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TENSOR PRODUCTS OF MEASURABLE OPERATORS

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ABSTRACT. We introduce and study a stability property for submodules of measurable operators and Calkin spaces and characterize the tensor stable singly generated Calkin spaces. Given semifinite von Neumann algebras $(\mathcal{M},\tau)$, $(\mathcal{N},\sigma)$ and corresponding measurable operators S, T, we provide a necessary and sufficient condition for the operator $S \otimes T$ to be measurable with respect to $(\mathcal{M} \otimes \mathcal{N}, \tau \otimes \sigma)$.

1. Introduction

Weiss considered in [14] a property for ideals of $\mathbf{B}(\mathcal{H})$, called "tensor product closure property", or "tensor stability". In a previous paper [1], we studied the analogous property for Calkin sequence spaces. In particular, we established a necessary and sufficient condition for the tensor stability of a singly generated Calkin sequence space.

In this paper, we study the analogous stability property for submodules of measurable operators and Calkin function spaces. Using results of O'Neil, we describe a large class of stable submodules of measurable operators. We then focus on singly generated Calkin function spaces. We give a necessary and sufficient condition for the tensor stability of a singly generated Calkin function space and provide examples of stable singly generated Calkin function spaces.

Let $(\mathcal{M},\tau)$, $(\mathcal{N},\sigma)$ be two semifinite von Neumann algebras and S, T measurable operators with respect to $(\mathcal{M},\tau)$, $(\mathcal{N},\sigma)$. In the first part of the paper, we give a necessary and sufficient condition for $S \otimes T$ to be a $\tau \otimes \sigma$-measurable operator. This characterization is used in an essential way in the study of stable submodules.

We now introduce some notation. If $\mathcal{H}$ is a Hilbert space, we denote by $\mathbf{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. If X is a set and $A \subseteq$

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X, we denote by χ_A the characteristic function of A. By m we denote the Lebesgue measure. If $\mathcal{M}$ is a von Neumann algebra, and $P \in \mathcal{M}$ we say that P is a projection if P is a selfadjoint idempotent.

2. The algebra $\overline{\mathcal{M}}$

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let $\widetilde{\mathcal{M}}$ be the set of all operators T in $\mathcal{H}$ which are densely defined, closed and affiliated to $\mathcal{M}$.

Assume that $\mathcal{M}$ admits a faithful semi-finite normal trace τ. Let $\overline{\mathcal{M}}$ be the subset of $\widetilde{\mathcal{M}}$ consisting of all $T \in \widetilde{\mathcal{M}}$ such that $\lim_{s\to+\infty} \tau(E^{|T|}(s,+\infty)) = 0$, where $|T| = (T^*T)^{1/2}$ and $E^{|T|}(s,+\infty)$ is the spectral projection of $|T|$ corresponding to the interval $(s,+\infty)$. Then $\overline{\mathcal{M}}$ is a $*$-algebra with respect to the operations $(S,T) \mapsto \overline{S+T}$, $(S,T) \mapsto \overline{ST}$, $T \mapsto T^*$, where $\overline{T}$ denotes the closure of an operator T [5, Paragraph 1.4]. The operators in $\overline{\mathcal{M}}$ are called *τ-measurable*. In the sequel, we shall write $S+T$ instead of $\overline{S+T}$ and ST instead of $\overline{ST}$.

On $\overline{\mathcal{M}}$, we consider the *measure topology*, introduced in [10], which is the translation invariant topology defined by the neighborhoods of 0 of the form

$$U_{\varepsilon,\delta} = \big\{T : \text{there exits a projection } P \in \mathcal{M} \\ \text{such that } \|TP\| < \varepsilon \text{ and } \tau\big(P^{\perp}\big) < \delta\big\},$$

where $\varepsilon, \delta > 0$ and, for a projection P, we have set $P^{\perp} = I - P$. Then $\overline{\mathcal{M}}$ becomes a complete topological $*$-algebra and $\mathcal{M}$ is a dense $*$-subalgebra of $\overline{\mathcal{M}}$ [10].

Let $T \in \overline{\mathcal{M}}$ and $t > 0$. We set

$$\mu_t(T) = \inf\big\{\|TP\| : P \text{ is a projection in } \mathcal{M} \text{ and } \tau(1-P) \leq t\big\}.$$

We will denote the function $t \to \mu_t(T)$ by $\mu(T)$.

We collect some properties of the function $\mu(T)$ in the following proposition [5, Proposition 2.2, Lemmata 2.5, 2.6].

Proposition 2.1. *Let T, S, R be τ-measurable operators. Then the following properties are satisfied*:

(i) *The map $t \to \mu_t(T)$ from $(0,+\infty)$ to $[0,+\infty]$ is non-increasing and right-continuous. Moreover,*

$$\lim_{t\to 0} \mu_t(T) = \|T\| \in [0,+\infty].$$

(ii) $\mu_t(T) = \mu_t(|T|) = \mu_t(T^*)$, *for* $t > 0$.

(iii) $\mu_t(T) = \inf\{s \geq 0 : \tau(E^{|T|}(s,+\infty)) \leq t\}$.

(iv) $\mu_{t+s}(T+S) \leq \mu_t(T) + \mu_s(S)$, *for* $t > 0$, $s > 0$.

(v) $\mu_t(RTS) \leq \|R\|\|S\|\mu_t(T)$, $t > 0$.

(vi) *For every $t > 0$ and for every projection $P \in \mathcal{M}$ with $\tau(P) \leq t$ we have that $\mu_t(TP) = 0$.*

(vii) *For every* $t > 0$ *and for every projection* $P \in \mathcal{M}$ *with* $\tau(P) > t$ *we have that* $\mu_t(P) = 1$.

Let T in $\overline{\mathcal{M}}$. We will say that T is a *τ-finite-rank operator* if there exists a projection $P \in \mathcal{M}$ such that $\tau(P) < +\infty$ and $T = PT$. It follows from Proposition 2.1(ii), (iii) and (vi) that T is a τ-finite rank operator if and only if $m(\operatorname{supp}(\mu(T))) < +\infty$. It follows from Proposition 2.1(i) that T is bounded if and only if $\mu(T)$ is bounded.

3. Tensor admissibility

Let (X, ν) be a σ-finite measure space. Let us denote by $\mathscr{M}(X)$ the linear space of all ν-measurable functions $f : X \to \mathbb{C}$, where we identify the functions which are equal ν-a.e. Let $f \in \mathscr{M}(X)$. The *distribution function* of f is the function $\delta_f : (0, +\infty) \to [0, +\infty]$ given by

$$\delta_f(s) = \nu\big(\{x \in X : |f(x)| > s\}\big).$$

It is trivial to verify that δ_f is a non-increasing right-continuous function. The *decreasing rearrangement* of f is the function $f^\star : (0, +\infty) \to [0, +\infty]$ given by:

$$f^\star(t) = \inf\big\{s \geq 0 : \delta_f(s) \leq t\big\}.$$

Note that, since δ_f is right continuous, the latter infimum is attained, and that $f^\star$ is right continuous [2, Chapter 2, Proposition 1.7].

The following are equivalent for a function $f \in \mathscr{M}(X)$:

(1) there exists a $t > 0$ such that $\delta_f(t) < +\infty$,
(2) $\lim_{t \to +\infty} \delta_f(t) = 0$,
(3) $f^\star(t) \neq +\infty$ for every $t > 0$.

We call a function $f \in \mathscr{M}(X)$ *admissible* if f satisfies the above conditions. The set of all admissible functions is a subspace of $\mathscr{M}(X)$ [2, Chapter 2, Proposition 1.3] which we will denote by $\mathscr{L}(X)$.

We equip the set $(0, +\infty)$ with the Lebesgue measure m and set $\mathscr{M} = \mathscr{M}((0, +\infty))$ and $\mathscr{L} = \mathscr{L}((0, +\infty))$.

Let $L^\infty(m)$ be the von Neumann algebra of all m-measurable essentially bounded functions $f : (0, +\infty) \to \mathbb{C}$. Then the map $\tau : L^\infty(m) \to \mathbb{C}$ defined by $\tau(f) = \int f \, dm$ is a semifinite normal trace on $L^\infty(m)$. The space $\widetilde{L^\infty(m)}$ of operators affiliated to $L^\infty(m)$ is $\mathscr{M}$, while the space of τ-measurable operators $\overline{L^\infty(m)}$ coincides with $\mathscr{L}$ and $\mu(f) = f^*$ for every $f \in \mathscr{L}$.

We will denote by $\mathscr{D}$ the cone of all decreasing and right continuous functions $f : (0, +\infty) \to [0, +\infty)$. Note that $\mathscr{L} = \{f \in \mathscr{M} : f^* \in \mathscr{D}\}$.

Let $f, g \in \mathscr{L}$. We denote by $f \otimes g$ the function $f \otimes g : (0, +\infty) \times (0, +\infty) \to \mathbb{C}$ defined by

$$(f \otimes g)(x, y) = f(x)g(y).$$

Definition 3.1. (1) A pair $(f, g) \in \mathscr{L} \times \mathscr{L}$ is called *tensor admissible* if $(f \otimes g)^\star \in \mathscr{D}$.

(2) A function $f \in \mathscr{L}$ is called *tensor admissible* if the pair (f,f) is tensor admissible.

Note that, by definition, a pair (f,g) is tensor admissible if and only if $(f \otimes g)^\star(t) < +\infty$ for every $t > 0$.

DEFINITION 3.2. Let $f \in \mathscr{D}$ and $\theta \in (0,1)$. For $n \in \mathbb{Z}$ we set

$$I_n(f,\theta) = \{t > 0 : \theta^{n+1} < f(t) \le \theta^n\}, \qquad J_n(f,\theta) = \bigcup_{i<n} I_i(f,\theta),$$

$$a_n(f,\theta) = m\big(I_n(f,\theta)\big), \qquad A_n(f,\theta) = \sum_{i<n} a_i(f,\theta) = m\big(J_n(f,\theta)\big),$$

$$L_\theta(f) = \sum_{n\in\mathbb{Z}} \theta^n \chi_{I_n(f,\theta)}.$$

We call the function $L_\theta(f)$ the *θ-approximation* of f.

Let f and g be two real valued functions in $\mathscr{L}$. We shall write $f \lesssim g$ if there exists a constant $C > 0$ such that for every $x \in (0,+\infty)$ we have $f(x) \le Cg(x)$. If $f \lesssim g$ and $g \lesssim f$ we will say that f and g are *equivalent* and we will write $f \sim g$.

The following remark follows directly from the definitions.

REMARK 3.3. Let $f \in \mathscr{D}$ and $\theta \in (0,1)$. Then

(1) $I_n(f,\theta) = [A_n(f,\theta), A_{n+1}(f,\theta)) = [A_n(f,\theta), A_n(f,\theta) + a_n(f,\theta))$.
(2) The θ-approximation $L_\theta(f)$ of f is decreasing and right continuous and hence belongs to $\mathscr{D}$.
(3) The θ-approximation $L_\theta(f)$ of f satisfies $\theta L_\theta(f) \le f \le L_\theta(f)$. Thus, $f \sim L_\theta(f)$.
(4) We have $I_n(f,\theta) = I_n(L_\theta(f),\theta)$ and $a_n(f,\theta) = a_n(L_\theta(f),\theta)$ for every $n \in \mathbb{Z}$.
(5) $m(\text{supp} f) < +\infty$ if and only if $\sum_{n\in\mathbb{Z}} a_n(f,\theta) < +\infty$.
(6) f is bounded if and only if there exists n_0 such that $a_n(f,\theta) = 0$ for $n \le n_0$.
(7) $L_\theta(L_\theta(f)) = L_\theta(f)$.

The proof of the following lemma is straightforward and we omit it.

LEMMA 3.4. *Let $f, g, f', g' \in \mathscr{D}$. If $f \lesssim f'$ and $g \lesssim g'$ then $(f \otimes g)^\star \lesssim (f' \otimes g')^\star$.*

In the sequel, we use the conventions $0 \cdot (+\infty) = 0$ and $[+\infty, +\infty) = \emptyset$.

THEOREM 3.5. *Let f, g be in $\mathscr{D}$ and $\theta \in (0,1)$. Let $a_n = a_n(f,\theta)$ and $b_n = a_n(g,\theta)$, $n \in \mathbb{Z}$. For every $k \in \mathbb{Z}$, we set $r_k = \sum_{i+j<k} a_i b_j$. Then the pair (f,g) is tensor admissible if and only if there exists k_0 such that $r_{k_0} < +\infty$. In that case, we have that*

$$(f \otimes g)^\star \sim \sum_{k\in\mathbb{Z}} \theta^k \chi_{[r_k, r_{k+1})}.$$

Proof. It follows from Lemma 3.4 and Remark 3.3 that the pair (f,g) is tensor admissible if and only if the pair $(L_\theta(f),L_\theta(g))$ is tensor admissible and also that $a_n(f,\theta)=a_n(L_\theta(f),\theta)$ and $a_n(g,\theta)=a_n(L_\theta(g),\theta)$. By (7) above, we may suppose that $f=L_\theta(f)$ and $g=L_\theta(g)$.

Let $t>0$.

Then $f(x)g(y)>t$ if and only if there exist i, j such that $x\in I_i(f,\theta)$, $y\in I_j(g,\theta)$ and $\theta^i\theta^j=\theta^{i+j}>t$. Therefore,

$$\big\{(x,y):f(x)g(y)>t\big\}=\bigcup_{i,j,\theta^{i+j}>t}\big\{(x,y):x\in I_i(f,\theta),y\in I_j(g,\theta)\big\}.$$

Let $k\in\mathbb{Z}$ be such that $\theta^k\le t<\theta^{k-1}$. We have

$$\begin{aligned}&\bigcup_{i,j,\theta^{i+j}>t}\big\{(x,y):x\in I_i(f,\theta),y\in I_j(g,\theta)\big\}\\&\quad=\bigcup_{i,j,\theta^{i+j}>\theta^k}\big\{(x,y):x\in I_i(f,\theta),y\in I_j(g,\theta)\big\}\\&\quad=\bigcup_{i,j,i+j<k}\big\{(x,y):x\in I_i(f,\theta),y\in I_j(g,\theta)\big\}.\end{aligned}$$

Thus

$$\begin{aligned}\delta_{f\otimes g}(t)&=m\big(\big\{(x,y):f(x)g(y)>t\big\}\big)\\&=m\Bigg(\bigcup_{i,j,i+j<k}\big\{(x,y):x\in I_i(f,\theta),y\in I_j(g,\theta)\big\}\Bigg)=r_k.\end{aligned}$$

The pair (f,g) is tensor admissible if and only if $\delta_{f\otimes g}(t)<+\infty$ for some t. Hence, the pair (f,g) is tensor admissible if and only if for some $k\in\mathbb{Z}$, $r_k<+\infty$. The proof of the first assertion is complete.

Let $k\in\mathbb{Z}$ be such that $r_k\le s<r_{k+1}$. Let $\varepsilon>0$ be such that $\theta^k+\varepsilon<\theta^{k-1}$. By the first part of the proof it follows that $\delta_{f\otimes g}(\theta^k+\varepsilon)=r_k\le s$ which implies that $(f\otimes g)^*(s)\le\theta^k+\varepsilon$. Let $\varepsilon>0$ be such that $\theta^k-\varepsilon>\theta^{k+1}$. Again by the first part of the proof we have that $\delta_{f\otimes g}(\theta^k-\varepsilon)=r_{k+1}>s$, which implies that $(f\otimes g)^*(s)\ge\theta^k-\varepsilon$. Hence, $(f\otimes g)^*(s)=\theta^k$. □

The proof of the following corollary is contained in the proof of Theorem 3.5.

Corollary 3.6. *Let f, g be in $\mathscr{D}$ and $\theta\in(0,1)$. Assume that $f=L_\theta(f)$ and that $g=L_\theta(g)$. Let $a_n=a_n(f,\theta)$, $b_n=a_n(g,\theta)$. For every $k\in\mathbb{Z}$, we set $r_k=\sum_{i+j<k}a_ib_j$. Then*

(1)

$$(f\otimes g)^\star=\sum_{k\in\mathbb{Z}}\theta^k\chi_{[r_k,r_{k+1})}.$$

(2)
$$a_n\big((f\otimes g)^*\big)=\sum_{i+j=n} a_i b_j$$

for every $n\in\mathbb{Z}$.

PROPOSITION 3.7. *Let* $f\in\mathscr{D}$. *If* f *is unbounded and tensor admissible, then* $\lim_{x\to+\infty} f(x)=0$.

Proof. Assume $\lim_{x\to+\infty} f(x)=a\neq 0$. Let $r>0$. Since f is unbounded the set $A=\{t: f(t)>r/a\}$ has positive measure. Then $\delta_{f\otimes f}(r)=(m\times m)(\{(t,s): |f(t)f(s)|>r\})\geq (m\times m)(\{(t,s): t\in A, s\in(0,+\infty)\})=+\infty$. □

All the von Neumann algebras, we consider from now on, are semifinite and atomless. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and $\mathcal{M}\subseteq B(\mathcal{H}_1)$ and $\mathcal{N}\subseteq B(\mathcal{H}_2)$ be von Neumann algebras. Assume that τ (resp. σ) is a faithful semi-finite normal trace on $\mathcal{M}$ (resp. $\mathcal{N}$).

Let $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ be the Hilbert tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$. We denote by $(\mathcal{M}\bar{\otimes}\mathcal{N},\tau\otimes\sigma)$ the spatial tensor product of $(\mathcal{M},\tau)$ and $(\mathcal{N},\sigma)$, that is, the von Neumann algebra acting on $\mathcal{H}_1\otimes\mathcal{H}_2$, generated by the operators $A\otimes B$, where $A\in\mathcal{M}$ and $B\in\mathcal{N}$, equipped with the trace $\tau\otimes\sigma$ defined by $(\tau\otimes\sigma)(A\otimes B)=\tau(A)\sigma(B)$, $A\in\mathcal{M}$, $B\in\mathcal{N}$. If $A\in\overline{\mathcal{M}}$ and $B\in\overline{\mathcal{N}}$ then $A\otimes B$ is a closed, densely defined operator affiliated to $\mathcal{M}\bar{\otimes}\mathcal{N}$ and $(A\otimes B)^*=A^*\otimes B^*$ [12, Theorem 8.1], but it is not true in general that $A\otimes B\in\overline{\mathcal{M}\bar{\otimes}\mathcal{N}}$. The next theorem provides a characterization of the pairs $(A,B)\in\overline{\mathcal{M}}\times\overline{\mathcal{N}}$ with the property that $A\otimes B\in\overline{\mathcal{M}\bar{\otimes}\mathcal{N}}$.

THEOREM 3.8. *Let* $\mathcal{H}_1$ (*resp.* $\mathcal{H}_2$) *be a Hilbert space, and* $\mathcal{M}$ (*resp.* $\mathcal{N}$) *be a Neumann algebra equipped with a faithful semi-finite normal trace* τ (*resp.* σ). *Let* $A\in\overline{\mathcal{M}}$ *and* $B\in\overline{\mathcal{N}}$. *Then* $A\otimes B\in\overline{\mathcal{M}\bar{\otimes}\mathcal{N}}$ *if and only if the pair* $(\mu(A),\mu(B))$ *is tensor admissible. In this case, we have that*

$$\mu(A\otimes B)=\big(\mu(A)\otimes\mu(B)\big)^{\star}.$$

Proof. Using polar decomposition, we may suppose that A, B are positive operators. Let E^A (resp. E^B) be the spectral measure of A (resp. B); thus, $A=\int_0^{+\infty} x\,dE^A(x)$ and $B=\int_0^{+\infty} y\,dE^B(y)$. Let $E^A\otimes E^B$ be the spectral measure on $\mathbb{R}\times\mathbb{R}$ with values in the projection lattice of $\mathcal{H}_1\otimes\mathcal{H}_2$ defined by

(1)
$$E^A\otimes E^B(\delta_1\times\delta_2)=E^A(\delta_1)\otimes E^B(\delta_2),$$

where δ_1, δ_2 are Borel subsets of $\mathbb{R}$. It follows from [12, Theorem 8.2] that

$$A\otimes B=\int_0^{+\infty}\int_0^{+\infty} xy\,d\big(E^A\otimes E^B\big)(x,y).$$

Let us denote by $E^{A,B}$ the spectral measure on $\mathbb{R}$ given by

(2)
$$E^{A,B}(\delta)=\big(E^A\otimes E^B\big)\big(\{(x,y): xy\in\delta\}\big).$$

Then

$$A \otimes B = \int_0^{+\infty} x\, dE^{A,B}(x).$$

For every $s > 0$, we set

$$\Delta_s = \big\{(x,y) \in (0,+\infty) \times (0,+\infty) : xy > s\big\};$$

note that $E^{A,B}(s,+\infty) = (E^A \otimes E^B)(\Delta_s)$.

Note that $A \otimes B \in \overline{\mathcal{M} \bar{\otimes} \mathcal{N}}$ if and only if for every $t > 0$ we have that $\mu_t(A \otimes B) < +\infty$. Since

$$\mu_t(A \otimes B) = \inf\big\{s \geq 0 : \tau \otimes \sigma\big(E^A \otimes E^B(\Delta_s)\big) \leq t\big\}$$

and

$$\big(\mu(A) \otimes \mu(B)\big)^\star(t) = \inf\big\{s \geq 0 : (m \times m)\big(\big\{(x,y) : \mu_x(A)\mu_y(B) > s\big\}\big) \leq t\big\}$$

the conclusion of the theorem will follow if we prove the following equality:

$$(\tau \otimes \sigma)\big(E^A \otimes E^B(\Delta_s)\big) = (m \times m)\big(\big\{(x,y) : \mu_x(A)\mu_y(B) > s\big\}\big). \tag{3}$$

Let $s > 0$. For every $i, n \in \mathbb{N}$, set

$$I_{(n,i)} = \left[\frac{i}{n}, \frac{i+1}{n}\right), \qquad J_{(n,i)} = \left(\frac{s}{i/n}, +\infty\right),$$

$$\delta_{(n,i)} = I_{(n,i)} \times J_{(n,i)}, \quad \text{and} \quad \delta_n = \bigcup_{i=1}^{+\infty} \delta_{(n,i)}.$$

Clearly,

$$\delta_1 \subseteq \delta_2 \subseteq \delta_3 \subseteq \cdots \quad \text{and} \quad \Delta_s = \bigcup_{n=1}^{+\infty} \delta_n. \tag{4}$$

By [5, Remark 3.3], for every Borel subset δ of $\mathbb{R}$ and every positive operator $T \in \overline{\mathcal{M}}$ we have that

$$\tau\big(E^T(\delta)\big) = \int_0^{+\infty} \chi_\delta\big(\mu_t(T)\big)\, dt = m\big(\big\{x : \mu_x(T) \in \delta\big\}\big). \tag{5}$$

By (4) and (5),

$$\begin{aligned}
&(\tau \otimes \sigma)\big(E^A \otimes E^B(\Delta_s)\big) \\
&\quad = \lim_{n\to\infty} (\tau \otimes \sigma)\big(E^A \otimes E^B(\delta_n)\big) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{+\infty} \tau \otimes \sigma\big(E^A \otimes E^B(\delta_{(n,i)})\big) \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{+\infty} \tau\big(E^A(I_{(n,i)})\big)\sigma\big(E^B(J_{(n,i)})\big)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\sum_{i=1}^{+\infty} m\big(\{x : \mu_x(A) \in I_{(n,i)}\}\big)m\big(\{y : \mu_y(B) \in J_{(n,i)}\}\big)\\
&= \lim_{n\to\infty}\sum_{i=1}^{+\infty} (m\times m)\big(\{x : \mu_x(A) \in I_{(n,i)}\} \times \{y : \mu_y(B) \in J_{(n,i)}\}\big)\\
&= \lim_{n\to\infty}(m\times m)\left(\bigcup_{i=1}^{+\infty}\{(x,y) : \mu_x(A) \in I_{(n,i)}, \mu_y(B) \in J_{(n,i)}\}\right)\\
&= (m\times m)\left(\bigcup_{n=1}^{+\infty}\bigcup_{i=1}^{+\infty}\{(x,y) : \mu_x(A) \in I_{(n,i)}, \mu_y(B) \in J_{(n,i)}\}\right)\\
&= (m\times m)\big(\{(x,y) : \mu_x(A)\mu_y(B) > s\}\big),
\end{aligned}
$$

and (3) is established. □

Recall that $L^\infty(m)$ is the von Neumann algebra of all m-measurable essentially bounded functions $f : (0,+\infty) \to \mathbb{C}$ equipped with the faithful semifinite normal trace τ given by $\tau(f) = \int f\,dm$. We also have that $\mathscr{L} = \overline{L^\infty(m)}$. Setting $\mathcal{M} = \mathcal{N} = L^\infty(m)$ in the above theorem we obtain the following.

COROLLARY 3.9. *Let $f \in \mathscr{L}$. If f is tensor admissible, then*

$$(f \otimes f)^\star = \big(f^\star \otimes f^\star\big)^\star.$$

DEFINITION 3.10. Let $(\mathcal{M},\tau)$, $(\mathcal{N},\sigma)$ be von Neumann algebras with faithful semi-finite normal traces τ, σ. Let $T \in \overline{\mathcal{M}}$, $S \in \overline{\mathcal{N}}$.

(1) We will call the pair (T,S) *tensor admissible* if $T \otimes S \in \overline{\mathcal{M}\,\bar{\otimes}\,\mathcal{N}}$.
(2) We will call T *tensor admissible* if $T \otimes T \in \overline{\mathcal{M}\,\bar{\otimes}\,\mathcal{M}}$.

REMARK 3.11. By Theorem 3.8, the pair (S,T) (resp. the operator T) is tensor admissible if and only if the pair $(\mu(S),\mu(T))$ (resp. the function $\mu(T)$) is tensor admissible.

Some properties of tensor admissible operators are described in the following theorem.

THEOREM 3.12. *Let $(\mathcal{M},\tau)$ be a von Neumann algebra with a faithful semi-finite normal trace τ. Let $S,T \in \overline{\mathcal{M}}$.*

(1) If S and T are bounded, then the pair (S,T) is tensor admissible.
(2) If S and T are τ-finite-rank operators, then the pair (S,T) is tensor admissible.
(3) If T is tensor admissible and S is a bounded τ-finite-rank operator, then $T+S$ is tensor admissible.

Proof. (1) is clear.

(2) It suffices to show that the pair $(\mu(S),\mu(T))$ is tensor admissible. Since S and T are τ-finite-rank operators, there exists r_0 such that $\mu_{r_0}(S) = 0$ and

$\mu_{r_0}(T)=0$. The pair $(\mu(S),\mu(T))$ is tensor admissible if, for some $r>0$, the set $\{(x,y):\mu_x(S)\mu_y(T)>r\}$ has finite Lebesgue measure. Since

$$\big\{(x,y):\mu_x(S)\mu_y(T)>r\big\}\subseteq\big\{(x,y):\mu_x(S)\mu_y(T)>0\big\}$$

and $(m\times m)(\{(x,y):\mu_x(S)\mu_y(T)>0\})\le r_0^2$, the assertion follows.

(3) If T is a τ-finite-rank operator, the assertion follows from (2). We assume that T is not a τ-finite-rank operator. Let s_0 be such that $\mu_{s_0}(S)=0$. Since T is tensor admissible, there exists r such that the measure of the set

$$F=\big\{(x,y):\mu_x(T)\mu_y(T)>r\big\}$$

is finite. By Proposition 2.1, for $x\le s_0$ we have that

$$\mu_x(T+S)\le\mu_x(T)+\|S\|.$$

Set $c=\frac{\|S\|}{\mu_{s_0}(T)}$ (note that, since T is not τ-finite-rank, $\mu_{s_0}(T)>0$). We have

$$\frac{\mu_x(T+S)}{\mu_x(T)}\le\frac{\mu_x(T)+\|S\|}{\mu_x(T)}\le 1+\frac{\|S\|}{\mu_x(T)}\le 1+\frac{\|S\|}{\mu_{s_0}(T)}=1+c.$$

It follows that

$$\mu_x(T+S)\le\mu_x(T)(1+c) \tag{6}$$

if $0<x\le s_0$.

For every $x>s_0$, by Proposition 2.1 we have that

$$\mu_x(T+S)\le\mu_{x-s_0}(T)+\mu_{s_0}(S)=\mu_{x-s_0}(T). \tag{7}$$

We show that the set

$$E=\big\{(x,y):\mu_x(T+S)\mu_y(T+S)>r(1+c)^2\big\}$$

has finite measure. Assume that $x\le s_0$, $y\le s_0$ and $(x,y)\in E$. Then

$$\mu_x(T+S)\mu_y(T+S)>r(1+c)^2$$

and hence, by (6), $\mu_x(T)\mu_y(T)>r$. Thus, the set

$$\big\{(x,y)\in E:x\le s_0,y\le s_0\big\}$$

is contained in F and therefore has finite measure.

Assume $x\le s_0$, $y>s_0$ and $(x,y)\in E$. We have that

$$\mu_x(T+S)\mu_y(T+S)>r(1+c)^2$$

and hence, by (6) and (7),

$$\mu_x(T)\mu_{y-s_0}(T)>r(1+c)>r.$$

Set

$$A=\big\{(x,y):x\le s_0,y>s_0,\mu_x(T)\mu_{y-s_0}(T)>r\big\}$$

and

$$B=\big\{(x,y):x\le s_0,y>0,\mu_x(T)\mu_y(T)>r\big\}.$$

Then B is contained in F and hence has finite measure; since A is the translate of B by the point $(0, s_0)$, we have that A has finite measure. Since $\{(x, y) \in E : x \leq s_0, y > s_0\}$ is contained in A, it has finite measure.

Similarly, we can show that the set $\{(x, y) \in E : x > s_0, y \leq s_0\}$ has finite measure.

Assume $x > s_0$, $y > s_0$ and $(x, y) \in E$. We have

$$\mu_x(T + S)\mu_y(T + S) > r(1 + c)^2$$

and hence, by (7),

$$\mu_{x-s_0}(T)\mu_{y-s_0}(T) > r(1 + c)^2 > r.$$

Set

$$A' = \{(x, y), x > s_0, y > s_0 : \mu_{x-s_0}(T)\mu_{y-s_0}(T) > r\}$$

and

$$B' = \{(x, y), x > 0, y > 0 : \mu_x(T)\mu_y(T) > r\}.$$

As before, B' has finite measure; since A' is the translate of B' by the point (s_0, s_0), we have that A' is of finite measure. As $\{(x, y) \in E : x > s_0, y > s_0\}$ is contained in A', it has finite measure. It follows that E has finite measure as the union of four sets of finite measure. The statement now follows from Remark 3.11. □

The following proposition is [4, Proposition 1 and Lemma 9]. Note that [4, Lemma 9] was formulated for bounded functions f, but its proof works in the case $f \in \mathscr{D}$ as well.

PROPOSITION 3.13. *Let $\mathcal{M}$ be a type II_∞ factor acting on a separable Hilbert space with a faithful semi-finite normal trace τ. There exists an increasing strongly continuous function $P : [0, +\infty) \to \mathcal{M}$, denoted $t \mapsto P_t$ such that*:

(1) *For every $t \in [0, +\infty)$, P_t is a projection in $\mathcal{M}$.*
(2) *For every $t \in [0, +\infty)$ we have that $\tau(P_t) = t$.*
(3) $\lim_{t\to\infty} P_t = I$.

Moreover, if f is a function in $\mathscr{D}$ and $T = \int_0^{+\infty} f(t)\, dP_t$, then T is a measurable operator and $\mu_t(T) = f(t)$.

EXAMPLE 3.14 (A measurable non-tensor admissible operator). We will construct a function in $\mathscr{D}$ which is not tensor admissible and then using Proposition 3.13 we will find a measurable non-tensor admissible operator.

Let $a > 0$. We consider the function $f : (0, +\infty) \to [0, +\infty)$ defined by $f(t) = t^{-a}$. We have $I_n(f, 1/2) = [2^{n/a}, 2^{(n+1)/a})$. Set $a_n = a_n(f, 1/2)$; we have

$$a_n = m(I_n(f, 1/2)) = 2^{n/a}(2^{1/a} - 1).$$

It follows from Theorem 3.5 that f is tensor admissible if and only if there exists k_0 such that $r_{k_0} < +\infty$, where $r_k = \sum_{i+j<k} a_i a_j$ for each $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, we have

$$r_k \geq \sum_{i+j=k-1} a_i a_j = \sum_{i+j=k-1} 2^{i/a} 2^{j/a} \big(2^{1/a} - 1\big)^2$$
$$= \sum_{i \in \mathbb{Z}} 2^{(k-1)/a} \big(2^{1/a} - 1\big)^2 = +\infty.$$

We conclude that f is not tensor admissible. Let $\mathcal{M}$, τ and P be as in Proposition 3.13, and $T = \int_0^{+\infty} f(t)\, dP_t$. The operator T is measurable and $\mu_t(T) = f(t)$. By Theorem 3.8, T is not tensor admissible.

Example 3.15 (Two tensor admissible positive measurable operators whose sum is not tensor admissible). Let $a > 0$ and $f : (0, +\infty) \to [0, +\infty)$ be the function defined by: $f(t) = t^{-a} - 1$ if $t \in (0, 1]$ and $f(t) = 0$ if $t > 1$ and $g : (0, +\infty) \to [0, +\infty)$ be the function defined by: $g(t) = 1$ if $t \in (0, 1]$ and $g(t) = t^{-a}$ if $t > 1$.

Let $\mathcal{M}$, τ and P be as in Proposition 3.13. Let $T_1 = \int_0^{+\infty} f(t)\, dP_t$, $T_2 = \int_0^{+\infty} g(t)\, dP_t$. The operators T_1 and T_2 are measurable and $\mu_t(T_1) = f(t)$, $\mu_t(T_2) = g(t)$. Since T_1 is a τ-finite-rank operator and T_2 is bounded, it follows from Theorem 3.12 that they are tensor admissible operators. However, $T_1 + T_2$ is not tensor admissible as we saw in the previous example.

4. Tensor stability

We start this section by recalling several well-known notions.

Definition 4.1. A linear subspace $\mathscr{S}$ of $\mathscr{L}$ is called a *Calkin function space* if it satisfies the following condition: for every $f \in \mathscr{S}$ and $g \in \mathscr{L}$ such that $g^\star \leq f^\star$ we have that $g \in \mathscr{S}$.

Let $\lambda > 0$. We consider the *dilation operator* $D_\lambda : \mathscr{L} \to \mathscr{L}$ defined by

$$D_\lambda(f)(t) = f\big(\lambda^{-1} t\big).$$

It follows from [9, p. 54] that if $\mathscr{S}$ is a Calkin function space, $\lambda > 0$ and $f \in \mathscr{S}$, then $D_\lambda f \in \mathscr{S}$.

Let $\mathscr{V}$ be a linear space. Recall that a quasi-norm on $\mathscr{V}$ is a non-negative function $x \mapsto \|x\|$ defined on $\mathscr{V}$ and satisfying the same axioms as a norm except for the triangle inequality which is replaced by the requirement: There exists a constant $c > 0$ such that

$$\|x + y\| \leq c\big(\|x\| + \|y\|\big),$$

for all $x, y \in \mathscr{V}$.

Definition 4.2. A Calkin function space $\mathscr{E}$ is called a *symmetric quasi-normed function space* (or a *symmetric quasi-normed space*) if there exists a quasi-norm ρ on $\mathscr{E}$ with the following property: If $f \in \mathscr{E}$, $g \in \mathscr{E}$ and $f^\star \leq g^\star$, then $\rho(f) \leq \rho(g)$. If ρ is a norm with this property, $\mathscr{E}$ is called *a symmetric normed function space* (or a *symmetric normed space*).

Definition 4.3. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a faithful semi-finite normal trace τ. A subspace $\mathcal{J}$ of $\overline{\mathcal{M}}$ such that for every $T \in \mathcal{J}$ and $A, B \in \mathcal{M}$ we have that $ATB \in \mathcal{J}$ is called a *submodule* of $\overline{\mathcal{M}}$.

Let $\mathscr{E}$ be a Calkin function space. Set $\mathscr{E}(\mathcal{M}) = \{T \in \overline{\mathcal{M}} : \mu_t(T) \in \mathscr{E}\}$. By Proposition 2.1(v), $\mathscr{E}(\mathcal{M})$ is a submodule of $\overline{\mathcal{M}}$. It is known that, when $\mathscr{M}$ is a semi-finite factor, the submodules of $\overline{\mathcal{M}}$ are in one-to-one correspondence with the Calkin spaces contained in $\mathscr{L}$ [7]. It follows that, in this case, every submodule of $\overline{\mathcal{M}}$ is of the form $\mathscr{E}(\mathcal{M})$ for some Calkin space $\mathscr{E}$.

Let $(\mathscr{E}, \rho)$ be a symmetric quasi-normed space and $(\mathcal{M}, \tau)$ be a von Neumann algebra with a faithful semi-finite normal trace τ. We define a function $\bar{\rho} : \mathscr{E}(\mathcal{M}) \to [0, +\infty)$ by

$$\bar{\rho}(T) = \rho\big(\mu(T)\big), \quad T \in \mathscr{E}(\mathcal{M}).$$

It is not hard to see that $\bar{\rho}$ is a quasi-norm on $\mathscr{E}(\mathcal{M})$ (see [8]). If $(\mathscr{E}, \rho)$ is a normed space then $(\mathscr{E}(\mathcal{M}), \bar{\rho})$ is a normed space. This important result was proved by Dodds, Dodds and de Pagter [3] in the case where ρ is a Fatou norm and by Kalton and Sukochev [8] in the general case. If $(\mathscr{E}, \rho)$ is a Banach space then $(\mathscr{E}(\mathcal{M}), \bar{\rho})$ is also a complete normed space ([3], [8]). We also note that if $(\mathscr{E}, \rho)$ is a complete symmetric quasi-normed space then $(\mathscr{E}(\mathcal{M}), \bar{\rho})$ is also a complete quasi-normed space ([13]).

The following theorem is a consequence of Theorem 3.8.

Theorem 4.4. *Let $(\mathcal{M}, \tau)$, $(\mathcal{N}, \sigma)$ be von Neumann algebras with faithful semi-finite normal traces τ and σ acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.*

(1) *Let $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{E}_3$ be Calkin function spaces. Assume that for every $f \in \mathscr{E}_1$ and $g \in \mathscr{E}_2$ we have that $(f \otimes g)^\star \in \mathscr{E}_3$. Then*

$$\mathscr{E}_1(\mathcal{M}) \otimes \mathscr{E}_2(\mathcal{N}) \subseteq \mathscr{E}_3(\mathcal{M} \bar{\otimes} \mathcal{N}).$$

(2) *Let $(\mathscr{E}_1, \rho_1)$, $(\mathscr{E}_2, \rho_2)$, $(\mathscr{E}_3, \rho_3)$ be symmetric quasi-normed spaces and $C > 0$. Assume that for every $f \in \mathscr{E}_1$ and every $g \in \mathscr{E}_2$, we have $(f \otimes g)^\star \in \mathscr{E}_3$ and*

$$\rho_3\big((f \otimes g)^\star\big) \leq C \rho_1(f) \rho_2(g).$$

Then

$$\mathscr{E}_1(\mathcal{M}) \otimes \mathscr{E}_2(\mathcal{N}) \subseteq \mathscr{E}_3(\mathcal{M} \bar{\otimes} \mathcal{N})$$

and for every $T \in \mathscr{E}_1(\mathcal{M})$ and $S \in \mathscr{E}_2(\mathcal{N})$ we have

$$\bar{\rho}_3(T \otimes S) \leq C \bar{\rho}_1(T) \bar{\rho}_2(S).$$

We combine the above theorem and results of O'Neil on tensor products of Lorentz spaces [11] to obtain Theorem 4.5 below. We first recall the definition of Lorentz spaces $\mathscr{L}_{(p,q)}$.

Let $f \in \mathscr{L}$ and $0 < p < +\infty$, $0 < q \leq +\infty$. We set

$$\|f\|_{p,q} = \begin{cases} (\int_0^{+\infty} (f^*(t)t^{\frac{1}{p}})^q \frac{dm(t)}{t})^{\frac{1}{q}} & \text{if } q < +\infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = +\infty \end{cases}$$

and

$$\mathscr{L}_{(p,q)} = \{f \in \mathscr{L} : \|f\|_{p,q} < +\infty\}$$

(see [11, Definition 6.5]). It is clear that the spaces $\mathscr{L}_{(p,q)}$ are Calkin spaces and it is known that they are complete symmetric quasi-normed function spaces [6, Theorem 1.4.11]. Theorem 4.5 below follows from [11, Theorem 7.7], Theorem 4.4 and [9, Theorem 2.4.4].

THEOREM 4.5. *Let $0 < p < +\infty$ and $0 < q, r, s \leq +\infty$ and let $(\mathcal{M}, \tau)$, $(\mathcal{N}, \sigma)$ be von Neumann algebras with faithful semi-finite normal traces τ and σ acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. A necessary and sufficient condition in order that for every $T \in \mathscr{L}_{(p,q)}(\mathcal{M})$ and $S \in \mathscr{L}_{(p,r)}(\mathcal{N})$ we have that $T \otimes S \in \mathscr{L}_{(p,s)}(\mathcal{M} \bar{\otimes} \mathcal{N})$ is that p, q, r, s satisfy the inequalities*:

$$q \leq s, \qquad r \leq s, \qquad \frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}. \tag{8}$$

In that case there exists a constant K which depends only on p, q, r, s such that for every $T \in \mathscr{L}_{(p,q)}(\mathcal{M})$ and $S \in \mathscr{L}_{(p,r)}(\mathcal{N})$ we have that

$$\|\mu(T \otimes S)\|_{(p,s)} \leq K \|\mu(T)\|_{(p,q)} \|\mu(S)\|_{(p,r)}.$$

From now on, we assume that $(\mathcal{M}, \tau)$ is a factor of type II_∞.

DEFINITION 4.6. (1) A Calkin function space $\mathscr{E}$ is called *tensor stable* if for every $f \in \mathscr{E}$ and $g \in \mathscr{E}$ we have that $(f \otimes g)^\star \in \mathscr{E}$.
(2) Let $\mathscr{E}$ be a Calkin function space. We shall say that the submodule $\mathscr{E}(\mathcal{M})$ is *tensor stable* if $\mathscr{E}$ is a tensor stable Calkin function space.

REMARK 4.7. Let $\mathscr{E}$ be a Calkin function space. It follows from Theorem 3.8 that the submodule $\mathscr{E}(\mathcal{M})$ is tensor stable if and only if

$$\mathscr{E}(\mathcal{M}) \otimes \mathscr{E}(\mathcal{M}) \subseteq \mathscr{E}(\mathcal{M} \otimes \mathcal{M}).$$

REMARK 4.8. Let $0 < p < +\infty$ and $0 < q \leq +\infty$. It follows from [11, Theorem 7.7] that the Calkin function space $\mathscr{L}_{(p,q)}$ is tensor stable if and only if $q \leq p$. It follows from Theorem 4.5 that the submodule $\mathscr{L}_{(p,q)}(\mathcal{M})$ is tensor stable if and only if $q \leq p$.

LEMMA 4.9. *Let $f, g \in \mathscr{D}$ and $\theta \in (0,1)$. Then there exists a constant $C > 0$ such that $f \leq Cg$ if and only if there exists an integer $r \geq 0$ such that for every $k \in \mathbb{Z}$ we have that*

$$A_k(f, \theta) \leq A_{k+r}(g, \theta).$$

Proof. Assume $f \le Cg$. We consider $r \in \mathbb{Z}$, $r \ge 0$ such that $\theta^r C \le 1$. Let $x \in J_n(f,\theta)$. Then $f(x) > \theta^n$ and so $g(x) \ge C^{-1} f(x) \ge \theta^r f(x) > \theta^{n+r}$. It follows that $x \in J_{n+r}(g,\theta)$ and $A_k(f,\theta) \le A_{k+r}(g,\theta)$.

Suppose now that there exists an integer $r \ge 0$ such that for every $k \in \mathbb{Z}$ we have that $A_k(f,\theta) \le A_{k+r}(g,\theta)$. Let n be such that $x \in I_n(f,\theta)$. Then $x \in [A_n(f,\theta), A_{n+1}(f,\theta))$ and we have

$$A_n(f,\theta) \le x < A_{n+1}(f,\theta) \le A_{n+r+1}(g,\theta)$$

by assumption. We thus have

$$g(x) \ge g\big(A_{n+r+1}(g,\theta)\big) > \theta^{n+r+2}.$$

But then $f(x) \le \theta^n = \theta^{n+r+2}\theta^{-r-2} \le g(A_{n+r+1}(g,\theta))\theta^{-r-2} \le g(x)\theta^{-r-2}$. □

Let $f \in \mathscr{D}$. Then it follows from [9, p. 54] that the set

$$\big\{g : \text{there exists a } C > 0, \text{and } \lambda > 0 \text{ such that } g^\star \le C D_\lambda f\big\}$$

is a Calkin function space and it is contained in every Calkin function space that contains f. Hence, it is the least Calkin space containing f. We will denote this space by $\mathscr{S}_f$. We will say that a Calkin space is *singly generated* if it is of the form $\mathscr{S}_f$ for some $f \in \mathscr{D}$.

THEOREM 4.10. *The Calkin space generated by $f \in \mathscr{D}$ is tensor stable if and only if there exists a constant $C > 0$ and $\lambda > 0$ such that $(f \otimes f)^\star \le C D_\lambda f$.*

Proof. If $\mathscr{S}_f$ is tensor stable, then $(f \otimes f)^\star \in \mathscr{S}_f$ and hence there exist $C > 0$ and $\lambda > 0$ such that $(f \otimes f)^\star \le C D_\lambda f$.

For the converse, assume that there exist $C > 0$ and $\lambda > 0$ such that $(f \otimes f)^\star \le C D_\lambda f$. Let $g_1, g_2 \in \mathscr{S}_f$. Then there exist $K > 0, M > 0$ and $\nu > 0, \kappa > 0$ such that $g_1^\star \le K D_\nu f$ and $g_2^\star \le M D_\kappa f$. We show that $(g_1 \otimes g_2)^\star \in \mathscr{S}_f$. We have

$$(g_1 \otimes g_2)^\star = \big(g_1^\star \otimes g_2^\star\big)^\star$$

by Corollary 3.9. Hence we may assume that $g_1, g_2 \in \mathscr{D}$. Let $\xi = \max\{\nu, \kappa\}$. Then, $g_1 \le K D_\nu f \le K D_\xi f$ and $g_2 \le M D_\kappa f \le M D_\xi f$. By Lemma 3.4, we have

$$(g_1 \otimes g_2)^\star \le KM(D_\xi f \otimes D_\xi f)^\star.$$

But

$$(D_\xi f \otimes D_\xi f)^\star = D_\xi (f \otimes f)^\star.$$

Hence,

$$(g_1 \otimes g_2)^\star \le KMC D_\xi D_\lambda f.$$

and $(g_1 \otimes g_2)^\star \in \mathscr{S}_f$. □

COROLLARY 4.11. *Let $f \in \mathscr{D}$ and $\theta \in (0,1)$. The Calkin space $\mathscr{S}_f$ is tensor stable if and only if there exist an integer $r \ge 0$ and $C > 0$ such that for every $k \in \mathbb{Z}$ we have*

$$A_k\big((f \otimes f)^\star, \theta\big) \le C A_{k+r}(f,\theta).$$

Proof. It follows from Lemma 4.9, Proposition 4.10 and the fact that $A_k(D_\lambda f,\theta)=\lambda A_k(f,\theta)$. □

DEFINITION 4.12. Let $(\mathcal{M},\tau)$ be a factor of type II_∞ and $\mathcal{J}$ be a submodule of $\overline{\mathcal{M}}$. We will say that $\mathcal{J}$ is *singly generated* if there exists $T\in\overline{\mathcal{M}}$ such that $\mathcal{J}$ is the least submodule of $\overline{\mathcal{M}}$ that contains T. In this case, we will say that $\mathcal{J}$ is generated by T.

REMARK 4.13. Let $(\mathcal{M},\tau)$ be a factor of type II_∞, $T\in\overline{\mathcal{M}}$ and $\mathcal{J}$ the submodule generated by T. Let $f=\mu(T)$. Then $\mathcal{J}=\mathscr{S}_f(\mathcal{M})$.

DEFINITION 4.14. Let $(\mathcal{M},\tau)$ be a factor of type II_∞. A function $f\in\mathscr{D}$ will be called *tensor stable* if $\mathscr{S}_f$ is tensor stable. An operator $T\in\overline{\mathcal{M}}$ will be called *tensor stable* if the submodule of $\overline{\mathcal{M}}$ generated by T is tensor stable.

REMARK 4.15. It follows from Remark 3.3 and Corollary 4.11 that a bounded τ-finite rank operator is tensor stable.

Let $\mathcal{M}$, τ and P be as in Proposition 3.13. Let $f\in\mathscr{D}$ and $T=\int_0^{+\infty} f(t)\,dP_t$. The operator T is measurable and $\mu_t(T)=f(t)$. By Remark 4.13, the operator T is tensor stable (resp. bounded, admissible) if and only if f is a tensor stable (resp. bounded, admissible) function. It follows that in order to construct an operator in $\overline{\mathcal{M}}$ with a certain property it is sufficient to construct a function in $\mathscr{D}$ with the corresponding property.

EXAMPLE 4.16 (A bounded not tensor stable operator). Let $f\in\mathscr{D}$ be the function defined by:

$$f=\sum_{n\in\mathbb{Z}}2^{-n}\chi_{I_n},$$

where: $I_n=[n,n+1)$ if $n\geq 1$, $I_0=(0,1)$ and $I_n=\emptyset$ if $n\leq -1$. Then $a_n(f,1/2)=m(I_n)=1$ for $n\geq 0$ and $a_n(f,1/2)=0$ for $n\leq -1$. By Corollary 3.6, for $n>0$ we have

$$A_n\big((f\otimes f)^\star,1/2\big)=\sum_{i+j<n}a_i(f,1/2)a_j(f,1/2)=n(n+1)/2.$$

Let $r\in\mathbb{Z}$, $r>0$. For $n>0$, we have

$$A_{n+r}(f,1/2)=\sum_{i=0}^{n+r-1}a_i(f,1/2)=n+r.$$

Since there are no $C>0$ and $r\in\mathbb{Z}$, $r>0$ such that $n(n+1)/2\leq C(n+r)$ for every n, it follows from Corollary 4.11 that f is not tensor stable.

EXAMPLE 4.17 (An unbounded tensor admissible not tensor stable operator). Let $f\in\mathscr{D}$ be the function defined by:

$$f=\sum_{n\in\mathbb{Z}}2^{-n}\chi_{I_n},$$

where $I_n = [2^{n/a}, 2^{(n+1)/a})$ if $n < 0$ and $I_n = \emptyset$ if $n \geq 0$, for some $a > 0$. Then $a_n(f, 1/2) = m(I_n) = 2^{n/a}(2^{1/a} - 1)$ if $n < 0$ and $a_n(f, 1/2) = 0$ if $n \geq 0$. Set $a_n = a_n(f, 1/2)$. It follows from Theorem 3.12 that f is tensor admissible.

It follows from Corollary 3.6 that

$$A_n\big((f \otimes f)^\star, 1/2\big) = \sum_{i+j<n} a_i a_j$$

for every $n \in \mathbb{Z}$. Since for $k < -1$ we have

$$\sum_{i+j=k} a_i a_j = \sum_{i<0, j<0, i+j=k} 2^{i/a} 2^{j/a} \big(2^{1/a} - 1\big)^2 = \big(|k| - 1\big) 2^{k/a} \big(2^{1/a} - 1\big)^2$$

we obtain for $n < 0$

$$A_n\big((f \otimes f)^\star, 1/2\big) = \sum_{i+j<n} a_i a_j = \sum_{k=-\infty}^{n-1} \big(|k| - 1\big) 2^{k/a} \big(2^{1/a} - 1\big)^2.$$

Assume that there exist r and C such that $A_n((f \otimes f)^\star, 1/2) \leq A_{n+r}(f, 1/2)$ for every $n \in \mathbb{Z}$. If $n \leq \min\{-1, -r\}$ we obtain

$$\begin{aligned}\sum_{k=-\infty}^{n-1} \big(|k| - 1\big) 2^{k/a} \big(2^{1/a} - 1\big)^2 &\leq \sum_{i=-\infty}^{n+r-1} 2^{i/a} \big(2^{1/a} - 1\big) \\ &= 2^{(n+r-1)/a} \big(2^{1/a} - 1\big) \frac{2^{1/a}}{2^{1/a} - 1} = 2^{(n+r)/a}.\end{aligned}$$

Hence

$$\begin{aligned}&\big(|n-1| - 1\big) 2^{(n-1)/a} \big(2^{1/a} - 1\big)^2 \leq 2^{(n+r)/a} \\ \Rightarrow \quad &\big(|n-1| - 1\big) \big(2^{1/a} - 1\big)^2 \leq 2^{(r+1)/a}\end{aligned}$$

for every $n \in \mathbb{Z}$ such that $n \leq \min\{-1, -r\}$ which is absurd. It follows from Corollary 4.11 that f is not tensor stable.

Example 4.18 (A bounded tensor stable operator). Let $a_0, a_1, \ldots$ be the sequence of Catalan numbers. They are defined as follows:

$$a_0 = 1, \qquad a_{n+1} = \sum_{i=1}^{n} a_i a_{n-i}.$$

We set $I_n = \emptyset$ for $n < 0$, $I_0 = (0, 1)$ and $I_n = [\sum_{i=0}^{n-1} a_i, \sum_{i=0}^{n} a_i)$ for $n > 0$. Let $f \in \mathscr{D}$ be the function defined by:

$$f = \sum_{n \in \mathbb{Z}} 2^{-n} \chi_{I_n}.$$

Then, $a_n(f, 1/2) = m(I_n) = a_n$ for $n \geq 0$ and $a_n(f, 1/2) = m(I_n) = 0$ for $n < 0$.

Let $n \in \mathbb{Z}$. Then

$$\begin{aligned} A_n\big((f\otimes f)^\star,1/2\big) &= \sum_{i+j<n} a_i(f,1/2)a_j(f,1/2) = \sum_{i\geq 0, j\geq 0, i+j<n} a_i a_j \\ &= \sum_{k=0}^{n-1}\sum_{i\geq 0,j\geq 0,i+j=k} a_i a_j = \sum_{k=0}^{n-1} a_{k+1} \leq \sum_{k=0}^{n} a_k \\ &= A_{n+1}(f,1/2). \end{aligned}$$

It follows from Corollary 4.11 that f is tensor stable.

Example 4.19. *An unbounded tensor stable operator.*

Let $0=n_0<n_1<n_2<\cdots$ a strictly increasing sequence of positive integers satisfying the following conditions:

(1) For every $k\geq 0$,
$$\sum_{i=0}^{k} n_i < n_{k+1}.$$
(2) There exists $C>0$ such that
$$n_{k+1}/n_k \leq C$$
for every $k\geq 1$.

We set $N_0=0$ and, $N_k=\sum_{i=0}^{k} n_k$ for $k=1,2,3,\ldots$, and

$$a_0=\frac{1}{2} \quad\text{and}\quad a_i=\frac{1}{n_{k+1}}\frac{1}{2^{k+2}}, \quad \text{if } i\in[N_k+1,N_{k+1}]. \tag{9}$$

We also set $I_n=[1-\sum_{i=0}^{|n|} a_i, 1-\sum_{i=0}^{|n|-1} a_i)$ for $n<0$, $I_0=[1-a_0,1)$ and $I_n=\emptyset$ for $n>0$.

Let $f\in\mathscr{D}$ be the function defined by: $f=\sum_{n\in\mathbb{Z}} 2^{-n}\chi_{I_n}=\sum_{n=-\infty}^{0} 2^{-n}\chi_{I_n}$. Then $a_n(f,1/2)=a_{|n|}$ if $n\leq 0$ and $a_n(f,1/2)=0$ if $n>0$.

Suppose that $n<0$, $|n|\in[N_k+1,N_{k+1}]$, that is $|n|=N_k+l$, with $1\leq l\leq n_{k+1}$.

We calculate $a_n((f\otimes f)^\star,1/2)$. By Corollary 3.6, we have

$$\begin{aligned} a_n\big((f\otimes f)^\star,1/2\big) &= \sum_{i+j=n} a_i(f,1/2)a_j(f,1/2) \\ &= \sum_{i+j=n} a_{|i|}a_{|j|} = \sum_{|i|+|j|=|n|} a_{|i|}a_{|j|}. \end{aligned}$$

Suppose that $|i|+|j|=|n|$; we claim that $|i|>N_{k-1}$ or $|j|>N_{k-1}$. Indeed, if $|i|\leq N_{k-1}$ and $|j|\leq N_{k-1}$, then, by (1) above, $|i|+|j|\leq 2N_{k-1}<N_k+1$. This is a contradiction since $|i|+|j|=|n|\in[N_k+1,N_{k+1}]$. Set $I=\{|i|\leq N_{k-1}\}$, $J=\{|i|>N_{k-1}\}$. We have

$$a_n\big((f\otimes f)^\star,1/2\big)=\sum_{|i|+|j|=|n|} a_{|i|}a_{|j|}=\sum_{|i|\in I} a_{|i|}a_{|n|-|i|}+\sum_{|i|\in J} a_{|i|}a_{|n|-|i|}.$$

If $|i| \in I$, then $|n| - |i| = |j| > N_{k-1}$ and

$$a_{|n|-|i|} \leq \frac{1}{n_k} \frac{1}{2^{k+1}}.$$

Since $\sum_{i\in\mathbb{Z}} a_i = 1$ we obtain

$$\sum_{|i|\in I} a_{|i|} a_{|n|-|i|} \leq \frac{1}{n_k} \frac{1}{2^{k+1}}.$$

If $|i| \in J$, then $|i| > N_{k-1}$ and

$$a_{|i|} \leq \frac{1}{n_k} \frac{1}{2^{k+1}}.$$

Since $\sum_{i\in\mathbb{Z}} a_i = 1$ we obtain

$$\sum_{|i|\in J} a_{|i|} a_{|n|-|i|} \leq \frac{1}{n_k} \frac{1}{2^{k+1}}.$$

Hence,

$$a_n\big((f \otimes f)^\star, 1/2\big) \leq 2\frac{1}{n_k} \frac{1}{2^{k+1}} \leq 4C\frac{1}{n_{k+1}} \frac{1}{2^{k+2}} = 4Ca_n(f, 1/2).$$

For $n = 0$, we have

$$a_0\big((f \otimes f)^\star, 1/2\big) = a_0^2 \leq 4Ca_0 = 4Ca_0(f, 1/2)$$

since $C > 1$. It follows from Corollary 4.11 that f is tensor stable.

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