CONFORMAL TRANSFORMATIONS OF COMPACT RIEMANNIAN MANIFOLDS

BY

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In his recent paper [3], H. Yamabe has proved that every Riemannian metric on a compact manifold of dimension $\geqq 3$ can be deformed conformally to a Riemannian metric of constant scalar curvature. This makes Riemannian manifolds of constant scalar curvature important in the study of conformal transformations of compact Riemannian manifolds. Indeed, a conformal transformation of a compact Riemannian manifold onto another can be considered as a conformal one between Riemannian manifolds of constant scalar curvature by suitable conformal change of metrics.

On the other hand, K. Yano and T. Nagano [6] have proved that every complete Einstein space of dimension $\geqq 3$ admitting a one-parameter group of global conformal transformations is isometric to an ordinary sphere. Furthermore similar results have been obtained by S. Ishihara and Y. Tashiro [1] in the case of so-called concircular transformations of complete Riemannian manifolds of constant scalar curvature.

The purpose of this paper is to obtain a necessary and sufficient condition for a conformal transformation of a compact Riemannian manifold onto another to be affine (or homothetic) with applications to the case of nonpositive constant scalar curvature. Namely the condition will be obtained in terms of scalar curvatures.

1. Preliminaries

Throughout this paper the dimensions of Riemannian manifolds are assumed to be greater than one and every quantity to be class C^∞ as well as the manifolds.

Let (M, g) and (M', g') be n-dimensional Riemannian manifolds with Riemannian metrics g and g' respectively. A homeomorphism f of M onto M' is called a *conformal* one of (M, g) onto (M', g') if the Riemannian metric $g^* = f_* g'$ induced from g' by f is conformally related with g, i.e., if there exists a scalar function ϕ on M such that $g^* = e^{2\phi} g$. If ϕ is constant, f is *homothetic*, and if especially ϕ is identically zero, f is an *isometry*.

Now if we denote by K_g the scalar curvature of g, it is known that the transform $f_* K_{g'}$ of the scalar curvature $K_{g'}$ of g' by f coincides with the scalar curvature of g^*, i.e., $f_* K_{g'} = K_{g^*}$. It is also a classical fact that K_{g^*} and K_g are related by the formula

$$(1) \qquad e^{2\phi} K_{g^*} - K_g = 2(n-1)\Delta\phi - (n-1)(n-2)|\, d\phi\,|^2,$$

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where $\Delta\phi$ is the Laplacian of ϕ and $|d\phi|$ the length of $d\phi$ with respect to the original Riemannian metric g.

If $f:(M, g) \to (M', g')$ is a conformal transformation, so also is the inverse $f^{-1}:(M', g') \to (M, g)$.

If M and M' are nonorientable, they have the orientable (double) coverings $\bar{M}$ and $\bar{M}'$ respectively. The natural projections $p:\bar{M} \to M$ and $p':\bar{M}' \to M'$ induce the Riemannian metrics $\bar{g} = p_* g$ and $\bar{g}' = p'_* g'$ on $\bar{M}$ and $\bar{M}'$ respectively. Given a homeomorphism $f:M \to M'$, there is a homeomorphism $f:\bar{M} \to \bar{M}'$ such that $p'\bar{f} = fp$. If f is conformal (homothetic or isometric), so also is $\bar{f}$ with respect to the naturally induced metrics, and vice versa.

2. Theorems

It will first be shown that the signs of scalar curvatures are preserved by a conformal transformation in the following sense:

THEOREM 1. *There is no conformal transformation between a compact Riemannian manifold of everywhere nonpositive scalar curvature and one of everywhere nonnegative scalar curvature except for the case where both scalar curvatures are identically zero.*

Proof. Using the notations as in §1, let $f:(M, g) \to (M', g')$ be a conformal transformation with $g^* = e^{2\phi}g$. Since f is conformal if and only if f^{-1} is conformal, we may assume, without loss of generality, that $K_g \leqq 0$ and $K_{g'} \geqq 0$ (or $K_{g^*} \geqq 0$). It follows from (1) that

$$2(n-1)\Delta\phi = e^{2\phi}K_{g^*} - K_g + (n-1)(n-2)|d\phi|^2 \geqq 0.$$

Then by Bochner's lemma [5, p. 30] we have $\Delta\phi = 0$ everywhere, and it implies $K_g = K_{g^*} = 0$, i.e., $K_g = K_{g'} = 0$.

If this is the case, and if M is connected, we may conclude that ϕ is a constant again by Bochner's lemma. Thus we have

THEOREM 2. *Every conformal transformation between connected compact Riemannian manifolds of zero scalar curvature is always homothetic.*

Next, the case of everywhere nonpositive scalar curvature will be considered. Since if one of the scalar curvatures is identically zero the problem has been solved in Theorems 1 and 2, we are mainly concerned with scalar curvatures which are everywhere nonpositive but not identically zero.

THEOREM 3. *Let (M, g) and (M', g') be compact Riemannian manifolds of everywhere nonpositive scalar curvatures which are not identically zero. Then a conformal transformation $f:(M, g) \to (M', g')$ with $g^* = f_*g' = e^{2\phi}g$ is homothetic if and only if $K_{g^*} = f_*K_{g'} = e^{-2k}K_g$ for some constant k. If this is the case, we have $\phi = k$.*

Proof. If f is homothetic with $\phi = k$, we have obviously $K_{g^*} = e^{-2k}K_g$.

To prove the converse we first assume that M and M' are orientable. If $K_{g^*} = e^{-2k}K_g$, we have, by virtue of (1),

$$K_g(e^{2(\phi-k)} - 1) = 2(n-1)\Delta\phi - (n-1)(n-2)|\, d\phi\,|^2. \tag{2}$$

Since M is compact and orientable, the global inner product with respect to the Riemannian metric g can be used. Taking the global inner product of each side of (2) with the scalar function $e^{2n(\phi-k)} - 1$, we have

$$L = (K_g(e^{2(\phi-k)} - 1), (e^{2n(\phi-k)} - 1)) = \int_M K_g(e^{2(\phi-k)} - 1)(e^{2n(\phi-k)} - 1)\, dv,$$

where dv denotes the volume element of M. The integrand is obviously nonpositive, and we obtain $L \leqq 0$. On the other hand since we have

$$(\Delta\phi, (e^{2n(\phi-k)} - 1)) = (d\phi, d(e^{2n(\phi-k)} - 1)) = 2n \int_M e^{2n(\phi-k)}\, |\, d\phi\,|^2\, dv,$$

the inner product R of the right-hand side of (2) with $e^{2n(\phi-k)} - 1$ is given by

$$R = (n-1)(3n+2) \int_M e^{2n(\phi-k)}\, |\, d\phi\,|^2\, dv$$

$$+ (n-1)(n-2) \int_M |\, d\phi\,|^2\, dv \geqq 0.$$

Therefore we must have $L = R = 0$. $L = 0$ implies

$$K_g(e^{2(\phi-k)} - 1)(e^{2n(\phi-k)} - 1) = 0.$$

Since K_g is not identically zero, we have $e^{2(\phi-k)} = 1$, which implies $\phi = k$.

If we take account of the remark on the nonorientable case given in §1, this completes the proof.

If, especially, $k = 0$, f becomes an isometry. Hence we obtain

THEOREM 4. *A conformal transformation between compact Riemannian manifolds of everywhere nonpositive but not identically zero scalar curvature is an isometry if and only if it preserves the scalar curvature.*

3. The case of constant scalar curvature

If K_g and $K_{g'}$ are both constant, then they are both zero or they have the same sign, by virtue of Theorem 1, provided that there is a conformal transformation $f: (M, g) \to (M', g')$. If they are both negative, $K_g/K_{g'}$ is a positive constant, and we can put $K_g/K_{g'} = e^{2k}$ for some constant k. Then by Theorem 3, f is a homothetic transformation with $f_* g' = (K_g/K_{g'})g$, which, together with Theorem 2, gives

THEOREM 5. *Let (M, g) and (M', g') be connected compact Riemannian manifolds of nonpositive constant scalar curvature. Then every conformal transformation $f: (M, g) \to (M', g')$ is homothetic with $K_{g'} f_* g' = K_g g$.*

Next, in the case of conformal transformations of a Riemannian manifold (onto itself) we must have a restriction on the function ϕ. In fact, if $f_* g = e^{2\phi} g$, we have $f_*(dv) = e^{n\phi}\, dv$. Hence if M is compact, we have

$$\int_M e^{n\phi}\, dv = \int_M f_*\,(dv) = \int_M dv.$$

If f is homothetic, i.e., if ϕ is constant, ϕ must be zero, and f is an isometry.

Now let f be a conformal transformation of a compact Riemannian manifold (M, g) of negative constant scalar curvature K_g. Since $f_* K_g = K_g$, we have $f_* g = g$ by Theorem 5, namely f is an isometry. If (M, g) is a connected compact Riemannian manifold of zero scalar curvature, f is homothetic by Theorem 2, from which together with the above remark it is an isometry. Thus we obtain

THEOREM 6. *A conformal transformation of a connected compact Riemannian manifold of nonpositive constant scalar curvature onto itself is always isometric.*

Remark. In Theorems 5 and 6 if the scalar curvatures are not identically zero, the assumption of connectedness is not needed.

Theorem 6 has been proved for an infinitesimal conformal transformation [2], [4].

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