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AMENABILITY PROPERTIES OF THE CENTRAL FOURIER ALGEBRA OF A COMPACT GROUP

MAHMOOD ALAGHMANDAN AND NICO SPRONK

ABSTRACT. We let the central Fourier algebra, $\mathrm{ZA}(G)$, be the subalgebra of functions u in the Fourier algebra $\mathrm{A}(G)$ of a compact group, for which $u(xyx^{-1}) = u(y)$ for all x, y in G. We show that this algebra admits bounded point derivations whenever G contains a non-Abelian closed connected subgroup. Conversely when G is virtually Abelian, then $\mathrm{ZA}(G)$ is amenable. Furthermore, for virtually Abelian G, we establish which closed ideals admit bounded approximate identities. We also show that $\mathrm{ZA}(G)$ is weakly amenable, in fact hyper-Tauberian, exactly when G admits no non-Abelian connected subgroup. We also study the amenability constant of $\mathrm{ZA}(G)$ for finite G and exhibit totally disconnected groups G for which $\mathrm{ZA}(G)$ is non-amenable. In passing, we establish some properties related to spectral synthesis of subsets of the spectrum of $\mathrm{ZA}(G)$.

1. Introduction

Let G be a compact group with Fourier algebra $\mathrm{A}(G)$ and B be a group of continuous automorphisms on G. We let

$$\mathrm{Z}_B\mathrm{A}(G) = \bigcap_{\beta\in B} \{u \in \mathrm{A}(G) : u \circ \beta = u\}$$

and call this the algebra the B-*centre* of $\mathrm{A}(G)$. In particular, we let $\mathrm{Inn}(G)$ be the group of inner automorphisms and let

$$\mathrm{ZA}(G) = \mathrm{Z}_{\mathrm{Inn}(G)}\mathrm{A}(G)$$

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and call this the *G-centre* of $\mathrm{A}(G)$, or the *central Fourier algebra* on G. Of course, since $\mathrm{A}(G)$ is a commutative algebra, this bears no relation to the centre of $\mathrm{A}(G)$ as an algebra.

We are motivated by the results of [7], where amenability properties of the centre of the group algebra, $\mathrm{ZL}^1(G)$, are studied. We note that this algebra is densely spanned by idempotents—that is, normalized characters of irreducible representations, $d_\pi\chi_\pi$—and hence is automatically weakly amenable, even hyper-Tauberian (see [54, Theo. 14]). Hence, those properties were not discussed for compact G. However, it was shown for G either non-Abelian and connected, or an infinite product of non-Abelian finite groups, that $\mathrm{ZL}^1(G)$ is non-amenable. It is conjectured in that paper that $\mathrm{ZL}^1(G)$ is amenable if and only if G has an open Abelian subgroup. Recently, the first named author and Crann [5] have verified sufficiency of this conjecture.

1.1. Plan. In the present article, we conduct a parallel investigation for $\mathrm{ZA}(G)$. Our techniques are different and some of our conclusions sharper. In the case that the connected component of the identity, G_e, is non-Abelian, we show that $\mathrm{ZA}(G)$ admits a point derivation; see Section 3. In Section 4, we show that when G is virtually Abelian then $\mathrm{ZA}(G)$ is amenable. In Section 5 we combine our knowledge of point derivations for non-Abelian connected groups with the structure of the central Fourier algebra for virtually Abelian compact groups, to further establish that $\mathrm{ZA}(G)$ is weakly amenable if and only if G_e is Abelian. We invest the extra effort to establish that this is exactly the case in which G is hyper-Tauberian, a condition identified and studied by Samei [54]. One of our major tools in Sections 4 and 5 is the relationship between these properties and certain conditions related to sets of spectral synthesis. In Section 6, we investigate the amenability constant $\mathrm{AM}(\mathrm{ZA}(G))$, and show that an infinite product, P, of non-Abelian finite groups, gives a non-amenable algebra $\mathrm{ZA}(P)$.

In the course of our investigation we gain, in Proposition 5.3, some results on spectral synthesis and weak synthesis of singleton and finite subsets of the spectrum of $\mathrm{ZA}(G)$, giving partial generalisations of results of Meany [45] and Ricci [49], with quite different proofs. We also gain a broad generalisation of an amenability result of Lasser [39]; see Remark 4.12. In Section 4.3, we give for virtually abelian compact groups, a full description of ideals in $\mathrm{ZA}(G)$ admitting bounded approximate identities

1.2. History. It was shown by Johnson [34] that $\mathrm{A}(G)$ may fail to be amenable for a compact G. This led Ruan [52] to define operator amenability and show for a locally compact group H that $\mathrm{A}(H)$ is operator amenable exactly when H is amenable; in particular this holds when $G = H$ is compact. One of the most interesting applications of this result is establishing, for amenable H, which closed ideals of $\mathrm{A}(H)$ admit bounded approximate identities, due to Forrest, Kaniuth, Lau and Spronk [19]. For Abelian H, this was established by Liu, van Rooij and Wang in [41]. That $\mathrm{A}(H)$ is always operator weakly amenable was established by Spronk [57], and, independantly Samei [53]. Both articles relied on identification of sets of spectral synthesis, and, in particular, ideas in the latter helped Samei establish the notion of hyper-Tauberian Banach algebras in [54]. The operator space structure of $\mathrm{ZA}(G)$ is maximal—see Section 2.2—and hence operator versions of amenability properties are automatic.

Forrest and Runde [20] established that $\mathrm{A}(H)$ is amenable exactly when H is virtually abelian, and further that if the connected component H_e is abelian, then $\mathrm{A}(H)$ is weakly amenable. Recently, Lee, Ludwig, Samei and Spronk [40] have established for a Lie group H, then $\mathrm{A}(H)$ can be weakly amenable only when H_e is Abelian. For general compact $G = H$, this fact was established by Forrest, Samei and Spronk in [21].

2. Preliminaries and notation

2.1. Amenability and weak amenability. We briefly note some fundamental definitions which go back to [32].

Let $\mathcal{A}$ be a commutative Banach algebra. A Banach $\mathcal{A}$-bimodule is any Banach space $\mathcal{X}$ which admits a pair of contractive homomorphisms $a \mapsto (x \mapsto ax)$ and $a \mapsto (x \mapsto xa)$, each from $\mathcal{A}$ into bounded operators $\mathcal{B}(\mathcal{X})$, such that the ranges of these maps commute: $a(xb) = (ax)b$. The homomorphisms given by pointwise adjoints of these maps make the dual, $\mathcal{X}^*$, into a *dual Banach $\mathcal{A}$-module*. We say $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual module, that is, $D : \mathcal{A} \to \mathcal{X}^*$ with $D(ab) = a(Db) + (Da)b$, is inner, that is, $Da = af - fa$ for some f in $\mathcal{X}^*$.

Following [8], we say $\mathcal{A}$ is *weakly amenable* if there are no non-zero bounded derivations for $\mathcal{A}$ into any *symmetric* Banach $\mathcal{A}$-module, that is, module $\mathcal{X}$ satisfying $ax = xa$. In particular, $\mathcal{A}$ is a symmetric Banach $\mathcal{A}$-bimodule as is its dual $\mathcal{A}^*$. It is sufficient to see that there are no non-zero bounded derivations from $\mathcal{A}$ into $\mathcal{A}^*$ to show that $\mathcal{A}$ is weakly amenable. Given a multiplicative functional χ on $\mathcal{A}$, a *point derivation* at χ is any linear functional D on $\mathcal{A}$ which satisfies $D(ab) = \chi(a)D(b) + D(a)\chi(b)$. The map $a \mapsto D(a)\chi : \mathcal{A} \to \mathcal{A}^*$ is then a derivation. Hence, a weakly amenable algebra admits no non-zero bounded point derivations.

2.2. The central Fourier algebra. Throughout this article, G will denote a compact group. We let $\widehat{G}$ denote the set of (equivalence classes of) continuous irreducible representations. For π in $\widehat{G}$, we let $\mathcal{H}_\pi$ denote the space on which it acts and let $d_\pi = \dim \mathcal{H}_\pi$. We denote normalized Haar integration by $\int_G \dots ds$. For integrable $u : G \to \mathbb{C}$ we let $\hat{u}(\pi) = \int_G u(s)\bar{\pi}(s)\,ds$, and [28, (34.4)] provides the following description of the *Fourier algebra*:

$$u \in \mathrm{A}(G) \quad \Leftrightarrow \quad \|u\|_{\mathrm{A}} = \sum_{\pi \in \widehat{G}} d_\pi \big\|\hat{u}(\pi)\big\|_1 < \infty,$$

where $\|\cdot\|_1$ denotes the trace norm. This is a special case of the general definition of the Fourier algebra, for locally compact groups, given in [16]. We let $\mathrm{VN}(G) = \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} \mathcal{B}(\mathcal{H}_\pi)$ and we have dual identification $\mathrm{A}(G)^* \cong \mathrm{VN}(G)$ via

$$\langle u, T\rangle = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\big(\hat{u}(\pi) T_\pi\big).$$

Let $Z_G : \mathrm{A}(G) \to \mathrm{ZA}(G)$ be given by $Z_G u(x) = \int_G u(sxs^{-1})\, ds$. This is easily verified to be a contractive linear projection. A well-known consequence of the Schur orthogonality relations is that for π in $\widehat{G}$ and ξ, η in $\mathcal{H}_\pi$ we have $Z_G\langle \pi(\cdot)\xi|\eta\rangle = \frac{\langle \xi|\eta\rangle}{d_\pi}\chi_\pi$, where χ_π is the character of π. Hence, $\mathrm{ZA}(G) = \overline{\mathrm{span}}\{\chi_\pi : \pi \in \widehat{G}\}$. Thus, since $\hat{\chi}_\pi(\pi) = \frac{1}{d_\pi} I_{d_\pi}$ and $\hat{\chi}_\pi(\pi') = 0$ for $\pi' \neq \pi$, we have $\|\hat{\chi}_\pi(\pi)\|_1 = 1$, and the description of the norm above gives that

$$u \in \mathrm{ZA}(G) \quad \Leftrightarrow \quad u = \sum_{\pi \in \widehat{G}} \alpha_\pi \chi_\pi \quad \text{with } \|u\|_{\mathrm{A}} = \sum_{\pi \in \widehat{G}} d_\pi |\alpha_\pi| < \infty. \tag{2.1}$$

Hence we have that $\mathrm{ZA}(G)^* \cong \mathrm{Z\,VN}(G) = \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} \mathbb{C} I_{d_\pi} \cong \ell^\infty(\widehat{G})$.

In particular, we see that $\mathrm{ZA}(G)$ is the predual of a commutative von Neumann algebra. Thus, generally, we will have no need to discuss the completely bounded theory of this space. However, we shall require, some knowledge of the operator space structure on $\mathrm{A}(G)$ in Section 6, which we shall simply reference therein.

Let $\sim_G$ denote the equivalence relation on G by conjugacy, i.e. $x \sim_G x'$ if and only if $x' = yxy^{-1}$ for some y in G. It may be deduced from [28, (34.37)] that each element of the Gelfand spectrum of $\mathrm{ZA}(G)$ is given by evaluation functionals from G, and hence may be identified with a point in $\mathrm{Conj}(G) = G/{\sim_G}$. See also Proposition 4.1, which gives a generalisation of this result. It is evident that Z_G has a certain expectation property: $Z_G(uv) = uZ_G v$ for u in $\mathrm{ZA}(G)$ and v in A.

We observe that $\mathrm{ZA}(G)$ is actually the *hypergroup algebra* $\ell^1(\widehat{G}, d^2)$, where $d^2(\pi) = d_\pi^2$, as verified in [1]. Notice that $\mathrm{ZL}^1(G)$, the algebra studied in [7], is the hypergroup algebra associated with the compact hypergroup $\mathrm{Conj}(G)$. In [31] it is established that $\widehat{G}$ is the dual hypergroup to $\mathrm{Conj}(G)$, and that $\mathrm{Conj}(G)$ is the dual hypergroup to $\widehat{G}$ is established in [3].

Let us end this section with a simple observation on quotient groups.

Proposition 2.1. *Let N be a closed normal subgroup of G. Then $P_N u(x) = \int_H u(xn)\, dn$ (normalized Haar integration on N) defines a surjective quotient map from $\mathrm{ZA}(G)$ to $\mathrm{ZA}(G/N)$.*

Proof. Since translation is an isometric action on $\mathrm{A}(G)$, continuous in G, it is clear that $P_N : \mathrm{ZA}(G) \to \mathrm{A}(G/N)$ is a contraction. Let $q : G \to G/N$ be the quotient map. Then there is an obvious embedding $\pi \mapsto \pi \circ q : \widehat{G/N} \to \widehat{G}$. An easy calculation shows that for π' in $\widehat{G}$, we have

$$P_N \chi_{\pi'} = \begin{cases} \chi_\pi \circ q & \text{if } \pi' = \pi \circ q, \\ 0 & \text{otherwise.} \end{cases}$$

See, for example, the proof of [43, Prop. 4.14]. Further, an application of (2.1) shows that the map

$$\sum_{\pi \in \widehat{G/N}} \alpha_\pi \chi_\pi \mapsto \sum_{\pi \in \widehat{G/N}} \alpha_\pi \chi_\pi \circ q : \mathrm{ZA}(G/N) \to \mathrm{ZA}(G)$$

is an isometry whose range is the image of P_N. □

3. Groups with non-Abelian connected component

We begin with the case of $G = \mathrm{SU}(2)$, the group of 2×2 unitary matrices of determinant one. We recall that the conjugacy classes in $\mathrm{SU}(2)$ are determined by the eigenvalues; that is, for each x in $\mathrm{SU}(2)$ there is y in $\mathrm{SU}(2)$ for which

$$x = y \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} y^{-1} \quad \text{where } \zeta \text{ in } \mathbb{T} \text{ satisfies } \operatorname{Im} \zeta \geq 0. \tag{3.1}$$

Hence we may label the conjugacy class by the specified eigenvalue: $C_x = C_\zeta$, so $C_\zeta = C_{\zeta^{-1}}$. It is well known that

$$\widehat{\mathrm{SU}}(2) = \{\pi_l : l = 0, 1, 2, \dots\},$$

where $d_{\pi_l} = l + 1$ and we have the associated character, evaluated at x as in (3.1), given by

$$\chi_l(x) = \chi_{\pi_l}(C_x) = \chi_{\pi_l}(C_\zeta) = \sum_{k=0}^{l} \zeta^{l-2k} = \begin{cases} \frac{\zeta^{l+1} - \zeta^{-l-1}}{\zeta - \zeta^{-1}} & \text{if } \zeta \in \mathbb{T} \setminus \{-1, 1\}, \\ \zeta^l (l+1) & \text{if } \zeta \in \{-1, 1\}. \end{cases}$$

We now recall that the 3×3 special orthogonal group $\mathrm{SO}(3)$ is isomorphic to $\mathrm{SU}(2)/\{-I, I\}$ and admits spectrum $\widehat{\mathrm{SO}}(3) = \{\pi_l : l = 0, 2, 4, \dots\} \subset \widehat{\mathrm{SU}}(2)$. The following is a sort of refinement of a result in [12], which is adapted specifically for our proof of Theorem 3.2.

PROPOSITION 3.1. *The algebra* $\mathrm{ZA}(\mathrm{SU}(2))$ *admits a bounded non-zero point derivation* D_z *at each class* C_z *of* $\mathrm{SU}(2)$ *for which* $\operatorname{Im} z > 0$; *while* $\mathrm{ZA}(\mathrm{SO}(3))$ *admits a bounded non-zero point derivation* D_z *at each class* C_z *of* $\mathrm{SU}(2)$ *for which* $\operatorname{Im} z > 0$ *and* $\operatorname{Re} z > 0$.

Proof. Let $\mathrm{Z}\mathcal{T} = \mathrm{span}\{\chi_l : l = 0, 1, 2, \dots\}$ which is dense in $\mathrm{ZA}(\mathrm{SU}(2))$. For u in $\mathrm{Z}\mathcal{T}$ and z in $\mathbb{T}$ with $\mathrm{Im}\, z > 0$ let $D_z u = z \frac{d}{d\zeta} u(C_\zeta)|_{\zeta = z}$. For such z we compute

$$D_z \chi_l = \frac{l(z^{l+2} - z^{-l-2}) - (l+2)(z^l - z^{-l})}{(z - z^{-1})^2}.$$

Notice that if we had $D_z \chi_l = 0$ for all l then a simple induction argument shows

$$z - z^{-1} = \frac{z^{2n+1} - z^{1-2n}}{2n+1} \quad \text{for each } n, \text{ so} \quad \left|z - z^{-1}\right| \leq \frac{2}{2n+1}$$

which is impossible if $\mathrm{Im}\, z > 0$, so $D_z \neq 0$ in such cases. Furthermore, we have

$$|D_z \chi_l| \leq \frac{4l + 4}{|z - z^{-1}|^2}.$$

Now if $u = \sum_{j=1}^n \alpha_j \chi_{l_j}$ for $l_1 < \cdots < l_n$, we can use the formula for the norm (2.1) to see that

$$|D_z u| \leq \sum_{j=1}^n |\alpha_j| \frac{4l_j + 4}{|z - z^{-1}|^2} = \frac{4}{|z - z^{-1}|^2} \sum_{j=1}^n |\alpha_j|(l_j + 1) = \frac{4}{|z - z^{-1}|^2} \|u\|_{\mathrm{A}}.$$

Hence D_z extends to a continuous derivation on $\mathrm{ZA}(\mathrm{SU}(2))$.

By virtue of Proposition 2.1 and the identification $\widehat{\mathrm{SO}}(3) \subset \widehat{\mathrm{SU}}(2)$, above, we have

$$\mathrm{ZA}\big(\mathrm{SO}(3)\big) = \overline{\mathrm{span}}\{\chi_l : 0, 2, 4, \dots\} \tilde{\subset} \mathrm{ZA}\big(\mathrm{SU}(2)\big).$$

By a similar argument as above each derivation D_z, with $\mathrm{Im}\, z > 0$ and $\mathrm{Im}\, z^2 > 0$, also defines a non-zero point derivation on $\mathrm{ZA}(\mathrm{SO}(3))$. □

THEOREM 3.2. *Let G have non-Abelian connected component G_e. Then* $\mathrm{ZA}(G)$ *admits a non-zero point derivation.*

Proof. According to the proof of [21, Theo. 2.1], G_e admits a closed subgroup S which is isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. [This uses a structure theorem for connected compact groups—see [47]—and the fact that any compact non-Abelian Lie algebra admits a copy of $\mathfrak{su}(2)$.] Since $\mathrm{ZA}(G)|_S \subset \mathrm{A}(G)|_S = \mathrm{A}(S)$ (see [26] or [28, (34.27)], for example), and since for u in $\mathrm{ZA}(G)$, $u(yxy^{-1}) = u(x)$ for x, y in G, a fortiori in S, we have $\mathrm{ZA}(G)|_S \subseteq \mathrm{ZA}(S)$. Further, since S is connected and $S \supsetneq \{e\}$, S is not contained in a single conjugacy class of G. Hence $\mathrm{ZA}(G)|_S \not\subset \mathbb{C}1$, and there is π in $\widehat{G}$ for which $\chi_\pi|_S \notin \mathbb{C}1$. Thus, we have

$$\pi|_S = \bigoplus_{j=1}^n m_j \pi_{l_j},$$

where each m_j is a non-zero multiplicity and $l_1 < \cdots < l_n$ with either $n > 1$ or $l_1 > 0$. It follows that $\chi_\pi|_S = \sum_{j=1}^n m_j \chi_{l_j}$ so

$$\chi_\pi(C_\zeta) = \sum_{j=1}^{n} m_j \sum_{k_j=0}^{l_j} \zeta^{l_j - 2k_j},$$

where C_ζ is the conjugacy class of elements with eigenvalue ζ in S. Thus for z in $\mathbb{T}$ with $\operatorname{Im} z > 0$, we have for the derivation D_z, defined in the proposition above, that

$$D_z(\chi_\pi|_S) = \sum_{j=1}^{n} \sum_{k_j=0}^{l_j} m_j(l_j - 2k_j) z^{l_j - 2k_j}.$$

Notice that the above expression is a non-zero polynomial in z of degree l_n. If z is transcendental over rationals, then $D_z(\chi_\pi|_S) \neq 0$. Thus for such z, $D = D_z \circ R_S : \mathrm{ZA}(G) \to \mathbb{C}$, where R_S is the restriction map, is a non-zero point derivation. □

REMARK 3.3. (i) For semisimple compact Lie G, a simplification of a result in [49] gives bounded non-zero point derivations at all regular points of G. This offers more precise data than does our Theorem 3.2 for such groups. However, our proof uses less Lie-theoretic machinery and returns results of sufficient strength for SU(2) and SO(3).

(ii) As mentioned in Section 2, $\mathrm{ZA}(G) \cong \ell^1(\widehat{G}, d^2)$ is a discrete hypergroup algebra. There are other discrete hypergroups, amongst a class which includes $\widehat{\mathrm{SU}}(2)$, whose hypergroup algebras are known to admit point derivations. See [39].

(iii) There are other known examples of B-central Fourier algebras which admit point derivations. For example, if $n \geq 3$, then the algebra of radial elements of $\mathrm{A}(\mathbb{R}^n)$, $\mathrm{Z}_{\mathrm{SO}(n)}\mathrm{A}(\mathbb{R}^n)$, admits a point derivation at each infinite orbit ([48, 2.6.10]). We remark that if we let $H_n = \mathbb{R}^n \rtimes \mathrm{SO}(n)_d$, $\mathbb{R}^n$ acted upon by the discretized special orthogonal group, then for odd n we have $\mathrm{ZA}(H_n) = \mathrm{Z}_{\mathrm{SO}(n)}\mathrm{A}(\mathbb{R}^n)$, while for even n, $\mathrm{ZA}(H_n) = \mathrm{Z}_{\mathrm{SO}(n)/\{\pm I\}}\mathrm{A}(\mathbb{R}^n \rtimes \{\pm I\})$.

A general analysis of algebras $\mathrm{ZA}(H)$ for locally compact H is beyond the scope of our present investigation. For groups with pre-compact conjugacy classes (a class which does not include examples H_n, above), there are some results for $\mathrm{ZL}^1(H)$ in [7].

4. Virtually Abelian groups

4.1. On sets of synthesis in certain fixed-point subalgebras. The purpose of this section is to gather some abstract results which will be useful for understanding $\mathrm{ZA}(G)$ for a virtually Abelian (locally) compact group G, in the next section.

Let $\mathcal{A}$ denote a commutative Banach algebra. *For the remainder of this section, we shall assume that $\mathcal{A}$ is unital, semisimple, regular and conjugate-closed on its Gelfand spectrum X.* Given a group of automorphisms B on $\mathcal{A}$ we let

$$\mathrm{Z}_B\mathcal{A} = \bigcap_{\beta\in B}\{u\in\mathcal{A} : \beta(u)=u\}$$

denote the fixed point algebra. If B is compact and acts continuously, we let $Z_B : \mathcal{A}\to \mathrm{Z}_B\mathcal{A}$ be given by $Z_B u = \int_B \beta(u)\,d\beta$ (normalized Haar measure), which may be understood as a Bochner integral. It is a surjective quotient map which satisfies the expectation property $Z_B(uv) = uZ_B v$ for u in $\mathrm{Z}_B\mathcal{A}$ and v in $\mathcal{A}$.

The following result is mostly known for $\mathrm{ZA}(G)$; it is partially established in [28, (34.37)], and we borrow aspects of the proof. As we require this result in a wider scope of applications, we give the simple proof.

PROPOSITION 4.1. *Let B be a compact group of continuous automorphisms on $\mathcal{A}$. The Gelfand spectrum of $\mathrm{Z}_B\mathcal{A}$ is the orbit space X/B, and this algebra is regular on its spectrum.*

Proof. Since X is the spectrum of $\mathcal{A}$, each β in B defines an automorphism $\beta^*|_X$ of X. The orbit space $X/B = \{B^*x : x\in X\}$, with quotient topology comprises a closed subset of the spectrum of $\mathrm{Z}_B\mathcal{A}$, and regularity of $\mathcal{A}$ passes immediately to the regularity of $\mathrm{Z}_B\mathcal{A}$ on X/B. Indeed, each of these facts is a consequence of the following observation. As regularity of $\mathcal{A}$ on its spectrum implies normality (see, for example, [28, (39.17)]), if E and F are B^*-invariant closed subsets of X, with $E\cap F=\varnothing$, then there is u in $\mathcal{A}$ for which $u|_E = 1$ and $u|_F = 0$. It is clear that that $Z_B u|_E = 1$ and $Z_B u|_F = 0$ too.

Let χ be any multiplicative character on $\mathrm{Z}_B\mathcal{A}$. Suppose that for some $u_1,\dots,u_n$ in $\ker\chi$, $\bigcap_{k=1}^n u_k^{-1}\{0\}\cap X/B = \varnothing$. Then $u = \sum_{k=1}^n |u_k|^2 > 0$ on X/B, hence, when regarded as an element of $\mathcal{A}$, is non-vanishing on X. Thus u admits an inverse u' in $\mathcal{A}$. But then $Z_B u'$ is the inverse of u in $\mathrm{Z}_B\mathcal{A}$, contradicting that $u\in\ker\chi$. We thus conclude that for any finite family $F\subset \mathrm{Z}_B\mathcal{A}$, $\bigcap_{u\in F} u^{-1}\{0\}\cap X/B\neq\varnothing$, and a compactness argument yields that $\bigcap_{u\in \mathrm{Z}_B\mathcal{A}} u^{-1}\{0\}\cap X/B\neq\varnothing$. It follows that $\chi\in X/B$. □

REMARK 4.2. We can recover the result of [18] that for a compact subgroup K of a locally compact group H, the algebra $\mathrm{A}(H:K) = \{u\in\mathrm{A}(H) : u(xk)=u(x)$ for u in H and k in $K\}$ has spectrum the coset space G/K. Indeed, consider the unitization $\mathrm{A}(H)\oplus\mathbb{C}1$ and let K act as automorphisms on this algebra by right translation; we obtain $\mathrm{A}(H:K)$ as a the subalgebra of $\mathrm{Z}_K(\mathrm{A}(H)\oplus\mathbb{C}1)$ of functions vanishing at infinity.

Now let E be a closed subset of X. We let

$$\mathrm{I}_{\mathcal{A}}(E) = \{u\in\mathcal{A} : u|_E = 0\},\quad\text{and}\quad \mathrm{I}^0_{\mathcal{A}}(E) = \{u\in\mathcal{A} : \operatorname{supp} u\cap E=\varnothing\}.$$

Definition 4.3. We say that E is

- *spectral* for $\mathcal{A}$ if $\overline{\mathrm{I}^0_{\mathcal{A}}(E)} = \mathrm{I}_{\mathcal{A}}(E)$;
- *Ditkin* for $\mathcal{A}$ provided each u in $\mathrm{I}_{\mathcal{A}}(E)$ satisfies that $u \in \overline{u\mathrm{I}^0_{\mathcal{A}}(E)}$;
- *ultra-strongly Ditkin* if $\mathrm{I}^0_{\mathcal{A}}(E)$ possesses a bounded approximate identity for $\mathrm{I}_{\mathcal{A}}(E)$; and
- *approximable* if $\mathrm{I}_{\mathcal{A}}(E)$ possesses a bounded approximate identity.

The definition of spectrality is well known, as is the Ditkin condition, also sometimes called the *Calderón* condition; see, for example, [61]. Following [63], [11], we will say that E is *strongly Ditkin* for $\mathcal{A}$ if $\mathrm{I}^0_{\mathcal{A}}(E)$ posseses a multiplier bounded approximate identity (u_α) for $\mathrm{I}_{\mathcal{A}}(E)$, i.e. so $\sup_\alpha \|u_\alpha u\| \leq C\|u\|$ for all u in $\mathrm{I}_{\mathcal{A}}(E)$. Note that a sequential approximate identity is automatically multiplier bounded, thanks to the uniform boundedness principle. We shall not require the last notion but mention it only for comparative purposes. The ultra-strongly Ditkin condition is defined in [11]. The term "approximable" is not in wide use as we use it, and has been used by the second named author in [58].

We have the following implications of properties for a closed subset E of X:

$$\begin{matrix}\text{approximable} \\ \text{\& spectral}\end{matrix} \Leftrightarrow \begin{matrix}\text{ultra-strongly} \\ \text{Ditkin}\end{matrix} \Rightarrow \begin{matrix}\text{strongly} \\ \text{Ditkin}\end{matrix} \Rightarrow \text{Ditkin} \Rightarrow \text{spectral.}$$

Remark 4.4. (i) It is not the case that approximable implies spectral. In [9], a remarkable example of a semi-simple, conjugate-closed, regular sequence algebra $\mathcal{A}$ is created which admits a contractive approximate identity, but for which the space of finitely supported elements $\mathcal{A}_c$ is not dense in $\mathcal{A}$. In particular, the unitization $\mathcal{A}\oplus\mathbb{C}1$ has spectrum the compactification $\mathbb{N}\cup\{\infty\}$, and hence $\{\infty\}$ is an approximable but non-spectral set for $\mathcal{A}\oplus\mathbb{C}1$.

(ii) It follows from Remark 4.5(ii), below that not every spectral set is Ditkin. In [46], a (strongly) Ditkin set is produced in $\mathrm{A}(\mathbb{T})$ with countable infinite boundary; so the boundary is an infinite Ditkin set of measure zero. Hence according to [63], this boundary set is not strongly Ditkin. The unitization of the pointwise algebra $\ell^1(\mathbb{N})$ has spectrum $\mathbb{N}\cup\{\infty\}$, and the set $\{\infty\}$ is strongly Ditkin for this algebra but not ultra-strongly Ditkin.

Remark 4.5. (i) It is known, due to [61] (see also [36, 5.2.1]) that a finite union of Ditkin sets is Ditkin. An easy variant of an argument of [63] tells us that the same is true for ultra-strongly Ditkin or approximable sets. Indeed, if E and F are approximable (respectively, ultra-strongly Ditkin), let (u_α) be a bounded approximate identity for $\mathrm{I}_{\mathcal{A}}(E)$ and (v_β) one for $\mathrm{I}_{\mathcal{A}}(F)$ (each contained in $\mathrm{I}^0_{\mathcal{A}}(E)$, respectively $\mathrm{I}^0_{\mathcal{A}}(F)$). Then $(u_\alpha v_\beta)$ (product directed set) is a bounded approximate identity for $\mathrm{I}_{\mathcal{A}}(E\cup F) = \mathrm{I}_{\mathcal{A}}(E)\cap\mathrm{I}_{\mathcal{A}}(F)$ (contained in $\mathrm{I}^0_{\mathcal{A}}(E\cup F) = \mathrm{I}^0_{\mathcal{A}}(E)\cap\mathrm{I}^0_{\mathcal{A}}(F)$), as is easily checked.

(ii) It is shown in [6] for the unitized Mirkil algebra, that there exists two spectral sets whose union is not spectral. In particular, spectral sets need not be Ditkin.

We now observe that finite groups of automorphisms preserve certain properties of sets which are stable under finite unions.

THEOREM 4.6. *Let B be a finite group of automophisms on $\mathcal{A}$ and E be a closed subset of X which is Ditkin, ultra-strongly Ditkin or approximable for $\mathcal{A}$. Then the subset B^*E of X/B enjoys the same property for $\mathrm{Z}_B\mathcal{A}$.*

Proof. We observe that for each automorphism β, we have $\beta(\mathrm{I}_{\mathcal{A}}(E)) = \mathrm{I}_{\mathcal{A}}(\beta^*E)$ and $\beta(\mathrm{I}^0_{\mathcal{A}}(E)) = \mathrm{I}^0_{\mathcal{A}}(\beta^*E)$. Hence, Remark 4.5 shows that $B^*E = \bigcup_{\beta\in B}\beta^*E$ is Ditkin, ultra-strongly Ditkin or approximable for $\mathcal{A}$, based on the respective assumption for E. Furthermore, we observe that for any u in $\mathcal{A}$ we have $Z_Bu = \frac{1}{|B|}\sum_{\beta\in B}\beta(u)$, and hence

$$Z_B\mathrm{I}^0_{\mathcal{A}}\big(B^*E\big) = \mathrm{I}^0_{\mathrm{Z}_B\mathcal{A}}\big(B^*E\big) \subseteq \mathrm{I}^0_{\mathcal{A}}\big(B^*E\big)$$

and the same sequence of inclusions holds for $\mathrm{I}_{\mathcal{A}}$. Suppose $u \in \mathrm{I}_{\mathrm{Z}_B\mathcal{A}}(B^*E)$ and (u_α) is a net from $\mathrm{I}^0_{\mathcal{A}}(E)$ for which $\|uu_\alpha - u\| \xrightarrow{\alpha} 0$. Then $uZ_Bu_\alpha - u = Z_B(uu_\alpha - u) \xrightarrow{\alpha} 0$. We immediately see that Ditkinness or ultra-strong Ditkinness is preserved. By merely picking (u_α) from within $\mathrm{I}_{\mathcal{A}}(E)$, we see that approximability is preserved. $\square$

PROPOSITION 4.7. *If the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is semisimple, and B is a compact group of automorphisms on $\mathcal{A}$, then $\mathrm{Z}_B\mathcal{A}\hat{\otimes}\mathrm{Z}_B\mathcal{A} = \mathrm{Z}_{B\times B}\mathcal{A}\hat{\otimes}\mathcal{A}$.*

Proof. Since Z_B is a quotient map, $\mathrm{Z}_B\mathcal{A}\hat{\otimes}\mathrm{Z}_B\mathcal{A}$ is isometrically a subspace of $\mathcal{A}\hat{\otimes}\mathcal{A}$. Moreover, $Z_B \otimes Z_B = Z_{B\times B}$. $\square$

Suppose $\mathcal{A}\hat{\otimes}\mathcal{A}$ is semisimple. With our assumptions $\mathcal{A}\hat{\otimes}\mathcal{A}$ is regular on its spectrum $X \times X$ ([60]). Then, following [54, Theo. 6], we call $\mathcal{A}$ *hyper-Tauberian* if the diagonal

$$X_D = \big\{(x, x) : x \in X\big\}$$

is spectral for $\mathcal{A}\hat{\otimes}\mathcal{A}$. It is a well-known interpretation of the splitting result of [25] (see also [14]) that approximability of X_D for $\mathcal{A}\hat{\otimes}\mathcal{A}$ is equivalent to amenability of $\mathcal{A}$. We further note that for a compact group of automorphisms on $\mathcal{A}$ that

$$(X/B)_D = \big\{\big(B^*x, B^*x\big) : x \in X\big\} = (B \times B)^*X_D.$$

These comments combine with the last two results to give us the following.

COROLLARY 4.8. *Suppose $\mathcal{A}\hat{\otimes}\mathcal{A}$ is semisimple and let B be a finite group of continuous automorphisms on $\mathcal{A}$.*

(i) *If X_D is Ditkin for $\mathcal{A}\hat{\otimes}\mathcal{A}$, then $\mathrm{Z}_B\mathcal{A}$ is hyper-Tauberian.*
(ii) *If $\mathcal{A}$ is amenable, then $\mathrm{Z}_B\mathcal{A}$ is amenable.*

We observe that (ii), above, follows from a more general result of [37].

We shall say that a closed subset E of X is *weakly spectral* for $\mathcal{A}$ if there is a fixed $n>0$ for which $\mathrm{I}_{\mathcal{A}}(E)^n=\{u^n : u\in \mathrm{I}_{\mathcal{A}}(E)\}\subseteq \overline{\mathrm{I}^0_{\mathcal{A}}(E)}$. We let the characteristic of E with respect to $\mathcal{A}$, $\xi_{\mathcal{A}}(E)$, denote the minimal such n, so $\xi(E)=1$ if E is spectral. These concepts were introduced in [62].

Proposition 4.9. *Suppose B is a compact group of automorphisms on $\mathcal{A}$ and $E=B^*E$ be weakly spectral for $\mathcal{A}$. Then E is weakly spectral for $\mathrm{Z}_B\mathcal{A}$ (with $\xi_{\mathrm{Z}_B\mathcal{A}}(E)\leq \xi_{\mathcal{A}}(E)$).*

Proof. It is evident that $Z_B\mathrm{I}_{\mathcal{A}}(E)=\mathrm{I}_{\mathrm{Z}_B\mathcal{A}}(E)$. It is also true that $Z_B\mathrm{I}^0_{\mathcal{A}}(E)=\mathrm{I}^0_{\mathrm{Z}_B\mathcal{A}}(E)$. Indeed, if $u\in \mathrm{I}^0_{\mathcal{A}}(E)$, then $\operatorname{supp} u\cap E=\varnothing$. Then there there is open $U=B^*U\supset E$ such that $\operatorname{supp} u\cap \overline{U}=\varnothing$. If not, then

$$\operatorname{supp} u\cap E=\operatorname{supp} u\cap \bigcap\big\{\overline{U}:U=B^*U \text{ open, } U\supset E\big\}\neq\varnothing$$

violating our initial assumption. Thus it follows that $\operatorname{supp}(Z_G u)\cap E=\varnothing$.

We note that $\mathrm{I}_{\mathrm{Z}_B\mathcal{A}}(E)\subset \mathrm{I}_{\mathcal{A}}(E)$. Hence, if E is weakly spectral for $\mathcal{A}$, then for $u\in \mathrm{I}_{\mathrm{Z}_B\mathcal{A}}(E)$, $u^{\xi_{\mathcal{A}}(E)}\in\overline{\mathrm{I}^0_{\mathcal{A}}(E)}$. But

$$u^{\xi_{\mathcal{A}}(E)}=Z_B u^{\xi_{\mathcal{A}}(E)}\in Z_B\overline{\mathrm{I}^0_{\mathcal{A}}(E)}\subseteq \overline{Z_B\mathrm{I}^0_{\mathcal{A}}(E)}=\overline{\mathrm{I}^0_{\mathrm{Z}_B\mathcal{A}}(E)}.$$

Hence, $\xi_{\mathrm{Z}_B\mathcal{A}}(E)\leq \xi_{\mathcal{A}}(E)$. ☐

4.2. Virtually Abelian groups. We say a locally compact group is *virtually Abelian* if it admits an Abelian subgroup of finite index, hence necessarily an open such subgroup. In the case of a compact G, an open Abelian subgroup is automatically of finite index. As with the article so far, *we assume that G is compact for the remainder of the section.*

Theorem 4.10. *Let G be virtually Abelian. Then there exists a normal open Abelian subgroup T. We then have that the T-centre, that is, when T acts on G by inner automorphisms, is given by an isomorphic identification*

$$\mathrm{Z}_T\mathrm{A}(G)=\bigoplus_{aT\in G/T}\mathrm{A}(aT:R_a),$$

where $R_a=R_{aT}$ is a closed subgroup of T and $\mathrm{A}(aT:R_a)=\{u\in 1_{aT}\mathrm{A}(G): u(atr)=u(at)$ for t in T and r in $R_a\}$. The algebra $\mathrm{Z}_T\mathrm{A}(G)$ admits spectrum $X=\bigsqcup_{aT\in G/T}aT/R_a$. We have

$$\mathrm{ZA}(G)=\mathrm{Z}_{G/T}\mathrm{Z}_T\mathrm{A}(G),$$

where the action of G, that is, of G/T, on an element of X is given by $bT\cdot atR_a=bab^{-1}btb^{-1}R_{bab^{-1}}$. For each a in G and t in T, we have conjugacy class

$$C_{at}=\big\{bab^{-1}btb^{-1}r : b\in G \text{ and } r\in R_{bab^{-1}T}\big\}.$$

Proof. Let S be an open Abelian subgroup and L a left transversal for S in G. Then

$$T = \bigcap_{b \in L} bSb^{-1}$$

is an open normal subgroup. Indeed, L is finite so T is the intersection of finitely many open subgroups. Furthermore, the definition of T is independent of choice of transversal and for any a in G, aL is another transversal, hence $aTa^{-1} = \bigcap_{b \in aL} bSb^{-1} = T$, so T is normal.

Since T is Abelian $s \mapsto s^{-1}$ is a homomorphism, and also since $s \mapsto a^{-1}sa$ is a homomorphism it is easy to see that

$$R_a = \left\{ s^{-1}a^{-1}sa : s \in T \right\}$$

is a subgroup of T, which is closed as T is compact. Observe that for a and a' in G, if $Ta = Ta'$, then $R_a = R_{a'}$. Hence, we may write $R_{Ta} = R_{aT} = R_a$. We recall that $\mathrm{A}(G) = \bigoplus_{aT \in G/T} \mathrm{A}(aT)$ where $\mathrm{A}(aT) = 1_{aT}\mathrm{A}(G) \cong \mathrm{A}(T)$. Now if a in G and t in T are fixed, then for s in T we have

$$sats^{-1} = a\left(a^{-1}sa\right)ts^{-1} = ats^{-1}\left(a^{-1}sa\right).$$

Hence orbits of the action of T on aT, by conjugation, are the same as orbits of the action of R_a on aT by right translation; we write $aT/\sim_T = aT/R_a$. We thus obtain the desired form for $\mathrm{Z}_T\mathrm{A}(G)$ and its spectrum $X = G/{\sim_T}$.

It is evident that $\mathrm{ZA}(G) \subseteq \mathrm{Z}_T\mathrm{A}(G)$, and the action of G by inner automorphisms on $X = G/{\sim_T}$ is really an action by G/T. In fact if we let $Z_T u(x) = \int_T u(sxs^{-1})\,ds$, then the Weyl integral formula tells us that $Z_G = Z_G \circ Z_T = Z_{G/T} \circ Z_T$. Hence, we gain the desired realization of $\mathrm{ZA}(G)$.

To see the action of G on X, and hence the structure of the conjugacy class C_{at}, we fix a and t as above, and for b in G and s in T we have

$$\begin{aligned} &bab^{-1}\left(btb^{-1}\right)\left[bs^{-1}b^{-1}\left(ba^{-1}b^{-1}bsb^{-1}bab^{-1}\right)\right] \\ &\quad = bab^{-1}\left(ba^{-1}b^{-1}bsb^{-1}bab^{-1}\right)\left(btb^{-1}\right)bs^{-1}b^{-1} = bs(at)(bs)^{-1}. \end{aligned}$$

Since each bsb^{-1} is a generic element of T, we get the desired result. □

THEOREM 4.11. *If G is virtually Abelian, then $\mathrm{ZA}(G)$ is hyper-Tauberian and amenable.*

Proof. We consider the algebra $\mathrm{Z}_T\mathrm{A}(G)$ and its spectrum X, whose form is described in Theorem 4.10. Each $\mathrm{A}(aT : R_a) \cong \mathrm{A}(T/R_a)$ (which is the Abelian group algebra $\mathrm{L}^1(\widehat{T/R_a})$). The diagonal $(T/R_a)_D$ in $T/R_a \times T/R_a$ is a subgroup and hence, thanks to [51], ultra-strongly Ditkin for $\mathrm{A}(T/R_a)\hat{\otimes}\mathrm{A}(T/R_a) \cong \mathrm{A}(T/R_a \times T/R_a)$. Thus, Remark 4.5 shows us that $X_D \cong \bigcup_{aT \in G/T}(T/R_a)_D$ is also ultra-strongly Ditkin. Letting $B = G/T$, we appeal to Corollary 4.8. □

REMARK 4.12. Let G and T be as in Theorem 4.10. Using reasoning above, we see that $1_T\,\mathrm{ZA}(G)=\mathrm{Z}_{G/T}\mathrm{A}(T)$, is amenable. In particular for $G=\mathbb{T}\rtimes\{\mathrm{id},\iota\}$, where $\iota(t)=t^{-1}$, we have that $\mathrm{Z}_{\{\mathrm{id},\iota\}}\mathrm{A}(\mathbb{T})\cong\mathrm{Z}_{\{\mathrm{id},\hat{\iota}\}}\ell^1(\mathbb{Z})\cong\ell^1(\mathbb{Z}/\{\mathrm{id},\hat{\iota}\})$. Here, $\mathbb{Z}/\{\mathrm{id},\iota\}\cong\mathbb{N}_0$ is the polynomial hypergroup with multiplication $\delta_n*\delta_m=\frac{1}{2}(\delta_{|n-m|}+\delta_{n+m})$. This hypergroup algebra is also proved to be amenable in [38].

In fact, we may define a class of hypergroups by letting F be any finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$ and considering the orbit space $\mathbb{Z}^n/F$. We let $\ell^1(\mathbb{Z}^n/F)$ denote the closed subalgebra of $\ell^1(\mathbb{Z}^n)$ generated by elements

$$\delta_{F(v)}=\frac{1}{|F|}\sum_{\alpha\in F}\delta_{\alpha(v)},\quad v\in\mathbb{Z}^n.$$

We have that $\ell^1(\mathbb{Z}^n/F)\cong\mathrm{Z}_F\ell^1(\mathbb{Z}^n)$ is amenable. Indeed, if $G=\mathbb{T}^n\rtimes F$, where F acts by dual action, then $\mathrm{Z}_F\ell^1(\mathbb{Z}^n)\cong\mathrm{Z}_{G/\mathbb{T}^n}\mathrm{A}(\mathbb{T}^n)$.

4.3. Approximable subsets of ZA(G)**.** We aim to give, for any compact group G, a characterisation of approximable subsets for ZA(G). When G is virtually Abelian this characterisation is especially satisfying. Recall that approximable sets were defined in Definition 4.3. See Section 1.2 for references on approximable subsets in $A(H)$, for amenable locally compact H.

A *coset* in a locally compact group H is any subset K of H which is closed under the ternary operation: $x,y,z\in K$ implies $xy^{-1}z\in K$. It is an exercise to see that this agrees with the "standard" notion of coset of some subgroup. We let $\Omega(H)$ denote the Boolean algebra generated by all cosets of H, and $\Omega_c(H)$ denote those elements of $\Omega(G)$ which are closed.

PROPOSITION 4.13. *Let E be a closed subset of* Conj(G). *Then E is approximable for* ZA(G) *if and only if $\tilde{E}\in\Omega_c(G)$, where $\tilde{E}=\bigcup_{C\in E}C$.*

Proof. If (u_α) is a bounded approximate identity for $\mathrm{I}_{\mathrm{ZA}}(E)$. Then (u_α) is a bounded net in $\mathrm{A}(G)\subseteq\mathrm{B}(G_d)$, where the latter space is the Fourier-Stieltjes algebra of the discretized group G_d. The embedding is an isometry thanks to [16]. Since ZA(G) is regular, (u_α) converges pointwise to the indicator function $1_{\mathrm{Conj}(G)\setminus E}$ on Conj(G), hence to $1_{G\setminus\tilde{E}}$ in the weak* topology of $\mathrm{B}(G_d)$. Hence by [29], $\tilde{E}\in\Omega(G)$. Since $\tilde{E}$, being the pre-image of E in G under the conjugation equivalence, is closed, we see that $\tilde{E}\in\Omega_c(G)$.

If $\tilde{E}\in\Omega_c(G)$, then by [19], $\tilde{E}$ is approximable for A(G). If (v_α) is a bounded approximate identity for $\mathrm{I}_{\mathrm{A}}(\tilde{E})$, then $(Z_G v_\alpha)$ is such for $\mathrm{I}_{\mathrm{ZA}}(E)$. □

We remark that the last proposition reduces the general question of amenability of ZA(G) into a question of whether the diagonal $E=\mathrm{Conj}(G)_D$ in $\mathrm{Conj}(G)\times\mathrm{Conj}(G)$ satisfies that $\tilde{E}\in\Omega(G\times G)$, hence is automatically in $\Omega_c(G\times G)$. (Indeed it follows from Lemma 6.1 that $\mathrm{ZA}(G)\hat{\otimes}\mathrm{ZA}(G)\cong\mathrm{ZA}(G\times G)$). We do not know how to determine this for a general, even totally disconnected, group, unless it is a product of finite groups; see Theorem 6.5, for example.

Theorem 4.14. *If G is virtually Abelian, and $E \in \Omega_c(G)$, then $G^*E = \bigcup_{x\in E} C_x \in \Omega_c(G)$. Hence the approximable subsets of* $\mathrm{ZA}(G)$ *are exactly sets of the form G^*E where $E \in \Omega_c(G)$.*

Proof. A result of [17] (after [24], [55]) shows that there are a finite number of closed subgroups $H_1, \dots, H_n$, elements $a_1, \dots, a_n$ of G, and for each k, open subgroups $K_{k1}, \dots, K_{km_k}$ of H_k and elements $b_{k1}, \dots, b_{km_k}$ of H_k such that $E = \bigcup_{k=1}^{n} a_k(H_k \setminus \bigcup_{j=1}^{m_k} b_{kj}K_{kj})$. Since each H_k is compact, each K_{kj} is of finite index. We can hence rearrange this result to show that E is simply a union of finitely many closed cosets of subgroups of G. Let such a coset be given by aH where H is a closed subgroup. By taking intersection, we may suppose H is a subgroup of an open normal Abelian subgroup T, hence $aH \subset aT$.

However, the calculations from the proof of Theorem 4.10 show that the orbit of aH under conjugation by G is $\bigcup_{bT\in G/T} bab^{-1}bHb^{-1}R_{bab^{-1}}$, which is clearly an element of $\Omega_c(G)$.

The characterization of approximable sets is now a direct consequence of Proposition 4.13 and the fact that $G^*(G^*E) = G^*E$. □

5. Hyper-Tauberian property and weak amenability

We can give a complete characterization of both hyper-Tauberianness and weak amenability for $\mathrm{ZA}(G)$. Recall that the definition of hyper-Tauberianness is given before Corollary 4.8.

Theorem 5.1. *For any compact group G the following are equivalent*:

(i) *the connected component of the identity, G_e, is Abelian*;
(ii) $\mathrm{ZA}(G)$ *is hyper-Tauberian*;
(iii) *all singleton sets of* $\mathrm{Conj}(G)$ *are spectral for* $\mathrm{ZA}(G)$;
(iv) $\mathrm{ZA}(G)$ *is weakly amenable*; *and*
(v) $\mathrm{ZA}(G)$ *admits no non-zero bounded point derivations.*

Proof. That (ii) implies (iii) and (iv) are both from [54, Theo. 5]. A well-known observation from [56] is that a commutative Banach algebra admits a non-zero bounded point derivation at a multiplicative functional χ if and only if $(\ker\chi)^2$ is not dense in $\ker\chi$. Hence, (iii) implies (v). That (iv) implies (v) follows from a well-known fact mentioned in Section 2. Theorem 3.2 provides that (v) implies (i).

Hence it remains to see that (i) implies (ii). In the case that G is virtually Abelian, this is from Theorem 4.11.

Now let us consider the general case where G_e is Abelian. We follow the proof of [20, Theo. 3.3]. We let $\mathcal{N}$ be a net of closed normal subgroups, ordered by reverse inclusion, such that for any neighbourhood U of e, we eventually have $N \subset U$ for some N in $\mathcal{N}$, and for which G/N is Lie for each N in $\mathcal{N}$. For example, we may let $N = N_F = \bigcap_{\pi \in F} \ker \pi$ for the increasing net of all finite subsets of $\widehat{G}$. Then each G/N is virtually Abelian. Indeed, it follows from [27, (7.12)], that each $(G/N)_e$ is connected, hence open as G/N is Lie. Hence, by Proposition 2.1, $\mathrm{ZA}(G/N) \cong P_N(\mathrm{ZA}(G))$, and hence is hyper-Tauberian. Also if $N \supset N'$ then $P_N(\mathrm{ZA}(G)) \subset P_{N'}(\mathrm{ZA}(G))$. It then is easy to check that for u in $\mathrm{ZA}(G)$, the difference $u - P_N u$ tends to 0 as N tends to e. Hence, $\bigcup_{N \in \mathcal{N}} P_N \,\mathrm{ZA}(G)$ is dense in $\mathrm{ZA}(G)$. We appeal to [54, Cor. 13] to see that ℓ^1-$\bigoplus_{N \in \mathcal{N}} P_N \,\mathrm{ZA}(G)$ is hyper-Tauberian, and hence the completion $\mathrm{ZA}_{\mathcal{N}}(G)$ of $\bigcup_{N \in \mathcal{N}} P_N \,\mathrm{ZA}(G)$, with respect to the norm

$$\|u\|_{\mathcal{N}} = \inf\left\{ \sum_{N \in \mathcal{N}} \|u_N\|_{\mathrm{A}} : u = \sum_{N \in \mathcal{N}} u_N, u_N \in P_N \,\mathrm{ZA}(G) \right\} \geq \|u\|_{\mathrm{A}}$$

is hyper-Tauberian, thanks to [54, Theo. 12]. Notice for $N \supseteq N'$ that $P_N \,\mathrm{ZA}(G) P_{N'} \,\mathrm{ZA}(G) \subseteq P_{N'} \,\mathrm{ZA}(G)$, so $\|\cdot\|_{\mathcal{N}}$ is indeed an algebra norm. But the continuous inclusion with dense range, $\mathrm{ZA}_{\mathcal{N}}(G) \hookrightarrow \mathrm{ZA}(G)$, shows again by [54, Theo. 12] that the latter algebra is hyper-Tauberian. □

REMARK 5.2. (i) We note that using the density of $\bigcup_{N \in \mathcal{N}} P_N \,\mathrm{ZA}(G)$ in $\mathrm{ZA}(G)$, above, we may have more easily shown that (i) implies (iv) directly. We found it more satisfying to obtain the stronger hyper-Tauberian property. We note that [54, Rem. 24(ii)] shows that hyper-Tauberianness is stronger than weak amenability.

(ii) A technique employed in the proof of [22, Lem. 3.6] and [7, Theo. 2.4] can be used to allow us to bypass the algebra $\mathrm{ZA}_{\mathcal{N}}(G)$, employed above. Our present technique allows us to avoid introducing local maps, which, admittedly, are used in the original definition of hyper-Tauberianness in [54].

The following partially recovers a non-spectral result of [45], but uses different methods. We defined weak spectrality just before Proposition 4.9.

PROPOSITION 5.3. *Let G be a non-Abelian compact connected Lie group.*

(i) *There exists a C in* $\mathrm{Conj}(G)$ *for which C is not spectral for* $\mathrm{A}(G)$.

(ii) *Any finite $F \subseteq \mathrm{Conj}(G)$ is weakly spectral for* $\mathrm{ZA}(G)$ *with* $\xi_{\mathrm{ZA}}(F) \leq |F| + \sum_{C \in F} \dim C/2$ *where* $\dim C$ *is the dimension of the manifold C.*

Proof. The failure of spectrality for some $\{C\}$ follows from (iii), above, and Proposition 4.9. On the other hand, [44, Cor. 4.9] shows that a single conjugacy class C is always weakly spectral for $\mathrm{A}(G)$ with $\xi_{\mathrm{A}}(C) \leq 1 + \dim C/2$. Hence, the subadditivity result in [62] shows that

$$\xi_{\mathrm{A}}\left(\bigcup_{C \in F} C \right) \leq \sum_{C \in F} \xi_{\mathrm{A}}(C) = |F| + \sum_{C \in F} \dim C/2.$$

Again we appeal to Proposition 4.9. □

6. Finite groups and their direct products

We recall from [33], that amenability of a Banach algebra $\mathcal{A}$ is equivalent to having a *bounded approximate diagonal* (b.a.d.): a bounded net $(d_\iota) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ which for each a in $\mathcal{A}$ satisfies

$$(a \otimes 1)d_\iota - d_\iota(1 \otimes a) \xrightarrow{\iota} 0 \quad \text{and} \quad m(d_\iota)a \xrightarrow{\iota} a,$$

where $m : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is the multiplication map. We let the *amenability constant* be given by

$$\mathrm{AM}(\mathcal{A}) = \inf\big\{M > 0 : \text{there is a b.a.d. } (d_\iota) \text{ for } \mathcal{A} \text{ with } \|d_\iota\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} \le M\big\},$$

where we adopt the convention that $\inf \varnothing = \infty$. It will be useful for us to understand the tensor product $\mathrm{ZA}(G)\hat{\otimes}\,\mathrm{ZA}(G')$, where G' is another compact group.

Lemma 6.1. *We have an isometric isomorphism*

$$\mathrm{ZA}(G)\hat{\otimes}\,\mathrm{ZA}\big(G'\big) \cong \mathrm{ZA}\big(G \times G'\big).$$

Proof. Let us give two proofs.

For the first, we recall the theorem of [15] that

$$\mathrm{A}(G)\hat{\otimes}^{\mathrm{op}}\mathrm{A}\big(G'\big) \cong \mathrm{A}\big(G \times G'\big), \tag{6.1}$$

where $\hat{\otimes}^{\mathrm{op}}$ denotes the operator projective tensor product. The map Z_G : $\mathrm{A}(G) \to \mathrm{ZA}(G)$ is easily verified to be a complete quotient map, so we have a completely isometric inclusion

$$\mathrm{ZA}(G)\hat{\otimes}^{\mathrm{op}}\,\mathrm{ZA}\big(G'\big) = Z_G \otimes Z_{G'}\big(\mathrm{A}(G)\hat{\otimes}^{\mathrm{op}}\mathrm{A}\big(G'\big)\big) \subset \mathrm{A}(G)\hat{\otimes}^{\mathrm{op}}\mathrm{A}\big(G'\big).$$

But in the identification (6.1), we have that $Z_G \otimes Z_{G'} \cong Z_{G\times G'}$, so $Z_G \otimes Z_{G'}(\mathrm{A}(G)\hat{\otimes}^{\mathrm{op}}\mathrm{A}(G')) \cong \mathrm{ZA}(G \times G')$. Since $\mathrm{ZA}(G)^* \cong \mathrm{Z\,VN}(G)$ is a commutative von Neuman algebra, we obtain an isometric identification $\mathrm{ZA}(G)\hat{\otimes}^{\mathrm{op}}\,\mathrm{ZA}(G') \cong \mathrm{ZA}(G)\hat{\otimes}\,\mathrm{ZA}(G')$.

For the second proof, we use the fact that $\mathrm{ZA}(G) \cong \ell^1(\widehat{G}, d^2)$ as noted in Section 2. Using that $\widehat{G \times G'} \cong \widehat{G} \times \widehat{G'}$ (irreducible representations of products are exactly the Kronecker products of irreducible representations) we see that $\mathrm{ZA}(G \times G') \cong \ell^1(\widehat{G} \times \widehat{G'}, d^2 \times d^2)$, where $d^2 \times d^2(\pi, \pi') = d_\pi^2 d_{\pi'}^2$. Hence, the usual tensor product formula shows that

$$\mathrm{ZA}(G)\hat{\otimes}\,\mathrm{ZA}\big(G'\big) \cong \ell^1\big(\widehat{G}, d^2\big)\hat{\otimes}\ell^1\big(\widehat{G'}, d^2\big) \cong \ell^1\big(\widehat{G} \times \widehat{G'}, d^2 \times d^2\big) \cong \mathrm{ZA}\big(G \times G'\big)$$

with isometric identifications. $\square$

The following computation on a finite group mirrors [7, Theo. 1.8], where for a finite group G it is shown that

$$\mathrm{AM}\big(\mathrm{ZL}^1(G)\big) = \frac{1}{|G|^2} \sum_{C,C' \in \mathrm{Conj}(G)} |C|\big|C'\big| \Bigg| \sum_{\pi \in \widehat{G}} d_\pi^2 \chi_\pi(C)\overline{\chi_\pi\big(C'\big)} \Bigg|.$$

PROPOSITION 6.2. *Let G be a finite group. Then*

$$\mathrm{AM}\big(\mathrm{ZA}(G)\big) = \frac{1}{|G|^2} \sum_{\pi,\pi' \in \widehat{G}\times\widehat{G}} d_\pi d_{\pi'} \left| \sum_{C\in\mathrm{Conj}(G)} |C|^2 \chi_\pi(C)\overline{\chi_{\pi'}(C)} \right|.$$

In particular we see that $1 \leq \mathrm{AM}(\mathrm{ZA}(G))$, *with the bound achieved exactly when G is Abelian.*

Proof. Any bounded approximate diagonal admits a cluster point, which is a diagonal; that is, d in $\mathrm{ZA}(G)\hat{\otimes}\mathrm{ZA}(G)$ for which $m(d) = 1$ and $(u \otimes 1)d = d(1 \otimes u)$. It was observed in [23] that for a finite dimensional amenable commutative algebra the diagonal is unique. In fact, Lemma 6.1 provides that this diagonal must be the indicator function of the diagonal of the spectrum of $\mathrm{ZA}(G\times G)$, $1_{\mathrm{Conj}(G)_D} = \sum_{C\in\mathrm{Conj}(G)} 1_{C\times C}$. The Schur orthogonality relations provide Fourier series

$$\begin{aligned} 1_{C\times C} &= \sum_{\pi,\pi' \in \widehat{G}\times\widehat{G}} \langle 1_{C\times C} | \chi_\pi \otimes \chi_{\pi'} \rangle \chi_\pi \otimes \chi_{\pi'} \\ &= \sum_{\pi,\pi' \in \widehat{G}\times\widehat{G}} \left(\frac{1}{|G|^2} \sum_{x,y\in G\times G} 1_{C\times C}(x,y)\overline{\chi_\pi(x)\chi_{\pi'}(y)} \right) \chi_\pi \otimes \chi_{\pi'} \\ &= \frac{1}{|G|^2} \sum_{\pi,\pi' \in \widehat{G}\times\widehat{G}} |C|^2 \overline{\chi_\pi(C)\chi_{\pi'}(C)} \chi_\pi \otimes \chi_{\pi'} \end{aligned}$$

and hence

$$1_{\mathrm{Conj}(G)_D} = \frac{1}{|G|^2} \sum_{\pi,\pi' \in \widehat{G}\times\widehat{G}} \left(\sum_{C\in\mathrm{Conj}(G)} |C|^2 \chi_\pi(C)\overline{\chi_{\pi'}(C)} \right) \chi_{\bar{\pi}} \otimes \chi_{\pi'},$$

where we have exchanged $\bar{\pi}$ for π to give our formula its “positive-definite” flavour. We again appeal to Lemma 6.1 and obtain

$$\mathrm{AM}\big(\mathrm{ZA}(G)\big) = \|1_{\mathrm{Conj}(G)_D}\|_{\mathrm{ZA}(G\times G)}$$

which, by (2.1) gives us the desired result.

Let us examine the lower bound. We restrict the outer sum to the diagonal to obtain

$$\begin{aligned} \mathrm{AM}\big(\mathrm{ZA}(G)\big) &\geq \frac{1}{|G|^2} \sum_{\pi\in\widehat{G}} d_\pi^2 \sum_{C\in\mathrm{Conj}(G)} |C|^2 \chi_\pi(C)\overline{\chi_\pi(C)} \\ &\geq \frac{1}{|G|} \sum_{\pi\in\widehat{G}} d_\pi^2 \sum_{C\in\mathrm{Conj}(G)} \frac{|C|}{|G|} \chi_\pi(C)\overline{\chi_\pi(C)} \\ &= \frac{1}{|G|} \sum_{\pi\in\widehat{G}} d_\pi^2 \langle \chi_\pi | \chi_\pi \rangle = \frac{1}{|G|} \sum_{\pi\in\widehat{G}} d_\pi^2 = 1. \end{aligned}$$

Notice that if G is non-Abelian, then at least one conjugacy class satisfies $|C|^2 > |C|$, and then for at least one π, say $\pi = 1$, $\chi_\pi(C) \neq 0$. Hence the second inequality, above, is strict. For an Abelian group, $\mathrm{ZA}(G) = \mathrm{A}(G) \cong \mathrm{L}^1(\widehat{G})$. The well-known diagonal $\frac{1}{|\widehat{G}|}\sum_{\chi\in\widehat{G}} \delta_{\bar{\chi}} \otimes \delta_\chi$ shows that $\mathrm{AM}(\mathrm{L}^1(\widehat{G})) = 1$. □

For a finite non-Abelian group, G, the lower bound of $\mathrm{AM}(\mathrm{ZL}^1(G)) \geq 1 + \frac{1}{300}$ was derived based on a result in [50]. Some improvements were made in the investigation [4]; and the generality of the lower bound of $\mathrm{AM}(\mathrm{ZL}^1(G)) \geq \frac{7}{4}$, which is sharp, was established in [13]. We have made no effort to establish if our lower bound, below, is sharp.

Corollary 6.3. *If G is a non-Abelian finite group, then* $\mathrm{AM}(\mathrm{ZA}(G)) \geq \frac{2}{\sqrt{3}}$.

Proof. Because G is compact, we have that $\mathrm{A}(G)$ is its own multiplier algebra, even its own completely bounded multiplier algebra. As such each u in $\mathrm{A}(G)$ induces a Schur multiplier on $G \times G$ matrices, $[a_{st}] \mapsto [u(s^{-1}t)a_{st}]$, with norm the same as $\|u\|_\mathrm{A}$. See [10], [35] for details of this.

The reasoning above also applies to $\mathrm{A}(G \times G)$. Consider the diagonal $w = 1_{\mathrm{Conj}(G)_D}$ element of $\mathrm{ZA}(G \times G) \subset \mathrm{A}(G \times G)$. It is an idempotent, that is, $w^2 = w$ with $\|w\|_\mathrm{A} > 1$. Hence by [59, Theo. 3.3] (using estimates which go back to [42]), we have that $\mathrm{AM}(\mathrm{ZA}(G)) = \|w\|_\mathrm{A} \geq \frac{2}{\sqrt{3}}$. □

Lemma 6.4. (i) *If $G_1, \ldots, G_n$ are finite groups and $P = \prod_{i=1}^n G_i$, then*

$$\mathrm{AM}\big(\mathrm{ZA}(P)\big) = \prod_{i=1}^n \mathrm{AM}\big(\mathrm{ZA}(G_i)\big).$$

(ii) *If $G = H \times F$ where H is compact and F is finite, then*

$$\mathrm{AM}\big(\mathrm{ZA}(G)\big) \geq \mathrm{AM}\big(\mathrm{ZA}(F)\big).$$

Proof. To see (i), we use Lemma 6.1 and the isomorphism $P \times P \cong \prod_{i=1}^n G_i \times G_i$ to see that

$$\mathrm{ZA}(P \times P) \cong \mathrm{ZA}(G_1 \times G_1)\hat{\otimes}\cdots\hat{\otimes}\,\mathrm{ZA}(G_n \times G_n).$$

Hence, the unique diagonal satisfies

$$1_{\mathrm{Conj}(P)_D} \cong 1_{\mathrm{Conj}(G_1)_D} \otimes \cdots \otimes 1_{\mathrm{Conj}(G_n)_D}.$$

We appeal to the fact that $\hat{\otimes}$ gives a cross-norm.

To see (ii) we have that the map $u \otimes v \mapsto u(e)v : \mathrm{ZA}(H)\hat{\otimes}\,\mathrm{ZA}(F) \to \mathrm{ZA}(F)$ extends to a contractive surjective homomorphism, and hence, again using Lemma 6.1, induces a contractive surjective homomorphism from $\mathrm{ZA}(G)$ onto $\mathrm{ZA}(F)$. It is standard and straightforward to check that if $\mathrm{AM}(\mathrm{ZA}(G)) < \infty$, then any bounded approximate diagonal for $\mathrm{ZA}(G)$ is carried to such for $\mathrm{ZA}(F)$, hence the diagonal for $\mathrm{ZA}(F)$ has norm bounded above by $\mathrm{AM}(\mathrm{ZA}(G))$. □

We lend the following evidence to our conjecture that $\mathrm{ZA}(G)$ is amenable if and only if G is virtually Abelian.

THEOREM 6.5. *Let $\{G_i\}_{i\in I}$ be an collection of finite groups and P be the compact product group $\prod_{i\in I} G_i$. Then $\mathrm{ZA}(P)$ is amenable if and only if all but finitely many groups G_i are Abelian.*

Proof. Suppose there is an infinite sequence of indices $i_1, i_2, \ldots$ for which each G_{i_k} is non-Abelian. Let $P_n = \prod_{k=1}^n G_{i_k}$ and $H_n = \prod_{i\in I\setminus\{i_1,\ldots,i_n\}} G_i$. We successively use parts (ii) and (i) of the lemma above, then Corollary 6.3 to see for each n that

$$\mathrm{AM}\big(\mathrm{ZA}(P)\big) \geq \mathrm{AM}\big(\mathrm{ZA}(P_n)\big) = \prod_{k=1}^{n} \mathrm{AM}\big(\mathrm{ZA}(G_{i_k})\big) \geq (2/\sqrt{3})^n.$$

Thus, we see that $\mathrm{AM}(\mathrm{ZA}(P)) = \infty$. □

Using techniques from the theory of hypergroups, the first-named author ([2]) has proved that if G is *tall*—that is, for each d, $\{\pi \in \widehat{G} : d_\pi = d\}$ is finite—then $\mathrm{ZA}(G)$ is non-amenable. There are examples of totally disconnected tall groups in [30]. Coupled with the last theorem, this gives two classes of totally disconnected and non-virtually Abelian G for which $\mathrm{ZA}(G)$ is non-amenable.

7. Open questions

In the course of this investigation, two open questions stand out. A third still remains from the paper [7], which motivated out investigation.

QUESTION 7.1. *For compact G, does amenability of $\mathrm{ZA}(G)$ imply that G is virtually Abelian?*

An approach to answering this is suggested in comments following Proposition 4.13. Thanks to Theorem 5.1, this question remains open only for compact groups with Abelian connected components of identity. Totally disconnected compact groups are pro-finite, and hence more refined qualitative version of Corollary 6.3, coupled with Proposition 2.1 may solve this.

For the next question, we use the assumptions and notation of Section 4.1. We state it in two equivalent forms.

QUESTION 7.2 ([58]). (i) *If $\mathcal{A}$ is amenable, must it be hyper-Tauberian?*
(ii) *If X_D is approximable for $\mathcal{A}\hat{\otimes}\mathcal{A}$, must it be spectral?*

As indicated in Remark 4.4, its is not generally true that approximability of a subset of the spectrum, for an a given algebra, implies spectrality.

The converse of the next question is answered in [5].

QUESTION 7.3 ([7]). *For a compact group G, if $\mathrm{ZL}^1(G)$ is amenable, must G be virtually Abelian?*

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Mahmood Alaghmandan, Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Gothenburg, SE-412 96, Sweden

E-mail address: mahala@calmers.se

Nico Spronk, Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

E-mail address: nspronk@uwaterloo.ca