# THE DIMENSION OF THE SET OF ZEROS AND THE GRAPH OF A SYMMETRIC STABLE PROCESS

BY

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## 1. Introduction

Let $\{X(t);\ t \geqq 0\}$ be the one-dimensional symmetric stable process of index $\alpha$ with $0 < \alpha \leqq 2$, that is, a process with stationary independent increments whose continuous transition density $f(t, x - y)$ is given by

$$(1.1) \qquad f(t, x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-t|\xi|^\alpha} e^{ix\xi}\, d\xi.$$

We assume throughout this paper that $X(0) = 0$ and that the sample functions are normalized to be right continuous and have left-hand limits everywhere. Furthermore we assume that $\{X(t); t \geqq 0\}$ is defined over some basic probability space $(\Omega, \mathfrak{F}, P)$ where $\mathfrak{F}$ is complete relative to $P$. Let

$$(1.2) \qquad Z(\omega) = \{t > 0 : X(t, \omega) = 0 \quad \text{or} \quad X(t-, \omega) = 0\}.$$

It is known, e.g. [8], that if $0 < \alpha \leqq 1$, then $Z(\omega)$ is empty for almost all $\omega$. Our first result is the following theorem.

THEOREM A. *$P[\dim Z(\omega) = 1 - 1/\alpha] = 1$ if $1 < \alpha \leqq 2$, where* "dim" *is the usual Hausdorff-Besicovitch dimension (see Section 2).*

If $Z'(\omega) = \{t : X(t, \omega) = 0\}$, then since for fixed $\omega$ the sets $Z(\omega)$ and $Z'(\omega)$ differ at most by a countable number of points, we have the following corollary to Theorem A.

COROLLARY. *$P[\dim Z'(\omega) = 1 - 1/\alpha] = 1$ if $1 < \alpha \leqq 2$.*

If $\alpha = 2$, our process is essentially Brownian motion, and in this case the above result is due to S. J. Taylor [9].

Our second result gives the dimension of the graph of $X(t)$.

THEOREM B. *Let $G(\omega) = \{(t, X(t, \omega)) : t \geqq 0\}$; then*

(i) $P[\dim G(\omega) = 2 - 1/\alpha] = 1$ *if* $1 < \alpha \leqq 2$

(ii) $P[\dim G(\omega) = 1] = 1$ *if* $0 < \alpha \leqq 1$.

Again in the case $\alpha = 2$ this result is due to S. J. Taylor [9]. However, there seems to be a lapse in his proof. In particular the equation in line $(-6)$ on page 270 is incorrect, but this is easily corrected.

In Section 2, following Lévy [6], we define the concept of *stochastic equivalence* for random sets and show that if two random sets $A$ and $B$ are stochastically equivalent, then for each $\beta > 0$ the $\beta$-dimensional Hausdorff measures

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$\Lambda^\beta(A)$ and $\Lambda^\beta(B)$ are random variables with the same distribution. In Section 3 we prove that $Z$ and the range of the stable subordinator of index $1 - 1/\alpha$ (defined in Section 3) are stochastically equivalent. Theorem A then follows from known results on the dimension of the range of a stable subordinator [2].

Finally in Section 4 we give a proof of Theorem B.

## 2. Random sets

Given a probability space $(\Omega, \mathfrak{F}, P)$ where $\mathfrak{F}$ is complete relative to $P$, and a function $A$ from $\Omega$ to subsets of the real line, $R$, we say that $A$ is a *random set* if

(i) $A(\omega)$ is compact for almost all $\omega$,
(ii) $\{\omega: A(\omega) \subset E\}$ is in $\mathfrak{F}$ for all open subsets $E$ of $R$.

Two random sets $A$ and $B$ (not necessarily defined over the same probability space) are *stochastically equivalent* if for every set $E$ that is a *finite* union of open intervals

$$P\{\omega: A(\omega) \subset E\} = P\{\omega: B(\omega) \subset E\}. \tag{2.1}$$

These definitions were suggested by Lévy [6, Ch. VI].

We now recall the definition of Hausdorff measure and dimension. Given $\alpha > 0$, $\varepsilon > 0$, and $K$ a subset of $R$, we set $\Lambda_\varepsilon^\alpha(K) = \inf \sum | I_j |^\alpha$ where the infimum is taken over all covers of $K$ by a countable union of intervals, $I_j$, none of which has a diameter exceeding $\varepsilon$. Here $| B |$ denotes the diameter of the set $B$. Moreover $\Lambda^\alpha(K) = \lim_{\varepsilon \to 0} \Lambda_\varepsilon^\alpha(K)$ exists, and

$$\inf \{\alpha > 0: \Lambda^\alpha(K) = 0\} = \sup \{\alpha \geqq 0: \Lambda^\alpha(K) = \infty\}. \tag{2.2}$$

The common value of the infimum and supremum in (2.2) is called the Hausdorff dimension of $K$ and is written dim $K$. Clearly if $K$ is compact, we may compute $\Lambda_\varepsilon^\alpha(K)$ by using only those covers of $K$ which are *finite* unions of open intervals with *rational* endpoints.

The following lemma is basic.

LEMMA 2.1. *If $A$ is a random set, then $\Lambda^\alpha(A)$ is a random variable (possibly taking on the value $+\infty$). If $A$ and $B$ are stochastically equivalent random sets, then $\Lambda^\alpha(A)$ and $\Lambda^\alpha(B)$ have the same distribution.*

*Proof.* Using a convenient, although incorrect, notation we will let $E$ denote a generic *finite* collection $I_1, \cdots, I_n$ of *open* intervals with *rational endpoints* and also let $E$ denote the (open) set $\bigcup_{j=1}^n I_j$. We define $d(E) = \max_{j \leqq n} | I_j |$ and $S_\alpha(E) = \sum_{j=1}^n | I_j |^\alpha$. For a fixed $\omega$ and $b \geqq 0$ we have $\Lambda_\varepsilon^\alpha(A(\omega)) < b$ if and only if there is an $E$ with $d(E) \leqq \varepsilon$ and $S_\alpha(E) < b$ such that $A(\omega) \subset E$. Let $E_1, E_2, \cdots$ be an enumeration of those $E$'s having the property that $d(E) \leqq \varepsilon$ and $S_\alpha(E) < b$. If $\Delta_i = \{\omega: A(\omega) \subset E_i\}$, then

$$\{\omega: \Lambda_\varepsilon^\alpha(A) < b\} = \bigcup_{i=1}^\infty \Delta_i .$$

By the definition of random set each $\Delta_i$ is in $\mathfrak{F}$, and hence $\Lambda_\varepsilon^\alpha(A)$ is a random variable. Letting $\varepsilon \to 0$ through a sequence of values yields the first assertion of Lemma 2.1.

Moreover we have

$$(2.3) \qquad P\{\omega : \Lambda_\varepsilon^\alpha(A) < b\} = \lim_{n\to\infty} P(\cup_{i=1}^n \Delta_i),$$

and for a fixed $n$ the inclusion-exclusion formula implies that

$$P(\cup_{i=1}^n \Delta_i) = \sum P(\Delta_i) - \sum P(\Delta_i \cap \Delta_j) + \sum P(\Delta_i \cap \Delta_j \cap \Delta_k) - \cdots.$$

Looking at a typical intersection we see that

$$P(\Delta_i \cap \cdots \cap \Delta_k) = P\{\omega : A(\omega) \subset E_i \cap \cdots \cap E_k\}.$$

Thus if $A$ and $B$ are stochastically equivalent random sets, the left side of (2.3) is unchanged if $A$ is replaced by $B$. Hence $\Lambda_\varepsilon^\alpha(A)$ and $\Lambda_\varepsilon^\alpha(B)$ have the same distribution. Again letting $\varepsilon \to 0$ through a sequence of values yields the second assertion of Lemma 2.1.

## 3. Proof of Theorem A

Let $\{T(t); t \geqq 0\}$ be the stable subordinator of index $\beta$, $0 < \beta < 1$, that is, a process with stationary independent and *positive* increments whose transition density $g(t, u)$ is given by

$$(3.1) \qquad e^{-ts^\beta} = \int_0^\infty e^{-su} g(t, u)\, du.$$

We assume that $T(0) = 0$, and that the sample functions of $T$ are normalized to be right continuous and have left-hand limits everywhere. The sample functions of $T$ are strictly monotone increasing with probability one. As in Section 1, $X = \{X(t); t \geqq 0\}$ is the symmetric stable process of index $\alpha$, and we will assume throughout this section that $1 < \alpha \leqq 2$. Moreover we will assume that the index $\beta$ of our stable subordinator $T$ is given by $\beta = 1 - 1/\alpha$.

Given a subset $E$ of $[0, \infty)$ we say that $X$ *touches* $a$ *in* $E$ if $X(t) = a$ or $X(t-) = a$ for some $t$ in $E$, and we say that $T$ *touches* $E$ if $T(t)$ is in $E$ or $T(t-)$ is in $E$ for some $t \geqq 0$.

LEMMA 3.1. *If* $I = [a, b]$ *where* $0 < a < b < \infty$, *then*

$$P[X \text{ touches } 0 \text{ in } I] = P[T \text{ touches } I] = [\Gamma(1/\alpha)\Gamma(1 - 1/\alpha)]^{-1} \int_0^{(b-a)/b} u^{1/\alpha - 1}(1 + u)^{-1}\, du.$$

*Proof.* We begin with the process $X$. Let $h(t, x)$ be the probability that a stable process of index $\alpha$, $1 < \alpha \leqq 2$, starting from $x$ touches 0 in $[0, t]$. Kac [5, Equation (5.4)] has shown that

$$(3.2) \qquad \int_0^\infty e^{-st} h(t, x)\, dt = [sK_s(0)]^{-1} K_s(x),$$

where

$$K_s(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} (s + |\xi|^\alpha)^{-1} \cos x\xi \, d\xi. \tag{3.3}$$

Thus if $p(a, t)$ denotes the probability that $X$ (starting from 0) touches 0 in $[a, t]$ we have for $t > a$

$$p(a, t) = \int_{-\infty}^{\infty} f(a, x) h(t - a, x) \, dx, \tag{3.4}$$

where $f$ is the transition density of $X$ defined in (1.1). If we set $p(a, t) = 0$ for $t \leqq a$ and take Laplace transforms on $t$, we obtain

$$\int_0^\infty e^{-st} p(a, t) \, dt = [2\pi s K_s(0)]^{-1} e^{-sa} \int_{-\infty}^{\infty} f(a, x) \, dx \int_{-\infty}^{\infty} \frac{\cos \xi x \, d\xi}{s + |\xi|^\alpha},$$

and the right-hand side, after a change of integration order and a change of variable, becomes

$$b_\alpha \int_0^\infty [s^{-1} e^{-sa(u+1)}] u^{1/\alpha - 1} (1 + u)^{-1} \, du, \tag{3.5}$$

where

$$b_\alpha = [\Gamma(1/\alpha)\Gamma(1 - 1/\alpha)]^{-1}. \tag{3.6}$$

The term in square brackets in (3.5) is the Laplace transform of the function which is 1 on $[a(u + 1), \infty)$ and 0 elsewhere. Thus

$$p(a, t) = b_\alpha \int_0^{(t-a)/a} u^{1/\alpha - 1} (1 + u)^{-1} \, du \tag{3.7}$$

provided $t > a$. If $t = b$, this is one half of the assertion of Lemma 3.1.

We turn our attention now to the subordinator $T$ of index $\beta = 1 - 1/\alpha$. Let $S_t = \inf \{\tau : T(\tau) \geqq t\}$, and let $F_t(A)$ be the probability that $T(S_t)$ is in $A$. Since $P\{T(\tau-) = a \text{ for some } \tau \geqq 0\} = 0$ for each fixed $a$, it follows that

$$P\{T \text{ touches } I\} = P\{T(S_a) \leqq b\} = F_a([a, b]).$$

From the fact that $T(\tau)$ has the same distribution as $\tau^{1/\beta} T(1)$, it follows easily that $E(S_t) = ct^\beta$ where $c$ is a positive constant and $E$ is the expectation operator. So the usual first passage time relationship (i.e., the strong Markov property) implies that

$$(t + a)^\beta = a^\beta + \int_a^{a+t} (a + t - x)^\beta F_a(dx), \qquad t > a.$$

This is an expression of convolution type, and so we can find $F_a$ by taking Laplace transforms. The result is

$$F_a([a, t]) = [\Gamma(\beta)\Gamma(1 - \beta)]^{-1} a^\beta \int_a^t x^{-1} (x - a)^{-\beta} \, dx.$$

Making the change of variable $u = a^{-1}(x - a)$, replacing $t$ by $b$, and recalling that $\beta = 1 - 1/\alpha$, we find that

$$F_a([a, b]) = p(a, b), \tag{3.8}$$

and thus the proof of Lemma 3.1 is complete.

LEMMA 3.2. *If $D$ is a finite disjoint union of closed intervals bounded away from* 0, *then*

$$P[X \text{ touches } 0 \text{ in } D] = P[T \text{ touches } D].$$

*Proof.* By the inclusion-exclusion formula it suffices to show that

$$P[\cap_{j=1}^n \{X \text{ touches } 0 \text{ in } [a_j, b_j]\}] = P[\cap_{j=1}^n \{T \text{ touches } [a_j, b_j]\}],$$

where $0 < a_1 < b_1 < a_2 < \cdots < b_n$. If $a > 0$, let

$$R_a = \inf \{t \geqq a : X(t) = 0 \quad \text{or} \quad X(t-) = 0\}$$

(or $+\infty$ if there are no such $t$); then $P\{R_a \leqq t\} = p(a, t)$. Hence $P\{R_a < \infty\} = 1$. If $R_a < \infty$, it follows from the discussion in Hunt [4, p. 54] that $X(R_a) = 0$ with probability one. Thus, using the strong Markov property repeatedly, we have

$$\begin{aligned} P[\cap_{j=1}^n \{X \text{ touches } 0 \text{ in } [a_j, b_j]\}] &= \int_{a_1}^{b_1} p(a_1, d\tau_1) \\ &\cdot \int_{a_2-\tau_1}^{b_2-\tau_1} p(a_2 - \tau_1, d\tau_2) \cdots \int_{a_{n-1}-\tau_{n-2}}^{b_{n-1}-\tau_{n-1}} p(a_{n-1} - \tau_{n-2}, d\tau_{n-1}) \\ &\cdot p(a_n - \tau_{n-1}, b_n - \tau_{n-1}). \end{aligned} \tag{3.9}$$

The same argument with $X$ replaced by $T$ and $R_a$ by $S_a$ shows that

$$\begin{aligned} P[\cap_{j=1}^n \{T \text{ touches } [a_j, b_j]\}] &= \int_{a_1}^{b_1} F_{a_1}(d\tau_1) \\ &\cdot \int_{a_2-\tau_1}^{b_2-\tau_1} F_{a_2-\tau_1}(d\tau_2) \cdots \int_{a_{n-1}-\tau_{n-2}}^{b_{n-1}-\tau_{n-2}} F_{a_{n-1}-\tau_{n-2}}(d\tau_{n-1}) \\ &\cdot F_{a_n-\tau_{n-1}}([a_n - \tau_{n-1}, b_n - \tau_{n-1}]). \end{aligned} \tag{3.10}$$

But Lemma 3.1, or more exactly (3.8), implies that the right-hand sides of (3.9) and (3.10) are equal, and hence Lemma 3.2 is established.

We are now ready to prove Theorem A.

*Proof of Theorem* A. Let $J$ be the closed interval $[c, d]$ with $0 < c < d < \infty$. We define

$$A(\omega) = \{t \in J : X(t, \omega) = 0 \quad \text{or} \quad X(t-, \omega) = 0\}$$

$$B(\omega) = \{t \in J : T(\tau, \omega) \text{ touches } t\}.$$

Both $A(\omega)$ and $B(\omega)$ are compact for almost all $\omega$ since the sample functions of $X$ and $T$ are right continuous and have left-hand limits. For a moment let $Y$ be the two-dimensional process $Y(t) = (t, X(t))$. If $E$ is an open set in $R$, let $D = J - E$, and define

$$Q = \inf \{t : Y(t) \in D \times \{0\} \quad \text{or} \quad Y(t-) \in D \times \{0\}\}$$

or $Q = \infty$ if there are no such $t$. Hunt [4, pp. 54–55] has shown that $Q$ is a random variable. Since $\{Q = \infty\} = \{A \subset E\}$, it follows that $A$ is a random set. A similar argument shows that $B$ is a random set.

We next show that $A$ and $B$ are stochastically equivalent. To this end let $E$ be a finite union of open intervals; then $D = J - E$ is a finite disjoint union of closed intervals bounded away from 0 since $c > 0$. Using Lemma 3.2 we have

$$P(A \subset E) = 1 - P[X \text{ touches } 0 \text{ in } D] = 1 - P[T \text{ touches } D] = P[B \subset E].$$

Thus $A$ and $B$ are stochastically equivalent, and therefore Lemma 2.1 implies that $\Lambda^\theta(A)$ and $\Lambda^\theta(B)$ have the same distribution for each fixed $\theta > 0$. If we let

$$Z(\omega) = \{t > 0 : X(t, \omega) = 0 \quad \text{or} \quad X(t-, \omega) = 0\}$$

and

$$R(\omega) = \{t > 0 : T(\tau, \omega) \text{ touches } t\},$$

then as $c \to 0$ and $d \to \infty$ the set $A$ swells out to $Z$, and $B$ to $R$. Thus $\Lambda^\theta(Z)$ and $\Lambda^\theta(R)$ have the same distribution. By the right continuity of the sample functions the sets $R(\omega)$ and $T([0, \infty), \omega)$ differ by at most a countable set for each fixed $\omega$, and so these sets have the same dimension. In [2, Theorem 3.2] we showed that $\dim T([0, \infty), \omega) = \beta = 1 - 1/\alpha$ for almost all $\omega$. (Actually we showed $\dim T([0, 1], \omega) = \beta$ for almost all $\omega$, but clearly this implies the preceding statement.) Combining this with the fact that for each fixed $\theta > 0$ the random variables $\Lambda^\theta(Z)$ and $\Lambda^\theta(R)$ have the same distribution yields

$$P[\dim Z = 1 - 1/\alpha] = 1.$$

Thus Theorem A is established.

## 4. Proof of Theorem B

Let us consider first the case $1 < \alpha \leqq 2$. If we define

$$T_x(\omega) = \inf \{t \geqq 0 : X(t, \omega) = x \quad \text{or} \quad X(t-, \omega) = x\},$$

then it follows from the results of Kac [5] that $P[T_x < \infty] = 1$, and from those of Hunt [4, pp. 54, 55] that $X(T_x) = x$ with probability one. Combining these facts with the strong Markov property and the corollary to Theorem A it follows that if we define

$$Z_x(\omega) = \{t : X(t, \omega) = x\}, \tag{4.1}$$

then for $1 < \alpha \leqq 2$

$$P[\dim Z_x(\omega) = 1 - 1/\alpha] = 1. \tag{4.2}$$

Given a probability measure $\mu$ on $\mathcal{B}(R)$, the Borel sets of $R$, the symmetric stable process of index $\alpha$ with initial distribution $\mu$ can be realized as $\{x + X(t, \omega); t \geqq 0\}$ over the probability space $(R \times \Omega, \mathcal{B}(R) \times \mathfrak{F}, \mu \times P)$ where $X(t, \omega); t \geqq 0\}$ is the symmetric stable process of index $\alpha$ with $X(0) = 0$ defined over $(\Omega, \mathfrak{F}, P)$. Let us put $Y(t, (x, \omega)) = x + X(t, \omega)$ for the moment. The measurability discussion in the proof of Theorem A, which depended only on the sample function properties and the regularity of the transition probabilities, implies that

$$\Delta = \{(x, \omega): \dim \{t: Y(t, (x, \omega)) = 0\} = 1 - 1/\alpha\}$$

is measurable relative to the completion of $\mathcal{B}(R) \times \mathfrak{F}$ with respect to $\mu \times P$ The set $\Delta_{-x} = \{\omega: (-x, \omega) \in \Delta\}$ is just $\{\omega: \dim Z_x(\omega) = 1 - 1/\alpha\}$, so by Fubini's theorem and (4.2) the set $\Delta$ has probability one. The probability measure meant is, of course, the completion of $\mu \times P$. Again by Fubini's theorem there is a set $\Omega_0 \in \mathfrak{F}$ with $P(\Omega_0) = 0$ such that if $\omega \notin \Omega_0$ then the set $\Delta^{\omega} = \{x: (x, \omega) \in \Delta\}$ is in the completion of $\mathcal{B}(R)$ with respect to $\mu$ and $\mu(\Delta^{\omega}) = 1$. (We are always assuming that $\mathfrak{F}$ is complete relative to $P$.) Finally taking $\mu$ to be equivalent (in the sense of absolute continuity) to Lebesgue measure we have that for all $\omega \notin \Omega_0$, where $P(\Omega_0) = 0$, $\dim Z_x(\omega) = 1 - 1/\alpha$ for almost all (Lebesgue measure) $x$.

J. M. Marstrand [7] has shown that if $E$ is a subset of the $(t, x)$ plane such that for every point $x$ in a given linear set $A$ we have $\Lambda^{\beta}\{t: (t, x) \in E\} > p$, then $\Lambda^{\beta+\lambda}(E) \geqq kp\Lambda^{\lambda}(A)$, where $k$ is a positive constant. Combining Marstrand's theorem with the observations following (4.2) we easily find that

$$P(\dim G(\omega) \geqq 2 - 1/\alpha] = 1 \tag{4.3}$$

provided $1 < \alpha \leqq 2$.

We now adapt an argument of Besicovitch and Ursell [1] to prove the opposite inequality. For each $\varepsilon > 0$ define as follows

$$M_{k\varepsilon} = \sup_{0 \leqq t \leqq \varepsilon} | X(t + (k-1)\varepsilon) - X((k-1)\varepsilon) |, \quad k = 1, 2, \cdots .$$

Since the process $X$ has stationary independent increments, the random variables $M_{1\varepsilon}, M_{2\varepsilon}, \cdots$ are independent and identically distributed. Moreover, since $X(rt)$ has the same distribution as $r^{1/\alpha}X(t)$ for any $r > 0$, we easily see that $M_{1\varepsilon}$ has the same distribution as $\varepsilon^{1/\alpha}M_{11}$. If $R(k, \varepsilon)$ is a rectangle with center at $((k-1)\varepsilon, X[(k-1)\varepsilon])$ and with sides $2\varepsilon$ and $2M_{k\varepsilon}$, then clearly $R(1, \varepsilon), \cdots, R([\varepsilon^{-1}], \varepsilon)$ is a cover of

$$G(\omega; 0, 1) = \{(t, X(t, \omega)): 0 \leqq t \leqq 1\}$$

for each $\omega$. Here $[\varepsilon^{-1}]$ is the greatest integer in $\varepsilon^{-1}$. However, each of the rectangles $R(k, \varepsilon)$ can be covered by $[\varepsilon^{-1}M_{k\varepsilon}] + 1$ squares of side $2\varepsilon$. Let us denote this cover of $G(\omega; 0, 1)$ by squares of side $2\varepsilon$ by $E(\varepsilon)$. If $E = (E_1, \cdots, E_n)$ is any finite cover of $G(\omega; 0, 1)$ and $\beta > 0$, let

$$S_{\beta}(E) = \sum_{i=1}^{n} | E_i |^{\beta}.$$

Thus if $\beta > 0$ we have

$$(4.4)\qquad \begin{aligned} S_\beta[E(\varepsilon)] &= \sum_{k=1}^{[\varepsilon^{-1}]} ([\varepsilon^{-1}M_{k\varepsilon}] + 1)(2\sqrt{2}\varepsilon)^\beta \\ &\leqq C\sum_{k=1}^{[\varepsilon^{-1}]} M_{k\varepsilon}\, \varepsilon^{\beta-1} + C\varepsilon^{\beta-1}, \end{aligned}$$

where $C$ is a positive constant depending only on $\beta$. If $\beta > 2 - 1/\alpha > 1$, then the second term above goes to zero as $\varepsilon \to 0$. On the other hand if we let $\varepsilon = n^{-1}$, then for any $x > 0$ we have

$$P\{ \sum_{k=1}^{n} \varepsilon^{\beta-1} M_{k\varepsilon} \leqq x\} = P\{n^{1-\beta-1/\alpha}[M_{11} + \cdots + M_{n1}] \leqq x\}.$$

Thus if $\beta > 2 - 1/\alpha$, and if we assume for the moment that $M_{11}$ has a finite expectation, the weak law of large numbers implies that the last displayed expression approaches one as $n \to \infty$. Therefore $S_\beta[E(n^{-1})] \to 0$ in probability, and hence a subsequence approaches zero with probability one provided $\beta > 2 - 1/\alpha$. This proves that

$$(4.5)\qquad P[\dim G(\omega; 0, 1) \leqq 2 - 1/\alpha] = 1,$$

subject to the finiteness of the expectation of $M_{11}$. Concerning this: pick a $C > 0$ such that for every $t \leqq 1$, $P\{| X(t) - X(1) | \geqq C\} \leqq \frac{1}{2}$. This can be done since almost all sample functions of $X$ are bounded on bounded intervals. A standard argument then shows that for every $\lambda > C$

$$P[M_{11} \geqq 2\lambda] \leqq 2P\{| X(1) | \geqq \lambda\}.$$

But $E\{| X(1)|\} < \infty$ since $\alpha > 1$, and hence $E(M_{11}) < \infty$. Clearly (4.3) and (4.5) taken together yield Theorem B (i).

Finally we consider the case $0 < \alpha \leqq 1$. Recall that if $f:[0, 1] \to R^N$, then $\beta - \operatorname{var} f = \sup \sum_{j=0}^{n-1} | f(t_{j+1}) - f(t_j) |^\beta$, where the supremum is taken over all finite subdivisions $0 \leqq t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$. If $Y(t)$ denotes the two-dimensional process $(t, X(t))$, then $Y([0, 1], \omega) = G(\omega; 0, 1)$. Clearly we have

$$\beta - \operatorname{var} Y(\cdot, \omega) \leqq 2^{\beta-1}[\beta - \operatorname{var} X(\cdot, \omega) + \beta - \operatorname{var} h],$$

where $h(t) = t$. If $\beta > 1$, then $\beta - \operatorname{var} h$ is finite, and if in addition $\alpha \leqq 1$, Theorem 4.1 of [2] implies that $\beta - \operatorname{var} X(\cdot, \omega)$ is finite for almost all $\omega$. Thus applying Theorem 8.4 of [3] we find that $\Lambda^\beta Y([0, 1], \omega) < \infty$ for almost all $\omega$ provided $\beta > 1$. Therefore

$$(4.6)\qquad P[\dim G(\omega; 0, 1) \leqq 1] = 1.$$

To prove the opposite inequality consider $r(t, \omega) = [X(t, \omega)^2 + t^2]^{1/2}$; then

$$P[r(t) \leqq u] = P[X^2(t) \leqq u^2 - t^2].$$

It follows easily that the random variable $r(t)$ has a probability density $g_t(u)$ given by

$$\begin{aligned} g_t(u) &= 2t^{-1/\alpha}u(u^2 - t^2)^{-1/2}f(1, t^{-1/\alpha}(u^2 - t^2)^{1/2}), && u > t, \\ &= 0, && u \leqq t, \end{aligned}$$

where $f(1, x)$ is the probability density of $X(1)$ given by (1.1). Therefore if $\beta > 0$,

$$E\{r(t)^{-\beta}\} = \int_0^\infty u^{-\beta} g_t(u)\, du$$

$$= 2t^{-\beta/\alpha} \int_0^\infty (t^{2-2/\alpha} + x^2)^{-\beta/2} f(1, x)\, dx,$$

where we have made the change of variable $x = t^{-1/\alpha}(u^2 - t^2)^{1/2}$. But $t^{2-2/\alpha} + x^2 \geqq t^{2-2/\alpha}$ for all $x$, and thus we obtain

$$E\{r(t)^{-\beta}\} \leqq Ct^{-\beta}, \tag{4.7}$$

where $C$ is a positive constant. Since $t^{-\beta}$ is integrable near $t = 0$ if $\beta < 1$, a standard argument using capacity (see [2], [3], or [9]) yields

$$P[\dim Y([0, 1], \omega) \geqq 1] = 1. \tag{4.8}$$

The reasoning leading to (4.7) is that of Taylor [9].

Combining (4.6) and (4.8) we find

$$P[\dim G(\omega; 0, 1) = 1] = 1, \tag{4.9}$$

and clearly this implies Theorem B (ii).

## References


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