ON THE MEAN VALUES OF CERTAIN TRIGONOMETRICAL POLYNOMIALS II

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

J. E. LITTLEWOOD

1. The present paper is a sequel to [1], which had the same title. We shall refer to this as I, and, e.g., to its Theorem 3 as Theorem 3(I).

The classes of trigonometrical polynomials

$$f = f_n(\theta) = \sum^n a_m e^{m\theta i}$$

(t.p. or t.p.(n) for short) with which we were and shall be concerned are as follows.

We have first two classes of $f = f_n$ (n arbitrary) with "unitary" coefficients, complex and real respectively

$$\mathcal{C}_u : \ f = \sum^n a_m e^{m\theta i}, \qquad | a_m | = 1;$$

$$\mathcal{R}_u : \ f = \sum^n \cos (m\theta + \alpha_m).$$

We consider further a wide generalization of $\mathcal{R}_u$, namely the class $\mathfrak{F}_k$ of $f = \sum^n a_m \cos (m\theta + \alpha_m)$ with *real* a_m satisfying

$$\text{(K)} \qquad \sum^n m^2 a_m^2 \geqq kn^2 \sum^n a_m^2 ,$$

where k is a positive constant (it is of course a positive absolute constant for $\mathcal{R}_u$).

We suppose usually, to avoid trivial exceptions, that $a_0 = 0$. When exceptionally we use an f with $a_0 \neq 0$ we could tacitly suppose that it was replaced by $f(\theta)e^{i\theta}$, with trivial differences in behaviour.

We distinguish also a class of "reasonable" t.p.: the function

$$\sqrt{n} \sum_{m=0}^{n} \frac{\sin (2m + 1)\theta}{2m + 1}$$

(which we shall meet below) is "unreasonable"; its factor $\sqrt{n}$ is spurious, and the important mean square $\sum a_m^2$ is of the same order, namely n, as a_1^2. We may insist that a "reasonable" t.p. should satisfy (at least) $| a_m | < a(m)$, where $a(m)$ is independent of n. In the context of our present subject it is

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appropriate to require

(i) $$\sum^n | a_m |^2 > Kn^\alpha \quad \text{for some positive } \alpha;$$

(ii) $$| a_m | < Km^\beta,$$

where in the light of (i) we may suppose $\beta > -\frac{1}{2}$ (generally we have $\beta \geqq 0$).

Notation. We denote positive absolute constants by A, and positive constants depending only on, e.g., α, β by $A(\alpha, \beta)$ or $A_{\alpha,\beta}$. Constants whose nature is irrelevant we denote by K. A's and K's are not in general the same from one occurrence to another: when we wish to identify A's [or $A(\alpha, \beta)$'s] temporarily we use suffixes 1, 2, $\cdots$. We use suffixes similarly to identify, temporarily, sets of points, etc. The use is temporary, and we restart suffixes at 1 when we proceed to new matter.

O's are absolute, and O_δ's depend only on δ.

The range $(-\pi, \pi)$, *qua* set of points, we abbreviate to E_0.

For a $g(\theta)$ of class L^λ, $\lambda > 0$, we use the usual notation

$$M_\lambda(g) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | g |^\lambda d\theta\right)^{1/\lambda} = \left(\frac{1}{| E_0 |}\int_{E_0} | g |^\lambda d\theta\right)^{1/\lambda},$$

and for a g defined in a subset E of E_0 we use

$$M_\lambda(g, E) = \left(\frac{1}{| E |}\int_{E} | g |^\lambda d\theta\right)^{1/\lambda}.$$

We consistently abbreviate e^{xi} to $E(x)$.

Other abbreviations which we shall use frequently are: ν for $\frac{1}{2}(n - 1)$, μ for $\sqrt{n}$, ω for $E(2\pi/n)$, α for $\frac{1}{2}\sqrt{\pi}$, η for $E(\frac{1}{4}\pi)$, γ for $\sqrt{(2/\pi)}$, and μ_1 for $\sqrt{(\frac{1}{2}n)}$.

2. The theorems of I with which we shall be concerned are as follows.

THEOREM 3(I). *For $f \in \mathfrak{F}_k$ we have:*

(i) *For every λ in $0 < \lambda < 2$,*

$$M_\lambda(f) \leqq (1 - A_{k,\lambda})M_2(f).$$

(ii) *For every $q > 2$,*

$$M_q(f) \geqq (1 + A_{k,q})M_2(f).$$

(iii) $| f | < (1 - A_k)M_2(f)$ *in (positive) measure A_k' of θ.*

The three results were shown to follow fairly easily from

THEOREM 3′(I). $M_1(f) \leqq (1 - A_k)M_2(f)$,

a particular case of (i). We will take the deduction for granted, and treat Theorem 3′(I) as the equivalent of Theorem 3(I).

THEOREM 4(I).[1] *Let q_1, κ, a be constants satisfying*

$$q_1 > 2, \qquad 0 < \kappa \leqq q_1, \qquad a \geqq 0,$$

and let $f \in \mathfrak{F}_k$. Suppose now that

$$M_{q_1}(f) \leqq (1 + a)M_2(f).$$

Then[2]

$$\frac{d}{d\lambda}\{M_\lambda(f)\} > BM_2(f) \quad (\kappa \leqq \lambda \leqq q_1), \quad \textit{where} \quad B = A(k, q_1, \kappa, a).$$

In connexion with Theorem 3(I) we study what happens for means $M_\lambda(f, E)$, where E is a *subinterval of* E_0.

Our concern with Theorem 4(I) is to prove by an example that we cannot replace κ by 0: the theorem is then shown to be best possible for $f \in \mathfrak{F}_k$ in the sense that none of its conditions can be relaxed. This was stated in I with a proof for the conditions other than $\kappa > 0$. A counterexample for "$\kappa = 0$" was given, but without proof (which needs in effect the new work in the present paper).

3. The other main theorem of I is about a particular $f(\theta) = F_\omega(\theta)$ with rather surprising properties. Let[3]

$$(1) \qquad n \equiv 1 \pmod 8, \qquad \nu = \tfrac{1}{2}(n-1), \qquad \mu = n^{1/2}.$$

$$(2) \qquad \omega = E(2\pi n^{-1}),$$

$$(3) \qquad F(\theta) = F_\omega(\theta) = \sum_{m=0}^{n-1} \omega^{m(m+1)/2} e^{m\theta i} \qquad (|\theta| \leqq \pi).$$

We will begin by a further study of this function.

The results are in terms of Fresnel integrals and allied functions. I will begin by giving a "dictionary," D, of what is relevant: proofs (of what was not already known) are to be found in I. I alter the original notation slightly.

We denote the range $(0, \pi)$ of θ by R, and for an arbitrarily small positive δ, $(\delta, \pi - \delta)$ by R_δ. It is necessary to treat the ranges $(0, \pi)$, $(-\pi, 0)$ separately. We accordingly adopt the convention that θ satisfies $0 \leqq \theta \leqq \pi$ [and distinguish $F(\theta)$ and $F(-\theta)$], using t for a variable in $(-\pi, \pi)$. We associate with θ [of $(0, \pi)$] a number $\lambda \geqq 0$ defined by

$$(\text{D1}) \qquad \lambda = \lambda(\theta) = \tfrac{1}{2}\pi^{-1/2}n^{1/2}\theta = \tfrac{1}{2}\pi^{-1/2}\mu\theta, \quad \text{or} \quad 4\pi\lambda^2 = n\theta^2.$$

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[1] What we state is a part result: we omit another part with which we are not here concerned.

[2] It is easily seen that $(d/d\lambda)\{M_\lambda(f)\}$ exists for a t.p. when $\lambda > 0$. In I the result was stated in "difference" form.

[3] In I, n was merely supposed to be odd; the stronger assumption has minor conveniences.

We write

(D2) $\theta_1 = \pi - \theta \quad (0 \leqq \theta \leqq \pi), \qquad \lambda_1 = \lambda(\theta_1) = \alpha\mu - \lambda, \quad \alpha = \tfrac{1}{2}\sqrt{\pi}$

[we abbreviate (the constantly occurring) $\tfrac{1}{2}\sqrt{\pi}$ to α], and note

(D3) $$n\theta = 4\alpha\mu\lambda, \qquad \lambda(\pi) = \alpha\mu.$$

We define, for the range $0 \leqq \lambda < \infty$,

(D4) $$Z = Z(\lambda) = V(\lambda) + iU(\lambda) = \gamma E(-\lambda^2)\int_\lambda^\infty E(x^2)\,dx, \quad \gamma = (2/\pi)^{1/2},$$

(D5) $U = U(\lambda)$, $V = V(\lambda)$ decrease from $U(0) = V(0) = \tfrac{1}{2}$ as λ increases from 0 to ∞.

(D6)[4] $$Z(0) = 2^{-1/2}\eta, \qquad \eta = E(\tfrac{1}{4}\pi).$$

(D7) $$Z'(\lambda) = -\gamma - 2\lambda i Z(\lambda).$$

U, V have the asymptotic expansions, for large λ (> 0),

(D8) $$\begin{cases} \dfrac{U}{\gamma} = \dfrac{1}{2\lambda} - \dfrac{1\cdot3}{2^3\cdot\lambda^5} + \dfrac{1\cdot3\cdot5\cdot7}{2^5\cdot\lambda^9} - \cdots, \\ \dfrac{V}{\gamma} = \dfrac{1}{2^2\cdot\lambda^2} - \dfrac{1\cdot3\cdot5}{2^4\cdot\lambda^6} + \dfrac{1\cdot3\cdot5\cdot7\cdot9}{2^6\cdot\lambda^{11}} - \cdots, \\ \text{and we have } Z = \tfrac{1}{2}i\gamma\lambda^{-1} + \tfrac{1}{4}\gamma\lambda^{-2} + O(\lambda^{-5}). \end{cases}$$

(D9) $$U = O\{(\lambda+1)^{-1}\}, \qquad Z = O\{(\lambda+1)^{-1}\}, \qquad Z' = O\{(\lambda+1)^{-2}\} \qquad (\lambda \geqq 0).$$

The definition (D4) of Z, U, V may be extended to negative values of λ, and we have

(D10) $$\begin{cases} Z(-\infty) = V(-\infty) + iU(-\infty) = \sqrt{2}\eta, \\ Z(-\lambda) = 1 + i - Z(\lambda) = \sqrt{2}\eta - Z(\lambda). \end{cases}$$

(D11) $$\begin{cases} \left(\dfrac{2}{\pi}\right)^{1/2} \displaystyle\int_0^T E(t^2 + 2kt)\,dt = E(-k^2)\{u(k) - u(k+T)\} \\ \qquad\qquad = E(-k^2)\{u(-k-T) - u(-k)\}, \\ \text{where } \quad u(k) = E(k^2)Z(k) = \gamma \displaystyle\int_k^\infty E(x^2)\,dx. \end{cases}$$

We observe finally that λ_1 and $Z(\lambda_1)$ occur only in the context $0 \leqq \lambda \leqq \alpha\mu$ (corresponding to $0 \leqq \theta \leqq \pi$): we abbreviate $Z(\lambda_1)$ to Z_1.

4. The theorem of I referred to is (in our present notation) as follows.

Let $n \equiv 1 \pmod 8$, *and, for* $|t| \leqq \pi$,

[4] $E(\tfrac{1}{4}\pi)$ occurs so often that it is convenient to abbreviate it to η.

$$F(t) = F_\omega(t) = \sum_{m=0}^{n-1} \omega_m e^{mti}, \qquad \omega_m = \omega^{m(m+1)/2}, \quad \omega = E(2\pi m^{-1}).$$

We have $M_2(F) = \mu = n^{1/2}$. *For* $n^{-1/2+\delta} \leqq \theta \leqq \pi$ *we have the following estimates for* $|F|^2$, *noting that* $F(-\theta) = E\{-(n-1)\theta\}F(\theta)$ *and so* $|F(-\theta)| = |F(\theta)|$:

$$(|F(\pm\theta)|/\mu)^2 = 1 - W\cos\chi + \tfrac{1}{2}W^2$$
$$+ \{W\sin\chi\sin n\theta - (W\cos\chi - \tfrac{1}{2}W^2)\cos n\theta\} + O_\delta(\mu^{-1/2+\delta}),$$

where

$$W^2 = 2\{U^2(\lambda) + V^2(\lambda)\} = 2|Z|^2, \qquad \chi = \lambda^2 + \arctan(U - V)/(U + V),$$

and

$$W(0) = 1, \qquad \chi(0) = 0.$$

The upper and lower bounds of $|F/\mu|^2$ *as* $n\theta$ *increases through* 2π (*which leaves* λ *unaltered to error* $O(\mu^{-1})$) *are, to error* $O_\delta(\mu^{-1+\delta})$,

$$\mathfrak{M}^2(\lambda), \qquad m^2(\lambda) = 1 - W\cos\chi + \tfrac{1}{2}W^2 \pm W(1 - W\cos\chi + \tfrac{1}{4}W^2)^{1/2}.$$

We have $m(0) = 0$, *and* $\max_{(\lambda)} \mathfrak{M}(\lambda) = 1.347 \cdots$ *at* $\lambda = 1.45 \cdots$, *so that* $|F(t)| \leqq 1.35\,\mu$ *for all t, and* $\min|F(t)| < O_\delta(\mu^\delta)$.

For $\mu^{-1+\delta} \leqq |t| \leqq \pi$ *we have* $|F(t)| = \mu + O_\delta(\mu^{1-\delta})$, *and for* $\mu^{-\delta} \leqq t \leqq \pi$ *we have* $|F(t)| = \mu + O_\delta(\mu^\delta)$.

We proceed now to extend this in several ways: (a) we estimate F instead of $|F|$; (b) we introduce a function ϕ which is more fundamental than F (F is a combination of $\phi(t)$ and $\phi(-t)$; (c) our estimate has an error $O(\mu^{-2}) = O(n^{-1})$, i.e., n times smaller than the individual terms. (c) is not a mere luxury, since it is required in developments; if it is found surprising, the explanation is that F (and ϕ) is a kind of cousin of the elliptic ϑ-functions. The approximation, in fact, could be carried in principle to $O(\mu^{-k})$, for any given numerical k.

5. We define

$$\phi(t) = \sum_0^\nu \omega_m e^{mti}, \quad \nu = \tfrac{1}{2}(n-1) \quad (|t| \leqq \pi).$$

It is then easily seen that $\omega_{n-1-m} = \omega_m$ ($m \leqq \nu$), and so

$$F(t) = \phi(t) + e^{(n-1)ti}\phi(-t) - \omega_\nu e^{\nu ti}, \qquad F(-t) = e^{-(n-1)ti}F(t) \quad (|t| \leqq \pi).$$

We recall that $\eta = E(\frac{1}{4}\pi)$, $\alpha = \frac{1}{2}\pi^{1/2}$, $Z = Z(\lambda)$, $Z_1 = Z(\lambda_1)$, where λ, λ_1 are defined in (D1, 2), Z and Z_1 in (D4, 11).

THEOREM 1. *We have*

(i) $\phi(\theta) = \frac{1}{2}\sqrt{2\mu}\{Z + \eta E(\frac{1}{2}n\theta - \frac{1}{2}\theta)Z_1\} + X + O(\mu^{-2}) \quad (0 \leqq \theta \leqq \pi)$

where

$$\text{(ii)}\quad X = \{\tfrac{1}{2} + \mathcal{P}_0(\theta) + \mathcal{P}(\theta)E(\tfrac{1}{2}n\theta) + \tfrac{1}{4}(2\pi)^{1/2}Z'\} + \mu^{-1}\{\mathcal{P}(\theta) + \mathcal{P}(\theta)E(\tfrac{1}{2}n\theta) + aZ''\}.$$

$$\text{(iii)}\quad \phi(-\theta) = \mu[\eta E(-\lambda^2 + \tfrac{1}{2}\theta) - \tfrac{1}{2}\sqrt{2}\{Z + \eta E(-\tfrac{1}{2}n\theta + \tfrac{1}{2}\theta)Z_1\}] + X_1 + O(\mu^{-2}) \quad (0 \leqq \theta \leqq \pi),$$

where

$$\text{(iv)}\quad X_1 = \{\tfrac{1}{2} + \mathcal{P}_0(\theta) + \mathcal{P}(\theta)E(-\tfrac{1}{2}n\theta) + \tfrac{1}{4}(2\pi)^{1/2}Z'\} + \mu^{-1}\{aZ'' + aE(-\tfrac{1}{2}n\theta + \tfrac{1}{2}\theta)Z_1 + aE(-\lambda^2 + \tfrac{1}{2}\theta) + \mathcal{P}(\theta)E(-\tfrac{1}{2}n\theta)\}.$$

In these formulae the $\mathcal{P}(\theta)$*'s (not the same at different occurrences) are elementary functions of* θ*, independent of* n*, regular in* $|\theta| < 2\pi$*; and they are expansible as* $\sum_0^\infty a_m \theta^m$ *or* $\sum_0^\infty b_m \mu^{-m}\lambda^m$*, with radii of convergence* 2π *in* θ *and* $2\alpha\mu$ *in* λ*, [twice the extreme value* $\lambda(\pi) = \alpha\mu$ *for* $\lambda(\theta)$*]. The* $\mathcal{P}_0(\theta)$*'s are* $\mathcal{P}$*'s with* $a_0 = 0$*. The* a*'s are absolute constants. We do not need the explicit (and elementary) forms of the* $\mathcal{P}$ *and* $\mathcal{P}_0$*, nor of the* a*'s.*

There are alternative forms of (i) *to* (iv) *in terms of* λ *instead of* θ*; i.e., with* $\pm 2\alpha\mu\lambda$ *for* $\pm\tfrac{1}{2}n\theta$*, and* $4\alpha\mu^{-1}\lambda$ *for* θ*.*

For $F(\theta)$ *we have* $F(-\theta) = E\{-(n-1)\theta\}F(\theta)$*, and*

$$\text{(v)}\quad F(\theta) = \mu T_1 + T_2 + \mu^{-1}T_3 + O(\mu^{-2}) \qquad (\theta \in R),$$

where

$$\text{(vi)}\quad T_1 = E\{(n-1)\theta\}[\eta E(-\lambda^2 + \tfrac{1}{2}\theta) - \tfrac{1}{2}\sqrt{2}Z\{1 - E(-n\theta + \theta)\}],$$

$$\text{(vii)}\quad T_2 = \tfrac{1}{2} + \mathcal{P}_0(\theta) + \mathcal{P}(\theta)E(\tfrac{1}{2}n\theta) + \mathcal{P}(\theta)E(n\theta) + \tfrac{1}{4}(2\pi)^{1/2}Z'\{1 + E(n\theta - \theta)\},$$

$$\text{(viii)}\quad T_3 = aE(n\theta - \tfrac{1}{2}\theta - \lambda^2) + Z''\{a + aE(n\theta - \theta)\} + aE(\tfrac{1}{2}n\theta - \tfrac{1}{2}\theta)Z_1 + \mathcal{P}(\theta) + \mathcal{P}(\theta)E(\tfrac{1}{2}n\theta).$$

The result $F(\theta) = \mu T_1 + o(\mu)$ was stated without proof in I; more precision in the error-term was treated as irrelevant for the time being; it would in fact have been $O_\delta(\mu^\delta)$.

Proof of Theorem 1. The results for F follow from §5(ii) and those for $\phi(\pm\theta)$.

We can take $\phi(\pm\theta)$ together up to a point: consider $\phi(\sigma\theta)$, where $\sigma = \pm 1$. We have for $\theta \in R$, by the Poisson Summation Formula,

$$\text{(1)}\quad \begin{cases} \phi(\sigma\theta) - (\tfrac{1}{2} + \tfrac{1}{2}\omega_\nu e^{\sigma\nu\theta i}) = G_0 + \sum_{m=1}^{\infty}(G_m + G_{-m}), \\ \text{where } G_M = G_M(\sigma\theta) = \int_0^\nu E\{\pi x(x+1)n^{-1} + x(\sigma\theta + 2M\pi)\}\,dx \\ \qquad\qquad (M = \pm m). \end{cases}$$

The series converges (by a well-known theorem), but not necessarily absolutely. Writing

(2) $$x = t(n/\pi)^{1/2}, \qquad T = \tfrac{1}{2}(\pi n)^{1/2}(1 - n^{-1}),$$

we have, for $M = \cdots, -2, -1, 0, 1, \cdots$

(3) $$G_M = \left(\frac{n}{\pi}\right)^{1/2} \int_0^T E(t^2 + 2kt)\,dt = (\tfrac{1}{2}n)^{1/2}\{u(k) - u(k+T)\} = (\tfrac{1}{2}n)^{1/2}\{u(-k-T) - u(-k)\},$$

by (D11), where

(4) $$k = k_M(\sigma\theta) = \tfrac{1}{2}(n/\pi)^{1/2}(\sigma\theta + 2M\pi + \pi n^{-1}) = (1 - n^{-1})T(\sigma\theta/\pi + 2M + n^{-1}),$$

the last equation being by (2). (4) and (2) yield

(5) $$k + T = \tfrac{1}{2}(n/\pi)^{1/2}(\sigma\theta + 2M\pi + \pi) \qquad \text{[exactly]}.$$

For $m \geqq 2$ we have from (4)

(6) $$k_m + T > k_m > Amn^{1/2}; \qquad k_{-m} < k_{-m} + T < -Amn^{1/2};$$

and this is true also for $m = 1$ when $\sigma = -1$ (but not when $\sigma = 1$). By (3), (6), (D8), and (D11) we have for $m \geqq 2$

(7) $$\begin{aligned} G_m + G_{-m} = (\tfrac{1}{2}n)^{1/2}\gamma\Bigg[\tfrac{1}{2}i\Bigg\{\left(\frac{1}{k_m} + \frac{1}{k_{-m}}\right) - \left(\frac{E\{(T+k_m)^2 - k_m^2\}}{k_m + T}\right. \\ \left. + \frac{E\{(T+k_{-m})^2 - k_{-m}^2\}}{k_{-m} + T}\right)\Bigg\}\Bigg] + (\tfrac{1}{2}n)^{1/2}\tfrac{1}{4}\gamma\left[\left(\frac{1}{k_m^2} + \frac{1}{k_{-m}^2}\right)\right. \\ \left. - \left(\frac{E\{(T+k_m)^2 - k_m^2\}}{(k_m + T)^2} + \frac{E\{(T+k_{-m})^2 - k_{-m}^2\}}{(k_{-m} + T)^2}\right)\right] + O(n^{1/2})(mn^{1/2})^{-5}. \end{aligned}$$

Calculations from (4) and (5) give

$$(T + k_{\pm m})^2 - k_{\pm m}^2 = \pm 2m\pi\nu + \tfrac{1}{4}\pi n + \nu\sigma\theta - \tfrac{1}{4}\pi n^{-1} \equiv \tfrac{1}{4}\pi + \nu\sigma\theta - \tfrac{1}{4}\pi n^{-1} = \psi,$$
$$E(\psi) = \eta E(\nu\sigma\theta) + O(n^{-1}),$$

and from this, (4), and (7) we find, after some reduction

(8) $$\begin{aligned} &\sum_2^\infty (G_M + G_{-M})(\sigma\theta) \\ &= i\sum_2^\infty \frac{2\sigma\theta}{4m^2\pi^2 - \theta^2} - \eta i E(\nu\sigma\theta)\sum_2^\infty \frac{2(\pi - \sigma\theta)}{4m^2\pi^2 - (\pi + \sigma\theta)^2} \\ &\quad + \mu^{-1}\Bigg[a\sum_2^\infty\left\{\frac{1}{(2m\pi + \sigma\theta)^2} + \frac{1}{(2m\pi - \sigma\theta)^2}\right\} \\ &\quad - a\eta E(\nu\sigma\theta)\left\{\frac{1}{(2m\pi + \pi + \sigma\theta)^2} + \frac{1}{(2m\pi - \pi - \sigma\theta)^2}\right\}\Bigg] + O(\mu^{-2}) \\ &= \{\mathcal{P}_0(\theta) + E(\tfrac{1}{2}n\sigma\theta)\mathcal{P}(\theta)\} + \mu^{-1}\{\mathcal{P}(\theta) + E(\tfrac{1}{2}n\sigma\theta)\mathcal{P}(\theta)\} + O(\mu^{-2}). \end{aligned}$$

6. We now consider $G_{\pm 1}(\sigma\theta)$ and distinguish $\sigma = \pm 1$, taking first $\sigma = -1$, for which, by the remark below (6) of §5, $G_1 + G_{-1}$ behaves like $G_M + G_{-M}$ for $m \geqq 2$, and is found to contribute an expression of the form (8) above, with $\sigma = -1$, so that

$$\sum_1^\infty \{G_M(-\theta) + G_{-M}(-\theta)\} = \{\mathcal{P}_0(\theta) + E(-\tfrac{1}{2}n\theta)\mathcal{P}(\theta)\} + \mu^{-1}\{\mathcal{P}(\theta) + E(-\tfrac{1}{2}n\theta)\mathcal{P}(\theta)\} + O(\mu^{-2}). \tag{1}$$

Taking now $\sigma = 1$, we see easily that $G_1(\theta)$ behaves like $G_M(\theta)$ for $m \geqq 2$, and contributes the form (8), with $\sigma = 1$; further, that

$$G_{-1}(\theta) = (\tfrac{1}{2}n)^{1/2}E(-k_{-1}^2)\{u(-k_{-1} - T) - u(-k_{-1})\} \qquad [u(x) = E(x^2)Z(x)]. \tag{2}$$

We find

$$-k_{-1} = \tfrac{1}{2}(n/\pi)^{1/2}(2\pi - \theta) - \tfrac{1}{2}\pi^{1/2}n^{-1/2}, \qquad -k_{-1} - T = \lambda_1\,;$$

$$k_{-1}^2 = (n-1)\pi - n\theta + \tfrac{1}{2}\theta + \lambda^2 + O(n^{-1}) \equiv \lambda^2 - n\theta + \tfrac{1}{2}\theta + O(n^{-1}); \tag{3}$$

$$(k_{-1} + T)^2 - k_{-1}^2 \equiv \lambda_1^2 - \lambda^2 + n\theta - \tfrac{1}{2}\theta + O(n^{-1})$$

$$= \tfrac{1}{4}n\pi + \nu\theta + O(n^{-1}) \equiv \tfrac{1}{4}\pi + \nu\theta + O(n^{-1}).$$

(2) and (3) lead, after reduction, to

$$G_{-1}(\theta) = \tfrac{1}{2}\sqrt{2}\mu\eta E(\tfrac{1}{2}n\theta - \tfrac{1}{2}\theta)Z_1 + E(\tfrac{1}{2}n\theta)\mathcal{P}(\theta) + \mu^{-1}E(\tfrac{1}{2}n\theta)\mathcal{P}(\theta) + O(\mu^{-2}).$$

Combining this with the known G_1 and $\sum_1^\infty (G_M + G_{-M})$ we have

$$\sum_1^\infty \{G_M(\theta) + G_{-M}(\theta)\} = \tfrac{1}{2}\sqrt{2}\mu\eta E(\tfrac{1}{2}n\theta - \tfrac{1}{2}\theta)Z_1 + \{\mathcal{P}_0(\theta) + E(\tfrac{1}{2}n\theta)\mathcal{P}(\theta)\} + \mu^{-1}\{\mathcal{P}(\theta) + E(\tfrac{1}{2}n\theta)\mathcal{P}(\theta)\} + O(\mu^{-2}). \tag{4}$$

7. It remains to estimate $G_0(\pm\theta)$. We have [exactly]

$$k = k_0 = k_0(\sigma\theta) = \sigma\lambda + \alpha\mu^{-1},$$

$$k + T = k_0 + T = \lambda(\pi + \sigma\theta)$$

(allowing ourselves when $\sigma = 1$ a momentary departure from the conventions about $\lambda(\theta)$). Hence

$$G_0(\theta) = \tfrac{1}{2}\sqrt{2}\,\mu E(-k^2)\,\{u(k) - u(k+T)\}$$

$$= \tfrac{1}{2}\sqrt{2}\mu\,[Z(k) - E\,\{(k+T)^2 - k^2\}\,Z(k+T)]$$

$$= \tfrac{1}{2}\sqrt{2}\mu\left[Z(\lambda + \alpha\mu^{-1}) - \eta E(\tfrac{1}{2}\,n\theta - \tfrac{1}{2}\,\theta) \cdot\left\{\frac{\gamma i}{2(k+T)} + \frac{\gamma}{4(k+T)^2} + O(\mu^{-3})\right\}\right],$$

and on reduction

$$G_0(\theta) = \tfrac{1}{2}\sqrt{2}\mu Z(\lambda) + \{\tfrac{1}{4}(2\pi)^{1/2}Z' + \mathcal{P}(\theta)E(\tfrac{1}{2}n\theta)\}$$
$$+ \mu^{-1}\{aZ'' + \mathcal{P}(\theta) + \mathcal{P}(\theta)E(\tfrac{1}{2}n\theta)\} + O(\mu^{-2}).$$

This, together with (4) of §6, gives the result of the theorem for $\phi(\theta)$.

For $G_0(-\theta)$ we have

$$k_0 = -\lambda + \alpha\mu^{-1}, \qquad k_0 + T = \lambda_1,$$

$$\begin{aligned}
G_0(-\theta) &= \tfrac{1}{2}\sqrt{2}\mu E(-k_0^2)\{u(k_0) - u(k_0 + T)\}\\
&= \tfrac{1}{2}\sqrt{2}\mu[Z(-\lambda + \alpha\mu^{-1}) - Z_1 E\{\lambda_1^2 - (\lambda - \alpha\mu^{-1})^2\}]\\
&= \tfrac{1}{2}\sqrt{2}\mu[\sqrt{2}\eta E\{-(\lambda - \alpha\mu^{-1})^2\} - Z(\lambda - \alpha\mu^{-1})\\
&\qquad - Z_1\,\eta E\{-2\alpha\mu\lambda + 2\alpha\mu^{-1}\lambda - \alpha^2\mu^{-2}\}] + O(\mu^{-2})\\
&= \mu[\eta E(-\lambda^2 + \tfrac{1}{2}\theta)(1 - i\alpha^2\mu^{-2}) + O(\mu^{-4})\\
&\qquad - \tfrac{1}{2}\sqrt{2}\{Z - \alpha\mu^{-1}Z' + \tfrac{1}{2}\alpha^2\mu^{-2}Z''\}\\
&\qquad - Z_1\,\eta E(-\tfrac{1}{2}n\theta + \tfrac{1}{2}\theta)(1 - i\alpha^2\mu^{-2})] + O(\mu^{-2}).
\end{aligned}$$

On reduction, and combination with (1) of §6, we obtain the result of the theorem for $\phi(-\theta)$, and this completes the proof of Theorem 1.

8. In view of possible applications I note the following.

THEOREM 1. COROLLARY. *The results for $\phi(\pm\theta)$, $F(\pm\theta)$ are valid over $-An^{-1} \leqq \theta \leqq \pi$ with errors $O(1)$.*

We have worked with the convention that $\lambda(t)$ is defined only for $\lambda \geqq 0$. We can however, for $t > 0$, define $\lambda(-t)$ as $-\lambda(t)$, and then we have

$$Z\{\lambda(-t)\} = \gamma E\{-\lambda^2(t)\}\int_{-\lambda(t)}^{\infty} E(x^2)\,dx \quad \text{and} \quad Z\{\lambda(-t)\} = 2Z(0) - Z\{\lambda(t)\}.$$

In the range $-An^{-1} \leqq t \leqq 0$, θ and $\lambda^2(t)$ are $O(n^{-1})$, and with $\lambda = \lambda(t)$

$$Z_1(\lambda) = Z_1(-\lambda) + 2\lambda Z_1'(0) + O(\lambda^2) = Z_1(-\lambda) + O(n^{-1}).$$

In

$$T = \phi(\theta) - \tfrac{1}{2}\sqrt{2}\{Z(\lambda) - \eta E(\tfrac{1}{2}n\theta - \tfrac{1}{2}\theta)\},$$

let us substitute $\theta = -\theta'$, $\lambda' = \lambda(\theta') = -\lambda(\theta)$, and substitute for $\phi(\theta)$ from

$$\begin{aligned}
\phi(\theta) &= \phi(-\theta')\\
&= \mu[\eta E(-\lambda'^2 + \tfrac{1}{2}\theta') - \tfrac{1}{2}\sqrt{2}\{Z(\lambda') + \eta E(-\tfrac{1}{2}n\theta' + \tfrac{1}{2}\theta')Z_1(\lambda')\}] + O(1).
\end{aligned}$$

We find easily that $T = O(1)$, so that the result for $\phi(\theta)$ in $(0, \pi)$ extends to $(-An^{-1}, \pi)$. Similarly for $\phi(-\theta)$: the corresponding results for $F(\pm\theta)$ then follow from the definition of F.

9. The results for $|F|$ in I follow (naturally) by straightforward calculation from those about F in Theorem 1. Since $Z = O\{(\lambda + 1)^{-1}\}$, $Z_1 = O\{(\lambda_1 + 1)^{-1}\}$, we can, however, now give much stronger forms. If we incorporate, for completeness, the upper bound for $|F|$ from I, we have the following theorem, which exhibits more clearly the behaviour at F at its critical point $\theta = 0$, and of $\phi(\theta)$ at its critical points $\theta = 0, \pi$. $\phi(-\theta)$ has no particular importance. We recall that R is $(0, \pi)$, R_δ is $(\delta, \pi - \delta)$, and that $\eta = E(\frac{1}{4}\pi)$.

THEOREM 2. *We have* $F(-\theta) = E\{-(n-1)\theta\}F(\theta) \quad (\theta \in R)$, *and*

(i) $F(\theta) = \mu\eta E(n\theta - \frac{1}{2}\theta - n^2\theta/4\pi) + O\{\mu/(\mu\theta + 1)\} \quad (\theta \in R)$.

(ii) *In particular the error term is* $O_\delta(1)$ *in* $(\delta \leqq \theta \leqq \pi)$.

(iii) *For all* θ, $|F(\pm\theta)| < (1.35)\mu \quad (n > n_0)$.

(iv) *For all* θ, $|\phi(\pm\theta)| < A_1\mu$.

(v) $\phi(\theta) = O\left(\dfrac{\mu}{\mu\theta + 1}\right) + O\left(\dfrac{\mu}{\mu(\pi - \theta) + 1}\right) \quad (\theta \in R)$.

(vi) *In particular,* $\phi(\theta) = O_\delta(1) \quad (\theta \in R_\delta)$.

10. The properties (iv) to (vi) of $\phi(\theta)$ enable us to construct new functions with interesting behaviour (F is of course the first of these).

In the first place, $\phi(t)$ $(|t| \leqq \pi)$ itself provides the counterexample to "$\kappa = 0$" in Theorem 4(I) mentioned in §2.

Let E be the range $(\frac{1}{4}\pi, \frac{3}{4}\pi)$; and let $\mu_1 = M_2(\phi) = (\frac{1}{2}n)^{1/2}$. Then $|\phi| < \varepsilon\mu_1$ in E for $n > n_0(\varepsilon)$, and we have

$$\left(\frac{M_\lambda(\phi)}{M_2(\phi)}\right)^\lambda \leqq \frac{1}{2\pi}\left(\int_E \varepsilon^\lambda\, dt + \int_{CE} A_1^\lambda\, dt\right) = \tfrac{1}{4}\varepsilon^\lambda + \tfrac{3}{4}A_1^\lambda .$$

If, e.g., $\lambda = \varepsilon^{1/2}$, this gives, for small enough ε, and so small enough λ,

$$\frac{M_\lambda}{M_2} < \varepsilon\left[1 + \frac{3}{4}\left\{\left(\frac{A_1}{\varepsilon}\right)^{\varepsilon^{-1/2}} - 1\right\}\right]^{\varepsilon^{-1/2}}$$

$$= \varepsilon\left\{1 + \left(\frac{3}{4} + \zeta\right)\log\frac{A_1}{\varepsilon}\right\}$$

(by straightforward calculation, where ζ is small with ε)

$$< \varepsilon^{3/4} = \lambda^{3/2}.$$

This is incompatible with the conclusion $(d/d\lambda)(M_\lambda) > BM_2$ of Theorem 4(I) for $\kappa > 0$.

Further remarks on Theorem 4(I).

(i) *If* $0 < \kappa < q_2 < q_1$, *and if a (quite general complex) g has* $M_{q_1}(g) \leqq (1 + a)M_2(g)$, *then*

$$D\{M_\lambda(g)\} \leqq CM_2(g) \quad (\kappa \leqq \lambda \leqq q_2), \quad C = A(\kappa, q_1, q_2, a),$$

where $D\{M_\lambda(g)\}$ *is* lim sup $\{M_{\lambda+\varepsilon}(g) - M_\lambda(g)\}/\varepsilon$ *as* $\varepsilon \to 0$.

Since $M_\lambda \geqq CM_2(g)$ in the range concerned, by the convexity of means M_λ, it is enough to prove that $Dm(\lambda) \leqq CM_2^\lambda$, where

$$m(\lambda) = M_\lambda^\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g|^\lambda d\theta.$$

Let E be the set in which

$$|g| \leqq 1, \qquad m_1(\lambda) = M_\lambda^\lambda(g, E), \qquad m_2(\lambda) = M_\lambda^\lambda(g, CE).$$

Then $Dm_1(\lambda) \leqq 0$, and so

$$2\pi\, Dm(\lambda) \leqq 2\pi m_2(\lambda) \leqq \limsup \int_{CE} |g|^\lambda \frac{\exp(\varepsilon \log|g|) - 1}{\varepsilon} d\theta$$

$$\leqq \limsup \int_{CE} |g|^\lambda \{\log|g| + \varepsilon(\log|g|)^2 |g|^\varepsilon\} d\theta.$$

Since $\log|g| \geqq 0$, the coefficient of ε is less than $A_\delta |g|^\delta$ for a fixed but arbitrarily small δ and small enough ε. For $\delta < q_1 - q_2$ the integral of $A_\delta |g|^{\lambda+\delta}$ is finite, and

$$\limsup = \int_{CE} |g|^\lambda \log|g|\, d\theta.$$

In this, $\log|g| \leqq C|g|^{(q_1-q_2)/2}$, and so

$$2\pi Dm(\lambda) \leqq C\int_{-\pi}^{\pi} |g|^{(q_1+q_2)/2} d\theta \leqq C,$$

as desired.

(ii) That f should be real in Theorem 4 (I) is essential. Thus, even in the narrowest complex class $\mathfrak{C}_u$ we can find f_n's behaving as follows.[5] Given any c in $0 \leqq c \leqq 1$ there is an f_n of $\mathfrak{C}_u$ such that, as $n \to \infty$,

$$M_q/M_2 \to \infty \quad (q > 2), \qquad M_\lambda/M_2 \to c \quad (0 < \lambda < 2).$$

Again, given any $q_1 > 2$, there are $f_n \in \mathfrak{C}_u$ such that, as $n \to \infty$,

$$M_q/M_2 \to \infty \quad (q > q_1), \qquad M_\lambda/M_2 \to 1 \quad (0 < \lambda < q_1).$$

In the first case we take $f_n = \sum_1^n e^{m\theta i}$ if $c = 0$; and it is easily verified that $F_n(\theta) + e^{n\theta i}\sum^{[cn]} e^{m\theta i}$, with $a = c^{-2} - 1$, has the desired properties when $0 < c \leqq 1$.

In the second case $\sum^{[n^k]} e^{m\theta i} + e^{([n^k]+1)\theta i}F_n(\theta)$ has the desired properties when $k = 1/(q_1 - 1)\ (< 1)$.

11. Theorem 3′(I) [representative of 3(I)] is about the entire interval E_0. It is natural to inquire whether anything similar is true for a sub-

[5] Part of this was stated in I without proof.

interval E. Is it, again, possible, to take an extreme case, for a *real* t.p.(n), not necessarily belonging to $\mathfrak{F}_k$, to be of the form $aM_2(f)\{1 + O(n^{-\alpha})\}$, $\alpha > 0$, in an E? Since

$$f = \sqrt{n} \sum^{n} \frac{\sin (2m + 1)\theta}{2m + 1}$$

is actually $M_2(f) + O(1)$ in the interval $(\frac{1}{4}\pi, \frac{3}{4}\pi)$, say, the question (of the extreme case) becomes whether such behaviour is possible with "reasonable" functions.

Our final answer, which leaves open the actual question, is that a reasonable real f_n with M_2 of the order of n^κ with positive κ, can be of the form $a_k n^{\kappa - 1/2} + O(1)$ in an E, and indeed $a_k n^{\kappa-1/2} + O_\delta(1)$ in $(\delta \leqq \theta \leqq \pi - \delta)$. κ can be arbitrarily large.

We shall, however, begin with a study of what *can* be proved, for an E, on the lines of Theorem 3(I). The arguments seem interesting in themselves even if the actual results appear rather unexciting; they have also the interest that, when adapted to the complete interval E_0, they give an alternative proof of Theorem 3′(I), and indeed extend it in a certain direction. Moreover, *both* proofs are rather queer.

12. We begin with a lemma about any function $g(\theta)$, not necessarily real or a t.p., defined in E and satisfying $M_1(g, E) > (1 - \varepsilon)M_2(g, E)$. Let us denote by ζ any number satisfying $0 \leqq \zeta \leqq B\varepsilon^A$, where A is an absolute constant, where the positive constant B depends only on the parameters of the context, and where in particular the B's in Lemma 1 below are A's (only a number like 10 of pairs B, A are involved in all). We denote by $\mathcal{E}$ any ("small") set satisfying $| \mathcal{E} | < \zeta$. We have now (with absolute O's)

LEMMA 1. *Suppose that g is defined in E and satisfies*

(1) $$M_1(E, g) \geqq (1 - \varepsilon)M_2(E, g) = (1 - \varepsilon)\mu.$$

Then, if E^* is a subinterval of E, we have

(i) $$\int_{E^*} | g |^2 \, d\theta = | E^* | \mu^2 + O(\zeta\mu^2);$$

(ii) $$\int_{E^*} | g | \, d\theta = | E^* | \mu + O(\zeta\mu).$$

In particular

(iii) $$\int_{\mathcal{E}} | g |^2 \, d\theta < \zeta\mu^2, \qquad \int_{\mathcal{E}} | g | \, d\theta < \zeta\mu.$$

Further, there exist ζ_1 and ζ_2 such that

(iv) $$1 - \zeta_1 < | g | < 1 + \zeta_2, \qquad 1 - \zeta_1 < | g |^2 < 1 + \zeta_2,$$

in E except for a set $\mathcal{E}$.

This is proved in Lemmas 5, 6, 7 of I.

13. We need next a form of Bernstein's theorem[6] for E, instead of the E_0 of the original.

LEMMA 2. *Let f be a t.p.(n), E a subinterval of E_0, and let E_l be E less a small*[7] *interval of length l at each end. Let M be the maximum of $|f|$ in E_0. Then for $k \geqq 1$*

(i) $$M_k(f', E_l) \leqq A\{nM_k(f, E) + Ml^{-1}\},$$

(ii)[8] $$M_k(f', E) \leqq A\{nM_k(f, E) + n^{1/2}M\}.$$

We have [2, I, 118, (13.18)]

$$|f'(\theta)| \leqq \left| 2 \int_{-\pi}^{\pi} \frac{\sin nt \sin^2 \frac{1}{2}(n+1)t}{\sin^2 \frac{1}{2} t} f(\theta + t)\, dt \right|,$$

(1) $$|f'(\theta)| \leqq A \int_{-l}^{l} |f(\theta + t)|\, \chi(t)\, dt + AM \int_{l}^{\infty} \frac{dt}{t^2} = A \int_{-l}^{l} + AMl^{-1},$$

where

$$\chi(t) = \frac{|\sin nt \sin^2 \frac{1}{2}(n+1)t|}{t^2}.$$

By Minkowski's inequality

$$M_k(f', E) \leqq M_k\left\{A \int_{-l}^{l} |f(\theta + t)|\, \chi(t)\, dt, E\right\} + M_k\{AMl^{-1}\},$$

$$\leqq A \left[\frac{1}{E} \int_E \left\{\left(\int_{-l}^{l} |f(\theta + t)|^k\, dt\right) \left(\int_{-l}^{l} \chi(t) \cdot 1\, dt\right)^{k-1}\right]^{1/k} + AMl^{-1},$$

by Hölder's inequality, and so

(2) $$M_k(f', E) \leqq An^{(k-1)/k} \left[\int_{-l}^{l} \chi(t) \left(\int_E |f(\theta + t)|^k\, d\theta\right) dt\right]^{1/k},$$

since $\int_{-\infty}^{\infty} \chi(t)\, dt$ is easily seen to be $O(n)$.

Now if E is $a \leqq \theta \leqq b$, we have

(3) $$\int_E |f(\theta + t)|^k\, d\theta \leqq \int_{a-l}^{b+l} |f(\phi)|^k\, d\phi \leqq |E|\, M_k^k(f, E) + 2l\, M^k,$$

(4) $$\int_{E_l} |f(\theta + t)|^k\, d\theta \leqq \int_a^b |f(\phi)|^k\, d\phi = |E|\, M_k^k(f, E).$$

Further, taking $E = E_l$ in (2) and substituting in it from (4), we have

$$M_k(f', E_l) \leqq An^{(k-1)/k} \left\{\int_{-l}^{l} \chi(t)\, dt \cdot M_k^k(f, E)\right\}^{1/k} + \frac{AM}{l}$$

$$\leqq AnM_k(f, E) + AMl^{-1}, \qquad \text{since } \int_{-l}^{l} \chi\, dt = O(n),$$

and this is (i).

---

[6] In I we used an extension of Bernstein's theorem of a different kind; this difference is only one of the differences in the two proofs of Theorem 3′ (I).

[7] l may depend on n, and be e.g. $n^{-\alpha}$.

[8] We do not use this, but it seems of interest in itself.

Again, substituting from (3) in (2), we have

$$M_k(f', E) \leqq An^{(k-1)/k}\left[\int_{-l}^{l} \chi\, dt\, \{M_k^k(f, E) + 2lM^k\}\right]^{1/k} + \frac{AM}{l}$$

$$\leqq An\{M_k(f, E) + lM + AM/ln\} \leqq An\, M_k(f, E) + An^{1/2} M$$

if we choose $l = n^{-1/2}$, and this is (ii).

14. THEOREM 3. *Let $f = f_n$ be any real trigonometrical polynomial of degree exactly n, E a subinterval of E_0, and let E_δ be E less intervals of small fixed length δ at each end. Let*

$$\mu = M_2(f, E), \qquad M = \max |f| \quad \text{in } E_0 .$$

Let $\eta = \alpha\pi n^{-1}$, where α is a positive constant, and let $f_\eta = f(\theta + \eta)$. Let H be the subset of E_δ in which $ff_\eta < 0$. Then there exists a positive absolute constant c, with the following properties. For large[9] n, either

(i) $$M_1(f, E_\delta) < (1 - c)\mu,$$

or else

(ii) $$M_1(f', E_\delta) < \mu\{o(n) + AM\mu^{-1}\delta^{-1}\}.$$

In the event of (ii) *we have also*

(iii) $$| H | < o(1) + A\alpha(\mu n)^{-1}M\delta^{-1};$$

and, if further $M_{q_0}(f, E) < K\mu$, $q_0 > 2$, we have also

(iv) $$M_2(f', E_\delta) < o(\mu n) + A_{q_0} KM\delta^{-1}.$$

In applications we usually have $M = o(\mu n)$, and then all the M-terms disappear.

We may normalize to $\mu = 1$. If the statement about the alternatives (i) and (ii) is false, there will exist, for any given ε, f_n with arbitrarily large n and satisfying

(1) $$M_1(f, E_\delta) > 1 - \varepsilon.$$

If ζ denotes any number of the form $A(\delta, E)\varepsilon^A$, it is enough to prove that for such f we have (recalling that $\mu = 1$), the equivalent (ii)′ of (ii), namely

(ii)′ $$M_1(f', E_\delta) < \zeta n + AM\delta^{-1} \qquad \text{for large } n.$$

Let $\mu_1 = M_2(f, E_\delta) \leqq M_2(f, E) = 1$. Then from (1)

(2) $$1 - \varepsilon < \mu_1 \leqq 1, \qquad M_1(f, E_\delta) > (1 - \varepsilon)\mu_1 .$$

By Lemma 1(iv) (with $g = f$, $\mu = \mu_1$, $E = E_\delta$), we have, except in an $\mathcal{E} \subset E_\delta$,

(3) $$\big|\, |f|^2 - 1 \,\big| < \zeta,$$

[9] It would be possible to prove that "large n" can be replaced by $n > n_0 = A(\varepsilon, \delta, E)$, and e.g. $o(1)$ by $\zeta(\varepsilon)$, ζ small with ε.

and since, by Lemma 1(iii)

$$\int_{\mathcal{E}} \left| \, |f|^2 - 1 \, \right| d\theta \leqq \int_{\mathcal{E}} |f|^2 \, d\theta + |\mathcal{E}| < \zeta,$$

we have

$$(4) \qquad \int_{E_\delta} \left| \, |f|^2 - 1 \, \right| d\theta < \zeta.$$

By (4) and Lemma 2, with $l = \delta$, $k = 1$, *and since*[10] $|f|^2 - 1 = f^2 - 1$ *is a t.p.*$(2n)$, we have

$$(5) \qquad \int_{E_\delta} \left| \frac{d}{d\theta}(f^2 - 1) \right| d\theta < \zeta n + AM\delta^{-1}, \qquad \int_{E_\delta} |ff'| < \zeta n + AM\delta^{-1}.$$

Since, by Lemma 1(iv), $|f| > 1 - \zeta$ except in an $\mathcal{E}_1 \subset E_\delta$, it follows from $(5)_2$ that

$$(6) \qquad \int_{E_\delta - \mathcal{E}_1} |f'| \, d\theta < \zeta n + AM\delta^{-1}/(1 - \zeta) < \zeta n + AM\delta^{-1}.$$

Now by Lemma 2 with $k = 2$ (say), and (4),

$$M_2(f', E_\delta) < n + M\delta^{-1},$$

so that

$$\int_{\mathcal{E}_1} |f'| \, d\theta \leqq |\mathcal{E}_1|^{1/2} M_2(f', \mathcal{E}) < \zeta(n + M\delta^{-1}).$$

Hence, from (6),

$$(7) \qquad \int_{E_\delta} |f'| < \zeta n + AM\delta^{-1},$$

the desired result (ii)$'$.

Taking next (iv) [to be deduced from (ii) or (ii)$'$] we have

$$M_{q_0}(f', E_\delta) \leqq M_{q_0}(f', E) < AKn + AM\delta^{-1},$$

and from this and (7), and the convexity of M_λ in $\log \lambda$,

$$M_2 = M_2(f', E_\delta) \leqq M_1^{\vartheta} M_{q_0}^{1-\vartheta} < K\zeta n + A_{q_0} KM\delta^{-1},$$

which proves (iv).

It remains to deduce (iii) from (ii)$'$. We have, for $\theta \in H$, except in an $\mathcal{E}_1 \subset H$,

$$1 - \zeta < |f|, \qquad |f_\eta| < 1 + \zeta,$$

and since f, f_η *have opposite signs*,

$$\int_{H - \mathcal{E}_1} |f - f_\eta| \, d\theta > 2|H - \mathcal{E}_1| - \zeta > 2|H| - \zeta,$$

---

[10] That $|f|^2 = f^2$ is the only use we make of the hypothesis that f is real in deducing (ii) from (1).

and so

$$\int_H |f - f_\eta|\, d\theta > 2\,|H| - \zeta, \qquad \text{since} \int_{\mathcal{E}_1} |f|\, d\theta \text{ and} \int |f_\eta|\, d\theta < \zeta .$$

Hence[11]

$$2\,|H| < \zeta + \int_H d\theta \int_0^\eta |f'(\theta + t)|\, dt < \zeta + \int_0^\eta dt \int_H |f'(\theta + t)|\, dt$$

$$< \zeta + \int_0^\eta dt \int_{E_{\delta/2}} |f'(\phi)|\, d\phi$$

$$< \zeta + \left(\int_0^\eta dt\right) \{\zeta\mu n + AM\delta^{-1}\},$$

by (iii) (with $\frac{1}{2}\delta$ for δ). This establishes (iii), and completes the proof of Theorem 3.

15. When E is E_0, the step at (1) in the proof of Lemma 2 becomes unnecessary, and the lemma becomes the known result $M_k(f') \leqq AnM_k(f)$, Bernstein's theorem with an extra A. The distinction between E and E_δ is thus unnecessary, and the final upshot is that the terms $M\delta^{-1}$ disappear altogether. If now we add a hypothesis:

$$M_2(f') \geqq k^{1/2} n M_2(f), \qquad k \text{ a positive constant,}$$

this excludes the alternative (ii) in Theorem 3. We therefore arrive at the following results: With c's positive absolute constants and $q_0 > 2$ we have:

(a) *With hypotheses $M_{q_0}(f) < KM_2(f)$, $M_2(f') \geqq k^{1/2} n\, M_2(f)$, we have, for large n,*

$$M_1(f) < (1 - c) M_2(f).$$

(b) *Without the hypothesis on M_{q_0} we have, for given ε and large n, either*

(i) $$M_1(f) < (1 - c) M_2(f),$$

or else[12]

(ii) $$|H| < \varepsilon$$

[where H is now the subset of E_0 in which $ff_\eta < 0$].

Now the hypothesis $M_2(f') \geqq k^{1/2} n M_2(f)$ is equivalent to the hypothesis $f \in \mathfrak{F}_k$ of Theorem 3′(I). Hence (a) is Theorem 3′(I) modified as follows. It is weakened by the hypothesis about M_{q_0}, weakened by requiring n to be large, but strengthened in that c is an absolute constant, instead of depending on k as it does in Theorem 3′(I).

[11] Observe that $\theta + t$ below need not belong to H or even E_δ; it belongs obviously, however, to $E_{\delta/2}$ for large n.

[12] This is true for all positive α; in I we had $\alpha < 1$ (though that sufficed).

The result (b) is parallel to a result at a certain stage in the proof of Theorem 3′(I). As we said above, the two proofs differ considerably. One main novelty here is the use of (4) §14. We go on now to complete the proof of what we will call Theorem 3″(I), namely 3′ modified by n being large but c an absolute constant. *This* argument, while different in detail from the parallel one in I, depends, as that does (and apparently inescapably), on the identity (3) below. There is incidentally nothing corresponding to this for an interval E.

16. Suppose "$f \in \mathfrak{F}_k$ implies that $M_1(f) > (1 - c)M_2(f)$ for large n" is false, so that there exist $f_n \in \mathfrak{F}_k$ with arbitrarily large n, and satisfying

$$M_1(f) > (1 - \varepsilon)M_2(f).$$

Then, always for such f and large n, we have $|H| < \zeta$. Since then both H and its translation by an amount η are $\mathcal{E}$'s, and, by Lemma 1 (iii), $\int |f|^2 d\theta < \zeta$, we have on the one hand

$$\int_H |f - f_\eta|^2 d\theta < \zeta. \tag{1}$$

On the other hand, *for $\theta \in CH$, f and f_η have the same sign.* Since by Lemma 1 (iv) we have in CH, except for $\mathcal{E}_1 \subset CH$,

$$1 - \zeta < |f|, \qquad |f_\eta| < 1 + \zeta,$$

we have[13] $|f - f_\eta|^2 < \zeta$ in $CH - \mathcal{E}_1$. Then

$$\int_{CH} |f - f_\eta|^2 d\theta \leqq \int_{CH-\mathcal{E}_1} \zeta\, d\theta + 2\left\{\int_{\mathcal{E}_1} |f|^2 d\theta + \int_{\mathcal{E}_1} |f_\eta|^2 d\theta\right\} < \zeta,$$

and

$$\int_{E_0} |f - f_\eta|^2 d\theta = \int_{CH} + \int_H = \int_{CH} + \int_{\mathcal{E}} < \zeta. \tag{2}$$

Now if $f = \sum a_m \cos(m\theta + \alpha_m)$, we have

$$\frac{1}{2\pi}\int_{E_0} |f - f_\eta|^2 d\theta = \sum |a_m|^2\, 2\sin^2(\tfrac{1}{2}\pi\alpha m n^{-1}). \tag{3}$$

If we choose, say, $\alpha\pi = 1$, we have $\frac{1}{2}\pi\alpha m n^{-1} \leqq \frac{1}{2}\pi\alpha < \frac{1}{2}\pi$, and so

$$\sin^2(\tfrac{1}{2}\pi\alpha m n^{-1}) \geqq A(\tfrac{1}{2}\pi\alpha m n^{-1})^2,$$

and then

$$\frac{1}{2\pi}\int_{E_0} |f - f_\eta|^2 d\theta \geqq A n^{-2} \sum^n m^2 a_m^2 > A_k,$$

since $f \in \mathfrak{F}_k$. This contradicts (3) and establishes Theorem 3″(I).

[13] Here, of course, the argument turns on f being real.

17. We proceed now to construct a "reasonable" real function with the behaviour mentioned in §11. The proof is inevitably rather long, with much detail that the reader can ignore if he wishes.

THEOREM 4. *Let $k \geqq 3$ be integral, and s a nonnegative integer. Let*

$$g(t) = g_{k,s}(t) = \int_0^t (\{\phi(t) - \tfrac{1}{2}\}^k - (\tfrac{1}{2})^k)\, e^{sti}\, dt \qquad (|t| \leqq \pi),$$

$$h(t) = g(t) - \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, dt,$$

so that g and h are trigonometrical polynomials of degree $\frac{1}{2}k(n-1)+s$, and h has constant term 0. Let $\mu = n^{1/2}$, $\eta = E(\frac{1}{4}\pi)$. Then in R_δ [or $\delta \leqq \theta \leqq \pi - \delta$]

$$h(\theta) = a_k \mu^{k-2} + \tau a_k' \log \mu \cdot \mu^{k-3} + O_\delta(\mu^{k-3}),$$

where

$$\tau = \tau(k) = \begin{cases} 1 & (k = 3) \\ 0 & (k > 3). \end{cases}$$

a_k' depends only on k, a_k (which depends on s) is given by

$$a_k = -i(1+\eta^k)\, k^{-1} + 2^{-3/2}\pi k \int_0^\infty \{\eta E(-\lambda^2) - 2^{-1/2} Z\}^{k-1} Z'\, d\lambda$$
$$+ 2\int_0^\infty \lambda[\{\eta E(-\lambda^2) - 2^{-1/2} Z\}^k - \eta^k E(-k\lambda^2)]\, d\lambda - s\beta_1(k),$$

where

$$\beta_1(k) = -\tfrac{1}{2}\pi \frac{k}{k-2}\{\eta^k + \eta^k i(k-1)^{-1/2} + 2k^{-1} - 4k^{-2}\}.$$

$\beta_1(k)$ is not 0, so that[14] *$a_k \neq 0$ for one of $s = 0, 1$. We select the appropriate s, with $a_k \neq 0$; let $\sigma = \operatorname{sgn} a_k$ and $f(t) = \eta\sigma h(t)$. The real and imaginary parts of f then have each the property of being a real polynomial with zero constant term, such that*

$$f(\theta) = a(k, n)\mu^{k-2} + O_\delta(1) \qquad (\theta \in R_\delta),$$

where $a(k, n) \sim |a_k|/\sqrt{2}$ as $n \to \infty$. $M_2(f)$ is of the order $\mu^{k-3/2}$ and $\mu^{k-2} = M_2^{(2k-4)/(2k-3)}$. O's depend only on k and s, and O_δ's in addition on δ.

We use α's for absolute (complex) constants, β's for constants depending only on k (b's are used consistently in a certain sense; see below).

18. We begin with

LEMMA 3. *In the range Λ, or $0 \leqq \lambda \leqq \alpha\mu$ [$\alpha = \frac{1}{2}\sqrt{\pi}$], let*

$$E = E(\lambda) = E(a_1 \mu\lambda + a_2 \lambda^2 + a_3 \mu^{-1}\lambda),$$

[14] This is incidentally a guarantee that the result $a_k \neq 0$ does not depend on some slip in the rather elaborate detail!

where a_1, a_2, a_3 are absolute constants satisfying either

$$a_1 > A, \quad a_1 + 2\alpha a_2 > A, \quad \text{or else} \quad a_1 < -A, \quad a_1 + 2\alpha a_2 < -A.$$

Let

$$D = \frac{d}{d\lambda}(a\mu\lambda + a_2\lambda^2 + a_3\mu^{-1}\lambda) = \mu(a_1 + 2a_2\mu^{-1}\lambda + a_3\mu^{-1}),$$

so that[15]

$$|D| > A\mu, \qquad D(0) = a_1\mu + a_3.$$

Suppose now that in Λ the function $H(\lambda)$ satisfies

$$H = H(\lambda) = O(1), \qquad H^{(r)}(\lambda) = O(l_\lambda) \qquad (r = 1, 2, 3).$$

Then we have upper bounds as follows.

$$\text{(i)} \qquad \int_0^\lambda EH\,d\lambda - \left\{\frac{EH}{iD} - \frac{H(0)}{iD(0)}\right\} = O(M_1), O(M_2), O(M_3),$$

where

$$M_1 = (1 + l_1)\mu^{-1}, \qquad M_2 = l_2\mu^{-1} + l_1\mu^{-2} + \mu^{-3},$$
$$M_3 = l_1\mu^{-2} + (1 + l_3)\mu^{-3} + l_2\mu^{-4};$$

$$\text{(ii)} \quad \int_0^{\alpha\mu} (\alpha\mu - \lambda)EH\,d\lambda - \mu\frac{i\alpha H(0)}{D(0)} = O(\mu M_1), O(\mu M_2), O(\mu M_3).$$

In particular

$$\text{(iii)} \quad \int_0^{\alpha\mu} (\alpha\mu - \lambda)EH\,d\lambda - \mu\frac{i\alpha H(0)}{D(0)} = O(\mu^{-2}) \quad \text{if} \quad l_1 = l_2 = l_3 = 1.$$

We have also the crude result

$$\text{(iv)} \quad \int_0^{\alpha\mu} (\alpha\mu - \lambda)EH\,d\lambda = O(1) \quad \text{if} \quad H = O(1), \quad H' = O(\mu).$$

(i) to (iii) are very powerful. In applications we always have $l_r \leqq \mu^{r-1}$, and often $l_1 = l_2 = l_3 = 1$.

The results (i) follow by straightforward calculation from the identities

$$\int_0^\lambda EH\,d\lambda = \left[\frac{EH}{iD}\right]_0^\lambda - \int_0^\lambda E\frac{d}{d\lambda}\left(\frac{H}{iD}\right)d\lambda;$$

$$\int_0^\lambda EH\,d\lambda = \left[\frac{EH}{iD}\right]_0^\lambda - \left[\frac{E}{iD}\frac{d}{d\lambda}\left(\frac{H}{iD}\right)\right]_0^\lambda + \int_0^\lambda E\frac{d}{d\lambda}\left\{\frac{1}{iD}\frac{d}{d\lambda}\left(\frac{H}{iD}\right)\right\}d\lambda;$$

$$\int_0^\lambda EH\,d\lambda = \left[\frac{EH}{iD}\right]_0^\lambda - \left[\frac{EH'}{i^2D^2}\right]_0^\lambda + 2a_2\left[\frac{EH}{i^2D^3}\right]_0^\lambda + \left[\frac{EH''}{i^3D^4}\right]_0^\lambda$$
$$+ (2a_2 + 1)\int_0^\lambda \frac{EH'}{i^2D^3}\,d\lambda - 2a_2\int_0^\lambda \frac{E}{i^2D}\frac{d}{d\lambda}\left(\frac{H}{D^2}\right)d\lambda$$
$$+ 4a_3^2\int_0^\lambda \frac{EH}{i^2D^4}\,d\lambda - \int_0^\lambda E\frac{d}{d\lambda}\left(\frac{H''}{i^3D^4}\right)d\lambda.$$

[15] This being true for the extreme values $\lambda = 0, \alpha\mu$.

The results (ii) follow from (i) when we observe that the l_r for $\lambda H/\mu$, l_r' say, satisfy

$$l_1' \leqq A(\mu^{-1} + l_1), \qquad l_2' \leqq A(l_1 \mu^{-1} + l_2), \qquad l_3' \leqq A(l_2 \mu^{-1} + l_3).$$

LEMMA 4.

$$\mu \int_0^{\alpha\mu} \frac{d\lambda}{(\lambda + 1)(\lambda_1 + 1)} = O(\log n).$$

For

$$\mu \int_0^{\alpha\mu} \leqq \mu \int_0^{\alpha\mu/2} \frac{d\lambda}{(\lambda + 1)A\mu} + \mu \int_{\alpha\mu/2}^{\alpha\mu} \frac{d\lambda}{A\mu\{(\alpha\mu - \lambda) + 1\}}.$$

LEMMA 5. *If a, b are absolute real constants, $a \neq 0$,*

$$\int_\lambda^\infty E(a\lambda^2 + 2b\mu^{-1}\lambda)\, d\lambda - \int_\lambda^\infty E(a\lambda^2)\, d\lambda = \frac{b}{a}\mu^{-1} \frac{E(2b\mu^{-1}\lambda) - 1}{2b\mu^{-1}\lambda} + O(\mu^{-2}).$$

If we write $\delta = b\mu^{-1}/a$, the left side is

$$E(-a\delta^2) \int_\lambda^\infty E\{a(\lambda + \delta)^2\}\, d\lambda - \int_\lambda^\infty E(a\lambda^2)\, d\lambda$$

$$= \int_\lambda^\infty [E\{a(\lambda + \delta)^2\} - E(a\lambda^2)]\, d\lambda + O(\mu^{-2})$$

$$= -\int_0^\delta E\{a(\lambda + x)^2\}\, dx + O(\mu^{-2})$$

$$= -E(a\lambda^2)\left[\int_0^\delta E(2a\lambda x)\, dx + O(\mu^{-2})\right] + O(\mu^{-2}),$$

which gives the result desired.

19. Let[16]

$$\psi = \psi(t) = E(st)(\{\phi(t) - \tfrac{1}{2}\}^k - (\tfrac{1}{2})^k), \qquad g(t) = \int_0^t \psi(t)\, dt,$$

$$h(t) = g(t) - \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, dt.$$

ψ has zero constant term, and g and h are t.p.$(k(n - 1) + s)$, and h has zero constant term. We calculate

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, dt = \frac{1}{2\pi}\left\{\int_0^\pi g(\theta)\, d\theta + \int_0^\pi g(-\theta)\, d\theta\right\},$$

and then $g(\theta)$ in R_δ. Since $\phi(\theta) = O_\delta(1)$ in R_δ, we must obviously have $h(\theta)$ constant to $O_\delta(1)$ there; but it is not obvious that $h(\theta)$ need be large. We do not calculate $g(-\theta)$ itself (its behaviour is not of interest), and we work with a formula for $\int_0^\pi g(-\theta)\, d\theta$ [by-passing $g(-\theta)$ itself].

We begin with $g(\theta)$ and $\int_0^\pi g(\theta)\, d\theta$.

[16] We use $\phi - \frac{1}{2}$ to get rid of the term $\frac{1}{2}$ in X and X_1 of Theorem 1.

20. Write $T = Z + \eta E(2\alpha\mu\lambda - 2\alpha\mu^{-1}\lambda)Z_1$, and recall that

$$T = a(\lambda + 1)^{-1} + aE(2\alpha\mu\lambda - 2\alpha\mu^{-1}\lambda)Z_1 + O\{(\lambda + 1)^{-2}\} = O(1),$$
$$Z', Z'' = O\{(\lambda + 1)^{-2}\}.$$

From Theorem 1, and with its notation, we have in $(0, \pi)$, after a little reduction [we can, and often do, absorb, e.g., $E(\frac{1}{2}\theta)$ or $E(s\theta)$ into a $\mathcal{P}$ or $\mathcal{P}_0$],

$$(1)\quad \begin{cases} g(\theta) = P_1 + P_2 + P_3 + O(\mu^{k-3}), \\ P_1 = \mu^{k-1}\cdot 2\pi^{1/2}\cdot 2^{-k/2}\left[\int_0^\lambda T^k\,d\lambda + \int_0^\lambda T^k\{E(s\theta) - 1\}\,d\lambda\right], \\ P_2 = \mu^{k-2}\int_0^\lambda T^{k-1}\{\beta\mathcal{P}_0 + \beta\mathcal{P}(2\alpha\mu\lambda)\}\,d\lambda \\ \qquad + \mu^{k-2}2^{-k/2}\pi^{1/2}k\left[\int_0^\lambda T^{k-1}Z'\,d\lambda + \int_0^\lambda T^{k-1}Z'\{E(s\theta) - 1\}\,d\lambda\right], \\ P_3 = \mu^{k-3}\left[\int_0^\lambda T^{k-1}\{\beta\mathcal{P}E(2\alpha\mu\lambda) + O(Z'')\}\,d\lambda\right. \\ \qquad \left. + \int_0^\lambda T^{k-2}\{\beta\mathcal{P}_0 + \beta\mathcal{P}(2\alpha\mu\lambda) + \beta\mathcal{P}(4\alpha\mu\lambda) + O(Z')\}\right]; \end{cases}$$

$$(2)\quad \begin{cases} \frac{1}{2\pi}\int_0^\pi g(\theta)\,d\theta = \pi^{-1/2}\mu^{-1}\int_0^{\alpha\mu}\left\{2\pi^{1/2}\int_0^\lambda \psi(\theta)\,d\lambda\right\}d\lambda \\ \qquad = 2\mu^{-1}\int_0^{\alpha\mu}(\alpha\mu - \lambda)\psi\,d\lambda; \\ \frac{1}{2\pi}\int_0^\pi g(\theta)\,d\theta = P_1' + P_2' + P_3' + O(\mu^{k-3}), \\ P_1' = \mu^{k-2}\cdot 2^{-k/2+1}\left[\int_0^{\alpha\mu}(\alpha\mu - \lambda)T^k\,d\lambda\right. \\ \qquad \left. + \int_0^{\alpha\mu}(\alpha\mu - \lambda)T^k\{E(s\theta) - 1\}\,d\lambda\right], \\ P_2' = \mu^{k-3}\left[\int_0^{\alpha\mu}(\alpha\mu - \lambda)T^{k-1}\{\beta\mathcal{P}_0 + \beta\mathcal{P}(2\alpha\mu\lambda)\}\,d\lambda\right. \\ \qquad + 2^{-k/2}\pi^{1/2}k\int_0^{\alpha\mu}(\alpha\mu - \lambda)Z'\,d\lambda \\ \qquad \left. + 2^{-k/2}\pi^{1/2}k\int_0^{\alpha\mu}(\alpha\mu - \lambda)Z'\{E(s\theta) - 1\}\,d\lambda\right], \\ P_3' = \mu^{k-4}\left[\int_0^{\alpha\mu}(\alpha\mu - \lambda)T^{k-1}\{\beta\mathcal{P}(2\alpha\mu\lambda) + O(Z'')\}\,d\lambda\right. \\ \qquad \left. + \int_0^{\alpha\mu}(\alpha\mu - \lambda)T^{k-2}\{\beta\mathcal{P}_0 + \beta\mathcal{P}E(2\alpha\mu\lambda) + \beta\mathcal{P}(4\alpha\mu\lambda) + O(Z')\}\,d\lambda\right]. \end{cases}$$

21. We begin by disposing of P_3 and P_3'. Abbreviate $E\{p(2\alpha\mu\lambda - 2\alpha\mu^{-1}\lambda)\}$ to E_p. Now

$$T^{k-2} = Z^{k-2} + \sum_{p=1}^{k-2} c_p Z^{k-p} Z_1^p E_p ,$$

where $c_p = O(1)$, and similarly for T^{k-1}. It is easily seen that in P_3, P_3' the terms in $E(2r\alpha\mu\lambda)$, both explicit and arising from the expansions of T^{k-2} and T^{k-1}, contribute, by Lemma 3, $O(\mu^{-1})$ to the $\int_0^\lambda$ and $\int_0^{\alpha\mu}$ concerned. Further since Z', $Z'' = O\{(\lambda + 1)^{-2}\}$, the terms in $O(Z')$, $O(Z'')$ also contribute $O(\mu^{-1})$ to these integrals. Thus

$$(1) \qquad \begin{cases} P_3 = O(\mu^{k-3}) + O(\mu^{k-3}) \displaystyle\int_0^\lambda Z^{k-2} \mathcal{P}_0 \, d\lambda, \\ P_3' = O(\mu^{k-3}) + O(\mu^{k-4}) \displaystyle\int_0^{\alpha\mu} (\alpha\mu - \lambda) Z^{k-2} \mathcal{P}_0 \, d\lambda. \end{cases}$$

Now for $p \geqq 1$, in particular for $p = k - 2$,

$$(2) \qquad \int_0^\lambda Z^p \mathcal{P}_0 \, d\lambda = \sum_1^\infty b_m \mu^{-m} \int_0^\lambda O\{(\lambda + 1)^{-1}\} \lambda^m \, d\lambda$$
$$= O(1) \sum_1 | b_m | \mu^{-m} \lambda^m = O(1),$$

and similarly

$$\int_0^\lambda (\alpha\mu - \lambda) Z^{k-2} \mathcal{P}_0 \, d\lambda = O(\mu) + O(1) \sum | b_m | \mu^{-m} \lambda^{m+1} = O(\mu).$$

From these and (1),

$$(3) \qquad P_3 = O(\mu^{k-3}), \qquad P_3' = O(\mu^{k-3}).$$

22. Consider now P_2 and P_2'. We expand T^{k-1} as before as

$$a Z^{k-1} + \sum_{p=1} c_p Z^{k-1-p} Z_1^p E(2p\alpha\mu\lambda - 2p\alpha\mu^{-1}\lambda),$$

and observe that the sum $\sum$ contributes $O(\mu^{-1})$ to the integrals $\int_0^\lambda$, $\int_0^{\alpha\mu}$ concerned, by Lemma $3(\text{i})_1$. Thus we may replace T^{k-1} by $(2^{-1/2}Z)^{k-1}$ in P_2 and P_2';

$$(1) \qquad P_2 = \mu^{k-2} \Bigg[\beta \int_0^\lambda Z^{k-1} \{\mathcal{P}_0 + \mathcal{P} E(2\alpha\mu\lambda)\} \, d\lambda$$
$$+ 2^{-k/2-1} k \pi^{1/2} \left\{ \int_0^\lambda Z^{k-1} Z' \, d\lambda + \int_0^\lambda Z^{k-1} Z' \{E(s\theta) - 1\} \, d\lambda \right\} \Bigg] + O(\mu^{k-3}),$$

$$(2) \qquad P_2' = \mu^{k-3} \Bigg[\beta \int_0^{\alpha\mu} (\alpha\mu - \lambda) Z^{k-1} \{\mathcal{P}_0 + \mathcal{P} E(2\alpha\mu\lambda)\} \, d\lambda$$
$$+ 2^{-k/2-1} k \pi^{1/2} \left\{ \int_0^{\alpha\mu} (\alpha\mu - \lambda) Z^{k-1} Z' \, d\lambda \right.$$
$$\left. + \int_0^{\alpha\mu} (\alpha\mu - \lambda) Z^{k-1} Z' \{E(s\theta) - 1\} \, d\lambda \right\} \Bigg] + O(\mu^{k-3}).$$

The terms in $\mathcal{P}(2\alpha\mu\lambda)$ in P_2 and P_2' may be suppressed, by means of an easy application of Lemma 3. Next,

$$\int_0^\lambda Z^{k-1}\mathcal{P}_0\,d\lambda = b_1\,\mu^{-1}\int_0^\lambda \lambda Z^{k-1}\,d\lambda + \sum_2^\infty b_m\,\mu^{-m}\int_0^\lambda Z^{k-1}\lambda^m\,d\lambda$$

(3)
$$= a_k'\,\tau\log(\lambda+1) + \sum_2 b_m\,\mu^{-m}\int_0^{\alpha\mu} O\{\lambda^m(\lambda+1)^{-2}\}\,d\lambda$$
$$= a_k'\,\tau\log(\lambda+1) + O(1);$$

and

(4)
$$\int_0^{\alpha\mu}(\alpha\mu-\lambda)Z^{k-1}\mathcal{P}_0\,d\lambda$$
$$= \alpha\mu\int_0^{\alpha\mu} Z^{k-1}\mathcal{P}_0\,d\lambda - \sum_1 b_m\,\mu^{-m}\int_0^{\alpha\mu} O\{\lambda^m(\lambda+1)^{-2}\}\,d\lambda$$
$$= \mu a_k'\,\tau\log(\lambda+1) + O(\mu).$$

In P_2 we have

(5)
$$\int_0^\lambda Z^{k-1}Z'\{E(s\theta)-1\}\,d\lambda = \sum_1 \frac{(4\alpha si)^m}{m!}\mu^{-m}\int_0^\lambda \lambda^m Z^{k-1}Z'\,d\lambda$$
$$= \sum_1 \frac{(4\alpha s)^m}{m!}O(\mu^{-m})\int_0^{\alpha\mu}\lambda^m(\lambda+1)^{-4}\,d\lambda = O(1),$$

and similarly the corresponding term in P_2' is $O(\mu)$.

Finally

(6)
$$\int_0^\lambda Z^{k-1}Z'\,d\lambda = k^{-1}\{Z^k - Z^k(0)\} = \begin{cases} O(1) & (\theta\in R), \\ -k^{-1}\eta^k2^{-k/2} + O_\delta(\mu^{-1}) & (\theta\in R_\delta); \end{cases}$$

(7)
$$\int_0^{\alpha\mu}(\alpha\mu-\lambda)\,Z^{k-1}Z'\,d\lambda = -\,\alpha\mu\,k^{-1}\eta^k\,2^{-k/2} + \int_0^{\alpha\mu}\lambda\,O\{(\lambda+1)^{-k-1}\}\,d\lambda$$
$$= -\,\eta^k\,2^{-k/2-1}\,\pi^{1/2}k^{-1} + O(1).$$

From (1) to (7) we find

(8)
$$P_2 = \begin{cases} O(\mu^{k-2}) & (\theta\in R) \\ (-\eta^k2^{-k-1}\pi)\mu^{k-2} + a_k'\,\tau\log\mu\cdot\mu^{k-3} + O_\delta(\mu^{k-3}) & (\theta\in R_\delta); \end{cases}$$

(9)
$$P_2' = (-\eta^k2^{-k-2}\pi)\mu^{k-2} + a_k'\,\tau\log\mu\cdot\mu^{k-3} + O(\mu^{k-3}).$$

23. We have now to discuss P_1 and P_1'. Let

(1)
$$P_1 = \mu^{k-1}\cdot2^{-k/2+1}\pi^{1/2}\left[\int_0^\lambda T^k\,d\lambda + \int_0^\lambda T^k\{E(4s\alpha\mu^{-1}\lambda)-1\}\,d\lambda\right]$$
$$= P_{11} + P_{12},$$

$$(2)\quad \begin{aligned} P_1' = \mu^{k-2}\cdot 2^{-k/2+1}\biggl[\int_0^{\alpha\mu} (\alpha\mu - \lambda)T^k\,d\lambda \\ + \int_0^{\alpha\mu} (\alpha\mu - \lambda)T^k\{E(4s\alpha\mu^{-1}\lambda) - 1\}\,d\lambda\biggr] = P_{11}' + P_{12}'. \end{aligned}$$

We have

$$(3)\qquad T^k = Z^k + \beta Z_1^k E_k + \sum_{p=1}^{k-1} c_p Z^{k-p} Z_1^p E_p, \qquad c_p = O(1).$$

Let

$$H = \mu Z^{k-p} Z_1^p \qquad (1 \leqq p \leqq k - 1).$$

H is $O(\mu Z Z_1) = O\{\mu(\lambda + 1)^{-1}(\lambda_1 + 1)^{-1}\} = O(1)$, and H' is *a fortiori* $O(1)$. Since $H(0) = O(\mu^{-p+1}) = O(1)$, we have $l_1 = 1$ in Lemma 3, and

$$\int_0^\lambda HE_p\,d\lambda = O\left\{\frac{H(\lambda)}{D}\right\} + O\left\{\frac{H(0)}{D}\right\} + O(M_1) = O(\mu^{-1}),$$

by Lemma 3(i)$_1$. Similarly, by Lemma 3(ii)$_1$,

$$\int_0^{\alpha\mu} (\alpha\mu - \lambda)HE_p\,d\lambda = O(1).$$

It follows that

$$(4)\qquad \begin{cases} \beta\mu^{k-1} \sum_{p=1}^{k-1} c_p \int_0^\lambda Z^{k-p} Z_1^p E_p\,d\lambda = O(\mu^{k-3}), \\ \beta\mu^{k-2} \sum_{p=1}^{k-1} c_p \int_0^{\alpha\mu} (\alpha\mu - \lambda) Z^{k-p} Z_1^p E_p\,d\lambda = O(\mu^{k-3}). \end{cases}$$

Next, Lemma 3(i)$_3$ and (ii)$_3$, with $H = Z_1^k$, and so $l_1 = l_2 = l_3 = 1$, give

$$\begin{aligned} \int_0^\lambda Z_1^k E_k\,d\lambda &= \frac{E_k Z_1^k(\lambda) - Z_1^k(0)}{i2k\alpha(\mu - \mu^{-1})} + O(\mu^{-2}) \\ &= O(\mu^{-1} Z_1^k) + O(\mu^{-2}) = \begin{cases} O(\mu^{-1}) & (\theta \in R), \\ O_\delta(\mu^{-2}) & (\theta \in R_\delta). \end{cases} \end{aligned}$$

Similarly for $\int_0^{\alpha\mu} (\alpha\mu - \lambda) Z_1^k E_k\,d\lambda$. From (1), (3), and (4), and since

$$\int_0^\lambda Z^k\,d\lambda = \int_0^{\alpha\mu} + O_\delta(\mu^{-2}) = \int_0^\infty + O_\delta(\mu^{-2})$$

in R_δ, we have

$$(5)\qquad P_{11} - \mu^{k-1}\cdot 2^{-k/2+1}\pi^{1/2} \int_0^\infty Z^k\,d\lambda = \begin{cases} O(\mu^{k-2}) & (\theta \in R), \\ O_\delta(\mu^{k-3}) & (\theta \in R_\delta). \end{cases}$$

Similarly

$$P'_{11} = \mu^{k-2} \cdot 2^{-k/2+1} \int_0^{\alpha\mu} (\alpha\mu - \lambda) Z^k \, d\lambda + O(\mu^{k-2}). \tag{6}$$

In (6) we have

$$\int_0^{\alpha\mu} \lambda Z^k \, d\lambda = \int_0^{\infty} \lambda Z^k \, d\lambda + O(1) \int_{\alpha\mu}^{\infty} \lambda^{-k+1} \, d\lambda = \int_0^{\infty} \lambda Z^k \, d\lambda + O(\mu^{-1}),$$

and so

$$P'_{11} = \mu^{k-1} \left(2^{-k/2} \pi^{1/2} \int_0^{\infty} Z^k \, d\lambda \right) + \mu^{k-2} \left(-2^{-k/2+1} \int_0^{\infty} \lambda Z^k \, d\lambda \right) + O(\mu^{k-3}). \tag{7}$$

In P_{12}, again, we may replace T^k by Z^k, and we have easily (after earlier work)

$$P_{12} = \mu^{k-1} \cdot 2^{-k/2+1} \pi^{1/2} \int_0^{\lambda} \sum_1^{\infty} \frac{(4s\alpha i)^m}{m!} \mu^{-m} \lambda^m Z^k \, d\lambda$$

$$P_{12} - \mu^{k-2} \left(i 2^{-k/2+2} \pi s \int_0^{\infty} Z^k \, d\lambda \right) = \begin{cases} O(\mu^{k-2}) & (\theta \in R), \\ O_\delta(\mu^{k-3}) & (\theta \in R_\delta). \end{cases}$$

Similarly

$$P'_{12} - \mu^{k-2} \left(i 2^{-k/2+1} \pi s \int_0^{\infty} Z^k \, d\lambda \right)$$
$$= \beta \mu^{k-2} \sum_1^{\infty} \frac{(4s\alpha i)^m \mu^{-m}}{m!} \int_0^{\alpha\mu} \lambda Z^k \, d\lambda + O(\mu^{k-3}) = O(\mu^{k-3}).$$

Collecting, we have

$$\left\{ \begin{aligned} & g(\theta) = \begin{cases} O(\mu^{k-1}) & (\theta \in R), \\ c_{k1} \mu^{k-2} + a_{k1} \mu^{k-2} + a'_{k1} \tau \log \mu \cdot \mu^{k-3} + O_\delta(\mu^{k-3}) & (\theta \in R_\delta), \end{cases} \\ & \frac{1}{2\pi} \int_0^{\pi} g(\theta) \, d\theta = c_{k2} \mu^{k-1} + a_{k2} \mu^{k-2} + a'_{k2} \tau \log \mu \cdot \mu^{k-3} + O(\mu^{k-3}); \\ & \qquad c_{k1} = 2^{-k/2+1} \pi^{1/2} \int_0^{\infty} Z^k \, d\lambda, \\ & \qquad a_{k1} = -\eta^k 2^{-k+1} \pi + s \cdot i 2^{-k/2+2} \int_0^{\infty} Z^k \, d\lambda, \\ & \qquad c_{k2} = \tfrac{1}{2} c_{k1}, \qquad a_{k2} = \tfrac{1}{2} a_{k1}. \end{aligned} \right. \tag{8}$$

This disposes of $g(\theta)$ and $(1/2\pi) \int_0^{\pi} g(\theta) \, d\theta$; we now take up the more complicated $g(-\theta)$ and $(1/2\pi) \int_0^{\pi} g(-\theta) \, d\theta$.

24. The abbreviation T has now served its purpose, and we may, without confusion, now abbreviate by (what corresponds in $g(-\theta)$ to the old T)

$$(1) \qquad T = \eta E(-\lambda^2 + 2\alpha\mu^{-1}\lambda) - 2^{-1/2}\{Z + \eta E_1 Z_1\}.$$

We will now also abbreviate $E\{-p(2\alpha\mu\lambda - 2\alpha\mu^{-1}\lambda)\}$ to E_p (the new form appropriate to $-\theta$). We have at once, for $\theta \epsilon R$,

$$(2) \qquad -g(-\theta) = \mu^{k-1}\cdot 2\pi^{1/2}\int_0^\lambda T^k E(s\theta)\, d\lambda + O(\mu^{k-2}) = O(\mu^{k-1}).$$

We have also, from Theorem 1(iii), (iv) [§5],

$$\frac{1}{2\pi}\int_0^\pi g(-\theta)\, d\theta = Q_1 + Q_2 + Q_3 + O(\mu^{k-3}),$$

$$Q_1 = -2\mu^{k-2}\int_0^{\alpha\mu} (\alpha\mu - \lambda)T^k\, d\lambda,$$

$$(3) \quad Q_2 = -2k\mu^{k-3}\int_0^{\alpha\mu} (\alpha\mu - \lambda)T^{k-1}(\beta\mathcal{P}_0 + \beta\mathcal{P}E_1 + \tfrac{1}{4}(2\pi)^{1/2}Z'E(s\theta))\, d\lambda$$
$$= Q_{21} + Q_{22} + Q_{23},$$

$$Q_3 = \mu^{k-4}\int_0^{\alpha\mu} (\alpha\mu - \lambda)[T^{k-1}(\beta Z'' + \beta\mathcal{P}E(-\lambda^2 + 2\alpha\mu^{-1}\lambda) + \beta\mathcal{P}E_1 Z_1)$$
$$+ T^{k-2}(\beta\mathcal{P}_0 + \beta\mathcal{P}E_1 + \beta\mathcal{P}E_2 + \beta Z' + \beta\{Z'\}^2)]\, d\lambda.$$

We begin with Q_3. We may in the first place suppress [with error $O(\mu^{k-3})$ always understood] the terms in Z', Z'', since Z', Z'' are each $O\{(\lambda + 1)^{-2}\}$.

We may next suppress the terms of T^{k-1}, T^{k-2}, expanded by the trinomial theorem, which are of the second or higher degree in Z and Z_1, since such terms are $O\{(\lambda + 1)^{-2}\}$, $O\{(\lambda + 1)^{-1}(\lambda_1 + 1)^{-1}\}$, or $O\{(\lambda_1 + 1)^{-2}\}$, which, integrated from 0 to $\alpha\mu$, give $O(1)$. Thus, writing E^* for $E(-\lambda + 2\alpha\mu^{-1}\lambda)$, and E_r^* for $(E^*)^r$, we have

$$Q_3 = \mu^{k-4}\int_0^{\alpha\mu} (\alpha\mu - \lambda)[\beta E_{k-1}^*(\mathcal{P}E^* + a\mathcal{P}E_1 Z_1)$$
$$(4) \qquad + \beta E_{k-2}^*(aZ + aE_1 Z_1)\mathcal{P}E^* + \beta E_{k-2}^*(\mathcal{P}_0 + \mathcal{P}E_1 + \mathcal{P}E_2)$$
$$+ E_{k-3}^*(\beta Z + \beta E_1 Z_1)(\mathcal{P}_0 + \mathcal{P}E_1 + \mathcal{P}E_2)]\, d\lambda + O(\mu^{k-3}).$$

All terms in Z_1 inside the square bracket are of the form $HE = Z_1\mathcal{P}E$, with H and E satisfying the conditions of Lemma 3(ii)$_3$, and since $Z_1(0) = O(\mu^{-1})$, they contribute $O(\mu^{k-4}) = O(\mu^{k-3})$ to Q_3. There then remain in the square bracket only terms of types

$$E(-u\lambda^2 + v\mu^{-1}\lambda)\mathcal{P} \quad \text{and} \quad E(-u\lambda^2 + v\mu^{-1}\lambda)\mathcal{P}Z, \qquad \text{with } u \geqq 1.$$

For these we integrate by parts on the E's in (4), and their contribution to

Q_3 is easily seen to be $O(\mu^{k-3})$. Thus

(5) $$Q_3 = O(\mu^{k-3}).$$

25. Consider now Q_2. We have first

$$Q_{22} = k\mu^{k-3} \int_0^{\alpha\mu} (\alpha\mu - \lambda)\mathcal{P}E_1(\sum c_{pqr} E_p^* Z^q E_r Z_1^r)\, d\lambda, \qquad c_{pqr} = O(1).$$

Now $E_1 \cdot E_p^* E_r$ is an E of the type of Lemma 3, and $\mathcal{P}Z^q Z_1^r$ is an H with $H, H' = O(1)$. By Lemma 3(iv) (the weakest form)

(1) $$Q_{22} = O(\mu^{k-3}).$$

Next,

(2) $$Q_{23} = -\tfrac{1}{2}(2\pi)^{1/2} k\mu^{k-3} \int_0^{\alpha\mu} (\alpha\mu - \lambda)(\eta E^* - 2^{-1/2}Z)^{k-1} Z' E(s\theta)\, d\lambda + \mu^{k-3} \sum_{r \geqq 1} \int_0^{\alpha\mu} (\alpha\mu - \lambda) Z' c_{pqr} E_p^* Z^q E_r Z_1^r E(s\theta)\, d\lambda.$$

In the second term the $E_r E_p^* E(s\theta)$ behave as in Q_{31} (where there was a μ^{-1} to spare), and Lemma 3 gives easily $O(\mu^{k-3})$. Thus

$$Q_{23} = -\tfrac{1}{2}(2\pi)^{1/2} k\mu^{k-3} \int_0^{\alpha\mu} (\alpha\mu - \lambda) \sum c_m E_m^* E(s\theta) Z^{k-1-m} Z'\, d\lambda + O(\mu^{k-3}),$$

where $\sum$ is the binomial expansion of $(\eta E^* - 2^{-1/2}Z)^{k-1} E(s\theta)$. In this we can replace $E_m^* E(s\theta)$ by $E(-m\lambda^2)$, since in the difference in the two integrals, which is

$$\int_0^{\alpha\mu} (\alpha\mu - \lambda) E(-m\lambda^2) Z^{k-1-m} Z' [E\{-(2m + 4s)\alpha\mu^{-1}\lambda\} - 1]\, d\lambda,$$

we can expand the square bracket as

$$\sum_1 \frac{\{-(2m + 4s)\alpha i\}^r}{r!} \mu^{-r} \lambda^r,$$

and integrate by parts on the $\lambda^2 E(-m\lambda^2)$, when we easily find the contribution to Q_{23} to be $O(\mu^{k-3})$. Next, the part integrand

$$-\lambda(\eta E^* - 2^{-1/2}Z)^{k-1} Z' E(s\theta)$$

in the first term of (2) is of the form

$$-\lambda\eta^{k-1} E_{k-1}^* E(s\theta) + O(\lambda Z Z'),$$

the integral $\int_0^\infty$ of each of these is convergent (integrate by parts on $\lambda E_{k-1}^* E(s\theta)$ in the first—the second is $O\{(\lambda + 1)^{-2}\}$), and $\int_0^{\alpha\mu}$ is $\beta + O(\mu^{-1})$. Thus the said part integral in (2) contributes $O(\mu^{k-3})$ to Q_{23}. We thus arrive at

$$Q_{23} = -\tfrac{1}{2}(2\pi)^{1/2}k\alpha\mu^{k-2}\int_0^{\alpha\mu}(\eta E(-\lambda^2) - 2^{-1/2}Z)^{k-1}Z'\,d\lambda + O(\mu^{k-3}).$$

In this the integral taken to ∞ converges like

$$\int^{\infty}|Z'| = \int^{\infty}O\{(\lambda+1)^2\}\,d\lambda,$$

and finally

$$(3)\quad Q_3 + Q_{23} = -2^{-3/2}\pi k\mu^{k-2}\int_0^{\infty}(\eta E(-\lambda^2) - 2^{-1/2}Z)^{k-1}Z'\,d\lambda + O(\mu^{k-3}).$$

26. In the remaining term Q_{21} of Q_2 write

$$Y = \eta E^* - 2^{-1/2}Z, \qquad W = -\eta 2^{-1/2}E_1 Z_1,$$

so that $T = Y + W$. Then

$$(1)\quad \begin{aligned} Q_{21} = {} & -2k\mu^{k-3}\int_0^{\alpha\mu}(\alpha\mu-\lambda)Y^{k-1}\mathcal{P}_0\,d\lambda \\ & + \sum_{p\geqq 1}\int_0^{\alpha\mu}(\alpha\mu-\lambda)c_{pqr}Z_1^pE_pE_q^*Z^r\mathcal{P}_0\,d\lambda. \end{aligned}$$

The second term is $O(\mu^{k-3})$ by the argument used for Q_{22}. Expanding Y^{k-1} in the first (and absorbing the $E(p2\alpha\mu^{-1}\lambda)$ in E^* into $\mathcal{P}_0$), we have [with $c_p = O(1)$ as always]

$$(2)\quad \begin{cases} Q_{21} = -2k\mu^{k-3}\int_0^{\alpha\mu}(\alpha\mu-\lambda)\sum_{p=0}^{k-1}c_pE(-p\lambda^2)(-2^{-1/2})^{k-1-p}Z^{k-1-p}\mathcal{P}_0\,d\lambda \\ \qquad\qquad + O(\mu^{k-3}) \\ \quad = -2k\mu^{k-3}\int_0^{\alpha\mu}(\alpha\mu-\lambda)Z^{k-1}\mathcal{P}_0\,d\lambda \\ \qquad\qquad + ak\mu^{k-3}\int_0^{\alpha\mu}(\alpha\mu-\lambda)\left(\sum_{p\geqq 1}\right)d\lambda + O(\mu^{k-3}). \end{cases}$$

We substitute $\mathcal{P}_0 = \sum_1^{\infty} b_m\mu^{-m}\lambda^m$. The first term in $(2)_2$ becomes

$$(3)\qquad -2k\mu^{k-3}\sum_1 b_m\mu^{-m}\int_0^{\alpha\mu}(\alpha\mu-\lambda)\lambda^mZ^{k-1}\,d\lambda.$$

Since

$$Z^{k-1} = O\{(\lambda+1)^{-(k-1)}\} = \begin{cases} O\{(\lambda+1)^{-3}\} & (k=3), \\ O\{(\lambda+1)^{-4}\} & (k>3), \end{cases}$$

the term $m = 1$ in (3) contributes $\mu^{k-3}\tau\alpha_k'\log\mu + O(\mu^{k-3})$; and the terms

$m > 1$ are easily seen to contribute $\mu^{k-3}O(1) \sum |b_m| = O(\mu^{k-3})$. Collecting, we obtain

(4) $$Q_{21} = \alpha_k' \tau \log \mu \cdot \mu^{k-3} + O(\mu^{k-3}).$$

From (5) §24, (1), (3) §25, (2), (3) §26, and the present (4), we find

(5) $$\begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^\pi g(-\theta)\, d\theta \\ \qquad = Q_1 - 2^{-3/2}\pi k \mu^{k-2} \displaystyle\int_0^\infty (\eta E(-\lambda^2) - 2^{-1/2}Z)^{k-1} Z' \, d\lambda + R, \\ \qquad R = a_k' \tau \log \mu \cdot \mu^{k-3} + O(\mu^{k-3}). \end{cases}$$

27. We have from (3) §24,

$$\begin{aligned} Q_1 = {} & -2\mu^{k-2} \int_0^{\alpha\mu} (\alpha\mu - \lambda) Y^k E(s\theta)\, d\lambda \\ & + \mu^{k-2} \sum_{p \geqq 1} c_p \int_0^{\alpha\mu} (\alpha\mu - \lambda) Z_1^{k-p} E_{k-p} E_p^* \, E(s\theta)\, d\lambda \\ & + \mu^{k-2} \sum_{p,q \geqq 1} c_{pqr} \int_0^{\alpha\mu} (\alpha\mu - \lambda) E_p Z_1^p Z^q E_r^* E(s\theta)\, d\lambda + O(\mu^{k-3}). \end{aligned}$$

In the third term let $H = Z_1^p Z^q \mu^{-1}$, $E = E_p E_r^* E(s\theta)$. Since $ZZ_1 = O(\mu^{-1})$, H and E satisfy the conditions of Lemma 3(iii), so that the term, which becomes

$$\mu^{k-1} \sum c \int_0^{\alpha\mu} (\alpha\mu - \lambda) H \, d\lambda,$$

is of the form

$$\mu^{k-1} \sum \left(O(\mu) \frac{Z_1^p(0)\, Z^q(0)\mu^{-1}}{\mu} + O(\mu^{-2}) \right) = \mu^{k-1} \sum_{p \geqq 1} O(\mu^{-p-1}) + O(\mu^{k-3}) = O(\mu^{k-3}).$$

Thus

(1) $$\begin{cases} Q_1 = Q_{11} + Q_{12}, \\ Q_{11} = -2\mu^{k-2} \displaystyle\int_0^{\alpha\mu} (\alpha\mu - \lambda)\, (\eta E(-\lambda^2 + 2\alpha\mu^{-1}\lambda) - 2^{-1/2}Z)^k \, d\lambda, \\ Q_{12} = -2\, \mu^{k-2} \displaystyle\int_0^{\alpha\mu} (\alpha\mu - \lambda)\, (\eta E(-\lambda^2 + 2\alpha\mu^{-1}\lambda) - 2^{-1/2}Z)^k \\ \qquad \cdot \{E(4s\alpha\mu^{-1}\lambda) - 1\}\, d\lambda. \end{cases}$$

28. We write (for an expression whose powers will now occur frequently)

(1) $$J = \eta E(-\lambda^2) - 2^{-1/2}Z.$$

Writing $\varepsilon = \alpha\mu^{-1}$, $t = \lambda - \varepsilon$, and for the moment Z for $Z(t)$ etc., we have

$$Q_{11} = -2\mu^{k-2}\int_{-\varepsilon}^{\alpha\mu-\varepsilon} (\alpha\mu - t + \varepsilon)\,(\eta E(-t^2)\,(1 + i\varepsilon^2) - 2^{-1/2}Z - 2^{-1/2}Z'\varepsilon$$
$$- 2^{-3/2}Z''\varepsilon^2 + O(\varepsilon^3))^k\,dt$$
$$= -2\mu^{k-2}\int_{-\varepsilon}^{\alpha\mu-\varepsilon} (\alpha\mu - t)\,[J^k + kJ^{k-1}\{i\varepsilon^2\eta E(-t^2) - 2^{-1/2}Z'\varepsilon - 2^{-3/2}Z''\varepsilon^2$$
$$+ \tfrac{1}{2}k(k-1)\,J^{k-2}\,\tfrac{1}{2}(Z')^2\varepsilon^2]\,dt$$
$$-2\mu^{k-2}\varepsilon\int_{-\varepsilon}^{\alpha\mu-\varepsilon} (J^k - kJ^{k-1}2^{-1/2}Z'\varepsilon)\,dt + O(\mu^{k-3}).$$

It is easily seen that the terms of the first integrand in ε^2 contribute $O(\mu^{k-3})$ to Q_{11}, and that the second integral also does this, so that

$$Q_{11} = -2\mu^{k-2}\int_{-\varepsilon}^{\alpha\mu-\varepsilon} (\alpha\mu - t)\,(J^k - 2^{-1/2}\varepsilon Z' kJ^{k-1})\,dt + O(\mu^{k-3}).$$

Writing momentarily $\chi(t)$ for the integrand and replacing, as we may,

$$\int_{-\varepsilon}^{\alpha\mu-\varepsilon} \chi(t)\,dt \quad \text{by} \quad \int_0^{\alpha\mu} \chi(t)\,dt + \varepsilon\{\chi(0) - \chi(\alpha\mu)\}$$
$$- \tfrac{1}{2}\varepsilon^2\{\chi'(0) - \chi'(\alpha\mu)\} + O(\varepsilon^3),$$

we find, on reduction (noting that $J(0) = 0$) and a return to λ as variable of integration,

$$Q_{11} = -2\mu^{k-2}\int_0^{\alpha\mu} (\alpha\mu - \lambda)\,(J^k - 2^{-1/2}\varepsilon Z'kJ^{k-1})\,d\lambda + O(\mu^{k-3})$$
$$= -\pi^{1/2}\mu^{k-1}\left(\int_0^\infty J^k\,d\lambda - \int_{\alpha\mu}^\infty J^k d\lambda\right) + 2\mu^{k-2}\int_0^{\alpha\mu} \lambda\,J^k\,d\lambda$$
$$+ k2^{1/2}\alpha^2\mu^{k-2}\left(\int_0^\infty Z'\,J^{k-1}\,d\lambda - \int_{\alpha\mu}^\infty\right)$$
$$- k2^{1/2}\alpha^2\mu^{k-3}\int_0^{\alpha\mu} \lambda Z'\,J^{k-1}d\lambda + O(\mu^{k-3}).$$

In this we have $\int_{\alpha\mu}^\infty Z'J^{k-1}\,d\lambda = O(\mu^{-1})$, and

$$\int_0^{\alpha\mu} \lambda Z'\,J^{k-1}\,d\lambda = \beta\int_0^{\alpha\mu} \lambda Z'\,E\{-(k-1)\lambda^2\}\,d\lambda + \int_0^{\alpha\mu} O(\lambda ZZ')\,d\lambda,$$

of which the second is $\int_0^{\alpha\mu} O\{(\lambda + 1)^{-2}\}\,d\lambda = O(1)$, and the first is easily shown to be $O(1)$ by integrating by parts on λE. Hence

$$(2)\quad \begin{cases} Q_{11} - \left(-\pi^{1/2}\mu^{k-1}\displaystyle\int_0^\infty J^k\,d\lambda + 2^{-3/2}\pi k\mu^{k-2}\int_0^\infty J^{k-1}Z'\,d\lambda\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = S_1 + S_2 + O(\mu^{k-3}), \\ S_1 = \pi^{1/2}\mu^{k-1}\displaystyle\int_{\alpha\mu}^\infty J^k\,d\lambda, \qquad S_2 = 2\mu^{k-2}\int_0^{\alpha\mu} \lambda J^k\,d\lambda. \end{cases}$$

$$(3)\quad J^k = \eta^k E(-k\lambda^2) - 2^{-1/2}\eta^{k-1}kE\{-(k-1)\lambda^2\}Z + \beta E\{-(k-2)\lambda^2\}Z^2 + O\{(\lambda+1)^{-3}\}.$$

Now $\int_{\alpha\mu}^{\infty} E\{-(k-2)\lambda^2 Z^2\, d\lambda = O(\mu^{-2})$, by partial summation on the E. Hence, from (3)

$$(4)\quad \begin{aligned} S_1 &= \pi^{1/2}\mu^{k-1}\left(\eta^k\left[\frac{E(-k\lambda^2)}{-2k\lambda i}\right]_{\alpha\mu}^{\infty} + \beta\int_{\alpha\mu}^{\infty}\frac{E(-k\lambda^2)}{\lambda^2}\,d\lambda\right) \\ &= -i\eta^k E(-\tfrac{1}{2}k\pi)k^{-1}\mu^{k-2} + O(\mu^{k-3}) = -ik^{-1}\mu^{k-2} + O(\mu^{k-3}). \end{aligned}$$

For S_2 we have

$$(5)\quad \begin{aligned} S_2 = 2\eta^k\mu^{k-2}\int_0^{\alpha\mu}\lambda E(-k\lambda^2)\,d\lambda &+ 2\mu^{k-2}\int_0^{\infty}\lambda(J^k - \eta^k E(-k\lambda^2))\,d\lambda \\ &- 2\mu^{k-2}\int_{\alpha\mu}^{\infty}\lambda\{J^k - \eta^k E(-k\lambda^2)\}\,d\lambda. \end{aligned}$$

In the last term the integrand is

$$\beta\lambda ZE\{-(k-1)\lambda^2\} + \beta Z^2E\{-(k-2)\lambda^2\} + O\{(\lambda+1)^{-3}\};$$

the integrals (from $\alpha\mu$ to ∞) of the first two are $O(\mu^{-1})$ by integration by parts on their λE's, and that of the last is $O(\mu^{-2})$.

The first term in S_2 is

$$2\eta^k\mu^{k-2}\left[\frac{E(-k\lambda^2)}{-2ki}\right]_0^{\alpha\mu} = -i\eta^k k^{-1}\mu^{k-2}.$$

From this and (2), (4), (5), we have

$$(6)\quad \begin{aligned} Q_{11} = c_{k3}\mu^{k-1} + \mu^{k-2}\Big(2^{-3/2}\pi k\int_0^{\infty} J^{k-1}Z'\,d\lambda - ik^{-1} - i\eta^k k^{-1} \\ + 2\int_0^{\infty}\lambda\{J^k - \eta^k E(-k\lambda^2)\}\,d\lambda\Big) + O(\mu^{k-3}). \end{aligned}$$

29. From (1) of §27 we have, writing $\varepsilon' = 2s\alpha\mu^{-1}$ (and $\varepsilon = \alpha\mu^{-1}$ as before)

$$(1)\quad \left\{ \begin{aligned} Q_{12} &= U_1 + U_2 + U_3, \\ U_1 &= -2\mu^{k-2}\int_0^{\alpha\mu}(\alpha\mu - \lambda)\eta^k E(-k\lambda^2)\,(E\{(2k\varepsilon + 2\varepsilon')\lambda\} - E\{2k\varepsilon\lambda\})\,d\lambda, \\ U_2 &= -2^{1/2}k\mu^{k-2}\int_0^{\alpha\mu}(\alpha\mu - \lambda)\eta^{k-1}E\{-(k-1)\lambda^2\}Z \\ &\qquad\cdot(E[\{2(k-1)\varepsilon + 2\varepsilon'\}\lambda] - E\{2(k-1)\varepsilon\lambda\})\,d\lambda, \\ U_3 &= \mu^{k-2}\sum_{p=2}^{k} c_p\int_0^{\alpha\mu}(\alpha\mu - \lambda)E\{-(k-p)\lambda^2\}Z^p \\ &\qquad\cdot(E[\{2(k-p)\varepsilon + 2\varepsilon'\}\lambda] - E\,[2(k-p)\varepsilon\lambda])\,d\lambda. \end{aligned} \right.$$

In the general term of U_3 write $2(k-p)\varepsilon + 2\varepsilon' = h_1\mu^{-1}$, $2(k-p)\varepsilon = h_2\mu^{-1}$, and $E = E\{-(k-p)\lambda^2\}$. Then the term is

$$c_p\,\mu^{k-2}\int_0^{\alpha\mu}(\alpha\mu-\lambda)\,EZ^p\left(\sum_{m=1}^{\infty}\frac{(h_1 i)^m-(h_2 i)^m}{m!}\,\mu^{-m}\lambda^m\right)d\lambda.$$

We split this into the two parts corresponding to $\alpha\mu$ and $-\lambda$, and except when $p = k$, integrate by parts on $\alpha\mu E\lambda^m$ and $-\lambda E\lambda^m$ respectively; it is easily seen (since $Z^p = O\{(\lambda+1)^{-2}\}$) that the result is $O(\mu^{k-3})$. The term $p = k$ is

$$\mu^{k-2}\int_0^{\alpha\mu}(\alpha\mu-\lambda)O\{(\lambda+1)^{-k}\}\left(\sum_{m=1}^{\infty}\frac{(h_1 i)^m-(h_2 i)^m}{m!}\,\mu^{-m}\lambda^m\right)d\lambda,$$

and since $k \geqq 3$ this is easily found to be $O(\mu^{k-3})$. Hence

(2) $$U_3 = O(\mu^{k-3}).$$

30. Consider now U_2. We have

$$U_2 = -2^{1/2}\eta^{k-1}k\mu^{k-2}\sum_{m=1}^{\infty}\int_0^{\alpha\mu}(\alpha\mu-\lambda)E\{-(k-1)\lambda^2\}\,Zu_m\,\mu^{-m}\lambda^m d\lambda,$$

$$u_m = \frac{(h_1 i)^m-(h_2 i)^m}{m!}$$

For $m \geqq 2$ the m^{th} term of $\sum$ is

$$\alpha\mu u_m\mu^{-m}\left(\left[\frac{E\lambda^{m-1}}{-2(k-1)i}Z\right]_0^{\alpha\mu}+\beta\int_0^{\alpha\mu}E\lambda^{m-1}Z'\,d\lambda + (m-1)\beta\int_0^{\alpha\mu}E\lambda^{m-2}Z\,d\lambda\right)$$

$$-\,u_m\,\mu^{-m}\left(\left[\frac{E\lambda^m}{-2(k-1)i}\right]_0^{\alpha\mu}+\beta\int_0^{\alpha\mu}E\lambda^m Z'\,d\lambda+m\beta\int_0^{\alpha\mu}E\lambda^{m-1}Z'\,d\lambda\right)$$

The integrated terms vanish at $\lambda = 0$ and $Z(\lambda)$ is $O(\mu^{-1})$ at $\lambda = \alpha\mu$; also $Z' = O\{(\lambda+1)^{-2}\}$; and it easily follows that the total is $O(u_m\,\mu^{-1})$, so that $\sum_{m=2}^{\infty}$ contributes $\mu^{k-3}\sum_2^{\infty}O(u_m) = O(\mu^{k-3})$ to U_2. Thus, abbreviating $E\{-(k-1)\lambda^2\}$ to E

$$\frac{U_2}{-2^{1/2}\eta^{k-1}k\mu^{k-2}} = (h_1-h_2)i\left(\alpha\int_0^{\alpha\mu}ZE\lambda\,d\lambda-\mu^{-1}\int_0^{\alpha\mu}ZE\lambda^2\,d\lambda\right)+O(\mu^{k-3})$$

$$= 4s\alpha i\left(\alpha\left[\frac{E}{-2(k-1)i}Z\right]_0^{\alpha\mu}+\frac{\alpha}{2(k-1)i}\int_0^{\alpha\mu}EZ'\,d\lambda\right)$$

$$-\,4s\alpha i\mu^{-1}\left(\left[\frac{E\lambda}{-2(k-1)i}Z\right]_0^{\alpha\mu}+\frac{1}{2(k-1)i}\int_0^{\alpha\mu}Z'E\lambda\,d\lambda+\frac{1}{2(k-1)i}\int_0^{\alpha\mu}ZE\,d\lambda\right)$$

$$= 4si\left(\alpha\frac{Z(0)}{2(k-1)i}+O(\mu^{-1})+\frac{\alpha}{2(k-1)i}\int_0^{\infty}EZ'\,d\lambda+O(\mu^{-1})\right)$$

$$-\,4s\alpha i\mu^{-1}\{O(1)+O(1)+O(1)\},$$

(1) $$U_2 = \mu^{k-2}\left[-s2^{-1/2}\eta^{k-1}\pi\frac{k}{k-1}\left(\eta 2^{-1/2}+\int_0^{\infty}E\{-(k-1)\lambda^2\}Z'\,d\lambda\right)\right]+O(\mu^{k-3}).$$

31. It remains to estimate U_1. We have first

$$U_1 = U_{11} + U_{12},$$

(1)
$$U_{11} = -\eta^k \pi^{1/2} \mu^{k-1} \int_0^{\alpha\mu} [E\{-k\lambda^2 + (2k\varepsilon + 2\varepsilon')\lambda\} - E\{-k\lambda^2 + 2k\varepsilon\lambda\}]\, d\lambda,$$

$$U_{12} = 2\eta^k \mu^{k-2} \int_0^{\alpha\mu} [E\{-k\lambda^2 + (2k\varepsilon + 2\varepsilon')\lambda\} - E\{-k\lambda^2 + 2k\varepsilon\lambda\}]\lambda\, d\lambda.$$

Now the square bracket in U_{11} and U_{12} is

$$E\{-k(\lambda - \varepsilon_1)^2\}\{1 + O(\mu^{-2})\} - E\{-k(\lambda - \varepsilon)^2\}\{1 + O(\mu^{-2})\},$$

where
$$\varepsilon_1 = \varepsilon + \varepsilon' k^{-1}.$$

Hence

$$\frac{U_{11}}{-\eta^k \pi^{1/2} \mu^{k-1}} = \left(\int_{-\varepsilon_1}^{\alpha\mu - \varepsilon_1} - \int_{-\varepsilon}^{\alpha\mu - \varepsilon} \right) E(-k\lambda^2)\, d\lambda + O(\mu^{-2})$$

$$= \left(\int_{-\varepsilon_1}^{-\varepsilon} - \int_{\alpha\mu - \varepsilon}^{\alpha\mu - \varepsilon_1} \right) E(-k\lambda^2)\, d\lambda + O(\mu^{-2})$$

$$= \int_{-\varepsilon_1}^{-\varepsilon} \{1 + O(\lambda^2)\}\, d\lambda - \int_{-\varepsilon}^{-\varepsilon_1} E\{-k(\alpha\mu + t)^2\}\, dt + O(\mu^{-2})$$

$$= (\varepsilon_1 - \varepsilon) + O(\mu^{-2}) - \int_{-\varepsilon}^{-\varepsilon_1} E(-k\alpha^2\mu^2)(1 - 2k\alpha\mu t i)\, dt + O(\mu^{-2}),$$

which reduces to

$$\mu^{-1} s \pi^{1/2} k^{-1} \{1 + E(-\tfrac{1}{4}\pi k n) + \pi i + \pi i s\} + O(\mu^{-2}).$$

Since $E(-\frac{1}{4}\pi k n) = E(-\frac{1}{4}\pi k) = \eta^{-k}$, we have

(2)
$$U_{11} = -\mu^{k-2} \cdot s\pi k^{-1} \{1 + \eta^k + \tfrac{1}{2}\pi i (s + k)\} + O(\mu^{k-3}).$$

U_{12} is similar. We have

$$\frac{U_{12}}{2\eta^k \mu^{k-2}} = \int_{-\varepsilon_1}^{-\varepsilon} \{1 + O(\lambda^2)\}\lambda\, d\lambda - E(-\tfrac{1}{4}\pi k n) \int_{-\varepsilon}^{-\varepsilon_1} (1 - 2k\alpha\mu t i)(\alpha\mu + t)\, dt + O(\mu^{-1}),$$

which reduces to

$$U_{12} = s\mu^{k-2} \cdot \pi k^{-1} \{1 + \tfrac{1}{2}\pi i (s + k)\} + O(\mu^{k-3}).$$

With (1) and (2) this gives

$$U_1 = -s\mu^{k-2} \cdot \pi k^{-1} + O(\mu^{k-3}).$$

32. From this, (1) §30, (1) and (2) §29, we have

$$(1)\quad Q_{12} = -s\mu^{k-2}\left[2^{-1/2}\eta^{k-1}\pi\frac{k}{k-1}\left(\eta 2^{-1/2} + \int_0^\infty E\{-(k-1)\lambda^2\}Z'\,d\lambda\right) + \pi k^{-1}\right] + O(\mu^{k-3}).$$

We need to evaluate the integral in this. Since $Z' = -\gamma + 2\lambda i Z$, we have

$$\begin{aligned} I &= \int_0^\infty E\{-(k-1)\lambda^2\}Z'\,d\lambda \\ &= -\gamma\int_0^\infty E\{-(k-1)\lambda^2\}\,d\lambda + \int_0^\infty E\{-(k-1)\lambda^2\}2\lambda i\,Z\,d\lambda \\ &= -\eta^{-1}2^{-1/2}(k-1)^{-1/2} + \left[\frac{EZ}{-(k-1)}\right]_0^\infty + \frac{1}{k-1}\int_0^\infty EZ'\,d\lambda, \end{aligned}$$

$$\frac{k-2}{k-1}I = -\eta^{-1}2^{-1/2}(k-1)^{-1/2} + \frac{2^{-1/2}\eta}{k-1}, \qquad I = \frac{2^{-1/2}\eta}{k-2}\{i(k-1)^{1/2} + 1\}.$$

Substituting from this in (1), and collecting from the result, (6) §28, and (5) §26, we obtain a result of the form

$$(2)\quad \frac{1}{2\pi}\int_0^\pi g(-\theta)\,d\theta = c_{k3}\,\mu^{k-1} + a_{k3}\,\mu^{k-2} + a_k'\,\tau\log\mu\cdot\mu^{k-3} + O(\mu^{k-3})$$

(with explicit c_{k3} and a_{k3}).

We recall now from (8) §23,

$$(3)\quad g(\theta) = c_{k1}\,\mu^{k-1} + a_{k1}\,\mu^{k-2} + a_k'\,\tau\log(\lambda+1)\cdot\mu^{k-3} + O(\mu^{k-3}),$$

$$(4)\quad \frac{1}{2\pi}\int_0^\pi g(\theta)\,d\theta = c_{k2}\,\mu^{k-1} + a_{k2}\,\mu^{k-2} + a_k'\,\tau\log\mu\cdot\mu^{k-3} + O(\mu^{k-3}),$$

$$c_{k2} = \tfrac{1}{2}c_{k1}.$$

From (2), (3), (4)

$$\begin{aligned} (5)\quad h(\theta) &= g(\theta) - \frac{1}{2\pi}\int_0^\pi g(\theta)\,d\theta - \frac{1}{2\pi}\int_0^\pi g(-\theta)\,d\theta \\ &= c_k\,\mu^{k-1} + a_k\,\mu^{k-2} + a_k'\,\tau\log(\lambda+1)\cdot\mu^{k-3} + a_k'\,\tau\log\mu\cdot\mu^{k-3} + O(\mu^{k-3}) \end{aligned}$$

with $c_k = c_{k2} - c_{k3}$, $a_k = a_{k1} - a_{k2} - a_{k3}$, and on reduction a_k has the form stated in Theorem 4.

For further reference we recall the values

$$(6)\quad \begin{cases} c_{k2} = \tfrac{1}{2}c_{k1} = \pi^{1/2}2^{-1-k/2}\displaystyle\int_0^\infty Z^k\,d\lambda, \\ c_{k3} = -\pi^{1/2}\displaystyle\int_0^\infty J^k\,d\lambda. \end{cases}$$

33. *We next prove that* $c_k = 0$, *or* $c_{k2} = c_{k3}$. This would follow at once from $M_2(h) = O(\mu^{k-3/2})$, a result stated for the equivalent f in Theorem 4. But we are unable to prove this result except indirectly *via* $c_k = 0$.

Let $g(t) = \sum_0^{k(n-1)} a_m e^{mti}$, so that $h = \sum_1 a_m e^{m\theta i}$. Then

$$\sum_1 m a_m e^{mti} = -ie^{-it}\psi(t), \qquad \psi(t) = \{\phi(t) - \tfrac{1}{2}\}^k - (\tfrac{1}{2})^k,$$

(1) $$\sum m^2 \mid a_m \mid^2 < A \int_{-\pi}^{\pi} \mid \phi(t) \mid^{2k} dt = O(\mu^{2k}),$$

since $\mid \phi \mid < A\mu$. Further, ψ is majorized by $(1 - e^{\theta i})^{-k}$, so that

(2) $$a_m = O(m^{k-1}).$$

It follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mid h \mid^2 dt = \sum_1^{k(n-1)} \mid a_m \mid^2 \leqq O(1) \sum^{\omega\mu} m^{2k-2} + (\omega\mu)^{-2} \sum_{\omega\mu} m^2 \mid a_m \mid^2$$
$$\leqq O\{(\omega\mu)^{2k-3}\} + O\{(\omega\mu)^{-2}\mu^{2k}\}.$$

If we choose $\omega = \mu^{1/(2k-1)}$, this gives

(3) $$M_2^2(h) = O(\mu^{2k-2-2/(2k-1)}) = o(\mu^{2k-2}).$$

Now if $c_k \neq 0$, we have $\mid h \mid > A(k)\mu^{k-1}$ in $(\frac{1}{4}\pi, \frac{3}{4}\pi)$, which is incompatible with (3).

34. We recall [from (3) of §23 and (2) §24] that, for $\theta \in R$,

(1) $$g(\theta) = \mu^{k-1} \cdot 2^{1-k/2}\pi^{1/2} \int_0^{\lambda} Z^k \, d\lambda + O(\mu^{-2}),$$

(2) $$g(-\theta) = -\mu^{k-1} 2\pi^{1/2} \int_0^{\lambda} T^k E(s\theta) \, d\lambda + O(\mu^{k-2});$$

and by arguments parallel to those for $(1/2\pi) \int_0^{\pi} g(-\theta) \, d\theta$ [but much simpler, since (i) the latter has an extra integral sign, (ii) we need only error $O(\mu^{k-2})$ instead of the much more exacting $O(\mu^{k-3})$] we can replace (2) by

(2)′ $$g(-\theta) = -\mu^{k-1} \cdot 2\pi^{1/2} \int_0^{\lambda} J^k \, d\lambda + O(\mu^{k-2}).$$

Now

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \, dt = (c_{k1} + c_{k2})\mu^{k-1} + O(\mu^{k-2}),$$

where

(3) $$c_{k1} = -\pi^{1/2} \int_0^{\infty} J^k \, d\lambda, \qquad c_{k2} = 2^{-k/2}\pi^{1/2} \int_0^{\infty} Z^k \, d\lambda.$$

From (1) and (3), and (2)′ and (3) respectively we get, for $\theta \epsilon R$,

$$(4) \qquad \begin{cases} h(\theta) = \mu^{k-1} A_k \int_\lambda^\infty Z^k \, d\lambda + O(\mu^{k-2}), \\ h(-\theta) = \mu^{k-1} A_k' \int_\lambda^\infty J^k \, d\lambda + O(\mu^{k-2}). \end{cases}$$

Since $\int_\lambda^\infty Z^k \, d\lambda = O\{(\lambda + 1)^{-(k-1)}\}$, and

$$\int_\lambda^\infty J^k \, d\lambda = O(1) \int_\lambda^\infty E(-k\lambda^2) \, d\lambda + O(1) \int_\lambda^\infty E\{-(k-1)\lambda^2\} Z \, d\lambda + O(1) \int_\lambda^\infty |Z|^2 \, d\lambda$$

$$= O\{(\lambda + 1)^{-1}\},$$

it follows from (4) that

$$\int_{-\pi}^{\pi} |h|^2 \, dt < \mu^{2k-3} A_k \int_0^\infty (\lambda + 1)^{-2} \, d\lambda = O(\mu^{2k-3}),$$

as desired.

Since (as we observed above) $\phi(\theta)$ is $O_\delta(1)$ in R_δ, we must similarly have $h(\theta)$ constant to error $O_\delta(1)$ in R_δ.

It is easily seen that the coefficient $\beta_1(k)$ of s is not 0; the proof of Theorem 4 is accordingly completed.

35. The relation $c_k = 0$, i.e.,

$$(1) \qquad \int_0^\infty (\{\eta E(-\lambda^2) - 2^{-1/2} Z\}^k + \{2^{-1/2} Z\}^k) \, d\lambda = 0,$$

establishes an identity connecting the various integrals $\int_0^\infty E(-p\lambda^2) Z^{k-p} \, d\lambda$ occurring in the expansion of the left side of (1); in particular, when k is even, this is of the form

$$\int_0^\infty Z^k \, d\lambda = \sum_{p=1}^{k-1} c_p \int_0^\infty E(-p\lambda^2) Z^{k-p} \, d\lambda + \tfrac{1}{2} \eta^{k-1} \pi^{1/2} k^{-1/2},$$

expressing the left side in terms of integrals with lower powers of Z (but of course with factors E).

The case $k = 2$ does not provide an example of the desired kind: it does, however, provide us with a curiosity. We shall find that the three definite integrals

$$\int_0^\infty U^2 \, d\lambda, \qquad \int_0^\infty V^2 \, d\lambda, \qquad \int_0^\infty UV \, d\lambda,$$

can be evaluated in finite terms.

By arguments similar to those used in the proof of Theorem 4, but naturally a good deal simpler, we arrive at the following result. $\theta = \pi$ is not singular for

$$g_2(t) = \int_0^t \phi^2(t)e^{ti}\,dt,$$

and we have for θ of R (not merely R_δ)

$$(2)\quad \begin{cases} g(\theta) = \pi^{1/2}\mu \int_0^\lambda Z^2\,d\lambda, \\ g(-\theta) = -2\pi^{1/2}\mu \int_0^\lambda \{iE(-2\lambda^2) - 2^{1/2}\eta ZE(-\lambda^2) + \tfrac{1}{2}Z^2\}\,d\lambda \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + O(\log n). \end{cases}$$

Since $g(\theta)$ is a t.p., we have $g(\pi) = g(-\pi)$, and since the two $\int_0^\lambda$ concerned in (2) are $\int_0^\infty + O(\mu^{-1})$ when $\lambda = \lambda(\pi) = \alpha\mu$, (2) yields the identity[17]

$$\int_0^\infty Z^2\,d\lambda = -2\int_0^\infty \{iE(-2\lambda^2) - 2^{1/2}\eta ZE(-\lambda^2) + \tfrac{1}{2}Z^2\}\,d\lambda,$$

which reduces, on substituting for $\int_0^\infty E(-2\lambda^2)\,d\lambda$, to

$$(3)\quad \int_0^\infty (V + iU)^2\,d\lambda = -(1+i)\tfrac{1}{4}\pi^{1/2} + (1+i)\int_0^\infty ZE(-\lambda^2)\,d\lambda.$$

It is not difficult to evaluate the integral on the right. We have

$$(4)\quad \begin{aligned} \int_0^\infty ZE(-\lambda^2)\,d\lambda &= \gamma \int_0^\infty \left(E(-2\lambda^2)\int_\lambda^\infty E(x^2)\,dx\right)d\lambda \\ &= \gamma\left(\int_0^\infty E(-2\lambda^2)\,d\lambda\right)\left(\int_0^\infty E(x^2)\,dx\right) \\ &\quad - \gamma\int_0^\infty \left(E(-2\lambda^2)\int_0^\lambda E(x^2)\,dx\right)d\lambda. \end{aligned}$$

The integral in the second term on the right is

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_0^\infty \left(E\{(-2+i\varepsilon)\lambda^2\}\int_0^\lambda E(x^2)\,dx\right)d\lambda \\ &= \lim \frac{1}{2}\int_0^\infty \left[E\{(-2+i\varepsilon)t\}t^{-1/2}\left\{\frac{1}{2}\int_0^t E(y)y^{-1/2}\,dy\right\}\right]dt \\ &= \lim \frac{1}{4}\int_0^\infty \left[E\{(-2+i\varepsilon)t\}t^{-1/2}\int_0^t \left(\sum_0^\infty \frac{(yi)^m}{m!}y^{-1/2}\,dy\right)\right]dt \\ &= \lim \frac{1}{4}\int_0^\infty \left[E\{(-2+i\varepsilon)t\}\left(\sum \frac{i^m t^m}{m!(m+\frac{1}{2})}\right)\right]dt \\ &= \lim \frac{1}{4}\sum \frac{i^m}{m!(m+\frac{1}{2})}\,m!(2i+\varepsilon)^{-m-1} = -\tfrac{1}{4}i\sum_0 \frac{1}{2m+1}\,2^{1/2}(2^{-1/2})^{2m+1} \\ &= -\tfrac{1}{8}i\sqrt{2}\log(\sqrt{2}+1). \end{aligned}$$

[17] This is, as might be expected, the identity (1) with $k = 2$.

From this, (3), and (4), we find

$$\int_0^\infty \{(V^2 - U^2) + 2iUV\}\, d\lambda = -\{\tfrac{1}{2}\sqrt{\pi} + \tfrac{1}{4}\pi^{-1/2} \log(\sqrt{2}+1)\} + i\tfrac{1}{4}\pi^{-1/2}\log(\sqrt{2}+1), \tag{5}$$

which gives

$$\int_0^\infty (U^2 - V^2)\, d\lambda \quad \text{and} \quad \int_0^\infty UV\, d\lambda.$$

(It is possible to give a direct proof of (5).) We need a third identity, and we proceed to give a direct proof that

$$\int_0^\infty (U^2 + V^2)\, d\lambda = \tfrac{1}{4}(2\pi)^{1/2}.$$

We have

$$\frac{d}{d\lambda}(U^2 + V^2) = -2\gamma V, \qquad U^2 + V^2 = 2\gamma \int_\lambda^\infty V\, d\lambda;$$

$$\int_0^\infty (U^2 + V^2)\, d\lambda = 2\gamma \int_0^\infty d\lambda \int_\lambda^\infty V(x)\, dx = 2\gamma \int_0^\infty xV(x)\, dx$$
$$= \lim_{X\to\infty} 2\gamma \int_0^X xV(x)\, dx = 2 \lim \int_0^X x \int_x^\infty \cos(x^2 - t^2)\, dt\, dx.$$

Now

$$\int_0^X x \int_x^\infty \cos(x^2 - t^2)\, dt\, dx = \int_0^X x\left(\int_x^X + \int_X^\infty\right)\cos(x^2 - t^2)\, dt\, dx$$
$$= \int_0^X dt \int_0^t x\cos(x^2 - t^2)\, dx + \int_0^X x\left(\frac{\sin(x^2 - X^2)}{2X} + O\left\{\frac{1}{X^3}\right\}\right) dx$$
$$= \int_0^X \tfrac{1}{2}\sin t^2\, dt + O\left(\frac{1}{X}\right) + O\left(\frac{1}{X}\right).$$

So

$$\int_0^\infty (U^2 + V^2)\, d\lambda = \int_0^\infty \sin t^2\, dt = \tfrac{1}{4}(2\pi)^{1/2}.$$

When $k = 3$ it is possible to evaluate a_k in finite terms.

I observe finally that there is one real identity available for any even index $k = 2\kappa$, namely

$$\Re\left(E(-\tfrac{1}{2}\kappa\pi - \tfrac{1}{4}\pi)\int_0^\infty \{(V^2 - U^2) + 2iUV\}^\kappa\, d\lambda\right) = 0.$$

For $k = 2$ we have in particular

$$\int_0^\infty (U^4 + V^4 - 6U^2V^2)\, d\lambda = -4\int_0^\infty (U^2 - V^2)UV\, d\lambda$$

This arises from the transformation

$$\int_0^\infty Z^4 \, d\lambda = \tfrac{3}{4}\pi^{5/2} E(\tfrac{5}{4}\pi) \cdot \int_0^1 \int_0^1 \int_0^1 \frac{u^2 v \, du \, dv \, dw}{(1 + u + uv + uvw)(1 + u^2 + u^2v^2 + u^2v^2w^2)^{3/2}},$$

which has an analogous form for general $k = 2\kappa$.

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TRINITY COLLEGE
CAMBRIDGE, ENGLAND