or*

(2) $M = \mathbf{P}^t(\mathbf{R})$, *and either* $N = \mathbf{S}^1 = \mathbf{P}^1(\mathbf{R})$, *or* $N = \mathbf{P}^r(\mathbf{R})$ $(2 \leqq r \leqq t)$; *or*

(3) $M = \mathbf{P}^t(\mathbf{C})$, *and either* $N = \mathbf{S}^r$ $(1 \leqq r \leqq 2)$, *or* $N = \mathbf{P}^r(\mathbf{R}$ *or* $\mathbf{C})$ $(2 \leqq r \leqq t)$; *or*

(4) $M = \mathbf{P}^t(\mathbf{K})$, *and either* $N = \mathbf{S}^r$ $(1 \leqq r \leqq 4)$, *or* $N = \mathbf{P}^r(\mathbf{R}, \mathbf{C},$ *or* $\mathbf{K})$ $(2 \leqq r \leqq t)$; *or*

(5) $M = \mathbf{P}^2(\mathbf{Cay})$, *and either* $N = \mathbf{S}^r$ $(1 \leqq r \leqq 8)$, *or* $N = \mathbf{P}^2(\mathbf{R}, \mathbf{C}, \mathbf{K},$ *or* $\mathbf{Cay})$.

These inclusions all exist; they are unique in the sense that, if two connected totally geodesic submanifolds of M are homeomorphic, then they are equivalent under an element of $\mathbf{I}_0(M)$.

Proof. By Lemmas 2, 3, and 4, we need only prove the uniqueness when N is not a sphere. Now let N_1 and N_2 be connected totally geodesic submanifolds of $M = \mathbf{P}^t(\mathbf{R}, \mathbf{C}, \mathbf{K}, \text{or } \mathbf{Cay})$, $N_i = \mathbf{P}^r(\mathbf{F})$. We may apply an element of $\mathbf{I}_0(M)$ to N_1, and assume that we have an element $x \in N_1 \cap N_2$. Let L, L_1, and L_2 be the respective polars (= focal sets) of x in M, N_1, and N_2. L_i is totally geodesic in N_i, thus in M, and thus in L, and $L_i = \mathbf{P}^{r-1}(\mathbf{F})$ or the L_i are spheres of the same dimension. By Lemma 2 or induction on t, an element of $\mathbf{I}_0(L)$ maps L_1 onto L_2. This element extends to an element of $\mathbf{I}_0(M)$ which maps N_1 onto N_2, Q.E.D.

4. Decomposition by projective lines

Let $\mathbf{B}$ be a connected totally geodesic submanifold of the Grassmann manifold $\mathbf{G}_{n,k}(\mathbf{F})$ of n-dimensional subspaces of $\mathbf{F}^k$, and assume that any two distinct elements of $\mathbf{B}$ have zero intersection as subspaces of $\mathbf{F}^k$. In the earlier paper [5] we saw that $\mathbf{B}$ is a compact Riemannian symmetric space of rank one, and we classified the possibilities where $\mathbf{B}$ is a sphere. Now suppose that $\mathbf{B}$ is not a sphere; thus $\mathbf{B}$ is a projective space $\mathbf{P}^t(\mathbf{R}, \mathbf{C}, \text{or } \mathbf{K})$ or $\mathbf{P}^2(\mathbf{Cay})$.

LEMMA 5. *Choose $B \in \mathbf{B}$, and let $\mathfrak{S}$ be the tangentspace to $\mathbf{B}$ at B. Then there is an orthogonal direct-sum decomposition*

$$\mathfrak{S} = \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_t$$

where $\mathfrak{S}_i$ is the tangentspace at B to a projective line L_i of $\mathbf{B}$ through B.

Remark. Counting dimensions, it is clear that $t = 2$ if $\mathbf{B} = \mathbf{P}^2(\mathbf{Cay})$, and $t = r$ if $\mathbf{B} = \mathbf{P}^r(\mathbf{R}, \mathbf{C}, \text{or } \mathbf{K})$.

Proof. If $\mathbf{B} = \mathbf{P}^r(\mathbf{F})$, view it as the set of one-dimensional subspaces of $\mathbf{F}^{r+1}$; we choose an orthonormal basis $\{x_0, \cdots, x_r\}$ of $\mathbf{F}^{r+1}$ such that x_0 spans B over $\mathbf{F}$, and we define L_i to be the set of $\mathbf{F}$-lines in $\mathbf{F}^{r+1}$ which lie in the space with $\mathbf{F}$-basis $\{x_0, x_i\}$. If $\mathbf{B} = \mathbf{P}^2(\mathbf{Cay})$, we choose a projective line L_1 through B, we define B' to be the antipodal of B on the 8-sphere L_1, and we define L_2 to be the polar of B' in $\mathbf{B}$; $B = L_1 \cap L_2$ because B is focal to B' and $L_1 \neq L_2$. In either case, the decomposition of $\mathfrak{S}$ is easily seen to be orthogonal, Q.E.D.

We will now see the relation between the transvections of $\mathbf{G}_{n,k}(\mathbf{F})$ and the decomposition of Lemma 5.

We have the orthogonal direct-sum decomposition $\mathfrak{U}(k, \mathbf{F}) = \mathfrak{K}_B + \mathfrak{P}_B$ of $\mathfrak{U}(k, \mathbf{F})$ at B, under the Killing form of $\mathfrak{U}(k, \mathbf{F})$, where $\mathbf{K}_B$ is the isotropy subgroup of $\mathbf{U}(k, \mathbf{F})$ at B. The tangentspace $\mathfrak{S}$ to $\mathbf{B}$ at B is identified as a subspace of $\mathfrak{P}_B$. Let $\{\alpha_q\}$ be a standard basis of $\mathbf{F}$ over $\mathbf{R}$:

$$\alpha_1 = 1 = -\alpha_i^2 \quad \text{and} \quad \alpha_i \alpha_j = -\alpha_j \alpha_i \in \{\alpha_q\} \quad \text{for} \quad 1 < i < j.$$

If $\mathbf{x} = \{x_1, \cdots, x_k\}$ is an orthonormal basis of $\mathbf{F}^k$ whose first n elements span B, then recall [5, Chapter II] that $\mathfrak{P}_B$ has basis consisting of the linear transformations of $\mathbf{F}^k$ with matrix $\alpha_q(E_{i,j} - \alpha_q^2 E_{j,i})$ $(1 \leqq i \leqq n < j \leqq k)$ relative to $\mathbf{x}$; given $X \in \mathfrak{S}$, $\mathbf{x}$ can be chosen such that the matrix of X is a real multiple of $\sum_{i=1}^{n} (E_{i,i+n} - E_{i+n,i})$.

L_i of Lemma 5 is an isoclinic sphere on a $2n$-dimensional subspace V_i of $\mathbf{F}^k$ [5, Theorem 3], and it is clear that $V_i = B \oplus B_i$ is an orthogonal direct-sum decomposition where B_i is the antipodal of B on L_i. Let V be the subspace $\sum_{A \in \mathbf{B}} A$ of $\mathbf{F}^k$. Then we have

LEMMA 6. *$V = B \oplus B_1 \oplus \cdots \oplus B_t$ is an orthogonal direct-sum decomposition.*

Proof. Let $\mathbf{x}$ be an orthonormal basis of $\mathbf{F}^k$ whose first n elements span

B, whose next n elements span B_1, and in which an element $X \in \mathfrak{S}_1$ has matrix $\sum_{i=1}^{n} (E_{i,i+n} - E_{i+n,i})$. Let $\mathfrak{P}_B'$ be the subspace of $\mathfrak{P}_B$ spanned by the transformations of matrix $\alpha_q(E_{i,n+j} - \alpha_q^2 E_{n+j,i})$ $(j \leqq n)$ relative to $\mathbf{x}$ (so $\mathfrak{S}_1 \subset \mathfrak{P}_B'$), let $\mathfrak{P}_B''$ be the subspace for $j > n$, and let $T : \mathfrak{P}_B \to \mathfrak{P}_B$ be the transformation $Y \to [X, [X, Y]]$. Then T is symmetric because X is skew, T preserves $\mathfrak{S}$, $\mathfrak{S}_1$, and $\mathfrak{P}_B'$ because they are Lie triple systems, and a short calculation shows that T induces multiplication by -1 on $\mathfrak{P}_B''$. An application of [5, Theorem 1] to L_1 shows (by the argument [5, §6] that an isoclinic sphere is totally geodesic) that $\mathfrak{S}_1$ is an orthogonal direct sum $\{X\} + \mathfrak{S}_1'$ and T induces multiplication by -4 on $\mathfrak{S}_1'$. As T is symmetric, there is an orthogonal direct-sum decomposition

$$\mathfrak{P}_B = \mathfrak{P}_{\beta_1} \oplus \cdots \oplus \mathfrak{P}_{\beta_c}$$

where $\mathfrak{P}_{\beta_i}$ is the eigenspace of some real β_i for T. As T preserves $\mathfrak{S}$ and $\mathfrak{S}_i$, it preserves $\mathfrak{S}' = \mathfrak{S}_2 \oplus \cdots \oplus \mathfrak{S}_t$. Thus $\mathfrak{S}' = \mathfrak{S}_1' \oplus \cdots \oplus \mathfrak{S}_c'$ where $\mathfrak{S}_i' = \mathfrak{S}' \cap \mathfrak{P}_{\beta_i}$. Let $Y \in \mathfrak{S}_i'$, so $T(Y) = \beta_i Y$. $Y = Y_1 + Y_2$ with $Y_1 \in \mathfrak{P}_B'$ and $Y_2 \in \mathfrak{P}_B''$. $T(Y_2) = -Y_2$, and $T(Y_1) = -4Y_1$ by [5, Theorem 1 and §6] because it is readily verified that every $\exp(\alpha Y_1)(B)$ is isoclinic to every element of L_1. Thus $Y \in \mathfrak{P}_B'$ or $Y \in \mathfrak{P}_B''$. It follows from Lemma 1 and [5, Theorem 3] that $Y \in \mathfrak{P}_B''$. This proves that $\mathfrak{S}' \subset \mathfrak{P}_B''$. In other words, we have proved that $B_1 \perp B_i$ for $i > 1$. Now observe that the elements of $\mathfrak{S}_1$ are zero on $V_1^{\perp}$. Similarly, $B_i \perp B_j$ for $i \neq j$, and the elements of $\mathfrak{S}_i$ are zero on $V_i^{\perp}$. The lemma follows, Q.E.D.

Lemma 6 results in a good description of $\mathbf{B}$:

PROPOSITION 1. *Let s be the real dimension of the projective lines of $\mathbf{B}$. Then there is an orthonormal basis $\mathbf{x} = \{x_1, \cdots, x_k\}$ of $\mathbf{F}^k$ such that $\{x_1, \cdots, x_n\}$ is an orthonormal basis of B and $\{x_{ni+1}, \cdots, x_{ni+n}\}$ is an orthonormal basis of B_i $(1 \leqq i \leqq t)$, there is a basis $\{X_{i,1}, \cdots, X_{i,s}\}$ of $\mathfrak{S}_i$ $(1 \leqq i \leqq t)$, and there are $n \times n$ $\mathbf{F}$-unitary matrices A_j $(1 \leqq j < s)$ with $A_i A_j + A_j A_i = -2\delta_{ij} I$, such that $X_{i,j}$ $(j < s)$ has matrix*

$$\underbrace{}_{n(i+1)}\underbrace{}_{k-n(i+1)}$$

$$\left.\begin{pmatrix} 0 & \cdots & 0 & A_j & 0 & \cdots & 0 \\ \vdots & & \vdots & 0 & & & \vdots \\ 0 & \cdots & 0 & \vdots & & & \vdots \\ A_j & 0 & \cdots & 0 & & \cdots & 0 \\ 0 & & & \vdots & & & \vdots \\ \vdots & & & \vdots & & & \vdots \\ 0 & & \cdots & 0 & & \cdots & 0 \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\} n(i+1) \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} k-n(i+1) \end{matrix}\right.$$

and $X_{i,s}$ has matrix

$$\begin{array}{c} \left.\begin{pmatrix} 0 & \cdots & 0 & I & 0 & \cdots & 0 \\ \vdots & & \vdots & 0 & & & \cdot \\ 0 & \cdots & 0 & \vdots & & & \vdots \\ -I & 0 & \cdots & 0 & & \cdots & 0 \\ 0 & & & \cdot & & & \cdot \\ \vdots & & & \vdots & & & \vdots \\ 0 & & \cdots & 0 & & \cdots & 0 \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\} n(i+1) \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} k-n(i+1) \end{matrix}\right. \\ \underbrace{\hphantom{0000000}}_{n(i+1)}\ \underbrace{\hphantom{0000000}}_{k-n(i+1)} \end{array}$$

relative to $\mathbf{x}$.

Proof. Let $\mathbf{x}_0$ be an orthonormal basis of B. We choose $X_{i,s} \in \mathfrak{S}_i$ and an orthonormal basis $\mathbf{x}_i$ of B_i such that the restriction $X_{i,s}|_{V_i}$ has matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ relative to the orthonormal basis $\{\mathbf{x}_0, \mathbf{x}_i\}$ of V_i. By [5, Theorems 1 and 3 and Remark 1], there are $\mathbf{F}$-unitary $n \times n$ matrices A_j $(1 \leqq j < t)$ such that $A_i A_j + A_j A_i = -2\delta_{ij} I$, and there is a basis $\{X_{1,1}, \cdots, X_{1,s}\}$ of $\mathfrak{S}_1$ such that $X_{1,j}|_{V_1}$ has matrix $\begin{pmatrix} 0 & A_j \\ A_j & 0 \end{pmatrix}$ in the basis $\{\mathbf{x}_0, \mathbf{x}_1\}$ of V_1. Let $Y_i = [X_{1,s}, X_{i,s}]$ $(1 < i \leqq t)$; the restriction of Y_i to $V_1 + V_i = B \oplus B_1 \oplus B_i$ has matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & 0 \end{pmatrix}$$

with respect to the orthonormal basis $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_i\}$ of $B \oplus B_1 \oplus B_i$. The transformation $Z \to [Y_i, Z]$ preserves $\mathfrak{S}$, for $\mathfrak{S}$ is a Lie triple system because $\mathbf{B}$ is totally geodesic; it sends $\mathfrak{S}_1$ onto $\mathfrak{S}_i$, $\mathfrak{S}_i$ onto $\mathfrak{S}_1$, and annihilates the other summands of $\mathfrak{S}$. It sends $X_{1,s}$ onto $X_{i,s}$, and thus sends $\{X_{1,1}, \cdots, X_{1,s}\}$ onto a basis $\{X_{i,1}, \cdots, X_{i,s}\}$ of $\mathfrak{S}_i$; $X_{i,j}|_{V_i}$ has matrix $\begin{pmatrix} 0 & A_j \\ A_j & 0 \end{pmatrix}$ in the basis $\{\mathbf{x}_0, \mathbf{x}_i\}$ of V_i. We complete the basis $\{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_t\}$ of $B \oplus B_1 \oplus \cdots \oplus B_t$ to an orthonormal basis $\mathbf{x}$ of $\mathbf{F}^k$, and the proposition follows, Q.E.D.

5. Elimination of the Cayley plane and the structure theorem for projective spaces

Retain the notation of Proposition 1, and suppose $s \geqq 4$. Let Y_{ij} $(1 \leqq i \leqq 2, 1 \leqq j \leqq s)$ be the restriction of $X_{i,j}$ to $W = B \oplus B_1 \oplus B_2$, and let $\mathbf{w} = \{w_1, \cdots, w_{3n}\}$ be the part of $\mathbf{x}$ which spans W. A short calculation shows that $Z = [[Y_{1,1}, Y_{2,2}], Y_{2,3}]$ has matrix

$$\begin{pmatrix} 0 & -A_1 A_2 A_3 & 0 \\ A_1 A_2 A_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the basis $\mathbf{w}$ of W. On the other hand, Z is a (real)-linear combination of the $Y_{1,j}$. A glance at Proposition 1 shows that $A_1 A_2 A_3 = \pm I$. If $s > 4$, then the same argument shows that $A_1 A_2 A_4 = \pm I$, whence $A_3 = \pm A_4$; this is impossible because A_3 and A_4 anticommute. We have proved

LEMMA 7. *In Proposition* 1, *either* $s = 1$, $s = 2$, *or* $s = 4$; *if* $s = 4$, *then* $A_1 A_2 A_3 = \pm I$.

As an immediate consequence, $\mathbf{B}$ cannot be the Cayley projective plane, for $s \neq 8$. But the other possibilities for $\mathbf{B}$ exist, subject to Proposition 1 and Lemma 7:

THEOREM 2. *Let* $\mathbf{F}$ *be a real division algebra, and let* s, t, n, *and* k *be positive integers such that* $t \geqq 2$, $k \geqq n(t+1)$, *and* $s = 1, 2$, *or* 4. *Let* $A_1, \cdots, A_{s-1}$ *be* $n \times n$ $\mathbf{F}$-*unitary matrices such that* $A_i A_j + A_j A_i = -2\delta_{ij} I$, *and suppose that* $A_1 A_2 A_3 = I$ *in case* $s = 4$. *Let* $\mathbf{x}$ *be an orthonormal basis of* $\mathbf{F}^k$, *let* $X_{i,j}$ $(1 \leqq i \leqq t, 1 \leqq j \leqq s)$ *be the linear transformation of* $\mathbf{F}^k$ *with matrix relative to* $\mathbf{x}$ *as given in Proposition* 1, *let* $\mathfrak{S}$ *be the real subspace of* $\mathfrak{U}(k, \mathbf{F})$ *spanned by the* $X_{i,j}$, *and define* $\mathfrak{T} = [\mathfrak{S}, \mathfrak{S}]$. *Then* $\mathfrak{S}$ *is a Lie triple system, so* $\mathfrak{G} = \mathfrak{T} + \mathfrak{S}$ *is a subalgebra of* $\mathfrak{U}(k, \mathbf{F})$. *Let* $\mathbf{G}$ *be the analytic subgroup of* $\mathbf{U}(k, \mathbf{F})$ *with Lie algebra* $\mathfrak{G}$, *and let* B *be the subspace of* $\mathbf{F}^k$ *spanned by the first* n *elements of* $\mathbf{x}$. *Then* $\mathbf{G}(B)$ *is a connected totally geodesic submanifold of the Grassmann manifold* $\mathbf{G}_{n,k}(\mathbf{F})$, *and any two distinct elements of* $\mathbf{G}(B)$ *have zero intersection as subspaces of* $\mathbf{F}^k$; $\mathbf{G}(B)$ *is isometric to a real* (*if* $s = 1$), *complex* (*if* $s = 2$) *or quaternionic* (*if* $s = 4$) *projective space of dimension* t (*topological dimension* st). *Conversely, if* $\mathbf{B}$ *is a connected totally geodesic submanifold of a Grassmann manifold* $\mathbf{G}_{n,k}(\mathbf{F})$, *if any two distinct elements of* $\mathbf{B}$ *have zero intersection as subspaces of* $\mathbf{F}^k$, *and if* $\mathbf{B}$ *is not isometric to a sphere, then* $k \geqq 3n$, *and* $\mathbf{B}$ *is one of the manifolds* $\mathbf{G}(B)$ *described above.*

Proof. Let $\mathfrak{S}_i$ be the subspace of $\mathfrak{S}$ with basis $\{X_{i,1}, \cdots, X_{i,s}\}$; $\mathfrak{S} = \sum \mathfrak{S}_i$. $[\mathfrak{S}_i, [\mathfrak{S}_i, \mathfrak{S}_i]] \subset \mathfrak{S}_i$ was observed in the proof of [5, Theorem 2], and it is obvious that $[\mathfrak{S}_i, [\mathfrak{S}_p, \mathfrak{S}_q]] = 0$ if i, p, and q are all different. A straightforward calculation shows $[\mathfrak{S}_i, [\mathfrak{S}_i, \mathfrak{S}_j]] \subset \mathfrak{S}_j$. By the Jacobi identity, it follows that $[\mathfrak{S}, [\mathfrak{S}, \mathfrak{S}]] \subset \mathfrak{S}$, i.e., $\mathfrak{S}$ is a Lie triple system.

Looking at matrices, we see that $\mathfrak{S} \subset \mathfrak{P}_B$ where $\mathfrak{U}(k, \mathbf{F}) = \mathfrak{K}_B + \mathfrak{P}_B$ is the decomposition at B; it follows that $\mathbf{G}(B)$ is totally geodesic in $\mathbf{G}_{n,k}(\mathbf{F})$. Let $B' \in \mathbf{G}(B)$, $B' \neq B$; we must show that $B \cap B' = 0$ as subspaces of $\mathbf{F}^k$. $\mathbf{G}(B) = \exp(\mathfrak{S})(B)$; thus $B' = \exp(X)(B)$ for some $X \in \mathfrak{S}$.

$$X = X_1 + \cdots + X_t, \qquad X_i \in \mathfrak{S}_i,$$

and we can conjugate by an element of $\mathbf{K}_B$, changing basis separately in each $\mathfrak{S}_i(B)$, and assume $X_i = \alpha_i X_{i,s}$ for real numbers α_i. Thus we may assume that X has matrix

$$\begin{pmatrix} 0 & \alpha_1 I & \cdots & \alpha_t I & 0 \\ -\alpha_1 I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -\alpha_t I & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

in the basis $\mathbf{x}$. Now it is clear, given $b \in B$, that $\exp(X)(b) \in B$ if and only if $\exp(X)(b) = \pm b$, and in that case $\exp(X)(b_1) = \pm b_1$ for every $b_1 \in B$, because we can change the basis of B without changing the matrix of X. Thus either $B = B'$ or $B \cap B' = 0$. It follows [5, Theorem 4] that $\mathbf{G}(B)$ is a real, complex, or quaternionic projective space, or the Cayley projective plane.

The remainder of the theorem follows from Lemma 7 and Proposition 1, Q.E.D.

As any two distinct elements of the totally geodesic submanifold $\mathbf{G}(B)$ have zero intersection as subspaces of $\mathbf{F}^k$, it follows [5, Remark 4] that any two elements of $\mathbf{G}(B)$ are isoclinic subspaces of $\mathbf{F}^k$. This leads us to

DEFINITION. *A submanifold of the form* $\mathbf{G}(B)$ *in Theorem* 2 *will be called an isoclinic projective space on the subspace of* $\mathbf{F}^k$ *with basis* $\{x_1, \cdots, x_{(t+1)n}\}$.

The main results of the earlier paper [5, Theorems 2 and 4] combined with Theorem 2 yield

THEOREM 2$'$. *Let* $\mathbf{B}$ *be a subset of* $\mathbf{G}_{n,k}(\mathbf{F})$. *Then these are equivalent:*

1. $\mathbf{B}$ *is a connected totally geodesic submanifold of* $\mathbf{G}_{n,k}(\mathbf{F})$, *and any two distinct elements of* $\mathbf{B}$ *have zero intersection as subspaces of* $\mathbf{F}^k$.

2. $\mathbf{B}$ *is an isoclinic sphere on a* $2n$*-dimensional subspace of* $\mathbf{F}^k$; *or* $\mathbf{B}$ *is a* t*-dimensional* ($t \geqq 2$) *real, complex, or quaternionic, isoclinic projective space on a* $(t+1)n$*-dimensional subspace of* $\mathbf{F}^k$.

6. The classification of isoclinic projective spaces

Consider the problem of existence and equivalence of the sets $\mathfrak{A}' = \{A_1, \cdots, A_{s-1}\}$ of Theorem 2. $\mathfrak{A}'$ is a subset of the $\mathbf{F}$-algebra $\mathfrak{M}_n(\mathbf{F})$ of all $n \times n$ matrices over $\mathbf{F}$; let $\mathfrak{M}_n(\mathbf{F})_R$ denote $\mathfrak{M}_n(\mathbf{F})$ viewed as an algebra over $\mathbf{R}$, and let $\mathfrak{A}$ denote the subalgebra of $\mathfrak{M}_n(\mathbf{F})_R$ generated by I and $\mathfrak{A}'$. It is clear that $\mathfrak{A}$ is isomorphic to $\mathbf{R}$ (if $s = 1$), to $\mathbf{C}$ (if $s = 2$), or to $\mathbf{K}$ (if $s = 4$; this depends on the fact that $A_1 A_2 A_3 = I$).

Now let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two such algebras, for the same $\mathbf{F}$, n, and s. Except for the case $s = 2$ and $\mathbf{F} = \mathbf{C}$, it is well known [1, Theorems 4.5 and 4.14] that $\mathfrak{M}_n(\mathbf{F})$ has a nonsingular element T such that $T\mathfrak{A}_2 T^{-1} = \mathfrak{A}_1$. By using the fact that $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$ generate isomorphic finite subgroups of $\mathbf{U}(n, \mathbf{F})$, it is not difficult to see that T may be chosen in $\mathbf{U}(n, \mathbf{F})$ and with the property that $T\mathfrak{A}_2' T^{-1} = \mathfrak{A}_1'$. If we view T as a change of orthonormal basis in the span of each $\{x_{in+1}, \cdots, x_{in+n}\}$, $0 \leqq i \leqq t$, then we have proved

LEMMA 8. *Except for the case* $s = 2$ *and* $\mathbf{F} = \mathbf{C}$, *the manifold* $\mathbf{G}(B)$ *of Theorem* 2 *is determined, up to a transformation of* $\mathbf{U}(k, \mathbf{F})$, *by* s, t, n, k, *and* $\mathbf{F}$. *In any case,* $\mathbf{G}(B)$ *exists* (*i.e., the* A_j *can be constructed*) *if and only if* n *satisfies the condition*:

	$s = 1$	$s = 2$	$s = 4$
$\mathbf{F} = \mathbf{R}$	no condition	$n \equiv 0 \pmod 2$	$n \equiv 0 \pmod 4$
$\mathbf{F} = \mathbf{C}$	no condition	no condition	$n \equiv 0 \pmod 2$
$\mathbf{F} = \mathbf{K}$	no condition	no condition	no condition

Remark. The condition can be expressed: $\dim_R \mathbf{F}^n \equiv 0 \pmod s$.

Now let $s = 2$ and $\mathbf{F} = \mathbf{C}$. $\mathfrak{A} \subset \mathfrak{M}_n(\mathbf{C})_R$ is completely determined by A_1. As A_1 is unitary with square $-I$, it is unitarily equivalent to a matrix

$$\begin{pmatrix} \sqrt{-1}\, I_u & 0 \\ 0 & -\sqrt{-1}\, I_v \end{pmatrix}, \qquad u + v = n.$$

The *nonordered* pair $\{u, v\}$ is an invariant of the unitary equivalence class of $\mathfrak{A}$, and completely determines that class. Together with k and t, $\{u, v\}$ determines $\mathbf{G}(B)$ up to a transformation of $\mathbf{U}(k, \mathbf{C})$. On the other hand, in the terminology of [5, §12], it is easily seen that each projective line of $\mathbf{G}(B)$ is an isoclinic 2-sphere of index $\{2u, 2v\}$ on a $2n$-dimensional subspace of $\mathbf{C}^k$. This index is invariant under every isometry of $\mathbf{G}_{n,k}(\mathbf{C})$ [5, Lemma 6], and is thus an invariant of $\mathbf{G}(B)$ in $\mathbf{G}_{n,k}(\mathbf{C})$.

DEFINITION. *The index $\nu(\mathbf{G}(B))$ is the nonordered pair $\{u, v\}$ in the discussion above.*

With Lemma 8, the above discussion yields

THEOREM 3. *Consider the Grassmann manifold $\mathbf{G}_{n,k}(\mathbf{F})$ where $\mathbf{F}$ is a real division algebra, and let $\mathbf{F}_s$ denote $\mathbf{R}$ if $s = 1$, $\mathbf{C}$ if $s = 2$, or $\mathbf{K}$ if $s = 4$. Then $\mathbf{G}_{n,k}(\mathbf{F})$ contains an isoclinic projective space $\mathbf{P}^t(\mathbf{F}_s)$ $(t \geqq 2)$ if and only if both $(t + 1)n \leqq k$ and $\dim_R \mathbf{F}^n \equiv 0 \pmod s$. Except for the case $\mathbf{F}_s = \mathbf{C} = \mathbf{F}$, any two isoclinic projective spaces $\mathbf{P}^t(\mathbf{F}_s)$ in $\mathbf{G}_{n,k}(\mathbf{F})$ are equivalent under an isometry of $\mathbf{G}_{n,k}(\mathbf{F})$. Two isoclinic projective spaces $\mathbf{P}^t(\mathbf{C})$ in $\mathbf{G}_{n,k}(\mathbf{C})$ are equivalent under an isometry of $\mathbf{G}_{n,k}(\mathbf{C})$ if and only if they have the same index; in this case there are $[n/2] + 1$ equivalence classes, the indices being $\{0, n\}$, $\{1, n - 1\}, \cdots, \{[n/2], n - [n/2]\}$, where $[\ ]$ denotes integral part.*

Theorem 3 classifies the isoclinic projective spaces. Together with Theorem 2 and [5, Theorems 4 and 8], it gives a complete description of the connected totally geodesic submanifolds of Grassmann manifolds $\mathbf{G}_{n,k}(\mathbf{F})$, for which any two distinct elements of the submanifold have zero intersection as subspaces of $\mathbf{F}^k$.

CHAPTER II. ISOCLINIC SUBSPACES OF ARBITRARY FIXED DIMENSION

We will see that every set of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$ can be enlarged to a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ in which any two distinct elements have zero intersection as subspaces of $\mathbf{F}^k$.

7. The closure operation for isoclinic sets

If U is a subspace of $\mathbf{F}^k$, then $\pi_U : \mathbf{F}^k \to U$ will denote the orthogonal projection. Recall that subspaces U and W of $\mathbf{F}^k$ are called *isoclinic* if the restrictions $\pi_U|_W : W \to U$ and $\pi_W|_U : U \to W$ are proportional to unitary transformations. We will consider only the case $\dim U = \dim W$, where the assumption that one of the restrictions be proportional to a unitary transformation automatically forces the same condition on the other restriction.

Let $\mathbf{B}$ be a set of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$. Define $\mathbf{B}_{(0)} = \mathbf{B}$, and suppose that we have constructed the sequence

$$\mathbf{B}_{(0)} \subset \mathbf{B}_{(1)} \subset \cdots \subset \mathbf{B}_{(i)}$$

of sets of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$. Given distinct nonorthogonal elements B and B' of $\mathbf{B}_{(i)}$, let $\mathbf{S}_{i,B,B'}$ be the isoclinic sphere on $B \oplus B'$ constructed as in [5, Chapter I] from the set of all elements of $\mathbf{B}_{(i)}$ which lie in $B \oplus B'$. The elements of $\mathbf{B}_{(i)}$ are pairwise isoclinic, as are the elements of $\mathbf{S}_{i,B,B'}$. Now let $X \in \mathbf{B}_{(i)}$ and $Y \in \mathbf{S}_{i,B,B'}$. $\pi_B|_X$ and $\pi_{B'}|_X$ are proportional to unitary maps; it follows that either $Z = \pi_{B \oplus B'}(X) = 0$, or that $\dim Z = n$ and X is isoclinic to a subspace of $B \oplus B'$ if and only if Z is isoclinic to that subspace. Suppose $\dim Z = n$. Now Z is isoclinic to every element of $\mathbf{B}_{(i)}$ lying in $B \oplus B'$; it follows from [5, Theorem 1] that Z is isoclinic to every element of $\mathbf{S}_{i,B,B'}$. Thus X and Y are isoclinic. We have just proved that the elements of $\mathbf{B}_{(i)} \cup \mathbf{S}_{i,B,B'}$ are pairwise isoclinic. Define

$$\mathbf{B}_{(i+1)} = \mathbf{B}_{(i)} \cup \bigcup_{\{B,B'\}} \mathbf{S}_{i,B,B'}$$

where $\{B, B'\}$ runs over all pairs of distinct nonorthogonal elements of $\mathbf{B}_{(i)}$. If $\{B, B'\}$ and $\{A, A'\}$ are two such pairs, then substitution of $\mathbf{B}_{(i)} \cup \mathbf{S}_{i,A,A'}$ for $\mathbf{B}_{(i)}$ in the above argument shows that the elements of $\mathbf{B}_{(i+1)}$ are pairwise isoclinic. Thus we have constructed a sequence

$$\mathbf{B}_{(0)} \subset \mathbf{B}_{(1)} \subset \cdots \subset \mathbf{B}_{(i)} \subset \mathbf{B}_{(i+1)}$$

of sets of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$.

DEFINITION. *The isoclinic closure* $\mathbf{B}_*$ *of* $\mathbf{B}$ *is defined by* $\mathbf{B}_* = \bigcup_{i=0}^{\infty} \mathbf{B}_{(i)}$. $\mathbf{B}$ *is said to be isoclinically closed if* $\mathbf{B} = \mathbf{B}_*$.

This definition is justified by

LEMMA 9. *Let* $\mathbf{B}$ *be a set of pairwise isoclinic n-dimensional subspaces of* $\mathbf{F}^k$, *and let* $\mathbf{B}_*$ *be its isoclinic closure. Then* $\mathbf{B}_*$ *is an isoclinically closed set of pairwise isoclinic n-dimensional subspaces of* $\mathbf{F}^k$.

Proof. Choose B and B' in $\mathbf{B}_*$. They lie in some $\mathbf{B}_{(i)}$, and are thus isoclinic. This proves that the elements of $\mathbf{B}_*$ are pairwise isoclinic.

Let $\mathbf{B}_* = (\mathbf{B}_*)_{(0)}$. We must prove that $(\mathbf{B}_*)_{(1)} = (\mathbf{B}_*)_{(0)}$. It will follow that $\mathbf{B}_* = (\mathbf{B}_*)_*$, proving $\mathbf{B}_*$ to be isoclinically closed. Let B and

B' be distinct nonorthogonal elements of $\mathbf{B}_*$, let $\mathbf{A}$ be the collection of all elements of $\mathbf{B}_*$ which lie in $B \oplus B'$, and let $\mathbf{S}$ be the isoclinic sphere on $B \oplus B'$ constructed from $\mathbf{A}$ as in [5, Theorem 1]. We must prove that $\mathbf{S} \subset \mathbf{B}_*$; it will follow that $(\mathbf{B}_*)_{(0)} = (\mathbf{B}_*)_{(1)}$. As $\mathbf{A} \subset \mathbf{B}_*$, it suffices to prove $\mathbf{S} = \mathbf{A}$. For this, we need only prove that $\mathbf{A}$ is an isoclinic sphere on $B \oplus B'$.

Let $A, A' \in \mathbf{A}$. For some integer m, $\mathbf{B}_{(m)}$ contains B, A, and A'. Thus $\mathbf{B}_{(m+1)}$ contains an isoclinic sphere on $B \oplus B'$ which contains A and A'. It follows that $\mathbf{A}$ is an isoclinic sphere on $B \oplus B'$, Q.E.D.

8. The notion of reducibility for isoclinic sets

Let $\mathbf{B}$ be a set of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$. Given $B, B' \in \mathbf{B}$, we say $B \sim B'$ if there is a sequence $\{B = B_1, B_2, \cdots, B_m = B'\}$ in $\mathbf{B}$ such that B_{i+1} is *not* orthogonal to B_i. This is easily seen to be an equivalence relation on $\mathbf{B}$.

DEFINITION. *The equivalence classes in* $\mathbf{B}$ *will be called the irreducible components of* $\mathbf{B}$. $\mathbf{B}$ *will be called irreducible if it has just one equivalence class. Given* $B \in \mathbf{B}$, *the equivalence class of* B *will be called the irreducible component of* B *in* $\mathbf{B}$.

DEFINITION. *The support* $\operatorname{supp} \mathbf{B}$ *of* $\mathbf{B}$ *is the subspace of* $\mathbf{F}^k$ *spanned by the union of the elements of* $\mathbf{B}$.

Suppose $B \in \mathbf{B} \subset \mathbf{B}'$, where $\mathbf{B}'$ is a set of pairwise isoclinic subspaces of $\mathbf{F}^k$. If $\mathbf{A}$ and $\mathbf{A}'$ are the respective irreducible components of B in $\mathbf{B}$ and $\mathbf{B}'$, then it is clear that $\mathbf{A} \subset \mathbf{A}'$ and thus $\operatorname{supp} \mathbf{A} \subset \operatorname{supp} \mathbf{A}'$.

Our definitions are justified by

LEMMA 10. *Let* $\mathbf{B}$ *be a set of pairwise isoclinic n-dimensional subspaces of* $\mathbf{F}^k$, *and let* $\mathbf{B}_*$ *be its isoclinic closure. Then* $\mathbf{B}$ *and* $\mathbf{B}_*$ *have finite and consistent decompositions*

$$\begin{array}{ccccccccc} \mathbf{B}_* & = & \mathbf{B}_*^1 & \cup & \mathbf{B}_*^2 & \cup & \cdots & \cup & \mathbf{B}_*^m \\ \cup & & \cup & & \cup & & & & \cup \\ \mathbf{B} & = & \mathbf{B}^1 & \cup & \mathbf{B}^2 & \cup & \cdots & \cup & \mathbf{B}^m \end{array}$$

into irreducible components, and $\mathbf{B}_*^i$ *is the isoclinic closure of* $\mathbf{B}^i$. *If* $i \neq j$, *then* $\operatorname{supp} \mathbf{B}^i = \operatorname{supp} \mathbf{B}_*^i \perp \operatorname{supp} \mathbf{B}_*^j = \operatorname{supp} \mathbf{B}^j$. *If we topologize* $\mathbf{B}_*$ *as a subset of the Grassmann manifold* $\mathbf{G}_{n,k}(\mathbf{F})$, *then its connected components are precisely its irreducible components.*

Proof. If two elements of $\mathbf{B}$ are not orthogonal, then they lie in the same irreducible component of $\mathbf{B}$; it follows that distinct irreducible components of $\mathbf{B}$ have supports orthogonal to each other. By finite-dimensionality of $\mathbf{F}^k$, $\mathbf{B}$ has only a finite number of irreducible components. Let

$$\mathbf{B} = \mathbf{B}^1 \cup \mathbf{B}^2 \cup \cdots \cup \mathbf{B}^m$$

be the decomposition of $\mathbf{B}$ into its irreducible components.

It is clear from §7 that $\mathbf{B}_* = (\mathbf{B}^1)_* \cup (\mathbf{B}^2)_* \cup \cdots \cup (\mathbf{B}^m)_*$, that each $\operatorname{supp} \mathbf{B}^i = \operatorname{supp} (\mathbf{B}^i)_*$, and that each $(\mathbf{B}^i)_*$ is irreducible. Setting $\mathbf{B}^i_* = (\mathbf{B}^i)_*$, the consistent decomposition follows easily, as does orthogonality of supports. The orthogonality of supports shows that, in the topology on $\mathbf{B}_*$ induced by $\mathbf{G}_{n,k}(\mathbf{F})$, each $\mathbf{B}^i_*$ is a closed subset of $\mathbf{B}_*$. Thus we need only prove that each $\mathbf{B}^i_*$ is a connected subset of $\mathbf{G}_{n,k}(\mathbf{F})$.

Let $B, B' \in \mathbf{B}^i_*$. As $\mathbf{B}^i_*$ is irreducible, we have a sequence

$$\{B = B_1, B_2, \cdots, B_t = B'\} \subset \mathbf{B}^i_*$$

such that B_{u+1} is not orthogonal to B_u $(1 \leqq u < t)$. Let

$$\mathbf{S}_u = \{B'' \in \mathbf{B}^i_* : B'' \subset B_u \oplus B_{u+1}\}.$$

$\mathbf{S}_u$ is an isoclinic sphere, thus homeomorphic to a sphere [5, Theorem 2]; it follows that $\mathbf{S}_u$ contains an arc from B_u to B_{u+1}. Joining these arcs, we have proved that $\mathbf{B}^i_*$ is arcwise connected. Thus $\mathbf{B}^i_*$ is connected, Q.E.D.

9. Isoclinic sets as submanifolds of Grassmann manifolds

The main result of Chapter II, a sort of converse to Theorem 2′, is

THEOREM 4. *Let* $\mathbf{B}$ *be an irreducible isoclinically closed set of pairwise isoclinic n-dimensional subspaces of* $\mathbf{F}^k$, *where* $\mathbf{F}$ *is a real division algebra, and view* $\mathbf{B}$ *as a subset of the Grassmann manifold* $\mathbf{G}_{n,k}(\mathbf{F})$. *Then* $\mathbf{B}$ *is a connected totally geodesic submanifold of* $\mathbf{G}_{n,k}(\mathbf{F})$ *in which any two distinct elements have zero intersection as subspaces of* $\mathbf{F}^k$.

In view of Lemma 10, it suffices to prove that $\mathbf{B}$ is a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$.

Proof. Choose $B \in \mathbf{B}$, let $\mathfrak{U}(k, \mathbf{F}) = \mathfrak{K}_B + \mathfrak{P}_B$ be the decomposition of $\mathfrak{U}(k, \mathbf{F})$ at B, and define open neighborhoods

$$\mathbf{V} = \{B' \in \mathbf{G}_{n,k}(\mathbf{F}) : B' \not\subset B^{\perp}\},$$

$$\mathbf{U} = \{B' \in \mathbf{B} : B' \not\subset B^{\perp}\} = \mathbf{V} \cap \mathbf{B}$$

of B in $\mathbf{G}_{n,k}(\mathbf{F})$ and in $\mathbf{B}$. We define

$$\mathfrak{S} = \{X \in \mathfrak{P}_B : \exp(tX)(B) \in \mathbf{U} \quad \text{for} \quad -1 \leqq t \leqq 1\}$$

and observe that $\mathbf{U} = \exp(\mathfrak{S})(B)$. Let $\mathfrak{T}$ be the real subspace of $\mathfrak{P}_B$ spanned by $\mathfrak{S}$. If we can prove that $\mathfrak{S}$ contains a neighborhood of zero in $\mathfrak{T}$, then it will follow that $\mathbf{B}$ is a regularly imbedded submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ and that $\mathfrak{T}$ is the tangentspace to $\mathbf{B}$ at B. When this is done, suppose $B \neq B' \in \mathbf{B}$, B' lying in a normal coordinate neighborhood of B in $\mathbf{V}$. $B' = \exp(X)(B)$ for some $X \in \mathfrak{S}$, and $\{\exp(tX)(B) : t \in \mathbf{R}\}$ is the minimizing geodesic in $\mathbf{G}_{n,k}(\mathbf{F})$ between B and B'. On the other hand, it is an isoclinic 1-sphere on $B \oplus B'$, and is thus contained in $\mathbf{B}$ because $\mathbf{B}$ is isoclinically closed. It follows that the submanifold $\mathbf{B}$ is totally geodesic.

Let X and Y be elements of $\mathfrak{S}$. Given small $t \in \mathbf{R}$, we will prove that $\exp(t(X + Y))(B) \in \mathbf{B}$. This suffices to show that $\mathfrak{S}$ contains a neighborhood of zero in $\mathfrak{T}$, proving the theorem. Define $B_1 = \exp(X)(B)$ and $B_2 = \exp(Y)(B)$. If $\dim(B + B_1 + B_2) \leqq 2n$, then $\mathbf{F}^k$ has a $2n$-dimensional subspace V which contains every $\exp(tX)(B)$ and every $\exp(tY)(B)$. The elements of $\mathbf{B}$ which lie in V form an isoclinic sphere $\mathbf{A}$, for $\mathbf{B}$ was assumed isoclinically closed. As X and Y are tangent to $\mathbf{A}$ at B, and as $\mathbf{A}$ is a totally geodesic submanifold of $\mathbf{G}_{n,k}(\mathbf{F})$ [5, Theorem 2], it follows that $\exp(t(X + Y))(B) \in \mathbf{A} \subset \mathbf{B}$ for every real t. Thus we may assume that $\dim(B + B_1 + B_2) > 2n$.

Let $W = B + B_1 + B_2$. B_2 has no nonzero element in common with any $\exp(tX)(B)$; it follows that $\dim W = 3n$. We may choose an orthonormal basis $\mathbf{w} = \{w_1, \cdots, w_{3n}\}$ of W, whose first n elements span B, such that the restriction $X|_W$ has matrix $\alpha\sum_{i=1}^{n}(E_{i,i+n} - E_{i+n,i})$ where $0 < \alpha < 1$. Let $B' = \exp(\alpha^{-1}X)(B)$; define $B'' = \exp(\beta^{-1}Y)(B)$ similarly.

If $B' \perp B''$, then we may assume that $\mathbf{w}$ was chosen such that $Y|_W$ has matrix

$$\beta\sum_{i=1}^{n}(E_{i,i+2n} - E_{i+2n,i}).$$

A short calculation shows that $\exp(t(X + Y))(B)$ lies in the isoclinic 1-sphere determined by $\exp(\sqrt{2}\,tX)(B)$ and $\exp(\sqrt{2}\,tY)(B)$, for small t, and is thus contained in $\mathbf{B}$. If B' is not orthogonal to B'', we examine the isoclinic 1-sphere determined by $\exp(\delta X)(B)$ and $\exp(\delta Y)(B)$ (δ small). It has an element B_3 such that $W = B + B_1 + B_3$ and $B_3 = \exp(Z)(B)$ where $Z \in \mathfrak{S}$ and the $B''' = \exp(\gamma^{-1}Z)(B)$ (defined in the same way as B') is orthogonal to B'. $X + Y = \sigma X + \tau Z$ for some real numbers σ and τ; from the case $B' \perp B''$, it follows that $\exp(t(X + Y))(B) \in \mathbf{B}$ for small t, Q.E.D.

10. Summary

We summarize the results of this paper and the earlier one [5].

If $\mathbf{B}$ is a connected totally geodesic submanifold of the Grassmann manifold $\mathbf{G}_{n,k}(\mathbf{F})$, and if any two distinct elements of $\mathbf{B}$ have zero intersection as subspaces of $\mathbf{F}^k$, then

1. $\mathbf{B}$ is isometric to a sphere (these manifolds are described in [5, Theorem 1] and classified in [5, Theorem 8]) or to a real, complex, or quaternionic projective space (these manifolds are described in Theorem 2 and classified in Theorem 3).
2. $\mathbf{B}$ is an irreducible isoclinically closed set of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$.

If $\mathbf{A}$ is a set of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$, then there is an orthogonal direct-sum decomposition $\mathbf{F}^k = V_1 \oplus \cdots \oplus V_m$, and there are connected totally geodesic submanifolds $\mathbf{B}^i$ of $\mathbf{G}_{n,k}(\mathbf{F})$, such that every element of $\mathbf{B}^i$ lies in V_i, any two distinct elements of $\mathbf{B}^i$ have zero intersection as subspaces of $\mathbf{F}^k$, and $\mathbf{A} \subset \bigcup_{i=1}^m \mathbf{B}^i$. Here $\bigcup_{i=1}^m \mathbf{B}^i$ is the isoclinic closure of $\mathbf{A}$.

This gives a complete analysis of the sets of pairwise isoclinic subspaces of any given dimension in $\mathbf{F}^k$, which, in turn, gives a complete analysis of the sets of Clifford-parallel linear subspaces of any given dimension in $\mathbf{P}^{k-1}(\mathbf{F})$.

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