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COEFFICIENT IDEALS IN DIMENSION TWO

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ABSTRACT. We describe coefficient ideals for both (x, y)-primary monomial ideals in $k[x, y]$ and $\mathfrak{m}$-primary ideals in two-dimensional regular local rings $(R, \mathfrak{m})$ by linking them to certain ideals of reduction number one. In the monomial case, we then explicitly determine the generators of a coefficient ideal by showing their symmetric relationship to the generators of the associated reduction number one ideal.

1. Introduction

In this paper, we describe coefficient ideals for both (x, y)-primary monomial ideals in $k[x, y]$ as well as $\mathfrak{m}$-primary ideals in regular local rings $(R, \mathfrak{m})$ of dimension two and their relationship to ideals of reduction number one. The coefficient ideal was first introduced by Aberbach and Huneke:

DEFINITION 1 ([2, 2.1]). Let I and J be two ideals. The *coefficient ideal of I and J* is the largest ideal $\mathfrak{a}$ such that $\mathfrak{a}I = \mathfrak{a}J$.

We show in the following that if I is either a 0-dimensional monomial ideal in $k[x, y]$ or a 0-dimensional ideal in a two-dimensional regular local ring, then the coefficient ideal of I and a minimal reduction J of I is independent of J, and we denote this ideal $\operatorname{coef}(I)$. In the process, we show that this coefficient ideal appears as a component of the canonical module of $R[It]$ and is therefore closely related to $\operatorname{core}(I)$, the intersection of all reductions of I. This extends similar results for different classes of ideals shown by I. Aberbach, A. Corso, A. Hosry, C. Huneke, E. Hyry, T. Järvilehto, S. Ohnishi, C. Polini, K. Smith, B. Ulrich, and K. Watanabe [1], [2], [5], [10], [11], [12], [16], [17].

To more fully understand $\operatorname{coef}(I)$, we use linkage theory to define another ideal I^*, the unique largest ideal containing I with the same coefficient ideal

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as I. The ideal I^* is also the unique smallest ideal of reduction number at most one containing I. If, additionally, $I^* = \check{I}$, where $\check{I}$ is the first coefficient ideal of I in the sense of [19], then I^* is also the largest ideal with the same core as I and in fact $\operatorname{core}(I) = I\operatorname{coef}(I) = I^*\operatorname{coef}(I^*)$.

Finally, by using coefficients of the Hilbert–Samuel function and the structure of monomial ideals, we show that, in the polynomial case, the exponent sets of $\operatorname{coef}(I)$ and I^* have an Alexander-like duality similar to that discussed by Miller and Sturmfels (see [15, 5.27]).

2. The coefficient ideal and its link

We begin with a lemma concerning the degree one component of the canonical module of $R[It, t^{-1}]$ described by Polini and Ulrich in [17]. The lemma uses the notion "general locally minimal reduction" as defined by Polini, Ulrich, and Vitulli in [18, 3.3], a locally minimal reduction of a monomial ideal I generated by d general linear combinations of the minimal monomial generators of I as well as appropriate powers of each of the variables to ensure it is also a reduction of I.

Lemma 1. *Let I be a 0-dimensional ideal either in a regular local ring $(R, \mathfrak{m})$ of dimension d or monomial in $k[x_1, \dots, x_d]$. Let J be a minimal reduction of I or a general locally minimal reduction of I, respectively. If $r = r_J(I)$, then $J^r : I^r = \operatorname{core}(I) : I$. If I is a monomial ideal in a polynomial ring $k[x_1, \dots, x_d]$ and J is a general locally minimal reduction of I, then additionally $J^r : I^r = \operatorname{mono}(J : I)$, where $\operatorname{mono}(J : I)$ is the largest monomial ideal contained in $J : I$.*

Proof. Under the hypotheses, $\operatorname{core}(I) = J^{r+1} : I^r$ (see, for instance, [17, 4.5] and [18, 2.3]). Notice that the colon ideal $J^t : I^t$ is stable for all $t \geq r$, so $\operatorname{core}(I) : I = (J^{r+1} : I^r) : I = J^{r+1} : I^{r+1} = J^r : I^r$. The equality $J^r : I^r = \operatorname{mono}(J : I)$ uses the characterization of the core of a monomial ideal, $\operatorname{core}(I) = \operatorname{mono}(J)$, given in [18, 3.6]. That is, $J^r : I^r = \operatorname{core}(I) : I = \operatorname{mono}(J) : I$. The inclusion $\operatorname{mono}(J) : I \subseteq J : I$ is clear, and since the first ideal is monomial, we have $\operatorname{mono}(J) : I \subseteq \operatorname{mono}(J : I)$. Finally, if m is a monomial generator of $\operatorname{mono}(J : I)$, then $mI \subseteq \operatorname{mono}(J)$ since I is also monomial. Thus, $\operatorname{mono}(J) : I = \operatorname{mono}(J : I)$. $\square$

Remark 1. Since $J^r : I^r \subseteq J^{r-1} : I^{r-1} \subseteq \cdots \subseteq J : I$, Lemma 1 shows that in fact $\operatorname{mono}(J^s : I^s) = \operatorname{mono}(J : I)$ for all $s \geq 1$.

The following results generalize the case presented in [18, 4.10] in which I is an ideal of $k[x, y]$ with a 2-generated monomial reduction.

Assumptions 1. For the remainder of the section, let I be an ideal in a regular local ring $(R, \mathfrak{m})$ of dimension two or a monomial ideal in $k[x, y]$. We can assume I is 0-dimensional by pulling out the greatest common divisor. Let J be a minimal reduction of I or a locally minimal reduction of I, respectively.

Proposition 2. *Define* $I^* := J : (J^r : I^r)$, *where* I *and* J *satisfy Assumptions* 1 *and* $r = r_J(I)$. *Then* I^* *does not depend on* J *and* $r(I^*) \leq 1$. *In addition*, $J : I^* = J^r : I^r$.

Proof. We will work with the characterization $J : (J^r : I^r) = J : (\mathrm{core}(I) : I)$ given by Lemma 1. If necessary, localize to assume J is generated by a regular sequence. By the Briançon–Skoda theorem [14, 1], $JI \subseteq I^2 \subseteq \mathrm{core}(I)$. That is, $J \subseteq \mathrm{core}(I) : I$. Since all of these ideals are $\mathfrak{m}$-primary and hence unmixed, $I^* = J : (\mathrm{core}(I) : I)$ is a link. Taking the colon with J again, we obtain $J : I^* = \mathrm{core}(I) : I$, which does not depend on J. Hence, the ideal I^* is 2-balanced [20, 3.6]. Therefore, $r(I^*) \leq 1$ according to [20, 4.8]. Now suppose K is another minimal reduction of I. Then by the discussion above, $J : I^* = K : I^* = \mathrm{core}(I) : I$. But $K : I^*$ is a link, so $I^* = K : (K : I^*) = K : (\mathrm{core}(I) : I) = K : (K^r : I^r)$. Thus, I^* is independent of the minimal reduction used to define it. □

Remark 2. If I as above is monomial, then I^* is also a monomial ideal. This is because I^* remains fixed under the torus action. See, for instance, [4, 5.1].

We can now equate the coefficient ideal of I and a minimal reduction J of I with the degree one component of the canonical module of $R[It, t^{-1}]$.

Proposition 3. *Let* I *and* J *satisfy Assumptions* 1 *and let* $r = r_J(I)$. *Then* $J^r : I^r = \mathrm{core}(I) : I$ *is the coefficient ideal of* I *and* J. *In particular, the coefficient ideal of* I *and* J *does not depend on* J.

Proof. If $\mathfrak{a}$ is an ideal of R, then $I\mathfrak{a} = J\mathfrak{a}$ implies $I^r\mathfrak{a} = J^r\mathfrak{a}$. Hence, the coefficient ideal of I and J is contained in $J^r : I^r$. For the other inclusion, consider I^* of Proposition 2. Then $J^r : I^r = J : I^*$, and because $r(I^*) \leq 1$, $\mathrm{core}(I^*) = J(J : I^*) = I(J : I^*)$ [5, 2.6]. By substitution, $J^r : I^r$ is contained in the coefficient ideal, as desired. □

Proposition 3 generalizes a result of Hyry, in which he assumed the Rees algebra $R[It]$ was Cohen–Macaulay [10, 3.4]. Since the coefficient ideal of I and J does not depend on the minimal reduction J, we can now simplify notation by defining the coefficient ideal of I.

Definition 2. Let I and J satisfy Assumptions 1. We define the *coefficient ideal of* I to be the coefficient ideal of I and J and denote this ideal by $\mathrm{coef}(I)$.

Corollary 4. *Let* I *satisfy Assumptions* 1, *and define* I^* *as in Proposition* 2. *Then* I^* *is the largest ideal between* I *and* $\overline{I}$ *with the same coefficient ideal as* I *and the smallest ideal between* I *and* $\overline{I}$ *with reduction number at most one.*

Proof. By definition, $I^* = J : \operatorname{coef}(I)$. Since a minimal reduction J of I is also a minimal reduction of I^*, $\operatorname{coef}(I^*) = J : I^*$ by Proposition 3. As all ideals are $\mathfrak{m}$-primary and locally I^* is linked to $\operatorname{coef}(I)$, indeed we have $J : I^* = J : (J : \operatorname{coef}(I)) = \operatorname{coef}(I)$.

Let L be an ideal with $I \subseteq L \subseteq \overline{I}$. If $\operatorname{coef}(L) = \operatorname{coef}(I^*) = J : I^*$, then $J(J : I^*) = L(J : I^*)$ by the definition of $\operatorname{coef}(L)$. Therefore, $L \subseteq J : (J : I^*) = I^*$. On the other hand, suppose $r(L) = 1$. Then since J is also a minimal reduction of L, we have $\operatorname{coef}(L) = (J : L)$ and so $J(J : L) = L(J : L) = I(J : L)$. But then $J : L \subseteq \operatorname{coef}(I) = \operatorname{coef}(I^*) = J : I^*$. By the reverse inclusion property of colons, $I^* \subseteq L$. □

Using Corollary 4, we now formalize the notion of I^* introduced in Proposition 2.

REMARK 3. Let I and J satisfy Assumptions 1, and let $r = r(I)$. Set $I^* := J : (J^r : I^r)$. By the above propositions, this ideal is the unique ideal of reduction number one between I and its integral closure with $\operatorname{coef}(I^*) = \operatorname{coef}(I)$. In addition, $\operatorname{core}(I^*) = I \operatorname{coef}(I) = J \operatorname{coef}(I)$.

REMARK 4. Notice if I is a 0-dimensional monomial ideal in $k[x, y]$, then $I^* = \operatorname{core}(I) : \operatorname{coef}(I)$. Indeed, if J is a general locally minimal reduction of I, then $I^* = J : \operatorname{coef}(I) = \operatorname{mono}(J : \operatorname{coef}(I))$ because I^* is monomial. Since $\operatorname{coef}(I)$ is monomial, this gives us $I^* = \operatorname{mono}(J) : \operatorname{coef}(I) = \operatorname{core}(I) : \operatorname{coef}(I)$. Furthermore, since $\operatorname{core}(I^*) = I^* \operatorname{coef}(I^*) = I^* \operatorname{coef}(I)$ by the previous remark, I^* satisfies the cancellation property $I^* \operatorname{coef}(I) : \operatorname{coef}(I) = I^*$.

If I satisfies Assumptions 1, then the ideal I^* associated to I is a reduction number one ideal with the same degree one component of the canonical module of its extended Rees algebra; namely, $\operatorname{coef}(I)$. Since $\operatorname{core}(I)$ is the degree two component of the canonical module, it is natural to ask whether $\operatorname{core}(I) = \operatorname{core}(I^*)$. We can even ask whether $\omega_{R[It,t^{-1}]}$ is generated in degrees less than or equal to one. Theorem 5 below shows that often this is the case. In fact, $\operatorname{core}(I) = \operatorname{core}(I^*)$ precisely when $I^* = \check{I}$, where $\check{I}$ is the first coefficient ideal of I, the largest ideal containing I with the same Hilbert–Samuel function coefficients e_0 and e_1 as I.

THEOREM 5. *Let I and J satisfy Assumptions 1 such that additionally $r_J(I) = r(I)$. Set $r = r(I)$. Then the following are equivalent*:

(a) $I^* = \check{I}$;
(b) $r(\check{I}) \leq 1$;
(c) $\lambda(R/\check{I}) = e_0(\check{I}) - e_1(\check{I})$;
(d) $R[\check{I}t]$ *is Cohen–Macaulay*;
(e) $J^i(J^s : I^s) = J^{s+i} : I^s$ *for all* $s \geq r$ *and* $i \geq 0$;
(f) $J^i(J^{s'} : L^{s'}) = J^{s'+i} : L^{s'}$ *for all* $s' \geq r_J(L)$ *and* $i \geq 0$, *where L is any ideal integral over I with the same core as I.*

If these equivalent conditions hold, then $\operatorname{core}(I) = J\operatorname{coef}(I) = I\operatorname{coef}(I) = \operatorname{core}(I^*)$.

Proof. Clearly, (a) implies (b). For the converse, $\omega_{R[It,t^{-1}]} = \omega_{R[\check{I}t,t^{-1}]}$ by [18, 4.1]. Specifically, the degree one components must be the same. Therefore, $\operatorname{coef}(I) = \operatorname{coef}(\check{I})$. But then $\check{I} \subseteq I^*$, since I^* is the largest ideal satisfying $\operatorname{coef}(I) = \operatorname{coef}(I^*)$. Since I^* is also the smallest ideal of reduction number at most one containing I, the condition $r(\check{I}) \leq 1$ ensures $I^* = \check{I}$.

The equivalence of (b) and (c) was proved by Huneke in [9, 2.1].

We can see the equivalence of (b) and (d) by combining [21, 3.1] with [7, 3.10].

To show (e) is equivalent to (a) and (b), let us first assume (e). Then we can take $u = 0$ in [18, 4.2]. Consequently, $\check{I} = J : (J^s : I^s)$ for all $s \gg 0$. But this is I^* by definition. For the converse, if $I^* = \check{I}$, then $r(\check{I}) \leq 1$. Since J is (locally) a minimal reduction of I, it must also (locally) be a minimal reduction of $\check{I}$. Thus we can apply [5, 2.6] to conclude $J^{i+s} : \check{I}^s = J^i(J^s : \check{I}^s) = I^i(J^s : \check{I}^s)$ for all $s \gg 0$ and $i \geq 0$. Thus for all $s \gg 0$ and $i \geq 0$, we have the following string of inequalities:

$$\begin{aligned} J^i\big(J^s : I^s\big) \subseteq J^{s+i} : I^s &= [\omega_{R[It,t^{-1}]}]_{d+i-1} \\ &= [\omega_{R[\check{I}t,t^{-1}]}]_{d+i-1} \\ &= J^{s+i} : \check{I}^s \\ &= J^i\big(J^s : \check{I}^s\big) \\ &= J^i\big(J^s : I^s\big). \end{aligned}$$

The final equality follows from $\operatorname{coef}(I) = \operatorname{coef}(\check{I})$ and Proposition 3. Hence, $J^i(J^s : I^s) = J^{s+i} : I^s$ for all $s \gg 0$ and $i \geq 0$. But $J^{s+i} : I^s$ is stable for all $s \geq r$. Indeed, for all $s \geq r$,

$$J^{s+1+i} : I^{s+1} = J^{s+1+i} : JI^s = \big(J^{s+1+i} : J\big) : I^s = J^{s+i} : I^s.$$

To show (f) is equivalent to (b) and (e), first assume (e). Polini, Ulrich, and Vitulli prove in [18, 4.9] that if $J^i(J^s : I^s) = J^{s+i} : I^s$ for $s \geq r$ and $i \geq 0$, then $\check{I}$ is largest ideal integral over I with the same core as I. Thus, $I \subseteq L \subseteq \check{I}$ for any L in the hypothesis. But then $\check{L} = \check{I}$, so by the equivalence of (b) and (e), $r(\check{L}) \leq 1$. We now apply (e) to L to see that $J^i(J^{s'} : L^{s'}) = J^{s'+i} : L^{s'}$ for any $s' \geq r_J(L)$ and $i \geq 0$. For the converse, we can take $L = I$ to get the desired result.

Finally, suppose the equivalent conditions hold. Recall $J^r : I^r = \operatorname{coef}(I)$ and $J^{r+1} : I^r = \operatorname{core}(I)$. Then (e) says $J\operatorname{coef}(I) = \operatorname{core}(I)$. Since $\operatorname{core}(I) = \operatorname{core}(\check{I})$, (a) finishes the proof. $\square$

The examples below show that the equivalent conditions of Theorem 5 do not hold for all monomial ideals in $k[x,y]$. Sometimes, $\check{I}$ is not even the largest ideal containing I with the same core as that of I.

EXAMPLE 1. Let $I=(x^6,x^3y^3,xy^6,y^7)$ and let J be a general locally minimal reduction of I. By CoCoA calculations following the method introduced in [18], $r_J(I)=r(I)=2$, but $I=\check{I}$ and

$$\operatorname{core}(I)=\big(y^{12},xy^{11},x^2y^{10},x^3y^8,x^4y^7,x^5y^6,x^6y^4,x^7y^3,x^9\big).$$

However, $x^9\notin I\operatorname{coef}(I)$. If it were, then we would have $x^3\in\operatorname{coef}(I)$ and, in turn, $x^3y^7\in I\operatorname{coef}(I)\subseteq\operatorname{core}(I)$. But $x^3y^7\notin\operatorname{core}(I)$, a contradiction. For this ideal, $I=\check{I}$ is still the unique largest monomial ideal integral over I with the same core as I. Any strictly larger monomial ideal with the same integral closure has a smaller core.

EXAMPLE 2. Let $I=(x^{15},x^9y^6,x^4y^{12},y^{17})$ and let J be a general locally minimal reduction of I. By CoCoA calculations, $r_J(I)=r(I)=8$, but $I(J^8:I^8)\neq(J^9:I^8)=\operatorname{core}(I)$. In particular, $x^{27}\in\operatorname{core}(I)$, but $x^{12}I\nsubseteq\operatorname{core}(I)$. Again by CoCoA calculations, the first coefficient ideal is

$$\check{I}=\big(x^{15},x^{12}y^5,x^9y^6,x^6y^{11},x^4y^{12},x^3y^{14},y^{17}\big),$$

and $r_J(\check{I})=6$. On the other hand, the largest ideal containing I with the same core as I is the reduction number two ideal

$$L=\big(x^{15},x^{14}y^3,x^{13}y^4,x^{12}y^5,x^9y^6,x^8y^9,x^7y^{10},\\ x^6y^{11},x^4y^{12},x^3y^{14},x^2y^{15},xy^{16},y^{17}\big).$$

The Hilbert coefficient $e_1(I)=e_1(\check{I})=97$, while $e_1(L)=100$.

3. The monomial case

For a 0-dimensional monomial ideal I in $k[x,y]$, we combine our results from the previous section with the structure of the exponent set of I to provide a combinatorial description for $\operatorname{coef}(I)$. Because we have shown that I has the same coefficient ideal as a possibly larger ideal of reduction number one, we may reduce to $r(I)\le 1$ and $\operatorname{coef}(I)=J:I$, where J is a locally minimal reduction of I. When J is monomial, the duality between the exponent sets of I and $J:I$ is well known and easy to show. In particular, if $J=(x^a,y^b)$, then $x^sy^t\in I-J$ if and only if $x^{a-s-1}y^{b-t-1}\notin J:I$. See, for example, Miller and Sturmfels [15, 5.27]. We will describe a surprising generalization of this duality when J is not monomial; that is, when I does not have a 2-generated monomial reduction.

ASSUMPTIONS 2. For the remainder of the paper, we assume I is a 0-dimensional monomial ideal in the polynomial ring $R=k[x,y]$ over a field of either characteristic zero or sufficiently large characteristic, and $r(I)\le 1$. Let

$I=(\mathbf{x}^{\mathbf{a}_0},\dots,\mathbf{x}^{\mathbf{a}_n})$, where $\mathbf{x}^{\mathbf{a}_i}=x^{a_i}y^{b_i}$, $a_0>\cdots>a_n=0$ and $0=b_0<\cdots<b_n$. Let J be a locally minimal reduction of I.

In addition, we denote the exponent set of an ideal I by $\Gamma(I)=\{(a,b)|x^ay^b\in I)\}$. The Newton polyhedron of I, denoted $\mathrm{NP}(I)$, is the convex hull of $\Gamma(I)$ in $\mathbb{R}^2$.

Our description of $\mathrm{coef}(I)=J:I$ relies on the Hilbert–Burch matrix for I. Recall that this matrix is given by the $(n+1)\times n$ matrix $M=\{m_{ij}\}$, where $m_{ii}=-y^{b_i-b_{i-1}}$, $m_{(i+1)i}=x^{a_{i-1}-a_i}$ for $i=1,\dots,n$, and $m_{ij}=0$ for $j\neq i-1,i$. That is,

$$M=\begin{array}{c} \\ 0\\ 1\\ 2\\ \vdots\\ n-1\\ n\end{array}\begin{array}{c}\begin{array}{cccccc} 0 & 1 & 2 & \cdots & n-2 & n-1\end{array}\\ \begin{pmatrix} -y^{b_1} & 0 & 0 & \cdots & 0 & 0\\ x^{a_0-a_1} & -y^{b_2-b_1} & 0 & \cdots & 0 & 0\\ 0 & x^{a_1-a_2} & -y^{b_3-b_2} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & x^{a_{n-2}-a_{n-1}} & -y^{b_n-b_{n-1}}\\ 0 & 0 & 0 & \cdots & 0 & x^{a_{n-1}}\end{pmatrix}\end{array}.$$

Because $J:I$ does not depend on the locally minimal reduction J, we can apply [6, 2.4] to conclude $J:I$ is generated by the $(n-1)\times(n-1)$ minors of M. Consider the minor formed by deleting column ℓ and rows j and k of M, where $0\le\ell\le n-1$ and $0\le j<k\le n$. Notice if either $\ell<j$ or $\ell>k-1$, then the minor generated by deleting column ℓ and rows j and k is zero. Hence, we may assume $0\le j\le\ell<k\le n$. If we delete column ℓ and rows j and k under these conditions, the resulting $(n-1)\times(n-1)$ matrix M' is a block diagonal matrix composed of four blocks, call them A, B, C, and D, defined as follows:

$$M'=\left(\begin{array}{cc|cc} A & 0 & 0 & 0\\ \hline 0 & B & 0 & 0\\ 0 & 0 & C & 0\\ \hline 0 & 0 & 0 & D\end{array}\right),$$

$$A=\begin{pmatrix} -y^{b_1} & 0 & \cdots & 0 & 0\\ x^{a_0-a_1} & -y^{b_2-b_1} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & -y^{b_{j-1}-b_{j-2}} & 0\\ 0 & 0 & \cdots & x^{a_{j-2}-a_{j-1}} & -y^{b_j-b_{j-1}}\end{pmatrix},$$

$$B=\begin{pmatrix} x^{a_j-a_{j+1}} & -y^{b_{j+2}-b_{j+1}} & \cdots & 0 & 0\\ 0 & x^{a_{j+1}-a_{j+2}} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & x^{a_{\ell-2}-a_{\ell-1}} & -y^{b_{j-1}-b_{j-2}}\\ 0 & 0 & \cdots & 0 & x^{a_{\ell-1}-a_\ell}\end{pmatrix},$$

$$C = \begin{pmatrix} -y^{b_{\ell+2}-b_{\ell+1}} & 0 & \dots & 0 & 0 \\ x^{a_{\ell+1}-a_{\ell+2}} & -y^{b_{\ell+3}-b_{\ell+2}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -y^{b_{k-1}-b_{k-2}} & 0 \\ 0 & 0 & \dots & x^{a_{k-2}-a_{k-1}} & -y^{b_k-b_{k-1}} \end{pmatrix},$$

$$D = \begin{pmatrix} x^{a_k-a_{k+1}} & -y^{b_{k+2}-b_{k+1}} & \dots & 0 & 0 \\ 0 & x^{a_{k+1}-a_{k+2}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & x^{a_{n-2}-a_{n-1}} & -y^{b_n-b_{n-1}} \\ 0 & 0 & \dots & 0 & x^{a_{n-1}} \end{pmatrix}.$$

Then, up to a factor of -1,

$$\begin{aligned} \det\bigl(M'\bigr) &= \det(A)\det(B)\det(C)\det(D) \\ &= y^{b_j}x^{a_j-a_\ell}y^{b_k-b_{\ell+1}}x^{a_k} \\ &= x^{a_j+a_k-a_\ell}y^{b_j+b_k-b_{\ell+1}}. \end{aligned}$$

Since every generator of $J:I$ can be obtained in this manner, the following lemma results.

Lemma 6. *Let I and J satisfy Assumptions 2. Then*

$$\operatorname{coef}(I) = J:I = \bigl(\bigl\{x^{a_j+a_k-a_\ell}y^{b_j+b_k-b_{\ell+1}}\bigr\}_{0\le j\le \ell<k\le n}\bigr).$$

By examining which of the generators of $\operatorname{coef}(I)$ given by Lemma 6 are minimal, we see the shape of $\Gamma(\operatorname{coef}(I))$ emerge. The following theorem details this shape and its symmetry with $\Gamma(I)$.

Theorem 7. *Let I and J satisfy Assumptions 2 with $\{t_0,\dots,t_s\} \subseteq \{0,\dots,n\}$ so that $\mathbf{a}_{t_0},\dots,\mathbf{a}_{t_s}$ minimally determine $\mathrm{NP}(I)$. Then $\Gamma(\operatorname{coef}(I)) = \Gamma(J:I)$ is minimally generated by all of the elements of the form $\mathbf{a}_{t_i}+\mathbf{a}_{t_{i+1}} - (a_\ell, b_{\ell+1})$, where $0 \le i \le s-1$ and $t_i \le \ell < t_{i+1}$.*

Before proving the theorem, we look carefully at elements of the form $\mathbf{a}_{t_i} + \mathbf{a}_{t_{i+1}} - (a_\ell, b_{\ell+1})$ and how they correspond to symmetry between $\Gamma(\operatorname{coef}(I))$ and $\Gamma(I)$. Fix $i \in \{0,\dots,s-1\}$ in order to focus on two consecutive minimal generators $\mathbf{a}_{t_i}$ and $\mathbf{a}_{t_{i+1}}$ of $\mathrm{NP}(I)$. Figures 1(A) and 1(D) illustrate this process for the reduction number one ideal $I = (x^{11}, x^9y^2, x^6y^3, x^5y^5, x^4y^6, x^2y^7, xy^9, y^{10})$. Fix ℓ so that $t_i \le \ell < t_{i+1}$. Notice since $\mathbf{a}_\ell$ and $\mathbf{a}_{\ell+1}$ are consecutive minimal generators of $\Gamma(I)$, the point $(a_\ell, b_{\ell+1})$ forms an inside corner of the region in $\mathbb{R}^2$ representing $\Gamma(I)$. See Figure 1(B) for an illustration. If we reflect the inside corner $(a_\ell, b_{\ell+1})$ through the point $(\mathbf{a}_{t_i} + \mathbf{a}_{t_{i+1}})/2$, the center of the box between $\mathbf{a}_{t_i}$ and $\mathbf{a}_{t_{i+1}}$, the resulting point is $\mathbf{a}_{t_i} + \mathbf{a}_{t_{i+1}} - (a_\ell, b_{\ell+1})$. See Figure 1(E) for an illustration.

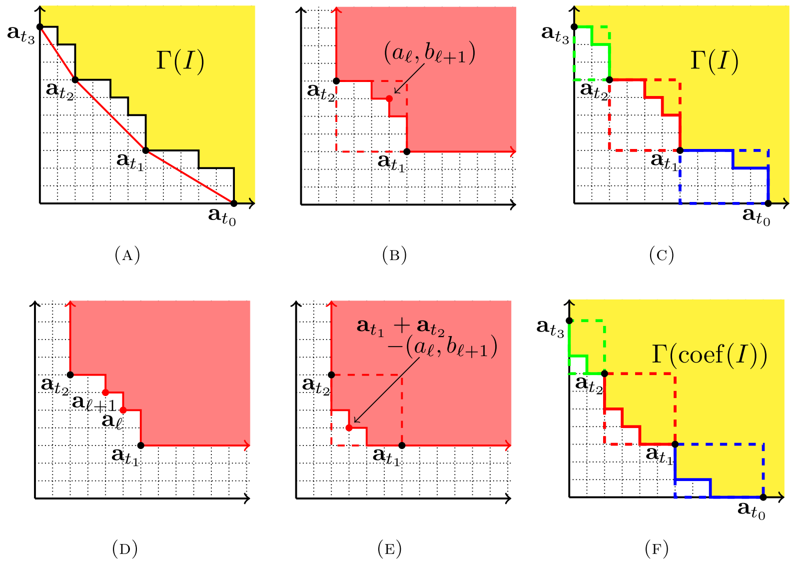

FIGURE 1. Duality of $\Gamma(I)$ and $\Gamma(\operatorname{coef}(I))$.

Theorem 7 shows that by repeating this process for all $i \in \{0, \dots, s-1\}$ and for all pairs of consecutive generators $\mathbf{a}_\ell$ and $\mathbf{a}_{\ell+1}$ for which $t_i \le \ell < t_{i+1}$, we actually generate $\Gamma(\operatorname{coef}(I))$. This creates a duality between $\Gamma(I)$ and $\Gamma(\operatorname{coef}(I))$ which locally (i.e., between generators $\mathbf{a}_{t_i}$ and $\mathbf{a}_{t_{i+1}}$) mirrors the Alexander duality of [15, 5.27]. Figures 1(C) and 1(F) exhibit this local duality.

In order to prove Theorem 7, we first relate the length of $J : I$ to the Hilbert coefficient $e_1(I)$.

LEMMA 8. *Let I and J satisfy Assumptions 2. Then $\lambda(R/J : I) = e_1(I)$, where λ denotes R-module length.*

Proof. In [8, 4.10], Huckaba and Marley showed that $R[It]$ is Cohen–Macaulay if and only if

$$e_1(I) = \sum_{n=1}^{d-1} \lambda_R\big(I^n / I^n \cap J\big),$$

where $d = \dim(R)$. Since $r(I) \le 1$, $R[It]$ is Cohen–Macaulay by Theorem 5. Then, because $d = 2$, we have

$$e_1(L) = \lambda(I/I \cap J) = \lambda(I/J) = \lambda\big(R/(J : I)\big),$$

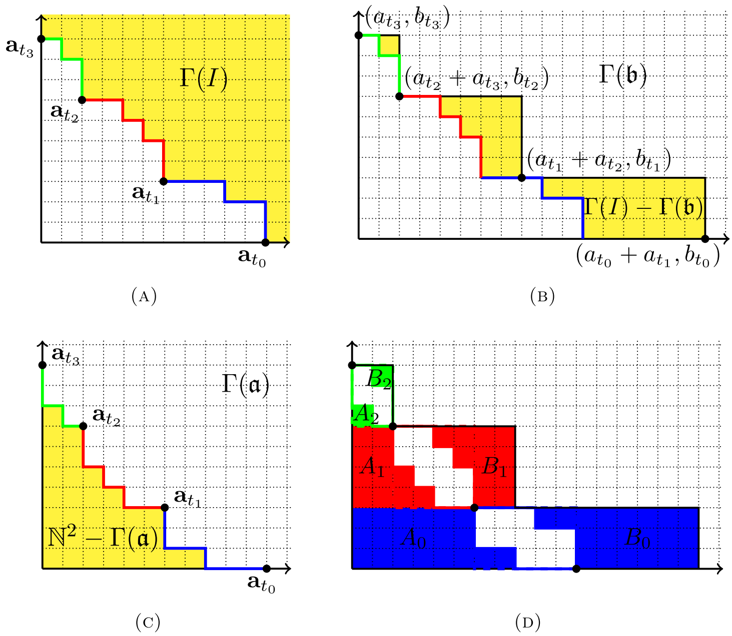

FIGURE 2. Illustration of the proof of Proposition 7.

where the final equality follows from linkage theory. That is, $J : I$ is linked to I via J, so $\omega_{R/(J:I)} = I/J$. Since $R/(J : I)$ is Artinian, $\lambda(\omega_{R/(J:I)}) = \lambda(R/(J : I))$ by [3, 3.2.12(e)(i)]. $\square$

We are now able to prove the theorem.

Proof of Theorem 7. Let $\mathfrak{a}$ be the ideal

$$\mathfrak{a} = \big(\mathbf{x}^{\mathbf{e}} | \mathbf{e} = \mathbf{a}_{t_{i-1}} + \mathbf{a}_{t_i} - (a_\ell, b_{\ell+1}), i \in [s], t_{i-1} \le \ell < t_i\big).$$

Each element of the form $\mathbf{a}_{t_{i-1}} + \mathbf{a}_{t_i} - (a_\ell, b_{\ell+1})$ is an element of $\Gamma(J : I)$ by Lemma 6. Thus, $\mathfrak{a} \subseteq J : I$. To show equality, we compare lengths. Using Lemma 8, we need only show $\lambda(R/\mathfrak{a}) = e_1(I)$.

Set $e_0 = e_0(I)$ and $e_1 = e_1(I)$. Recall $\mathbf{a}_{t_i} = (a_{t_i}, b_{t_i})$. Consider the ideal $\mathfrak{b} = \sum_{i=0}^{s}(x^{a_{t_i}+a_{t_{i+1}}} y^{b_{t_i}})$, where $b_{t_{s+1}}$ is taken to be 0. Clearly $\mathfrak{b} \subseteq I$, as pictured in Figure 2(B) above, so we may consider $\lambda(I/\mathfrak{b})$.

CLAIM 1. $\lambda(I/\mathfrak{b}) = e_1$.

Proof of Claim 1. Since $r(I) \le 1$, we can apply Theorem 5 to conclude $\lambda(R/I) = e_0 - e_1$. Then by the additivity of lengths, $\lambda(I/\mathfrak{b}) = \lambda(R/\mathfrak{b}) - \lambda(R/I) = \lambda(R/\mathfrak{b}) - (e_0 - e_1)$. Thus, to show $\lambda(I/\mathfrak{b}) = e_1$, it suffices to show $\lambda(R/\mathfrak{b}) = e_0$. Now according to [22, 7.35], $\operatorname{covol}(\operatorname{NP}(I)) = e_0/2$. On the other hand, adding up areas of trapezoids, $\operatorname{covol}(\mathrm{NP}(I)) = \sum_{i=1}^{n} \frac{1}{2}(a_{t_i} + a_t)(b_{t_i} - b_{t_{i-1}})$. Then $e_0 = \sum_{i=1}^{n}(a_{t_{i-1}} + a_{t_i})(b_{t_i} - b_{t_{i-1}})$. But this is precisely $\operatorname{covol}(\Gamma(\mathfrak{b})) = \lambda(R/\mathfrak{b})$, as desired. □

CLAIM 2. $\lambda(R/\mathfrak{a}) = \lambda(I/\mathfrak{b})$.

Proof of Claim 2. Notice that we can identify $\lambda(R/\mathfrak{a})$ with $\#(\mathbb{N}^2 - \Gamma(\mathfrak{a}))$ and $\lambda(L/\mathfrak{b})$ with $\#(\Gamma(I) - \Gamma(\mathfrak{b}))$. We further identify a point (m_1, m_2) in $\mathbb{N}^2$ with the square of area one in $\mathbb{R}^2$ with vertices at $(m_1, m_2), (m_1+1, m_2), (m_1+1, m_2+1)$, and (m_1, m_2+1). By this identification, the areas of the regions in $\mathbb{R}^2$ represented by $\mathbb{N}^2 - \Gamma(\mathfrak{a})$ and $\Gamma(I) - \Gamma(\mathfrak{b})$ are equal to the number of points in the respective sets. Therefore, to prove the claim, it is sufficient to show that the area of the region represented by $\mathbb{N}^2 - \Gamma(\mathfrak{a})$ is equal to the area of the region represented by $\Gamma(I) - \Gamma(\mathfrak{b})$.

To show this equality, we first divide the two regions into pieces. Figure 2 illustrates this process. Define $A_i := \{(m_1, m_2) \in \mathbb{N}^2 - \Gamma(\mathbf{a}) | q_i \le m_2 < q_{i+1}\}$ for $i = 0, \dots, s-1$. Notice the A_i are disjoint sets whose union is $\mathbb{N}^2 - \Gamma(\mathfrak{a})$. Define $B_i := \{(n_1, n_2) \in \Gamma(I) - \Gamma(\mathbf{b}) | q_i \le n_2 < q_{i+1}\}$ for $i = 0, \dots, s-1$. Similarly, the B_i are disjoint sets whose union is $\Gamma(I) - \Gamma(\mathbf{b})$. Fix $i \in \{0, \dots, s-1\}$. We identify A_i and B_i with the respective regions in $\mathbb{R}^2$ that the two sets represent. We will show the area of A_i is equal to the area of B_i.

Let φ be the map which takes a point in $\mathbb{R}^2$ to its reflection through the point $(\mathbf{a}_{t_i} + \mathbf{a}_{t_{i+1}})/2$. That is, $\varphi((m_1, m_2)) = \mathbf{a}_{t_i} + \mathbf{a}_{t_{i+1}} - (m_1, m_2)$. Then by construction, $\varphi(B_i) = A_i$, when A_i and B_i are thought of as regions in $\mathbb{R}^2$. See Figure 2(D) for an illustration. Since area is invariant under reflection, the area of A_i must equal the area of B_i. Because i was arbitrarily chosen, it follows that the sum of the areas of the A_i for $i = 0, \dots, s-1$ must equal the sum of the areas of the B_i. Therefore, the area of the region representing $\mathbb{N}^2 - \Gamma(\mathfrak{a})$ equals the area of the region representing $\Gamma(I) - \Gamma(\mathbf{b})$, proving the claim. □

Combining Claim 1 and Claim 2 yields the desired result. □

REMARK 5. Let I satisfy Assumptions 2. Then the number of minimal generators $\mu(\operatorname{coef}(I)) = \mu(I) - 1$. Indeed, by Theorem 7 and its subsequent discussion, we see that two consecutive minimal generators of $\Gamma(I)$ correspond to exactly one minimal generator of $\Gamma(\operatorname{coef}(I))$.

Given the close connection between $\operatorname{coef}(I)$ and $\operatorname{core}(I)$, it is natural to wonder whether $\Gamma(\operatorname{core}(I))$ also has a symmetric relationship to $\Gamma(I)$, especially when $\operatorname{core}(I) = I \operatorname{coef}(I)$ under the equivalent conditions of Theorem 5. This is indeed shown to be the case in [13].

The results in this paper depend heavily on dimension two, which raises questions concerning higher dimensions. Under what conditions is the coefficient ideal still $J^r : I^r$? In dimension $d > 2$, reduction number one ideals are not as prevalent. Are there natural substitutes, such as reduction number $d-1$ ideals? What can be said about the shape of the exponent set of monomial coefficient ideals in this broader setting?

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