

ON SOME SPAN THEOREMS¹

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1. In recent years the method of the extremal metric has proved to be the most consistently effective method for the treatment of problems in the theory of univalent functions. Among its consequences is the General Coefficient Theorem [2], [4], [5], which in a single statement includes a large part of the explicit results on univalent functions. Up to the present, however, one class of problems for univalent functions has not been successfully treated by this method, namely those associated with the span of multiply-connected domains. In the present paper we show that by a slight modification of the method of the extremal metric we are able to treat these problems. Indeed this approach provides both great generality and perhaps the most penetrating analysis of these problems and at the same time the greatest simplicity consonant with the former. There are many results of this type which can be treated by this method, but to avoid an excessively lengthy exposition we will confine ourselves to two of them. First we prove the most familiar span theorem in a form more general than that previously given. Then we prove a theorem of this type for functions regular in a domain, in distinction to the usual situation where the functions are required to exhibit a prescribed singularity. To the best of our knowledge this is the first general result of this kind.

2. Let first D be a plane domain of finite connectivity containing the point at infinity with boundary continua $C_j, j = 1, \dots, n$. We consider several families of functions for this domain. We may without loss of generality assume that each C_j is an analytic curve. Let m, p, q be nonnegative integers such that $m + p + q = n$. Let $\Sigma(D; m, p)$ denote the class of functions $f(z)$ univalent in D , regular apart from a simple pole at the point at infinity where each has a development of the form

$$(1) \quad z + \sum_{n=1}^{\infty} a_n z^{-n}$$

and such that under the mapping $w = f(z)$, $C_j, j = 1, \dots, m$, correspond to horizontal slits, $C_j, j = m + 1, \dots, m + p$, correspond to vertical slits. The function $f(z)$ will automatically be regular on $C_j, j = 1, \dots, m + p$. Let $F(D; m, p)$ denote the class of functions $f(z)$ regular in D apart from a simple pole at the point at infinity where each has a development of the form (1), regular on $C_j, j = 1, \dots, n$, and such that the values of $f(z)$ for z on $C_j, j = 1, \dots, m$, have constant imaginary part, for z on $C_j, j = m + 1, \dots,$

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$m + p$, constant real part. The following canonical mappings are of importance.

LEMMA 1. *Given α , $\alpha = 0, \pi/2$, there exists a unique function*

$$f(z; D, m, p, \alpha) \in \Sigma(D; m, p)$$

such that under the mapping $w = f(z; D, m, p, \alpha)$ the boundary curves C_j , $j = m + p + 1, \dots, n$, correspond to slits on lines of slope α .

This result appears in a paper of Komatu and Ozawa [6]. It is easily proved by the methods of [3].

3. Let $f(z)$ be a function belonging to $\Sigma(D; m, p)$ or to $F(D; m, p)$, and let $\alpha = 0, \pi/2$. We now define the (f, α) -star, $S(f, \alpha)$, a subdomain of D associated with f . On the Riemann image of D under the mapping $w = f(z)$ we regard the trace $\tau(\lambda)$ of each line λ of inclination α . This is the image of an at most countable set of arcs in D which divide D into certain subdomains, among them two which contain points in the neighborhood of the point at infinity. We denote these domains by $D_1(\lambda)$, $D_2(\lambda)$ containing respectively the directions $ie^{i\alpha}$, $-ie^{i\alpha}$. One component $\Delta(\lambda)$ of $D - D_1(\lambda)$ contains $D_2(\lambda)$. Let us denote the common boundary of $D_1(\lambda)$ and $\Delta(\lambda)$ by $\sigma(\lambda)$. As λ varies through all lines of inclination α , $\sigma(\lambda)$ sweeps out a set which we denote by $S(f, \alpha)$ and call the (f, α) -star of D . The following lemma is useful.

LEMMA 2. *If $f \in F(D; m, p)$ or $\Sigma(D; m, p)$ and the (f, α) -star of D coincides with D , $\alpha = 0, \pi/2$, then f is univalent in D , and each complementary component of $f(D)$ is met by each horizontal and each vertical line in at most a single segment (or point).*

In the proof of univalence we evidently may assume that $f \in F(D; m, p)$. On every C_j each function $\Re f(z)$ and $\Im f(z)$ has at least one arc or point where it assumes a maximum and a minimum relative to values on C_j . In the case of an arc this value must be assumed constantly on C_j . If this occurs for either $\Re f(z)$ or $\Im f(z)$ on each boundary component, the univalence of f is immediate. Suppose then there are boundary components where this does not occur. We will show that if f is not univalent, then at least one such point is a point of relative maximum or minimum for values of $\Re f(z)$ or $\Im f(z)$ on \bar{D} . From this we see that either the $(f, 0)$ -star or the $(f, \pi/2)$ -star fails to coincide with D .

To complete the proof we regard a curve C in D lying in a neighborhood of the point at infinity in which f is univalent and which is mapped by $w = f(z)$ onto a circle. Let Δ be the subdomain of D bounded by C and not containing the point at infinity. We will now regard the relationship between the number of boundary curves, minimum points relative to $\bar{\Delta}$, interior and boundary saddle points for $\pm \Re f(z)$, $\pm \Im f(z)$ in $\bar{\Delta}$. We will denote the num-

bers of these respectively by v , m , S , and s , where in each case we disregard those boundary components on which the function considered is constant. If none of the latter were present, we would have by a result of Morse [7; Theorem 11.2] the relation

$$2 - v = m - S - s.$$

It is easily seen that for any of these functions a boundary curve on which it is constant can be replaced by an adjacent curve which satisfies the conditions required for the validity of Morse's result, and which would have on it no minimum point and at least one boundary saddle point. Thus we have the inequality

$$2 - v \leq m - S - s.$$

Now evidently $S \geq 0$, and the boundary curve C contributes 1 to v , 1 to m , and 0 to s . Any other minimum point must lie on a boundary C_j , and our proof is completed by proving $m > 1$ for one of the functions in question. On any boundary component C_j not excluded above, the points of relative maximum for values on C_j each contribute at least 1 to s . Thus if there is more than one such for any boundary component, we have $m > 1$. If this does not occur for any of the four functions, then the image of C_j is met by each horizontal and each vertical line in at most two points. If, as we describe the boundary curves of D in the positive sense, all of their images under $w = f(z)$ are described in the clockwise sense, $f(z)$ is clearly univalent. Let us assume then that one of these images is described in the counterclockwise sense, and that on the corresponding boundary component $\Re f(z)$ assumes its maximum at A , its minimum at B , and $\Im f(z)$ assumes its maximum at E , its minimum at F , in each case relative to values on this component. If A is not a point of relative maximum of $\Re f(z)$ for values on \bar{D} , it contributes at least 1 to s for the functions $\Re f(z)$ and $\pm \Im f(z)$. Similar remarks apply to the points B , E , F . Thus in any case we must have $m > 1$ for at least one of these functions. This completes the proof of univalence.

For functions in $\Sigma(D; m, p)$ the final statement of Lemma 2 is obvious.

4. Let us denote the functions $f(z; D, m, p, 0)$, $f(z; D, m, p, \pi/2)$ for brevity by $h(z)$, $k(z)$, and let $f \in F(D; m, p)$ or $\Sigma(D; m, p)$. Let h, k, f have the developments of form (1) with a_i respectively equal to a, b, c . Let L be so large that the square with sides $2L$ parallel to the axes centred at the origin lies in the part of the Riemann image \mathcal{R} of D under the mapping $w = f(z)$ which lies schlicht above the w -plane. Let $S'(f, \alpha)$ denote the image of $S(f, \alpha)$ under this mapping, let $D(L)$ denote the portion of \mathcal{R} enclosed by the above square (not containing the point at infinity), and let $D'(L)$ be the intersection of $D(L)$ and $S'(f, 0)$. Let $4L^2$ minus the area of $D'(L)$ be denoted by A^* , $4L^2$ minus the area of $D(L)$ by A . Evidently these are independent of L and may be negative. The latter may be called

the complementary area of $f(D)$. Let the mapping of \mathfrak{R} given by kf^{-1} be denoted by V . On $S'(f, 0)$ we can use w as local uniformizing parameter and denote the function by $V(w)$ (not necessarily globally single-valued). In particular this is valid in a neighborhood of the point at infinity where $V(w)$ has the expansion

$$V(w) = w + \mu w^{-1} + \text{higher powers of } w^{-1}$$

with $\mu = b - c$.

If $\sigma(\lambda, L)$ denotes the subset of $\tau(\lambda)$ for slope 0 which is the intercept by $D(L)$ on the image of $s(\lambda)$ under f and $w = u + iv$, we have, apart from a finite number of choices for λ ,

$$(2) \quad \int_{\sigma(\lambda, L)} \Re V'(w) du = \Re(V(P(\lambda)) - V(Q(\lambda))),$$

where $P(\lambda), Q(\lambda)$ are the points on $\tau(\lambda)$ at which $\Re w = L, \Re w = -L$. Let $E(f, \alpha)$ be the complement relative to D of $S(f, \alpha)$, and let E^* be the area of the image of $E(f, 0)$ under k . Integrating (2) with respect to v over the interval $-L < v < L$ we obtain

$$\iint_{D'(L)} \Re V'(w) dA_w = 4L^2 + \pi \Re \mu + O(L^{-1}),$$

that is,

$$(3) \quad \iint_{D'(L)} \Re V'(w) dA_w = \iint_{D'(L)} dA_w + A^* + \pi \Re \mu + O(L^{-1}),$$

where dA_w denotes the element of area over the w -plane. On the other hand, a standard argument shows (see [2], [4], [5])

$$(4) \quad \iint_{D'(L)} dA_w + A^* = \iint_{D'(L)} |V'(w)|^2 dA_w + E^* + O(L^{-1}).$$

Combining (3) and (4) we obtain

$$(5) \quad \iint_{D'(L)} |V'(w) - 1|^2 dA_w + E^* = -2\pi \Re \mu - A^* + O(L^{-1}).$$

Letting L tend to infinity we find

$$(6) \quad \iint_{S'(f, 0)} |V'(w) - 1|^2 dA_w + E^* = -2\pi \Re \mu - A^*,$$

and transferring the integral to the z -plane we get

$$(7) \quad \iint_{S(f, 0)} |k'(z) - f'(z)|^2 dA_z + E^* = 2\pi \Re(c - b) - A^*,$$

where dA_z denotes the element of area in the z -plane.

Consider now the image Δ of D in the ζ -plane ($\zeta = \xi + i\eta$) under the

mapping $\zeta = h(z)$ and a square of sides $2L$ parallel to the axes centred at the origin with L so large that this square encloses the complement of Δ . Let $\Delta(L)$ be the portion of Δ enclosed by this square. Let the function fh^{-1} defined in Δ be denoted by $H(\zeta)$. This function has expansion in a neighborhood of the point at infinity given by

$$H(\zeta) = \zeta + \nu\zeta^{-1} + \text{higher powers of } \zeta^{-1},$$

where $\nu = c - a$. For all but a finite number of values of η , $-L < \eta < L$, the intersection $\iota(\eta, L)$ of the line $g\zeta = \eta$ with $\Delta(L)$ will consist of a finite number of segments (in some cases one) of total length $2L$. We see at once that for these values of η

$$(8) \quad \int_{\iota(\eta, L)} \Re H'(\zeta) d\xi = \Re(H(P'(\eta)) - H(Q'(\eta))),$$

where $P'(\eta)$, $Q'(\eta)$ are the points $L + i\eta$, $L - i\eta$. Integrating (8) with respect to η over the interval $-L < \eta < L$ we obtain

$$\iint_{\Delta(L)} \Re H'(\zeta) dA_\zeta = 4L^2 + \pi\Re\nu + O(L^{-1}),$$

that is,

$$(9) \quad \iint_{\Delta(L)} \Re H'(\zeta) dA_\zeta = \iint_{\Delta(L)} dA_\zeta + \pi\Re\nu + O(L^{-1}),$$

where dA_ζ denotes the element of area in the ζ -plane. On the other hand, in the same way as for (4) we obtain

$$(10) \quad \iint_{\Delta(L)} |H'(\zeta)|^2 dA_\zeta + A = \iint_{\Delta(L)} dA_\zeta + O(L^{-1}).$$

Combining (9) and (10) we obtain

$$(11) \quad \iint_{\Delta(L)} |H'(\zeta) - 1|^2 dA_\zeta = -2\pi\Re\nu - A + O(L^{-1}).$$

Letting L tend to infinity we find

$$(12) \quad \iint_{\Delta} |H'(\zeta) - 1|^2 dA_\zeta = -2\pi\Re\nu - A,$$

and transferring the integral to the z -plane we get

$$(13) \quad \iint_D |f'(z) - h'(z)|^2 dA_z = 2\pi\Re(a - c) - A.$$

These formulae constitute our basic result.

THEOREM 1. *Let $f \in F(D; m, p)$ or $\Sigma(D; m, p)$, and let $f(z; D, m, p, 0)$, $f(z; D, m, p, \pi/2)$ be denoted respectively by $h(z)$, $k(z)$. Let h, k, f have*

developments of form (1) with a_1 respectively equal to a, b, c . Let A be the complementary area of $f(D)$. Then

$$(14) \quad \iint_{S(f,0)} |k'(z) - f'(z)|^2 dA_z + \iint_{E(f,0)} |k'(z)|^2 dA_z = 2\pi\Re(c - b) - A - \iint_{E(f,0)} |f'(z)|^2 dA_z,$$

$$(15) \quad \iint_D |f'(z) - h'(z)|^2 dA_z = 2\pi\Re(a - c) - A.$$

Analogous formulae hold interchanging the roles of $h(z), k(z)$, namely

$$(16) \quad \iint_{S(f,\pi/2)} |h'(z) - f'(z)|^2 dA_z + \iint_{E(f,\pi/2)} |h'(z)|^2 dA_z = 2\pi\Re(a - c) - A - \iint_{E(f,\pi/2)} |f'(z)|^2 dA_z,$$

$$(17) \quad \iint_D |f'(z) - k'(z)|^2 dA_z = 2\pi\Re(c - b) - A.$$

Formula (14) is a trivial modification of (12). Formulae (16) and (17) are obtained by completely parallel arguments.

LEMMA 3. In the notation of Theorem 1, if $m = p = 0$, we have

$$\Re(h'(z) \bar{k}'(z)) > 0 \quad \text{for } z \text{ in } D.$$

It is well known that the function

$$e^{i\alpha}(\cos \alpha h(z) - i \sin \alpha k(z))$$

is univalent in D for $0 \leq \alpha \leq \pi$ and thus has a nonzero derivative in this domain. If we set

$$h = u_1 + iw_1, \quad k = u_2 + iw_2$$

as the division into real and imaginary parts, the vanishing of the above derivative for some α would be equivalent to there being a nontrivial solution to the equations in ξ, η

$$\frac{\partial u_1}{\partial x} \xi + \frac{\partial u_2}{\partial x} \eta = 0, \quad \frac{\partial v_2}{\partial x} \xi - \frac{\partial v_1}{\partial x} \eta = 0,$$

that is,

$$\frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} \frac{\partial u_2}{\partial x} = 0.$$

The left-hand side of the last equation is precisely $\Re(h'(z)\bar{k}'(z))$ which is thus nonzero in D . However this expression is positive in the neighborhood of the point at infinity and thus everywhere in D .

The result of Lemma 3 could also be deduced from one step in the argument on p. 182 in [1]. However the present proof seems simpler and more natural.

COROLLARY 1. *Let A, h, k, a, b, c be as in Theorem 1. Let $\beta + \gamma = 1$, $\beta, \gamma \geq 0$. Then*

$$A + 2\pi(\beta - \gamma)\Re c$$

is maximized in the class $F(D; m, p)$ uniquely in the case of the function $\beta h + \gamma k$. The value of this maximum is $2\pi(\beta^2\Re a - \gamma^2\Re b)$. If $m = p = 0$, the function $\beta h + \gamma k$ is univalent, so the same result is valid for the class $\Sigma(D; m, p)$. Moreover in this case $\beta h + \gamma k$ maps D onto a domain bounded by convex continua.

Setting $f = k$ in (15) we find

$$(18) \quad \iint_D |k'(z) - h'(z)|^2 dA_z = 2\pi\Re(a - b).$$

Now multiplying (15) by β , (17) by γ , and adding we find

$$(19) \quad \beta \iint_D |f'(z) - h'(z)|^2 dA_z + \gamma \iint_D |f'(z) - k'(z)|^2 dA_z \\ = 2\pi\Re(\beta a - \gamma b) - 2\pi(\beta - \gamma)\Re c - A.$$

If we use (18), this is equivalent to

$$(20) \quad \beta \iint_D |f'(z) - h'(z)|^2 dA_z + \gamma \iint_D |f'(z) - k'(z)|^2 dA_z \\ - \beta\gamma \iint_D |k'(z) - h'(z)|^2 dA_z + A + 2\pi(\beta - \gamma)\Re c \\ = 2\pi\Re(\beta a - \gamma b) - \beta\gamma \iint_D |k'(z) - h'(z)|^2 dA_z.$$

Now

$$\beta \iint_D |f'(z) - h'(z)|^2 dA_z + \gamma \iint_D |f'(z) - k'(z)|^2 dA_z \\ - \beta\gamma \iint_D |k'(z) - h'(z)|^2 dA_z \geq 0,$$

with equality only if

$$f \equiv \beta h + \gamma k,$$

while the right-hand side of (20) is equal to

$$2\pi(\beta - \gamma)\Re(\beta a + \gamma b) + A_{\beta\gamma},$$

where $A_{\beta\gamma}$ is the value of A for the function $\beta h + \gamma k$. This completes the proof of the first statement in Corollary 1.

The value of this maximum is derived by setting $f = \beta h + \gamma k$ in (19) and

using (18). In this connection we observe the equality

$$A_{\beta\gamma} = 2\pi\beta\gamma\Re(a - b) = \beta\gamma \iint_D |k'(z) - h'(z)|^2 dA_z.$$

Taking f to be the function $\beta h + \gamma k$ in the formulae of Theorem 1, multiplying (14) by γ , (15) by β , and adding we obtain

$$\begin{aligned} \gamma\beta^2 \iint_S |h'(z) - k'(z)|^2 dA_z + \gamma \iint_E |k'(z)|^2 dA_z \\ + \beta\gamma^2 \iint_D |h'(z) - k'(z)|^2 dA_z + \gamma \iint_E |\beta h'(z) + \gamma k'(z)|^2 dA_z \\ = -A_{\beta\gamma} + 4\pi\beta\gamma\Re(a - b), \end{aligned}$$

where S, E stand respectively for $S(\beta h + \gamma k, 0)$, $E(\beta h + \gamma k, 0)$. This reduces to

$$\gamma \iint_E |k'(z)|^2 dA_z + \gamma \iint_E |\beta h'(z) + \gamma k'(z)|^2 dA_z - \beta^2\gamma \iint_E |h'(z) - k'(z)|^2 dA_z = 0,$$

and further to

$$(21) \quad (1 + \gamma^2 - \beta^2)\gamma \iint_E |k'(z)|^2 dA_z + 2\beta\gamma \iint_E \Re(h'(z)\bar{k}'(z)) dA_z = 0.$$

In the case $m = p = 0$ we have by Lemma 3 that both integrals on the left-hand side of (21) are positive unless $E(\beta h + \gamma k, 0)$ is void. Thus $E(\beta h + \gamma k, 0)$ is void. A similar application of (16) and (17) shows that $E(\beta h + \gamma k, \pi/2)$ is likewise void under these conditions. Thus by Lemma 2 the function $\beta h + \gamma k$ is univalent. Finally, similar considerations can be applied to any pair of orthogonal directions, so that it follows by the last statement in Lemma 2 that the mapping $w = \beta h(z) + \gamma k(z)$ maps D onto a domain whose complementary continua are convex.

5. Let D be a plane domain of finite connectivity with boundary continua $C_j, j = 1, \dots, n + 1$, and let four boundary elements P_1, P_2, P_3, P_4 be given in natural cyclic order on C_1 . We may without loss of generality assume that D lies in the finite z -plane, and that each C_j is an analytic curve, so that the points P_j are boundary points. Let m, p, q be nonnegative integers such that $m + p + q = n$. Let $\hat{\Sigma}(D; m, p)$ denote the class of functions $f(z)$ regular and univalent in D such that under the mapping $w = f(z)$ ($w = u + iv$), C_1 corresponds for some $L > 0$ to the perimeter of the rectangle

$$0 < u < L, \quad 0 < v < 1,$$

with P_1, P_2, P_3, P_4 going into the points $0, L, L + i, i, C_j, j = 2, \dots, m + 1$, correspond to horizontal slits, and $C_j, j = m + 2, \dots, p + 1$, correspond

to vertical slits. The function $f(z)$ will automatically be regular on C_j , $j = 2, \dots, m + p + 1$. Let $\hat{F}(D; m, p)$ denote the class of functions $f(z)$ regular in D such that under the mapping $w = f(z)$, C_1 corresponds for some $L > 0$ to the perimeter of the rectangle

$$0 < u < L, \quad 0 < v < 1,$$

with P_1, P_2, P_3, P_4 going into the points $0, L, L + i, i$, such that $f(z)$ is regular on $C_j, j = 2, \dots, n + 1$ with

$$0 < \Re f(z) < L, \quad 0 < \Im f(z) < 1$$

on each such boundary component, and such that the values of $f(z)$ for z on $C_j, j = 2, \dots, m + 1$, have constant imaginary part, for z on $C_j, j = m + 2, \dots, m + p + 1$, constant real part. The existence of the following canonical conformal mappings follows from [3, Theorem 1].

LEMMA 4. *Given $\alpha, \alpha = 0, \pi/2$, there exists a unique function*

$$\hat{f}(z; D, m, p, \alpha) \in \hat{\Sigma}(D; m, p)$$

such that under the mapping $w = \hat{f}(z; D, m, p, \alpha)$ the boundary curves $C_j, j = m + p + 2, \dots, n + 1$, correspond to slits on lines of slope α .

6. Let $f(z)$ be a function belonging to $\hat{\Sigma}(D; m, p)$ or to $\hat{F}(D; m, p)$. We will define the $(f, 0)$ -star, $S(f, 0)$, a subdomain of D associated with f . On the Riemann image of D under the mapping $w = f(z)$ we regard the trace $\tau(v)$ of each line $\Re w = v$ with $0 < v < 1$. This is the image of an at most countable set of arcs in D which divide D into certain subdomains, among them two special ones: $D_1(v)$ on whose boundary is the arc P_3P_4 , and $D_2(v)$ on whose boundary is the arc P_1P_2 . One component $\Delta(v)$ of $D - D_1(v)$ contains $D_2(v)$. Let us denote the common boundary of $D_1(v)$ and $\Delta(v)$ by $s(v)$. As v runs through the values $0 < v < 1$, $s(v)$ sweeps out a set which we denote by $S(f, 0)$. The $(f, \pi/2)$ -star, $S(f, \pi/2)$, is defined similarly except we use lines $\Re w = u, 0 < u < L$, and domains with P_4P_1 and P_2P_3 on their respective boundaries.

LEMMA 5. *If $f \in \hat{F}(D; m, p)$ or $\hat{\Sigma}(D; m, p)$ and the (f, α) -star of D coincides with $D, \alpha = 0, \pi/2$, then f is univalent in D , and each bounded complementary component of $f(D)$ is met by each horizontal and each vertical line in at most a single segment (or point).*

The proof of this result is not essentially different from that of Lemma 2.

7. Let us denote the functions $\hat{f}(z; D, m, p, 0), \hat{f}(z; D, m, p, \pi/2)$ for brevity by $l(z), q(z)$, and let $f \in \hat{F}(D; m, p)$ or $\hat{\Sigma}(D; m, p)$. Let the values of the length L corresponding to l, q, f be denoted by L_1, L_2, L . Let $S'(f, \alpha)$ denote the image of $S(f, \alpha)$ under the mapping $w = f(z)$. Let L minus the

area of $f(D)$ be denoted by A , L minus the area of $S'(f, 0)$ by A^* . The former will be called the complementary area of $f(D)$. Let the mapping defined on the Riemann image \mathfrak{R} of D under f by qf^{-1} be denoted by V . On $S'(f, 0)$ we can use the plane variable w as local uniformizing parameter and denote the function by $V(w)$ (not necessarily globally single-valued).

If $\sigma(v)$ denotes the subset of $\tau(v)$ which is the image of $s(v)$ under f , we have, apart from a finite number of values of v

$$(22) \quad \int_{\sigma(v)} \mathfrak{R}V'(w) \, du = L_2.$$

Let $E(f, \alpha)$ be the complement relative to D of $S(f, \alpha)$, and E^* the area of the image of $E(f, 0)$ under q . Integrating (22) with respect to v over the interval $0 < v < 1$ we obtain

$$(23) \quad \iint_{S'(f,0)} \mathfrak{R}V'(w) \, dA_w = L_2,$$

where dA_w has its usual meaning. On the other hand,

$$(24) \quad \iint_{S'(f,0)} dA_w + A^* = L,$$

$$(25) \quad \iint_{S'(f,0)} |V'(w)|^2 \, dA_w + E^* = L_2.$$

Combining (23), (24), and (25) we obtain

$$(26) \quad \iint_{S'(f,0)} |V'(w) - 1|^2 \, dA_w + E^* = L - L_2 - A^*,$$

and transferring the integral to the z -plane we get

$$(27) \quad \iint_{S(f,0)} |q'(z) - f'(z)|^2 \, dA_z + E^* = L - L_2 - A^*,$$

where dA_z has its usual meaning.

Let the function fl^{-1} defined in $l(D)$ in the ζ -plane be denoted by $H(\zeta)$. For all but a finite number of values of η , $0 < \eta < 1$, the intersection $\iota(\eta)$ of the line $g\zeta = \eta$ with $l(D)$ will consist of a finite number of segments of total length L_1 . We see at once for these values of η ($\zeta = \xi + i\eta$)

$$(28) \quad \int_{\iota(\eta)} \mathfrak{R}H'(\zeta) \, d\xi = L.$$

Integrating (28) with respect to η over the interval $0 < \eta < 1$ we obtain

$$(29) \quad \iint_{l(D)} \mathfrak{R}H'(\zeta) \, dA_\zeta = L,$$

where dA_ζ has its usual meaning. On the other hand,

$$(30) \quad \iint_{l(D)} dA_{\zeta} = L_1,$$

$$(31) \quad \iint_{l(D)} |H'(\zeta)|^2 dA_{\zeta} = L - A.$$

Combining (29), (30), and (31) we obtain

$$(32) \quad \iint_{l(D)} |H'(\zeta) - 1|^2 dA_{\zeta} = L_1 - L - A,$$

and transferring the integral to the z -plane we get

$$(33) \quad \iint_D |f'(z) - l'(z)|^2 dA_z = L_1 - L - A.$$

Summarizing these results we have

THEOREM 2. *Let $f \in \hat{F}(D; m, p)$ or $\hat{\Sigma}(D; m, p)$, and let $\hat{f}(z; D, m, p, 0)$, $\hat{f}(z; D, m, p, \pi/2)$ be denoted respectively by $l(z)$, $q(z)$. Let the values of L corresponding to l , q , f be denoted respectively by L_1 , L_2 , L . Let A be the complementary area of $f(D)$. Then*

$$(34) \quad \begin{aligned} \iint_{S(f,0)} |q'(z) - f'(z)|^2 dA_z + \iint_{B(f,0)} |q'(z)|^2 dA_z \\ = L - L_2 - A - \iint_{B(f,0)} |f'(z)|^2 dA_z, \end{aligned}$$

$$(35) \quad \iint_D |f'(z) - l'(z)|^2 dA_z = L_1 - L - A.$$

Analogous formulae hold interchanging the roles of $l(z)$, $q(z)$, namely

$$(36) \quad \begin{aligned} \iint_{S(f,\pi/2)} |l'(z) - f'(z)|^2 dA_z + \iint_{B(f,\pi/2)} |l'(z)|^2 dA_z \\ = L_1 - L - A - \iint_{B(f,\pi/2)} |f'(z)|^2 dA_z, \end{aligned}$$

$$(37) \quad \iint_D |f'(z) - q'(z)|^2 dA_z = L - L_2 - A.$$

Formula (34) is a trivial modification of (27). Formulae (36) and (37) are obtained by completely parallel arguments.

LEMMA 6. *In the notation of Theorem 2, if $m = p = 0$, we have*

$$\Re(l'(z)\bar{q}'(z)) > 0 \quad \text{for } z \text{ in } D.$$

The first step in our proof is to verify that the function

$$\cos \alpha l(z) - i \sin \alpha q(z)$$

is univalent for each value of α , $0 \leq \alpha < 2\pi$. The image of a boundary component C_j , $j = 2, \dots, n + 1$, under this function lies on a segment parallel to the real axis. We will examine now the image of C_1 , treating for definiteness the case $0 < \alpha < \pi/2$. The situation in other cases is completely analogous. As z describes the arc P_1P_2 , its image point w describes an arc σ_1 which meets each horizontal and each vertical line at most once and runs from the origin to the point $L_1 \cos \alpha - iL_2 \sin \alpha$. As z describes the arc P_2P_3 , w describes an arc σ_2 which meets each horizontal and each vertical line at most once and runs from the preceding point to the point $(L_1 \cos \alpha + \sin \alpha) + i(\cos \alpha - L_2 \sin \alpha)$. As z describes the arc P_3P_4 , w describes an arc σ_3 which meets each horizontal and each vertical line at most once and runs from the preceding point to the point $\sin \alpha + i \cos \alpha$. As z describes the arc P_4P_1 , w describes an arc σ_4 which meets each horizontal and each vertical line at most once and runs from the preceding point back to the origin. The arcs σ_j described in succession form a closed curve γ which lies in the rectangle R :

$$0 \leq u \leq L_1 \cos \alpha + \sin \alpha, \quad -L_2 \sin \alpha \leq v \leq \cos \alpha.$$

For $j = 1, 2, 3, 4$, let τ_j denote the arc composed of parts of two sides of this rectangle joining the end points of σ_j . The degree of γ about a point exterior to R is evidently zero. For a point w interior to R

$$\Delta_{\sigma_j}(\arg(w - w_0)) \leq \Delta_{\tau_j}(\arg(w - w_0)), \quad j = 1, 2, 3, 4,$$

this being the usual notation for the increment in the argument. Thus the degree of γ about any point is at most one. From these facts about the images of the C_j it is immediate that the given function is univalent.

Now as in the proof of Lemma 3 it follows that $\Re(l'(z)\bar{q}'(z)) \neq 0$ for z in D .

Finally we observe that $\Re(l'(z)\bar{q}'(z)) > 0$ for points in D near the boundary and thus everywhere in D .

COROLLARY 2. *Let A, l, q, L_1, L_2, L be as in Theorem 2. Let $\beta + \gamma = 1$, $\beta, \gamma \geq 0$. Then*

$$A + (\beta - \gamma)L$$

is maximized in the class $\hat{F}(D; m, p)$ uniquely for the function $\beta l + \gamma q$. The value of this maximum is $\beta^2 L_1 - \gamma^2 L_2$. If $m = p = 0$, the function $\beta l + \gamma q$ is univalent, so the same result is valid for the class $\hat{\Sigma}(D; m, p)$. Moreover in this case $\beta l + \gamma q$ maps D onto a domain each of whose finite complementary continua is met by each horizontal and each vertical line in at most a single segment (or point).

Setting $f = q$ in (35) we find

$$(38) \quad \iint_D |l'(z) - q'(z)|^2 dA_z = L_1 - L_2.$$

Now multiplying (35) by β , (37) by γ , and adding we find

$$(39) \quad \beta \iint_D |f'(z) - v'(z)|^2 dA_z + \gamma \iint_D |f'(z) - q'(z)|^2 dA_z \\ = \beta L_1 - \gamma L_2 - (\beta - \gamma) L - A.$$

If we use (38), this is equivalent to

$$(40) \quad \beta \iint_D |f'(z) - v'(z)|^2 dA_z + \gamma \iint_D |f'(z) - q'(z)|^2 dA_z \\ - \beta\gamma \iint_D |v'(z) - q'(z)|^2 dA_z + A + (\beta - \gamma) L \\ = \beta L_1 - \gamma L_2 - \beta\gamma \iint_D |v'(z) - q'(z)|^2 dA_z.$$

Now

$$\beta \iint_D |f'(z) - v'(z)|^2 dA_z + \gamma \iint_D |f'(z) - q'(z)|^2 dA_z \\ - \beta\gamma \iint_D |v'(z) - q'(z)|^2 dA_z \geq 0,$$

with equality only if

$$f \equiv \beta l + \gamma q,$$

while the right-hand side of (40) is equal to

$$A_{\beta\gamma} = (\beta - \gamma) (\beta L_1 + \gamma L_2),$$

where $A_{\beta\gamma}$ is the value of A for the function $\beta l + \gamma q$. This completes the proof of the first statement in Corollary 2.

The value of this maximum is derived by setting $f = \beta l + \gamma q$ in (39) and using (38). In this connection we observe the equality

$$A_{\beta\gamma} = \beta\gamma(L_1 - L_2) = \beta\gamma \iint_D |v'(z) - q'(z)|^2 dA_z.$$

Taking f to be the function $\beta l + \gamma q$ in the formulae of Theorem 2, multiplying (34) by γ , (35) by β , and adding we obtain

$$\gamma\beta^2 \iint_S |v'(z) - q'(z)|^2 dA_z + \gamma \iint_E |q'(z)|^2 dA_z \\ + \beta\gamma^2 \iint_D |v'(z) - q'(z)|^2 dA_z + \gamma \iint_E |\beta v'(z) + \gamma q'(z)|^2 dA_z \\ = -A_{\beta\gamma} + 2\beta\gamma(L_1 - L_2),$$

where S, E stand respectively for $S(\beta l + \gamma q, 0), E(\beta l + \gamma q, 0)$. This reduces to

$$\begin{aligned} \gamma \iint_E |q'(z)|^2 dA_z + \gamma \iint_E |\beta l'(z) + \gamma q'(z)|^2 dA_z \\ - \beta^2 \gamma \iint_E |l'(z) - q'(z)|^2 dA_z = 0, \end{aligned}$$

and further to

$$(41) \quad (1 + \gamma^2 - \beta^2) \gamma \iint_E |q'(z)|^2 dA_z + 2\beta\gamma \iint_E \Re(l'(z)\bar{q}'(z)) dA_z = 0.$$

In the case $m = p = 0$ we have by Lemma 6 that both integrals on the left-hand side of (41) are positive unless $E(\beta l + \gamma q, 0)$ is void. Thus this set must be void. A similar application of (36) and (37) shows that $E(\beta l + \gamma q, \pi/2)$ is likewise void under these conditions. Thus by Lemma 5 the function $\beta l + \gamma q$ is univalent. Moreover the mapping $w = \beta l(z) + \gamma q(z)$ carries D onto a domain each of whose finite complementary continua is met by each horizontal and each vertical line in at most a single segment (or point).

Finally we remark that we could have normalized the mappings of $\hat{F}(D; m, p)$ and $\hat{Z}(D; m, p)$ so that the exterior horizontal dimension of the image rather than the exterior vertical dimension was fixed.

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