REPRESENTATION OF INVARIANT MEASURES

BY

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1. Introduction

The problem of representing invariant probability measures as mixtures of ergodic probability measures has been treated by a number of different authors in differing contexts. V. Neumann [13] considered the problem in relation to continuous-parameter semigroups of measure-preserving transformations when the underlying space X is a complete separable metric space. De Finetti [6] and Hewitt and Savage [9] treat the representation question for symmetrically invariant measures on product spaces. Choquet [3] shows that his representation theorems give an easy proof of the existence of a unique representation when the underlying space X is a compact metric space and the measure-preserving transformations are a group of homeomorphisms of X onto X.

In a recent paper Blum and Hanson [1] show that if the measure-preserving transformations $\mathfrak{G}$ form a free group on a single generator T, then the representation (if there is one) of an invariant probability measure μ is determined by the restriction of μ to the invariant measurable sets. In this paper we take the methods of Blum and Hanson [1] as a starting point. We show that representation of invariant measures may be constructed, using the methods of Blum and Hanson, under broad enough conditions that we are able to show how to construct the representation in each of the cases mentioned above.

Throughout this paper X will be a set, $\mathfrak{F}$ a σ-algebra of subsets of X. $(X, \mathfrak{F}, \mu)$ will be called a probability space if μ is a nonnegative countably additive measure defined on $\mathfrak{F}$ such that $\mu(X) = 1$. A transformation

$$T : X \rightarrow X$$

will be called measurable if $A \in \mathfrak{F}$ implies $T^{-1}A \in \mathfrak{F}$. A measurable transformation T will be called measure-preserving (relative to $(X, \mathfrak{F}, \mu)$) if $A \in \mathfrak{F}$ implies $\mu(A) = \mu(T^{-1}A)$. Throughout, $\mathfrak{G}$ will be a set of measurable transformations of X into X. A probability measure μ on $\mathfrak{F}$ will be called invariant (relative to $(X, \mathfrak{F}, \mathfrak{G})$) if $A \in \mathfrak{F}$, $T \in \mathfrak{G}$ imply $\mu(A) = \mu(T^{-1}A)$. Relative to $(X, \mathfrak{F}, \mathfrak{G})$ we let $\mathfrak{P}$ be the set of invariant probability measures, and $\mathfrak{P}_1$ the set of extreme points of $\mathfrak{P}$. The convex set $\mathfrak{P}$ may not have any extreme points, but in the situations discussed in this paper, if $\mathfrak{P}$ is nonempty, then $\mathfrak{P}_1$ is also nonempty. In the following, $\mathfrak{F}_0$ will be the σ-algebra of measurable subsets invariant under the transformations in $\mathfrak{G}$, that is, $A \in \mathfrak{F}_0$ if and only if $A \in \mathfrak{F}$ and for all $T \in \mathfrak{G}$, $A = T^{-1}A$. A measure $\mu \in \mathfrak{P}$ will be

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called ergodic if $A \in \mathfrak{F}_0$ implies $\mu(A) = 0$ or $\mu(A) = 1$. It can be shown that if $\mu \in \mathfrak{P}_1$, then μ is ergodic. See for example Blum and Hanson [1]. In certain cases it can be shown that if $\mu \in \mathfrak{P}$ and is ergodic, then $\mu \in \mathfrak{P}_1$. This question is discussed again in Section 2, Corollary 1.

We construct a σ-algebra $\mathfrak{B}$ of subsets of $\mathfrak{P}_1$ as follows. If $A \in \mathfrak{F}$ and $0 \leqq \alpha \leqq 1$, then $\{\pi \mid \pi \in \mathfrak{P}_1, \pi(A) \leqq \alpha\} \in \mathfrak{B}$. $\mathfrak{B}$ is the least σ-algebra of sets containing all sets of this form. It is easily seen that this definition of $\mathfrak{B}$ makes $\mathfrak{B}$ the least σ-algebra of subsets of $\mathfrak{P}_1$ such that for each $A \in \mathfrak{F}$ the map $\pi \longrightarrow \pi(A)$ is a $(\mathfrak{P}_1, \mathfrak{B})$-measurable function. A measure $\mu \in \mathfrak{P}$ is said to be representable if there exists a probability measure λ defined on $\mathfrak{B}$ such that if $A \in \mathfrak{F}$, then

$$\mu(\mathrm{A}) = \int_{\mathfrak{P}_1} \pi(\mathrm{A}) \, d\lambda(\pi).$$

In Section 2 three conditions (a), (b), and (c) are stated. If in a given problem these conditions are satisfied, then an invariant measure μ has a uniquely determined representation which may be constructed using the methods of Blum and Hanson, op. cit. In Section 2 sufficient conditions are developed that (a) and (b) be satisfied.

Condition (c) is a statement about the existence of extreme points. This question is discussed in Section 3. In Section 3 a representation theorem is stated for the case X is a σ-compact locally compact space, $\mathfrak{F}$ the Baire sets of X, and $\mathfrak{S}$ a set of continuous functions on X into X, each $T \in \mathfrak{S}$ being continuous at ∞.

In Section 4 representation theorems are obtained in the case X is a complete separable metric space, $\mathfrak{F}$ the Borel sets of X.

Some of the results of this paper are obtained using the hypothesis that $\mathfrak{S}$ is a locally compact semigroup having a countable base for the open sets of $\mathfrak{S}$. In order to show that these results are not vacuous, in Section 5 it is shown that given such a semigroup $\mathfrak{S}$ there exist a probability space $(X, \mathfrak{F}, \mu)$, X a complete separable metric space, $\mathfrak{F}$ the Borel sets of X, μ a nonatomic probability measure, and a semigroup $\mathfrak{S}^*$ homeomorphic and algebraically isomorphic to $\mathfrak{S}$ such that each $T \in \mathfrak{S}^*$ is a measurable and measure-preserving map of X to X, and such that the map $(T, x) \longrightarrow T(x)$ is jointly continuous in the product topology of $\mathfrak{S}^* \times X$.

In order to establish continuity properties of $\mathfrak{S}$ acting on $(X, \mathfrak{F}, \mu)$, if $\mathfrak{S}$ is a locally compact semigroup, we suppose throughout that $\mathfrak{S}$ is a Baire subset of $\mathfrak{S}$, and that $\mathfrak{H}$ is the σ-algebra of Baire subsets of $\mathfrak{S}$. If $\mathfrak{S}$ is not a group, if $A \in \mathfrak{H}$, $\{S \mid TS \in A\} = T^{-1}(A)$ may not be a set of $\mathfrak{H}$. Nor is it clear that if $A \in \mathfrak{H}$, then $T(A) = \{TS \mid S \in A\} \in \mathfrak{H}$. To get around these questions of measurability we suppose in the sequel that if $\mathfrak{S}$ is not a group, then $\mathfrak{S}$ has a countable base for the open sets of $\mathfrak{S}$. The topology of $\mathfrak{S}$ is then metrizable. The Baire sets of $\mathfrak{S}$ and the Borel sets of $\mathfrak{S}$ coincide. Since the map $S \longrightarrow TS$ is continuous, it follows that if $A \in \mathfrak{H}$, $T^{-1}(A) \in \mathfrak{H}$. Further, if C is a compact subset, then $TC \in \mathfrak{H}$ since TC is compact. We will say a measure η on $(\mathfrak{S}, \mathfrak{H})$ is a regular semi-invariant measure if $\eta(C) < \infty$ for every compact Baire set C and $\eta(T^{-1}A) \leqq \eta(A)$ for every $T \in \mathfrak{S}$, $A \in \mathfrak{H}$. It follows that a regular semi-invariant measure η is a regular Baire measure on $\mathfrak{H}$. See Halmos [7]. Further, if $\mathfrak{S}$ is a compact group, then since $T^{-1}\mathfrak{S} = \mathfrak{S}$, it follows that $\eta(T^{-1}(\cdot))$ is a finite measure on $\mathfrak{S}$, $\eta(T^{-1}(A)) \leqq \eta(A)$ for every $A \in \mathfrak{H}$, $\eta(T^{-1}\mathfrak{S}) = \eta(\mathfrak{S})$, and therefore $\eta(T^{-1}(A)) = \eta(A)$ for every $A \in \mathfrak{H}$. A regular semi-invariant measure has the property that if $g \in L_1(\mathfrak{S}, \mathfrak{H}, \eta)$, $g \geqq 0$, $T \in \mathfrak{S}$, then

$$\int g(S)\, d\eta(S) \geqq \int g(TS)\, d\eta(S),$$

as follows from standard approximation arguments. If $C \in \mathfrak{H}$ is a compact set and $T \in \mathfrak{S}$, then $\chi_C(S) = 1$ implies $\chi_{TC}(TS) = 1$, so that for all $S \in \mathfrak{S}$, $\chi_C(S) \leqq \chi_{TC}(TS)$. Integration gives

$$\eta(C) \leqq \int \chi_{TC}(TS)\, d\eta(S) \leqq \int \chi_{TC}(S)\, d\eta(S) = \eta(TC).$$

If the cancellation law holds in $\mathfrak{S}$, then the map $S \rightarrow TS$ is a 1-1 map. If in addition $\mathfrak{S}$ has a countable base for the open sets of $\mathfrak{S}$, it follows $T(A) \in \mathfrak{H}$ for every $A \in \mathfrak{H}$. For since T is 1-1,

$$T \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} TA_i \quad \text{and} \quad T \bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} TA_i .$$

The set of sets A such that $T(A) \in \mathfrak{H}$ is therefore a monotone class which contains all sets which are G_δ's, and hence every set of $\mathfrak{H}$ is included.

We will say $\mathfrak{S}$ is a locally compact semigroup of jointly measurable transformations if the map $(T, x) \rightarrow T(x)$ is measurable in the product space $(\mathfrak{S} \times X, \mathfrak{H} \times \mathfrak{F})$. We always assume the identity of $\mathfrak{S}$ is the identity transformation of X.

We assume throughout that the reader is familiar with the basic results of measure theory such as presented by Halmos [7].

2. A theorem of Blum and Hanson

If $\mathfrak{S}$ is a group generated by a single transformation T, Blum and Hanson [1] give an explicit construction for the measure λ which gives the representation of μ. If μ is an invariant probability measure on $(X, \mathfrak{F})$ and $\mu \mid \mathfrak{F}_0$ is the restriction of μ to the invariant sets, then Blum and Hanson, op. cit., show that if μ has a representation, it is unique, and the measure λ is determined in a natural way by $\mu \mid \mathfrak{F}_0$.

A reading of the paper by Blum and Hanson, op. cit., shows their construction remains valid if the following conditions are satisfied.

(a) If μ_1 and μ_2 are in $\mathfrak{P}$, and if for all $A \in \mathfrak{F}_0$, $\mu_1(A) = \mu_2(A)$, then for all $A \in \mathfrak{F}$, $\mu_1(A) = \mu_2(A)$.

(b) If $A \in \mathfrak{F}$ and $0 \leqq \alpha \leqq 1$, there is a $B \in \mathfrak{F}_0$ such that

$$\{\pi \mid \pi \in \mathfrak{P}_1, \pi(A) \leqq \alpha\} = \{\pi \mid \pi \in \mathfrak{P}_1, \pi(B) = 1\}.$$

(c) If $\mu \in \mathfrak{P}$ and $A \in \mathfrak{F}_0$ such that $\mu(A) > 0$, then there is a $\pi \in \mathfrak{P}_1$ such that $\pi(A) = 1$.

Given that (a), (b), and (c) are satisfied, the construction of Blum and Hanson, op. cit., defines a map $\psi : \mathfrak{F}_0 \to \mathfrak{B}$ by $\psi(A) = \{\pi \mid \pi(A) = 1\}$. It is shown that ψ is a σ-algebra homomorphism of $\mathfrak{F}_0$ onto $\mathfrak{B}$. The measure λ is defined, if $B \in \mathfrak{B}$, by $\lambda(B) = \mu(\psi^{-1}B)$.

It turns out that representations constructed as described above relative to $\mathfrak{F}_0$ do not cover all cases which can be treated by the methods of this paper. It is necessary to define a σ-algebra $\bar{\mathfrak{F}}_0 \supset \mathfrak{F}_0$ as follows. Let $\mathfrak{N}$ be the σ-algebra of sets $A \in \mathfrak{F}$ such that $A \in \mathfrak{N}$ if and only if $\nu(A) = 0$ for every $\nu \in \mathfrak{P}$. If $T \in \mathfrak{S}$, let $\mathfrak{F}_{0,T}$ be the σ-algebra of all sets $A \in \mathfrak{F}$ such that $A \in \mathfrak{F}_{0,T}$ if and only if $A = T^{-1}A$. Let $\bar{\mathfrak{F}}_{0,T}$ be the least σ-algebra containing all sets in $\mathfrak{F}_{0,T}$ and $\mathfrak{N}$. Define

$$\bar{\mathfrak{F}}_0 = \bigcap_{T \in \mathfrak{S}} \bar{\mathfrak{F}}_{0,T}.$$

We restate (a), (b), and (c) for $\bar{\mathfrak{F}}_0$.

($\bar{a}$) If μ_1 and $\mu_2 \in \mathfrak{P}$, and if $\mu_1(A) = \mu_2(A)$ for every $A \in \bar{\mathfrak{F}}_0$, then for all $A \in \mathfrak{F}$, $\mu_1(A) = \mu_2(A)$.

($\bar{b}$) If $A \in \mathfrak{F}$, $0 \leqq \alpha \leqq 1$, there is a $B \in \bar{\mathfrak{F}}_0$ such that

$$\{\pi \mid \pi \in \mathfrak{P}_1, \pi(A) \leqq \alpha\} = \{\pi \mid \pi \in \mathfrak{P}_1, \pi(B) = 1\}.$$

($\bar{c}$) If $A \in \bar{\mathfrak{F}}_0$, $\mu \in \mathfrak{P}$, and $\mu(A) > 0$, there exists $\pi \in \mathfrak{P}_1$ such that $\pi(A) = 1$.

If ($\bar{a}$), ($\bar{b}$), and ($\bar{c}$) hold, and if $\mu \in \mathfrak{P}$, $\mu \mid \bar{\mathfrak{F}}_0$ determines uniquely the representation of μ in a manner similar to the way $\mu \mid \mathfrak{F}_0$ determines a representation of μ if (a), (b), and (c) hold.

In this section we will develop sufficient conditions for (a), (b), ($\bar{a}$), or ($\bar{b}$) to be valid. The existence statements of (c) and ($\bar{c}$) are investigated in later sections.

A sub σ-algebra $\mathfrak{F}_1 \subset \mathfrak{F}$ is said to be sufficient for the family $\mathfrak{P}$ of probability measures if and only if the following holds. If $f(\cdot)$ is a bounded $\mathfrak{F}$-measurable function, there exists a bounded $\mathfrak{F}_1$-measurable function $\phi(f)(\cdot)$ such that if $A \in \mathfrak{F}_1$, $\nu \in \mathfrak{P}$,

$$\int_A f(x)\, d\nu(x) = \int_A \phi(f)(x)\, d\nu(x).$$

That is, there is a single function $\phi(f)(\cdot)$ which acts as a conditional expectation of f for every $\nu \in \mathfrak{P}$. We will show below in Theorem 1 that if $\mathfrak{F}_0$ is a sufficient sub σ-algebra for $\mathfrak{P}$, then (a) and (b) are valid; if $\bar{\mathfrak{F}}_0$ is a sufficient sub σ-algebra for $\mathfrak{P}$, then ($\bar{a}$) and ($\bar{b}$) are valid. From Theorem 2 below it will follow that if $\mathfrak{S}$ is countable, then ($\bar{\text{a}}$) and ($\bar{\text{b}}$) are valid. Theorem 3 below gives conditions under which $\mathfrak{F}_0$ is a sufficient sub σ-algebra.

In order to obtain these results we need the following lemma.

LEMMA 1. *If* $\pi \in \mathfrak{P}_1$ *and* $A \in \bar{\mathfrak{F}}_0$, *then* $\pi(A) = 0$ *or* $\pi(A) = 1$.

To show this we will need to know the form of sets in $\bar{\mathfrak{F}}_{0,T}$. It can be shown that $A \in \bar{\mathfrak{F}}_{0,T}$ if and only if there exist $B \in \mathfrak{F}_{0,T}$, $U \in \mathfrak{N}$, $V \in \mathfrak{N}$ such that $A = (B - U) \cup V$. It is a matter of calculation to show that the sets of this form are a σ-algebra and therefore all of $\bar{\mathfrak{F}}_{0,T}$.

Suppose then that $\pi \in \mathfrak{P}_1$ and that for some $B_0 \in \bar{\mathfrak{F}}_0$, $0 < \pi(B_0) < 1$. Let $X - B_0 = B_1$. Then if $A \in \mathfrak{F}$,

$$\pi(A) = \pi(B_0)\pi_0(A) + \pi(B_1)\pi_1(A),$$

where $\pi_i(A) = \pi(A \cap B_i)/\pi(B_i)$, $i = 0, 1$. If we show $\pi_i \in \mathfrak{P}$, $i = 0, 1$, a contradiction will be obtained. Suppose $T \in \mathfrak{S}$. Since $B_i \in \bar{\mathfrak{F}}_0$, $B_i \in \bar{\mathfrak{F}}_{0,T}$, and this implies $B_i = (C_i - U_i) \cup V_i$, $C_i \in \mathfrak{F}_{0,T}$, U_i, $V_i \in \mathfrak{N}$, $i = 0, 1$. Then

$$\begin{aligned} T^{-1}B_i &= ((T^{-1}C_i) - (T^{-1}U_i)) \cup (T^{-1}V_i) \\ &= (C_i - T^{-1}U_i) \cup (T^{-1}V_i), \qquad i = 0, 1. \end{aligned}$$

And $T^{-1}U_i$, $T^{-1}V_i \in \mathfrak{N}$, $i = 0, 1$. Then

$$\begin{aligned} \pi((T^{-1}A) \cap B_i) &= \pi((T^{-1}A) \cap C_i) = \pi((T^{-1}A) \cap (T^{-1}B_i)) \\ &= \pi(T^{-1}(A \cap B_i)) = \pi(A \cap B_i), \qquad i = 0, 1. \end{aligned}$$

It follows that $\pi_i \in \mathfrak{P}$, $i = 0, 1$. This contradiction shows if $\pi \in \mathfrak{P}_1$, then $\pi(B) = 0$ or $\pi(B) = 1$ for every $B \in \bar{\mathfrak{F}}_0$.

Suppose then $\mathfrak{F}_0$ is a sufficient sub σ-algebra for $\mathfrak{P}$. If $A \in \mathfrak{F}$, let χ_A be the characteristic function of the set A. By hypothesis there is an $\mathfrak{F}_0$-measurable function $\phi(\chi_A)(\cdot)$ which acts as a conditional expectation for each $\nu \in \mathfrak{P}$. To prove (a), suppose ν_1, $\nu_2 \in \mathfrak{P}$ and if $B \in \mathfrak{F}_0$, $\nu_1(B) = \nu_2(B)$. Then

$$\nu_1(A) = \int \phi(\chi_A)(x)\, d\nu_1(x) = \int \phi(\chi_A)(x)\, d\nu_2(x) = \nu_2(A).$$

To prove (b), suppose $\pi \in \mathfrak{P}_1$. Then for each real a, define

$$F(a) = \pi\{x \mid \phi(\chi_A)(x) \leqq a\}.$$

Since $\{x \mid \phi(\chi_A)(x) \leqq a\} \in \mathfrak{F}_0$, by Lemma 1 it follows that $F(a) = 0$ or $F(a) = 1$. Since F is a right continuous function, there is an a_0 such that $F(a_0-) = 0$, $F(a_0) = 1$. That is, $\phi(\chi_A)(x) = a_0$ a.e. π. Consequently, $\pi(A) = 1$. If $0 \leqq \alpha \leqq 1$, let $B_\alpha = \{x \mid \phi(\chi_A)(x) \leqq \alpha\}$. Then it follows that $\{\pi \mid \pi \in \mathfrak{P}_1, \pi(A) \leqq \alpha\} = \{\pi \mid \pi \in \mathfrak{P}_1, \pi(B_\alpha) = 1\}$. Therefore (b) is proved.

If $\bar{\mathfrak{F}}_0$ is a sufficient sub σ-algebra for $\mathfrak{P}$, then by using Lemma 1 and arguments similar to those above, the validity of ($\bar{\text{a}}$) and ($\bar{\text{b}}$) may be established.

THEOREM 1. *If* $\mathfrak{F}_0$ *is a sufficient sub* σ*-algebra for* $\mathfrak{P}$, *then* (a) *and* (b) *are valid. If* $\bar{\mathfrak{F}}_0$ *is a sufficient sub* σ*-algebra for* $\mathfrak{P}$, *then* ($\bar{\text{a}}$) *and* ($\bar{\text{b}}$) *are valid.*

An immediate corollary is the following.

COROLLARY 1. *If* $\mathfrak{F}_0$ *is a sufficient sub* σ*-algebra for* $\mathfrak{P}$, *then* $\pi \in \mathfrak{P}$ *is ergodic if and only if* $\pi \in \mathfrak{P}_1$. *If* $\bar{\mathfrak{F}}_0$ *is a sufficient sub* σ*-algebra for* $\mathfrak{P}$, *then* $\pi \in \mathfrak{P}_1$ *if and only if for every* $A \in \bar{\mathfrak{F}}_0$, $\pi(A) = 0$ *or* $\pi(A) = 1$.

We will now obtain sufficient conditions that $\mathfrak{F}_0$ or $\bar{\mathfrak{F}}_0$ be sufficient sub σ-algebras.

LEMMA 2. *If* $T \in \mathfrak{G}$, *then the sub* σ*-algebra* $\mathfrak{F}_{0,T}$ *is sufficient for* $\mathfrak{P}$.

Proof. If f is a bounded $\mathfrak{F}$-measurable function, then by the pointwise ergodic theorem, if $\nu \in \mathfrak{P}$, $\nu\{x \mid \lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} f(T^i x) \text{ exists}\} = 1$. Define

$$\phi(f)(x) = \lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} f(T^i x) \quad \text{if this limit exists,}$$

$$\phi(f)(x) = 0 \quad \text{otherwise.}$$

Then $\phi(f)$ is $\mathfrak{F}_{0,T}$-measurable, and if $A \in \mathfrak{F}_{0,T}$, $\nu \in \mathfrak{P}$,

$$\int_A \phi(f)(x)\, d\nu(x) = \int_A f(x)\, d\nu(x).$$

THEOREM 2. *If* $\mathfrak{G}$ *is a countable set of transformations, then* $\bar{\mathfrak{F}}_0$ *is a sufficient sub* σ*-algebra.*

Proof. By Lemma 2, $\mathfrak{F}_{0,T}$, and therefore $\bar{\mathfrak{F}}_{0,T}$, is a sufficient sub σ-algebra. By a theorem of Burkholder [2], $\bar{\mathfrak{F}}_0 = \bigcap_{T\in\mathfrak{G}} \bar{\mathfrak{F}}_{0,T}$ is a sufficient sub σ-algebra.

That, in the conclusion of Theorem 2, $\bar{\mathfrak{F}}_0$ cannot be replaced by $\mathfrak{F}_0$, is shown in the following example. Let $R = (-\infty, \infty)$ and $X = R \times R$, the Cartesian product of R and R. Let $\mathfrak{F}$ be the Borel sets of X. Define transformations T_1, T_2 by $T_1((x_1, x_2)) = (x_1, x_1)$ and $T_2((x_1, x_2)) = (x_2, x_2)$. It is easily verified that the only invariant sets are X and the null set. Therefore every invariant measure is ergodic. Every measure concentrated on the diagonal set of X is invariant. The extreme points are those measures π such that for some x, $\pi(\{(x, x)\}) = 1$.

THEOREM 3. *Let* $(X, \mathfrak{F}, \mu)$ *be a probability space. Suppose* $\mathfrak{G}$ *is a set of measurable and measure-preserving transformations relative to* $(X, \mathfrak{F}, \mu)$. *Suppose* $\mathfrak{G}$ *satisfies one of the following*:

(d) $\mathfrak{G}$ *is a compact group of jointly measurable onto transformations*;

(e) $\mathfrak{G}$ *is a countable group*;

(f) $\mathfrak{G}$ *is a locally compact group of jointly measurable onto transformations, and* $\mathfrak{G}$ *has a countable base for the open sets of* $\mathfrak{G}$;

(g) $\mathfrak{G}$ *is a countable set of transformations* (*not necessarily* 1-1 *or onto*) *such that if* $T_1, T_2 \in \mathfrak{G}$, *then* $T_1 T_2 = T_2 T_1$;

(h) $\mathfrak{G}$ *is a locally compact Abelian semigroup of jointly measurable trans-*

formations such that $\mathfrak{S}$ *has a countable base for the open sets of* $\mathfrak{S}$ *and* $\mathfrak{H}$ *has a nonzero regular semi-invariant measure* η.
Then $\mathfrak{F}_0$ *is a sufficient sub* σ*-algebra for* $\mathfrak{P}$.

Before proving Theorem 3 we prove two lemmas.

LEMMA 3. *Suppose* $\mathfrak{S}$ *is a locally compact semigroup,* $\mathfrak{S}$ *has a countable base for the open sets of* $\mathfrak{S}$, *and* $\mathfrak{H}$ *has a nonzero regular semi-invariant measure* η. *If* f *is a bounded* $\mathfrak{H}$*-measurable function and* $C \epsilon \mathfrak{H}$ *is compact, then*

$$\lim_{T\to T_0} \int_C |f(TS) - f(T_0S)| \, d\eta(S) = 0.$$

Proof. Suppose g is a real-valued continuous function on $\mathfrak{S}$. Then $\lim_{T\to T_0} g(TS) = g(T_0 S)$, $S \epsilon \mathfrak{S}$. By the bounded convergence theorem it follows that if $C \subset \mathfrak{S}$ is compact, $\lim_{T\to T_0} g(TS)\chi_C(S) = g(T_0 S)\chi_C(S)$ in $L_1(\mathfrak{S}, \mathfrak{H}, \eta)$. For suppose D is a compact neighborhood of T_0. Then DC is compact, and if $T \epsilon D$, $S \epsilon C$, then $TS \epsilon DC$. Since g is continuous,

$$\sup \{|g(S)| \mid S \epsilon DC\} < \infty.$$

But $\eta(DC) < \infty$. Therefore the family of functions in question are uniformly bounded by a function in $L_1(\mathfrak{S}, \mathfrak{H}, \eta)$.

Suppose g_1 is a bounded and $\mathfrak{H}$-measurable function. Then given $\varepsilon > 0$ there is a continuous function g_ε such that

$$\int_{DC} |g_1(S) - g_\varepsilon(S)| \, d\eta(S) < \varepsilon.$$

Then if $T \epsilon D$,

$$\int_C |g_1(TS) - g_\varepsilon(TS)| \, d\eta(S) \leqq \int \chi_{TC}(TS) |g_1(TS) - g_\varepsilon(TS)| \, d\eta(S)$$

$$\leqq \int \chi_{TC}(S) |g_1(S) - g_\varepsilon(S)| \, d\eta(S) \leqq \int_{DC} |g_1(S) - g_\varepsilon(S)| \, d\eta(S) < \varepsilon.$$

By the usual approximation arguments it follows that

$$\lim_{T\to T_0} \int_C |g_1(TS) - g_1(T_0S)| \, d\eta(S) = 0.$$

LEMMA 4. *Suppose* $(X, \mathfrak{F}, \nu)$ *is a probability space and* $\mathfrak{S}$ *is a locally compact semigroup of jointly measurable and measure-preserving transformations relative to* $(X, \mathfrak{F}, \nu)$ *such that* $\mathfrak{S}$ *has a countable base for the open sets of* $\mathfrak{S}$. *Suppose* $\mathfrak{H}$ *has a nonzero regular semi-invariant measure* η. *Then if* $f \epsilon L_1(X, \mathfrak{F}, \nu)$ *and* $T_0 \epsilon \mathfrak{S}$,

$$\lim_{T\to T_0} \int_X |f(Tx) - f(T_0x)| \, d\nu(x) = 0.$$

Proof. Suppose C is compact, $\eta(C) > 0$, and D is a compact neighborhood of T_0. Suppose f is a bounded and $\mathfrak{F}$-measurable function. If $T \in D$, then

$$\eta(C) \int_X |f(Tx) - f(T_0 x)| \, d\nu(x) = \int_C \int_X |f(Tx) - f(T_0 x)| \, d\nu(x) \, d\eta(S)$$

$$= \int_C \int_X |f(TSx) - f(T_0 Sx)| \, d\nu(x) \, d\eta(S)$$

$$= \int_X \int_C |f(TSx) - f(T_0 Sx)| \, d\eta(S) \, d\nu(x),$$

justification for the steps being the invariance of ν and Fubini's theorem. If $x \in X$, by Lemma 3

$$0 = \lim_{T \to T_0} \int_C |f(TSx) - f(T_0 Sx)| \, d\eta(S),$$

and these integrals are uniformly bounded by $2\eta(C) \sup_x |f(x)|$. Since $\eta(C) > 0$, by the bounded convergence theorem,

$$\lim_{T \to T_0} \int_X |f(Tx) - f(T_0 x)| \, d\nu(x) = 0.$$

Finally, if $f \in L_1(X, \mathfrak{F}, \nu)$, given $\varepsilon > 0$ there is a bounded $\mathfrak{F}$-measurable function f_ε such that

$$\int_X |f(x) - f_\varepsilon(x)| \, d\nu(x) < \varepsilon.$$

Since ν is invariant, if $T \in \mathfrak{G}$, then

$$\int_X |f(Tx) - f_\varepsilon(Tx)| \, d\nu(x) < \varepsilon.$$

It follows at once that

$$\lim_{T \to T_0} \int_X |f(Tx) - f(T_0 x)| \, d\nu(x) = 0.$$

Proof of Theorem 3. In each of the proofs following we will suppose f is a bounded $\mathfrak{F}$-measurable function. The proof then consists of construction of an $\mathfrak{F}_0$-measurable function $\phi(f)$ satisfying, if $A \in \mathfrak{F}_0$, $\nu \in \mathfrak{P}$,

$$\int_A f(x) \, d\nu(x) = \int_A \phi(f)(x) \, d\nu(x).$$

To prove (d) suppose η is the Haar measure on $\mathfrak{G}$ such that $\eta(\mathfrak{G}) = 1$. Define $\phi(f)$ by

$$\phi(f)(x) = \int_{\mathfrak{G}} f(Tx) \, d\eta(T).$$

It follows at once that $\phi(f)$ is bounded, by Fubini's theorem that $\phi(f)$ is $\mathfrak{F}$-measurable, and from the invariance of Haar measure that $\phi(f)$ is an invariant function. If $A \in \mathfrak{F}_0$ and $\nu \in \mathfrak{P}$, then

$$\int_A \phi(f)(x)\, d\nu(x) = \int_A \int_{\mathfrak{G}} f(Tx)\, d\eta(T)\, d\nu(x) = \int_{\mathfrak{G}} \left(\int_A f(Tx)\, d\nu(x) \right) d\eta(T)$$

$$= \int_{\mathfrak{G}} \left(\int_{TA} f(x)\, d\nu(x) \right) d\eta(T) = \int_A f(x)\, d\nu(x),$$

since $TA = A$.

To prove (e), we use Theorem 2. Since $\bar{\mathfrak{F}}_0$ is a sufficient sub σ-algebra, there exists an $\bar{\mathfrak{F}}_0$-measurable function $\phi(f)(\cdot)$ such that if $A \in \bar{\mathfrak{F}}_0$, $\nu \in \mathfrak{P}$, then

$$\int_A \phi(f)(x)\, d\nu(x) = \int_A f(x)\, d\nu(x).$$

By definition of $\bar{\mathfrak{F}}_0$, $\phi(f)(\cdot)$ is $\bar{\mathfrak{F}}_{0,T}$-measurable, $T \in \mathfrak{G}$. Therefore if $T \in \mathfrak{G}$, $\nu \in \mathfrak{P}$,

$$\phi(f)(Tx) = \phi(f)(x) \quad \text{a.e. } \nu.$$

Let $A = \{x \mid \phi(f)(Tx) = \phi(f)(x), \text{ all } T \in \mathfrak{G}\}$. Then since $\mathfrak{G}$ is countable, if $\nu \in \mathfrak{P}$, $\nu(A) = 1$. Define

$$\phi^*(f)(x) = 0; \qquad \text{if } x \notin A,$$

$$\phi^*(f)(x) = \phi(f)(x) \quad \text{if } x \in A.$$

Since $\mathfrak{G}$ is a group, $\phi^*(f)(\cdot)$ is invariant. Clearly if $A \in \mathfrak{F}_0$, $\nu \in \mathfrak{P}$,

$$\int_A f(x)\, d\nu(x) = \int_A \phi^*(f)(x)\, d\nu(x).$$

Therefore $\mathfrak{F}_0$ is a sufficient sub σ-algebra for $\mathfrak{P}$.

To prove (f) let $\mathfrak{G}_0$ be a countable dense subgroup of $\mathfrak{G}$. By (e) just proved there is a function $\phi(f)(\cdot)$ measurable in $\bigcap_{T \in \mathfrak{G}_0} \mathfrak{F}_{0,T}$ such that if $A \in \bigcap_{T \in \mathfrak{G}_0} \mathfrak{F}_{0,T}$, $\nu \in \mathfrak{P}$, then

$$\int_A f(x)\, d\nu(x) = \int_A \phi(f)(x)\, d\nu(x).$$

Let $T \in \mathfrak{G}$ and $\{T_n, n \geqq 1\} \subset \mathfrak{G}_0$ be a sequence such that $T = \lim_{n\to\infty} T_n$. By Lemma 4, if $\nu \in \mathfrak{P}$,

$$0 = \lim_{n\to\infty} \int_X | \phi(f)(T_n x) - \phi(f)(Tx) |\, d\nu(x)$$

$$= \int_X | \phi(f)(x) - \phi(f)(Tx) |\, d\nu(x).$$

By a well-known theorem it follows there is a bounded $\mathfrak{F}_0$-measurable function $\phi^*(f)(\cdot)$ such that if $\nu \in \mathfrak{P}$, $\phi^*(f)(x) = \phi(f)(x)$ a.e. ν. See for example Lehmann [12, p. 225]. Since $\mathfrak{F}_0 \subset \bigcap_{T \in \mathfrak{G}_0} \mathfrak{F}_{0,T}$, part (f) now follows.

To prove (g) let $\mathfrak{S} = \{T_i, i \geqq 1\}$. If $m \geqq 1$, define $\phi_1(f)$ by

$$\phi_1(f)(x) = \lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} f(T_1^i x),$$

and $\phi_m(f)$ inductively by

$$\phi_{m+1}(f)(x) = \lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} \phi_m(f)(T_{m+1}^i x).$$

It is easily verified that if $m \geqq 1$, $\phi_m(f)$ is invariant under $T_1, \cdots, T_m$. We use here the hypothesis that $T_1, \cdots, T_m$ commute. If $\mathfrak{F}_0^{(m)}$ is the set of sets $A \in \mathfrak{F}$ invariant under $T_1, \cdots, T_m$, then if $\nu \in \mathfrak{P}$,

$$\phi_m(f) = E(f \mid \mathfrak{F}_0^{(m)}) \quad \text{a.e. } \nu,$$

as follows from the pointwise ergodic theorem. Therefore $\{\phi_m(f), m \geqq 1\}$ form a martingale, and by the martingale theorems, Doob [4], if $\nu \in \mathfrak{P}$,

$$\lim_{m\to\infty} \phi_m(f) = E(f \mid \textstyle\bigcap_{m=1}^{\infty} \mathfrak{F}_0^{(m)}) \quad \text{a.e. } \nu.$$

The proof is complete.

To prove (h) suppose $\mathfrak{S}_0$ is a countable dense subsemigroup of $\mathfrak{S}$. If f is a bounded $\mathfrak{F}$-measurable function, then by part (g) of Theorem 3 there is a function g invariant under $\mathfrak{S}_0$ such that if $A \in \mathfrak{F}$ is invariant under $\mathfrak{S}_0$, then

$$\int_A f(x)\, d\nu(x) = \int_A g(x)\, d\nu(x), \qquad \nu \in \mathfrak{P}.$$

The function g is bounded. If $x \in X$, $T \in \mathfrak{S}$, and $\{T_i, i \geqq 1\}$ is a sequence of transformations in $\mathfrak{S}_0$ such that $T = \lim_{i\to\infty} T_i$, then by Lemma 3, if $C \subset \mathfrak{S}$ is compact,

$$\lim_{i\to\infty} \int_C |g(T_i Sx) - g(TSx)|\, d\eta(S) = 0.$$

Since g is invariant under T_i, $i \geqq 1$, if $x \in X$,

$$\int_C |g(Sx) - g(TSx)|\, d\eta(S) = 0.$$

This holds for every compact $C \subset \mathfrak{S}$. Therefore, if $x \in X$,

$$\int_{\mathfrak{S}} |g(Sx) - g(TSx)|\, d\eta(S) = 0.$$

Let $h \geqq 0$ be an $\mathfrak{H}$-measurable function,

$$\int_{\mathfrak{S}} h(S)\, d\eta(S) = 1.$$

Define $\phi(f)$ by

$$\phi(f)(x) = \int_{\mathfrak{S}} g(Sx)h(S)\, d\eta(S).$$

$\phi(f)(\cdot)$ is clearly a bounded function, and by Fubini's theorem is $\mathfrak{F}$-measurable. $\mathfrak{S}$ is Abelian. If $T \in \mathfrak{S}$, $x \in X$, since $g(TSx) = g(Sx)$ a.e. η,

$$\phi(f)(Tx) = \int_{\mathfrak{S}} g(STx)h(S)\, d\eta(S) = \int_{\mathfrak{S}} g(Sx)h(S)\, d\eta(S) = \phi(f)(x).$$

Therefore $\phi(f)$ is invariant. If $A \in \mathfrak{F}_0$, then by Fubini's theorem, if $\nu \in \mathfrak{P}$,

$$\begin{aligned}\int_A \phi(f)(x)\, d\nu(x) &= \int_{\mathfrak{S}} \int_X \chi_A(x) g(Sx) h(S)\, d\nu(x)\, d\eta(S) \\ &= \int_{\mathfrak{S}} \int_X \chi_A(Sx) g(Sx) h(S)\, d\nu(x)\, d\eta(S) \\ &= \int_{\mathfrak{S}} \left(\int_A g(x)\, d\nu(x) \right) h(S)\, d\eta(S) = \int_A g(x)\, d\nu(x) \\ &= \int_A f(x)\, d\nu(x).\end{aligned}$$

The proof is complete.

COROLLARY 2. *Suppose $\mathfrak{S}$ is a semigroup of measurable transformations on $(X, \mathfrak{F})$. Let M_X^+ be the set of all finite nonnegative countably additive measures defined on $\mathfrak{F}$, and $M_{\mathfrak{S}}^+ \subset M_X^+$ the set of measures invariant under $\mathfrak{S}$. Let M_X^+ be partially ordered by $\mu_1 \leqq \mu_2$ if and only if $\mu_1(A) \leqq \mu_2(A)$ for all $A \in \mathfrak{F}$. Then $M_{\mathfrak{S}}^+$ is a sublattice of M_X^+.*

Proof. Let $\mathfrak{S}_0$ be a countable Abelian subsemigroup of $\mathfrak{S}$. Suppose μ_1 and $\mu_2 \in M_{\mathfrak{S}}^+$. Let μ be defined by $\mu(A) = \mu_1(A) + \mu_2(A)$, $A \in \mathfrak{F}$. By the Radon-Nikodym theorem there is an $\mathfrak{F}$-measurable function f such that if $A \in \mathfrak{F}$,

$$\mu_1(A) = \int_A f(x)\, d\mu(x).$$

By Theorem 3 there is a function $\phi(f)$ invariant under $\mathfrak{S}_0$ such that if A is invariant under $\mathfrak{S}_0$, $A \in \mathfrak{F}$, then

$$\int_A f(x)\, d\mu(x) = \int_A \phi(f)(x)\, d\mu(x).$$

Since μ is invariant under $\mathfrak{S}_0$, the measure ν, defined if $B \in \mathfrak{F}$, by

$$\nu(B) = \int_B \phi(f)(x)\, d\mu(x),$$

is invariant under $\mathfrak{S}_0$. For if $T \in \mathfrak{S}_0$,

$$\begin{aligned}\nu(T^{-1}B) &= \int_{T^{-1}B} \phi(f)(x)\, d\mu(x) = \int \chi_B(Tx)\phi(f)(x)\, d\mu(x) \\ &= \int \chi_B(Tx)\phi(f)(Tx)\, d\mu(x) = \int_B \phi(f)(x)\, d\mu(x) = \nu(B).\end{aligned}$$

Since condition (a) is satisfied by $\mathfrak{G}_0$, it follows that $\nu \equiv \mu_1$. It follows that $f = \phi(f)$ a.e. μ, and that if $T \in \mathfrak{G}_0$, $f(T(\cdot)) = f(\cdot)$ a.e. μ. Repeating this argument over all possible Abelian subsemigroups of $\mathfrak{G}$ shows that if $T \in \mathfrak{G}$, then $f(T(\cdot)) = f(\cdot)$ a.e. μ. It is easily verified that the least upper bound $\mu_1 \vee \mu_2$ and the greatest lower bound $\mu_1 \wedge \mu_2$ satisfy, if $A \in \mathfrak{F}$,

$$(\mu_1 \vee \mu_2)(A) = \int_A \max(f(x), 1 - f(x))\, d\mu(x),$$

$$(\mu_1 \wedge \mu_2)(A) = \int_A \min(f(x), 1 - f(x))\, d\mu(x).$$

Since μ is invariant under $\mathfrak{G}$, and since if $T \in \mathfrak{G}$, $f(T(\cdot)) = f(\cdot)$ a.e. μ, it follows that $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are invariant measures and therefore in $M_{\mathfrak{G}}^{+}$.

3. Existence of extreme points

In this section we consider conditions under which condition (c) is satisfied. Condition (c) states the existence of countably additive measures. This strongly suggests some kind of compactness argument is required. We begin by proving an existence theorem given that X is a compact Hausdorff space, $\mathfrak{F}$ the Baire sets of X, and then relax this restriction somewhat to the case X is a σ-compact locally compact Hausdorff space, and in Section 4 to the case X is a complete separable metric space. In addition it is necessary for our arguments to restrict the set of transformations $\mathfrak{G}$.

LEMMA 5. *Suppose $(X, \mathfrak{F}, \mu)$ is a probability space, X a compact Hausdorff space, and $\mathfrak{F}$ the Baire sets of X. Suppose $\mathfrak{G}$ is a family of continuous and measure-preserving transformations relative to $(X, \mathfrak{F}, \mu)$. Then each $T \in \mathfrak{G}$ is a measurable transformation, and $\mathfrak{P}$ is a weak* compact set of linear functionals on $C(X)$.*

Proof. If $T \in \mathfrak{G}$ and $f \in C(X)$, then $f(T(\cdot))$ is a real-valued continuous function on X. Therefore $f(T(\cdot))$ is Baire measurable. If C is a compact Baire set, there is a sequence $\{f_n, n \geqq 1\} \subset C(X)$ such that for all $x \in X$, $\lim_{n\to\infty} f_n(x) = \chi_C(x)$, the characteristic function of C. Therefore $\chi_C(T(\cdot))$ is Baire measurable, that is, $T^{-1}C \in \mathfrak{F}$. From this it follows at once that T is a (Baire) measurable transformation.

Suppose μ is a weak limit of $\mathfrak{P}$. If $f \in C(X)$, $T \in \mathfrak{G}$, and $\varepsilon > 0$, there is a $\nu \in \mathfrak{P}$ such that

$$\left| \int f(x)\, d\mu(x) - \int f(x)\, d\nu(x) \right| < \varepsilon/2$$

and

$$\left| \int f(T(x))\, d\mu(x) - \int f(T(x))\, d\nu(x) \right| < \varepsilon/2.$$

Since ν is an invariant measure,

$$\left| \int f(T(x))\, d\mu(x) - \int f(x)\, d\mu(x) \right| < \varepsilon.$$

Since this holds for every $\varepsilon > 0$,

$$\int f(T(x))\, d\mu(x) = \int f(x)\, d\mu(x).$$

Since this holds if $f \in C(X)$, $T \in \mathfrak{S}$, it follows that μ is invariant; therefore $\mu \in \mathfrak{P}$. Since $\mathfrak{P}$ is closed in the weak* topology and $\mathfrak{P}$ is bounded, $\mathfrak{P}$ is a weak* compact set.

LEMMA 6. *Suppose $(X, \mathfrak{F}, \mu)$ is a probability space, X a σ-compact locally compact Hausdorff space, $\mathfrak{F}$ the Baire sets of X. Suppose $\mathfrak{S}$ is a set of continuous and measure-preserving transformations relative to $(X, \mathfrak{F}, \mu)$ such that each $T \in \mathfrak{S}$ is continuous at ∞.*

If X is compact and $C \subset X$ is compact, and if $\mu(C) > 0$, there exists $\pi \in \mathfrak{P}_1$ such that $\pi(C) > 0$. If X is σ-compact and $A \in \bigcap_{T \in \mathfrak{S}} \bar{\mathfrak{F}}_{0,T}$, $\mu(A) > 0$, there exists $\pi \in \mathfrak{P}_1$ such that $\pi(A) = 1$.

Proof. If X is compact, then $\mathfrak{P}$ is a weak* compact set of linear functionals on $C(X)$, by Lemma 5. Below we use Lemma V8.2, Dunford and Schwartz [5], to show that certain closed convex subsets of $\mathfrak{P}$ have extreme points.

Suppose X is compact, $C \subset X$ is compact, $\mu(C) > 0$. If $0 \leqq \alpha \leqq 1$, $\{\nu \mid \nu \in \mathfrak{P}, \nu(C) \geqq \alpha\}$ is a convex and weakly compact set. For suppose $\{f_n, n \geqq 1\} \subset C(X)$, if $n \geqq 1$, $f_{n+1} \leqq f_n$, and $\lim_{n\to\infty} f_n = \chi_C$, the characteristic function of C. Then

$$\{\nu \mid \nu \in \mathfrak{P}, \nu(C) \geqq \alpha\} = \bigcap_{n=1}^{\infty} \left\{\nu \mid \nu \in \mathfrak{P}, \int f_n(x)\, d\nu(x) \geqq \alpha\right\}$$

which is weakly compact since $\mathfrak{P}$ is weakly compact. Let

$$\alpha_0 = \sup\{\alpha \mid \{\nu \mid \nu \in \mathfrak{P}, \nu(C) \geqq \alpha\} \neq \phi\}.$$

Then $\{\nu \mid \nu \in \mathfrak{P}, \nu(C) \geqq \alpha_0\}$ is a convex weakly compact nonempty set. Let π be an extreme point of this set. Then $\pi \in \mathfrak{P}_1$. If not, then there are $\nu_1, \nu_2 \in \mathfrak{P}$ and $0 < \beta < 1$ such that $\pi = \beta\nu_1 + (1 - \beta)\nu_2$. By construction $\nu_1(C) = \nu_2(C) = \alpha_0$ follows. Therefore $\nu_1, \nu_2 \in \{\nu \mid \nu \in \mathfrak{P}, \nu(C) \geqq \alpha_0\}$, and π is not an extreme point of this set. This contradiction shows $\pi \in \mathfrak{P}_1$. That completes the proof if X is compact.

If X is locally compact, let q be a point not in X, and let $X^* = \{q\} \cup X$ be given the one-point compactification topology. See for example Kelley [10]. Let $\mathfrak{F}^*$ be the σ-algebra of Baire sets of X^*.

X is σ-compact. In this case $\{q\}$ is a compact G_δ and therefore a Baire set. Also the complement $\{q\}' = X$ is a Baire set. Further, $\mathfrak{F} \subset \mathfrak{F}^*$. We extend μ and the transformations $T \in \mathfrak{G}$ to μ^* and $T^* \in \mathfrak{G}^*$ by the definitions, if $B \in \mathfrak{F}^*$, $\mu^*(B) = \mu(B \cap X)$. If $x \in X$, $T^*(x) = T(x)$, $T^*(q) = \lim_{x \to q} T(x)$. Then the transformations in $\mathfrak{G}^*$ are continuous on X.

Suppose $A \in \mathfrak{N}$, that is, $\nu(A) = 0$ for all $\nu \in \mathfrak{P}$. If ν^* is an invariant probability measure on $\mathfrak{F}^*$, then $(\nu^* \mid \mathfrak{F})$ is clearly an invariant measure on $\mathfrak{F}$. Let $(\nu^* \mid \mathfrak{F})(X) = \alpha$. If $\alpha \neq 0$, then $(1/\alpha)(\nu^* \mid \mathfrak{F}) \in \mathfrak{P}$, and therefore $(1/\alpha)(\nu^* \mid \mathfrak{F})(A) = 0$. Therefore $\nu^*(A) = 0$ if $\alpha \neq 0$, $\nu^*(A) = 0$ if $\alpha = 0$. It follows that if an invariant measure π on $\mathfrak{F}^*$ is extremal, and if $A \in \bigcap_{T \in \mathfrak{G}} \bar{\mathfrak{F}}_{0,T}$, $\pi(A) > 0$, then $\pi(A) = 1$.

Suppose then $\mu(A) > 0$, $A \in \bigcap_{T \in \mathfrak{G}} \bar{\mathfrak{F}}_{0,T}$. By the regularity of μ there is a compact set $C \subset A \subset X$ such that $\mu^*(C) = \mu(C) > 0$. By the first part of the proof there is an extremal measure π on $\mathfrak{F}^*$ such that $0 < \pi(C) \leqq \pi(A)$. Therefore $\pi(A) = 1$ and $\pi(X) = 1$. It follows that $(\pi \mid \mathfrak{F})(X) = 1$, $(\pi \mid \mathfrak{F})(A) = 1$, and $\pi \mid \mathfrak{F} \in \mathfrak{P}_1$.

THEOREM 4. *Suppose $(X, \mathfrak{F}, \mu)$ is a probability space, X a locally compact and σ-compact Hausdorff space, $\mathfrak{F}$ the Baire sets of X. If $\mathfrak{G}$ is a set of continuous maps of X to X such that each $T \in \mathfrak{G}$ is continuous at ∞, and if $\mathfrak{G}$ is countable or $\mathfrak{G}$ satisfies the hypotheses of Theorem 3, then μ has a unique representation.*

Proof. If $\mathfrak{G}$ is countable, then conditions ($\bar{a}$) and ($\bar{b}$) are satisfied, while if $\mathfrak{G}$ satisfies the hypotheses of Theorem 3, conditions (a) and (b) are satisfied. Suppose $A \in \bar{\mathfrak{F}}_0$ and $\mu(A) > 0$. By Lemma 6, there is $\pi \in \mathfrak{P}_1$ such that $\pi(A) = 1$. Conditions (c) and ($\bar{c}$) are both satisfied. The proof is complete.

4. Complete separable metric spaces

In this section we generalize Theorem 4 to a form that covers many practical applications. The main result, Theorem 5, is for countable sets of transformations $\mathfrak{G}$. Since the semigroup closure of a countable set of measure-preserving transformations is again a countable set of measure-preserving transformations, we assume in Theorem 5 that $\mathfrak{G}$ is a semigroup. Further we may suppose $\mathfrak{G}$ has an identity element. The proof of Theorem 5 in the case $\mathfrak{G}$ is a countable semigroup requires the construction of a representation relative to $\bar{\mathfrak{F}}_0$ rather than $\mathfrak{F}_0$, $\bar{\mathfrak{F}}_0$ defined as in the beginning of Section 2. In special cases, if $\mathfrak{G}$ is a group or if $\mathfrak{G}$ is Abelian, a representation may be constructed relative to $\mathfrak{F}_0$.

By continuity considerations the applicability of Theorem 5 may be extended. Corollary 3 covers the case $\mathfrak{G}$ has a countable subset $\mathfrak{G}_0$ dense in $\mathfrak{G}$ under pointwise convergence, that is, if $T \in \mathfrak{G}$, then there is a sequence $\{T_i, i \geqq 1\}$ of transformations in $\mathfrak{G}_0$ such that for all $x \in X$, $T(x) = \lim_{n \to \infty} T_n(x)$. At the end of this section we apply Corollary 3 to the problem of a set $\mathfrak{G}$ of continuous transformations of a compact metric space. Corollary 4 covers the case $\mathfrak{G}$ is a locally compact group having a countable base for the open sets of $\mathfrak{G}$. Corollary 5 covers the case $\mathfrak{G}$ is a locally compact Abelian semigroup having a countable base for the open sets of $\mathfrak{G}$ and having a nonzero regular semi-invariant measure η (see Introduction for a definition).

THEOREM 5. *Suppose* $(X, \mathfrak{F}, \mu)$ *is a probability space,* X *a complete separable metric space,* $\mathfrak{F}$ *the set of Borel subsets of* X. *If* $\mathfrak{G}$ *is a countable semigroup of measurable and measure-preserving transformations, then* μ *has a unique representation.*

Proof. To prove this theorem we work with the sets in $\bar{\mathfrak{F}}_0$. As shown in Section 2, since $\mathfrak{G}$ is countable, conditions ($\bar{\text{a}}$) and ($\bar{\text{b}}$) are satisfied by $\bar{\mathfrak{F}}_0$, $\mathfrak{G}$. It remains to verify condition ($\bar{\text{c}}$). To do this we embed X in a compact metric space $Q^{\mathfrak{G}}$ in such a way that the transformations in $\mathfrak{G}$ induce a set of continuous transformations on $Q^{\mathfrak{G}}$. The results in Section 3 can then be applied. To obtain the embedding we show first there is a 1-1 measurable transformation $\psi\colon X \to [0, 1]$. Let Q^{ω} be the Cartesian product of [0, 1] with itself a countable number of times, Q^{ω} given the Cartesian product topology. Then the topology of Q^{ω} is metrizable, so we consider Q^{ω} to be a compact metric space. By Kelley [10] there is a homeomorphism $\psi_1\colon X \to Q^{\omega}$ mapping X into Q^{ω}. As is well known there exists a 1-1 measurable transformation $\psi_2\colon Q^{\omega} \to [0, 1]$ onto. Consequently, the mapping $\psi_2 \psi_1$ is a 1-1 measurable transformation of X into [0, 1].

Let $Q^{\mathfrak{G}}$ be the set of all functions on $\mathfrak{G}$ to [0, 1] in the Cartesian product topology. Then $Q^{\mathfrak{G}}$ is homeomorphic to Q^{ω} and is a compact metric space. We map X into $Q^{\mathfrak{G}}$ as follows. Let $\psi = \psi_2 \psi_1$, and let ϕ be defined by

$$\phi(x) = \{\psi(Tx),\ T \in \mathfrak{G}\}.$$

Since ψ is a 1-1 measurable transformation, the map ϕ is a 1-1 measurable transformation of X into $Q^{\mathfrak{G}}$ (since $\mathfrak{G}$ has an identity element).

Let $\psi_3\colon Q^{\mathfrak{G}} \to [0, 1]$ be a 1-1 measurable transformation of $Q^{\mathfrak{G}}$ onto [0, 1]. By Hausdorff [8, p. 269], ψ_3^{-1} is a measurable transformation. Further by Hausdorff, op. cit., $\psi_3 \phi(A)$ is a Borel subset of [0, 1], and therefore $\phi(A)$ is a Borel subset of $Q^{\mathfrak{G}}$, if $A \in \mathfrak{F}$.

Each $T \in \mathfrak{G}$ induces a transformation T^* on $Q^{\mathfrak{G}}$ by the definition $(T^*y)(S) = y(ST)$. Recall $y \in Q^{\mathfrak{G}}$ is a function on $\mathfrak{G}$. The mappings T^* are continuous, and if $\mathfrak{G}$ is a group, the mappings T^* are 1-1. We let $\mathfrak{G}^*$ be the set of mappings induced by the transformations in $\mathfrak{G}$.

Let $\mathfrak{F}^*$ be the Borel subsets of $Q^{\mathfrak{G}}$. If ν is a probability measure defined on $(X, \mathfrak{F})$, we define a probability measure $\nu^* = (\nu\phi^{-1})$ by $(\nu\phi^{-1})(A) = \nu(\phi^{-1}A)$, $A \in \mathfrak{F}^*$. We observe that $\phi(X) \in \mathfrak{F}^*$ and $\nu^*(\phi(X)) = \nu(X) = 1$. Further if ν is an invariant measure relative to $\mathfrak{G}$, then $(\nu\phi^{-1})$ is invariant relative to $\mathfrak{G}^*$. By definition,

$$T^*(\phi(x)) = T^*(\{\psi(Sx),\ S \in \mathfrak{G}\}) = \{\psi(STx),\ S \in \mathfrak{G}\} = \phi(Tx).$$

Therefore

$$(\nu\phi^{-1})(T^{*-1}A) = \nu(\phi^{-1}T^{*-1}A) = \nu(T^{-1}\phi^{-1}A) = \nu(\phi^{-1}A) = (\nu\phi^{-1})(A).$$

Conversely if ν^* is a probability measure defined on $\mathfrak{F}^*$ and $\nu^*(\phi(X)) = 1$, we define a probability measure $(\nu^*\phi)$ on $\mathfrak{F}$ by, if $A \in \mathfrak{F}$, $(\nu^*\phi)(A) = \nu^*(\phi(A))$. This is possible since, as observed earlier, if $A \in \mathfrak{F}$, then $\phi(A) \in \mathfrak{F}^*$.

Suppose then $\mathfrak{P}^*$ is the set of invariant measures on $\mathfrak{F}^*$, and $\mathfrak{P}_1^*$ the set of extreme points of $\mathfrak{P}^*$. We let $\bar{\mathfrak{F}}_0^*$ be the extension of the $\mathfrak{F}_0^*$ invariant sets as described in Section 2. We show every $\nu^* \in \mathfrak{P}^*$ can be represented. Since conditions $(\bar{a})$ and $(\bar{b})$ are satisfied, it suffices to verify $(\bar{c})$. For this purpose let $A \in \bar{\mathfrak{F}}_0^*$, $\nu^*(A) > 0$. By Lemma 6 there is $\pi^* \in \mathfrak{P}_1^*$ such that $\pi^*(A) = 1$. It follows ν^* has a unique representation. In particular if $\mu \in \mathfrak{P}$, then $(\mu\phi^{-1})$ is invariant under $\mathfrak{G}^*$ and can be represented.

We now show that if $A \in \mathfrak{F}$ and if $T^* \in \mathfrak{G}^*$,

$$(T^{*-1}\phi(A)) \cap \phi(X) = \phi(T^{-1}A).$$

To see this, $y = \phi(x) \in T^{*-1}\phi(A)$ if and only if $T^*\phi(x) \in \phi(A)$ if and only if $\phi(Tx) \in \phi(A)$ if and only if $x \in T^{-1}A$ if and only if $\phi(x) \in \phi(T^{-1}A)$. It follows that if $\nu^* \in \mathfrak{P}^*$, $\nu^*(\phi(X)) = 1$, then, if $A \in \mathfrak{F}$, $T \in \mathfrak{G}$,

$$\begin{aligned}(\nu^*\phi)(T^{-1}A) &= \nu^*(\phi(T^{-1}A)) = \nu^*((T^{*-1}\phi(A)) \cap \phi(X)) \\ &= \nu^*(T^{*-1}\phi(A)) = \nu^*(\phi(A)) = (\nu^*\phi)(A).\end{aligned}$$

Therefore $(\nu^*\phi) \in \mathfrak{P}$.

If $A \in \bar{\mathfrak{F}}_0$ and $\pi^* \in \mathfrak{P}_1^*$ and $\pi^*(\phi(X)) = 1$, then $(\pi^*\phi) \in \mathfrak{P}_1$, and

$$(\pi^*\phi)(A) = 0 \quad \text{or} \quad (\pi^*\phi)(A) = 1.$$

For if $(\pi^*\phi) \notin \mathfrak{P}_1$, then there are $0 < \beta < 1$ and $\nu_1, \nu_2 \in \mathfrak{P}$ such that $(\pi^*\phi) = \beta\nu_1 + (1 - \beta)\nu_2$. Therefore since $\pi^*(\phi(X)) = 1$, $\pi^* = ((\pi^*\phi)\phi^{-1}) = \beta(\nu_1\phi^{-1}) + (1 - \beta)(\nu_2\phi^{-1})$, a contradiction. Therefore $(\pi^*\phi) \in \mathfrak{P}_1$, and as shown in Corollary 1, $(\pi^*\phi)(A) = 0$ or $(\pi^*\phi)(A) = 1$, $A \in \bar{\mathfrak{F}}_0$.

Then, if $\mu \in \mathfrak{P}$, $\mu(A) > 0$ for some $A \in \bar{\mathfrak{F}}_0$, since $(\mu\phi^{-1})$ can be represented and $(\mu\phi^{-1})(\phi A) > 0$, there is a $\pi^* \in \mathfrak{P}_1^*$ such that $\pi^*(\phi(X)) = 1$ and $\pi^*(\phi(A)) > 0$. Then $(\pi^*\phi) \in \mathfrak{P}_1$, and since $(\pi^*\phi)(A) > 0$, $(\pi^*\phi)(A) = 1$. Condition $(\bar{c})$ is therefore satisfied, and the proof is complete.

COROLLARY 3. *Suppose $(X, \mathfrak{F}, \mu)$ is as in Theorem 5. Let $\mathfrak{G}$ be a set of measurable and measure-preserving transformations. Suppose $\mathfrak{G}$ has a countable subset $\mathfrak{G}_0$ dense in $\mathfrak{G}$ under pointwise convergence. Then $\bar{\mathfrak{F}}_0$ is a sufficient sub σ-algebra for $\mathfrak{P}$. Each $\mu \in \mathfrak{P}$ has a unique representation determined by $\bar{\mathfrak{F}}_0$.*

Proof. We prove first that $\bar{\mathfrak{F}}_0$ is a sufficient sub σ-algebra for $\mathfrak{P}$. Suppose $\nu \in \mathfrak{P}$, g is a bounded function measurable in $\bigcap_{T \in \mathfrak{G}_0} \bar{\mathfrak{F}}_{0,T}$. Then if $T \in \mathfrak{G}_0$, $g(Tx) = g(x)$ a.e. ν. Suppose $T \in \mathfrak{G}$ and $\{T_i, i \geqq 1\}$ is a sequence of transformations in $\mathfrak{G}_0$ such that for all $x \in X$, $\lim_{i\to\infty} T_i(x) = T(x)$. Let $\{h_j, j \geqq 1\}$

be a sequence of bounded real-valued continuous functions on X such that

$$\lim_{j\to\infty} \int | h_j(x) - g(x) | \, d\nu(x) = 0.$$

By hypothesis on g, invariance of ν, continuity of h, and the bounded convergence theorem,

$$\begin{aligned}\int | h_j(x) - g(x) | \, d\nu(x) &= \lim_{i\to\infty} \int | h_j(T_i x) - g(T_i x) | \, d\nu(x) \\ &= \lim_{i\to\infty} \int | h_j(T_i x) - g(x) | \, d\nu(x) \\ &= \int | h_j(Tx) - g(x) | \, d\nu(x).\end{aligned}$$

Since

$$\lim_{j\to\infty} \int | h_j(Tx) - g(Tx) | \, d\nu(x) = 0,$$

it now follows that

$$\int | g(Tx) - g(x) | \, d\nu(x) = 0.$$

This holds for every $T \in \mathfrak{G}$, $\nu \in \mathfrak{P}$. Consequently if $\nu \in \mathfrak{P}$,

$$\nu(\{x \mid g(x) = g(T^i x), i \geqq 0\}) = 1.$$

Since $\lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} g(T^i x)$ is an invariant function, it follows g is measurable in $\bar{\mathfrak{F}}_{0,T}$. Consequently g is measurable in $\bar{\mathfrak{F}}_0 = \bigcap_{T\in\mathfrak{G}} \bar{\mathfrak{F}}_{0,T}$. By Theorem 2 if f is a bounded measurable function, there is a bounded function $\phi(f)$ measurable in $\bigcap_{T\in\mathfrak{G}_0} \bar{\mathfrak{F}}_{0,T}$ such that if $A \in \bigcap_{T\in\mathfrak{G}_0} \bar{\mathfrak{F}}_{0,T}$, then

$$\int_A f(x) \, d\nu(x) = \int_A \phi(f)(x) \, d\nu(x), \qquad \text{all } \nu \in \mathfrak{P}.$$

Then this holds if $A \in \bar{\mathfrak{F}}_0$. As just shown $\phi(f)$ is measurable in $\bar{\mathfrak{F}}_0$. Therefore $\bar{\mathfrak{F}}_0$ is a sufficient sub σ-algebra for $\mathfrak{P}$.

Let $\mu \in \mathfrak{P}$, $A \in \bar{\mathfrak{F}}_0$, $\mu(A) > 0$. By Theorem 5 there is a measure π invariant under $\mathfrak{G}_0$ such that $\pi(A) = 1$, and if $B \in \bigcap_{T\in\mathfrak{G}_0} \bar{\mathfrak{F}}_{0,T}$, then $\pi(B) = 0$ or $\pi(B) = 1$. Let $T \in \mathfrak{G}$. As shown above,

$$\int h(Tx) \, d\pi(x) = \int h(x) \, d\pi(x)$$

for all bounded continuous functions. It follows that for every open set U, $\pi(T^{-1}U) = \pi(U)$. Since the measures $\pi(T^{-1}(\cdot))$ and $\pi(\cdot)$ are regular, it follows $\pi(T^{-1}B) = \pi(B)$ for every Borel set B. Therefore $\pi \in \mathfrak{P}$, and by Corollary 1, since $\bar{\mathfrak{F}}_0$ is a sufficient sub σ-algebra, $\pi \in \mathfrak{P}_1$.

Therefore ($\bar{a}$), ($\bar{b}$), and ($\bar{c}$) are satisfied, and the proof is complete.

COROLLARY 4. *Suppose* $(X, \mathfrak{F}, \mu)$ *is as in Theorem* 5. *Let* $\mathfrak{G}$ *be a locally compact group of jointly measurable and measure-preserving transformations. If* $\mathfrak{G}$ *has a countable base for the open sets of* $\mathfrak{G}$, *then* μ *has a unique representation.*

Proof. Conditions (a) and (b) are satisfied. We verify condition (c). Let $\mathfrak{G}_0$ be a countable dense subgroup of $\mathfrak{G}$. If $A \epsilon \mathfrak{F}_0$ and $\mu(A) > 0$, then by Theorem 5 there is a measure π invariant and ergodic relative to $\mathfrak{G}_0$ such that $\pi(A) = 1$. We show that π is already an invariant measure relative to $\mathfrak{G}$. Since $\mathfrak{G}$ satisfies the hypotheses of Theorem 3, $\pi \epsilon \mathfrak{P}_1$ follows by Corollary 1. Therefore condition (c) would be verified.

Suppose then that π is invariant under the transformations in $\mathfrak{G}_0$ and $\{T_i, i \geqq 1\}$ is a sequence of transformations in $\mathfrak{G}_0$ such that $T_i \rightarrow T_0$ as $i \rightarrow \infty$. By Fubini's theorem, if $B \epsilon \mathfrak{F}$, then

$$\int \chi_B(S^{-1}x)\, d\pi(x) = \pi(SB)$$

is a bounded $\mathfrak{H}$-measurable function. By Lemma 3, if C is a compact subset of $\mathfrak{G}$, then

$$\lim_{i \to \infty} \int_C |\pi(T_i SB) - \pi(T_0 SB)|\, d\eta(S) = 0.$$

By the invariance of π, $\pi(T_i SB) = \pi(SB)$. Therefore

$$\int_C |\pi(SB) - \pi(T_0 SB)|\, d\eta(S) = 0,$$

and since this holds for every compact set C,

$$\int_{\mathfrak{G}} |\pi(SB) - \pi(T_0 SB)|\, d\eta(S) = 0.$$

Since X is a complete separable metric space, it has a countable base $\mathfrak{U}_1$ for its open sets. Let $\mathfrak{U}$ be the set of all open sets which are finite unions of sets in $\mathfrak{U}_1$. It follows from the above that there is an $S \epsilon \mathfrak{G}$ such that for every $U \epsilon \mathfrak{U}$, $\pi(SU) = \pi(T_0 SU)$. Since $\pi(S(\cdot))$ and $\pi(T_0 S(\cdot))$ are countably additive measures, it follows at once by taking limits on sequences of sets in $\mathfrak{U}$ that if $U \subset X$ is an open set, $\pi(SU) = \pi(T_0 SU)$. It then follows from the regularity of $\pi(S(\cdot))$ and $\pi(T_0 S(\cdot))$ that if $A \epsilon \mathfrak{F}$, $\pi(SA) = \pi(T_0 SA)$. Since S is 1-1 onto and S^{-1} is also measurable, it follows for every $A \epsilon \mathfrak{F}$ that $\pi(A) = \pi(T_0 A)$. The proof is complete.

COROLLARY 5. *Suppose* $(X, \mathfrak{F}, \mu)$ *is as in Theorem* 5. *If* $\mathfrak{G}$ *is a locally compact Abelian semigroup such that* $\mathfrak{G}$ *has a countable base for the open sets of* $\mathfrak{G}$, *and* $\mathfrak{H}$ *has a nonzero regular semi-invariant measure, then* μ *has a unique representation.*

Proof. Suppose $\mathfrak{S}_0$ is a dense countable subsemigroup of $\mathfrak{S}$ and π is invariant relative to $\mathfrak{S}_0$. As in the proof of Corollary 4 we may show

$$\int_{\mathfrak{S}} |\pi(S^{-1}B) - \pi(T_0^{-1}S^{-1}B)| \, d\eta(S) = 0.$$

We use here the fact that the transformations of $\mathfrak{S}$ commute to obtain

$$\pi((TS)^{-1}B) = \pi(S^{-1}T^{-1}B) = \pi(T^{-1}S^{-1}B).$$

Suppose $g \geqq 0$, $g \in L_1(\mathfrak{S}, \mathfrak{H}, \eta)$, $\int_{\mathfrak{S}} g(S) \, d\eta(S) = 1$. Then π^* defined by

$$\int_{\mathfrak{S}} \pi(S^{-1}B) g(S) \, d\eta(S) = \pi^*(B)$$

is clearly an invariant probability measure. Suppose $A \in \mathfrak{F}_0$, that is, A is invariant under all $T \in \mathfrak{S}$. It follows that $\pi(A) = \pi^*(A)$. Suppose $A \in \mathfrak{F}_0$ and $\mu(A) > 0$. By Theorem 5, if $\mathfrak{S}_0$ is a dense countable Abelian subsemigroup of $\mathfrak{S}$, there is an ergodic invariant measure π, invariant under the transformations of $\mathfrak{S}_0$, such that $\pi(A) = 1$. From the above it follows the corresponding π^* is ergodic and $\pi^*(A) = 1$. It then follows from Corollary 1, Section 2, that $\pi^* \in \mathfrak{P}_1$, and therefore that condition (c) is satisfied. Since $\mathfrak{S}$ satisfies the hypotheses of Theorem 3, conditions (a) and (b) are also satisfied. Therefore μ has a unique representation determined by $\mathfrak{F}_0$.

A classical theorem in representation theory says that if X is a compact metric space, $\mathfrak{F}$ the Borel subsets of X, μ a probability measure on $(X, \mathfrak{F})$, and $\mathfrak{S}$ a group of homeomorphisms of X onto X, each $T \in \mathfrak{S}$ measure-preserving, then the measure μ is representable.

Suppose ρ is the metric on X, and $C(X, X)$ the set of continuous functions on X to X. If f and $g \in C(X, X)$, define $\rho(f, g) = \sup_x \rho(f(x), g(x))$. It is known that $C(X, X)$ is a separable metric space. See for example Kuratowski [11, pp. 120, 315]. By Corollary 3 it follows that μ has a unique representation. For $\mathfrak{S} \subset C(X, X)$, and therefore $\mathfrak{S}$ is a separable metric space under the restriction of ρ to $\mathfrak{S}$. $\mathfrak{S}$ therefore has a countable subgroup which is dense in $\mathfrak{S}$ under pointwise convergence.

It is not, however, necessary to assume the transformations in $\mathfrak{S}$ are homeomorphisms.

THEOREM 6. *Suppose $(X, \mathfrak{F}, \mu)$ is a probability space, X compact metric, $\mathfrak{F}$ the Borel subsets of X. Let $\mathfrak{S}$ be a semigroup of continuous and measure-preserving transformations of X into X. Then μ has a unique representation determined by $\bar{\mathfrak{F}}_0$.*

Remark. The existence of a unique representation asserted in Theorem 6 may be proved using the results of Choquet [3]. That Choquet's theorems may be validly applied follows from Corollary 2, Section 2 and Lemma 5, Section 3.

5. Examples

Suppose $\mathfrak{S}$ is a locally compact semigroup having a countable base for the open sets of $\mathfrak{S}$. Let q be a point not in $\mathfrak{S}$, $\mathfrak{S}' = \{q\} \cup \mathfrak{S}$, and let $\mathfrak{S}'$ have the one-point compactification topology. If $T \in \mathfrak{S}$, define $Tq = qT = q$. Then it is easily seen that the map $(T, S') \to TS'$ is jointly continuous in the product topology on $\mathfrak{S} \times \mathfrak{S}'$. If μ is the probability measure supported on $\{q\}$, then μ is invariant under the transformations in $\mathfrak{S}$.

It is clear that if $(X, \mathfrak{F})$ is any measurable space such that $\mathfrak{S}$ acts on X as a semigroup of measurable transformations, then if $x \in X$ is a fixed point under $\mathfrak{S}$, there exists an invariant probability measure on $(X, \mathfrak{F})$. We use this idea to construct nontrivial examples.

Let $\mathfrak{S}'$ be as above, and $C(\mathfrak{S}', [0, 1])$ the set of all continuous functions on $\mathfrak{S}'$ to $[0, 1]$ taken with the sup norm. $C(\mathfrak{S}', [0, 1])$ is a complete separable metric space. For ($\mathfrak{S}$ and hence) $\mathfrak{S}'$ has a countable base for the open sets of ($\mathfrak{S}$) $\mathfrak{S}'$, and therefore the topology of $\mathfrak{S}'$ is metrizable. Therefore $\mathfrak{S}'$ is a compact metric space, and it follows that $C(\mathfrak{S}', [0, 1])$ is a separable metric space.

$\mathfrak{S}$ induces a semigroup $\mathfrak{S}^*$ of transformations on $C(\mathfrak{S}', [0, 1])$, $T \to T^* \in \mathfrak{S}^*$ defined by $(T^*f)(S) = f(ST)$, $f \in C(\mathfrak{S}', [0, 1])$, and $S \in \mathfrak{S}'$. Since

$$((T_1 T_2)^* f)(S) = f(ST_1 T_2) = (T_2^* f)(ST_1) = (T_1^*(T_2^* f))(S),$$

the map $T \to T^*$ is a homomorphism. If $T_1^* = T_2^*$, then for all $f \in C(\mathfrak{S}', [0, 1])$, $S \in \mathfrak{S}'$, $f(ST_1) = f(ST_2)$. Since the functions in $C(\mathfrak{S}', [0, 1])$ separate the points of $\mathfrak{S}'$, $ST_1 = ST_2$ for all $S \in \mathfrak{S}'$. Taking S = identity of $\mathfrak{S}$ gives $T_1 = T_2$. Therefore, $T \to T^*$ is an isomorphism. We take on $\mathfrak{S}^*$ the topology which makes this a homeomorphism.

The semigroup $\mathfrak{S}^*$ then acts in a jointly continuous way. For if $\{T_i^*, i \geqq 1\}$ is a sequence of elements in $\mathfrak{S}^*$, $\lim_{i\to\infty} T_i^* = T^*$, and if $\{f_i, i \geqq 1\}$ is a sequence of functions in $C(\mathfrak{S}', [0, 1])$, $\lim_{i\to\infty} f_i = f$, then f_i converges to f uniformly as $i \to \infty$, and

$$\lim_{i\to\infty} (T_i^* f_i)(S) = \lim_{i\to\infty} f_i(ST_i) = f(ST) = (T^*f)(S).$$

$\mathfrak{S}^*$ acting on $C(\mathfrak{S}', [0, 1])$ has the constant functions as fixed points. The set $\mathfrak{A}$ of constant functions is clearly homeomorphic to $[0, 1]$, and therefore $\mathfrak{A}$ is a compact, therefore a Borel, subset of $C(\mathfrak{S}', [0, 1])$. Let μ^* be any probability measure defined on the Borel subsets of $[0, 1]$. Let μ^{**} be the corresponding measure on the Borel subsets of $\mathfrak{A}$. Extend μ^{**} to a measure μ on the Borel subsets $\mathfrak{F}$ of $C(\mathfrak{S}', [0, 1])$ by, if $A \in \mathfrak{F}$, $\mu(A) = \mu^{**}(A \cap \mathfrak{A})$. μ is clearly an invariant probability measure.

THEOREM 7. *Let $\mathfrak{S}$ be a locally compact semigroup having a countable base for the open sets of $\mathfrak{S}$. Then there are a probability space $(X, \mathfrak{F}, \mu)$, X a complete separable metric space, $\mathfrak{F}$ the Borel subsets of X, and a locally compact semigroup $\mathfrak{S}^*$ of jointly continuous and measure-preserving transformations* *relative to* $(X, \mathfrak{F}, \mu)$ *such that* $\mathfrak{S}$ *and* $\mathfrak{S}^*$ *are isomorphic. The probability measure* μ *is nonatomic.*

Suppose $X = [0, 1]$, $\mathfrak{F}$ are Borel subsets of X, and μ^* is Lebesgue measure on $(X, \mathfrak{F})$. Let $\mathfrak{T}$ be the set of all measurable and measure-preserving transformations relative to $(X, \mathfrak{F}, \mu^*)$. $\mathfrak{T}$ is clearly a semigroup.

THEOREM 8. *Let* $\mathfrak{S}$ *be a countable semigroup such that the cancellation law holds for* $\mathfrak{S}$. *There is a subsemigroup of* $\mathfrak{T}$ *which is algebraically isomorphic to* $\mathfrak{S}$.

Proof. Let $Q^{\mathfrak{S}}$ be the set of all functions on $\mathfrak{S}$ to $[0, 1]$, $Q^{\mathfrak{S}}$ given the product topology. As is well known, $Q^{\mathfrak{S}}$, as the Cartesian product of $[0, 1]$ with itself χ_0 times, may be made into a product measure space, starting from Lebesgue measure μ^* on $[0, 1]$. As observed in the proof of Theorem 5, the elements of $\mathfrak{S}$ naturally induce a semigroup $\mathfrak{S}^*$ of continuous maps of $Q^{\mathfrak{S}}$, defined by $(T^*y)(S) = y(ST)$, $T \to T^*$, $y \in Q^{\mathfrak{S}}$, $S \in \mathfrak{S}$. Since the cancellation law holds for $\mathfrak{S}$, one may readily verify that each $T^* \in \mathfrak{S}^*$ is a measure-preserving transformation. Finally, as is well known, there is a map $\psi_4 : Q^{\mathfrak{S}} \to [0, 1]$ which is 1-1, onto, and such that ψ_4 and ψ_4^{-1} are each measurable. Further if μ is the product measure on $Q^{\mathfrak{S}}$, and if A is a Borel subset of $[0, 1]$, $\mu(\psi_4^{-1}A) = \mu^*(A)$. Let $\mathfrak{S}^{**}$ be the transformations in $\mathfrak{T}$ given by $S \in \mathfrak{S}^{**}$ if and only if $S = \psi_4 T^* \psi_4^{-1}$, $T^* \in \mathfrak{S}^*$. It is easily verified that $\mathfrak{S}$ and $\mathfrak{S}^{**}$ are algebraically isomorphic.

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