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ISOMETRIES ON THE VECTOR VALUED LITTLE BLOCH SPACE

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ABSTRACT. In this paper, we describe the surjective linear isometries on a vector valued little Bloch space with range space a smooth, strictly convex and reflexive complex Banach space. We also describe the hermitian operators and the generalized bi-circular projections supported by these spaces.

1. Introduction

The type of linear surjective isometries supported by a given Banach space depends largely on the geometric properties of the space, see [21], [22] and [25]. Often, these operators are described from their induced actions on the set of extreme points of the unit ball of the dual space, see [9] and [14]. In addition of being a class of operators of great intrinsic interest, linear surjective isometries play a crucial role in the definition of other important classes of operators such as the hermitian operators and the generalized bi-circular projections, see [23]. In this paper, we give a characterization of the surjective isometries on a class of vector valued little Bloch spaces and then derive the form of the hermitian operators and the generalized bi-circular projections.

The little Bloch space consists of all analytic functions f defined on the open unit disc, $\triangle = \{z \in \mathbb{C} : |z| < 1\}$, with values in a Banach space E with norm $\|\cdot\|_E$, which satisfy the condition

$$\lim_{|z|\to 1} \left(1 - |z|^2\right) \left\| f'(z) \right\|_E = 0.$$

This space with the norm $\|f\|_{\mathcal{B}} = \|f(0)\|_E + \sup_{z\in\triangle}(1 - |z|^2)\|f'(z)\|_E$ is a Banach space and will be denoted by $\mathcal{B}_*(\triangle, E)$. Towards a characterization of the surjective linear isometries on this setting, we start by considering surjective isometries on $\mathcal{B}_0(\triangle, E)$, the subspace consisting of all functions in

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$\mathcal{B}_*(\triangle, E)$ vanishing at zero. The reason for this restriction is that $\mathcal{B}_*(\triangle, E)$ is isometrically isomorphic to $\mathcal{B}_0(\triangle, E) \oplus_1 E$, and when the range space E does not support L_1-projections (see [1] and also [13]), $\mathcal{B}_0(\triangle, E)$ also does not support L_1-projections. This implies that an isometry on $\mathcal{B}_*(\triangle, E)$ admits a natural decomposition into an isometry on $\mathcal{B}_0(\triangle, E)$ and an isometry on E, cf. [1] and [18].

In order to derive a representation for the surjective isometries on $\mathcal{B}_0(\triangle, E)$, we define an embedding of $\mathcal{B}_0(\triangle, E)$ onto $\mathcal{Y}$, a closed subspace of $\mathcal{C}_0(\triangle, E)$. Then we use that the adjoint of a surjective isometry on $\mathcal{Y}$ defines a permutation on the set of extreme points of $\mathcal{Y}_1^*$. In this process we employ a result due to Brosowski and Deutsch (see [19, Corollary 2.3.6, p. 33]) stating that any extreme point of $\mathcal{Y}_1^*$ is of the form $e^*\delta_z$, with e^* a norm one functional in E^* and δ_z a point evaluation functional. The forthcoming Corollary 2.2 states that all such functionals are extreme points of $\mathcal{Y}_1^*$. This allows us to derive the form for the surjective isometries as described in Theorem 3.5.

It was shown by Vidav in [31], [32] that hermitian operators are essentially the generators of strongly continuous one parameter groups of surjective isometries. The knowledge of the surjective isometries defines naturally a class of operators containing the hermitian operators. In particular, we will show that bounded hermitian operators on $\mathcal{B}_0(\triangle, E)$ are in a one-to-one correspondence with the bounded hermitian operators of the range space. Another class of operators considered here and directly linked to surjective isometries are the generalized bi-circular projections, introduced in [20]. These projections have been studied and characterized in a variety of spaces. In most known cases, generalized bi-circular projections can be expressed as the average of the identity with an isometric reflection, see for example [10], [11], [26] and also [30]. In the last section of this paper, we extend this representation to generalized bi-circular projections on this new collection of spaces.

Throughout this paper, we assume that the range space E is a smooth, strictly convex and reflexive Banach space, however some results hold under weaker conditions.

Given a Banach space X, X_1^* denotes the unit ball of its dual space, and $\text{ext}(X_1^*)$ denotes the set of extreme points of X_1^*.

2. Extreme points of $\mathcal{B}_0(\triangle, E)_1^*$

We consider the following embedding of $\mathcal{B}_0(\triangle, E)$ into $\mathcal{C}_0(\triangle, E)$

$$\Phi : \mathcal{B}_0(\triangle, E) \to \mathcal{C}_0(\triangle, E),$$
$$f \to F = \Phi(f) : \triangle \to E,$$

given by $\Phi(f)(z) = (1 - |z|^2)f'(z)$. The map Φ is a linear isometry onto a closed subspace of $\mathcal{C}_0(\triangle, E)$, denoted by $\mathcal{Y}$. We recall that $\mathcal{C}_0(\triangle, E)$ is the set of all E-valued continuous functions defined on $\triangle$ such that $\lim_{|z|\to 1} F(z) = 0$.

A result due to Brosowski and Deutsch (see [19], Corollary 2.3.6) implies that extreme points of the unit ball of the dual space of $\mathcal{Y}$ are functionals of the form $e^\star\delta_z$, with $e^\star \in \operatorname{ext}(E_1^*)$, $z \in \triangle$ and $\delta_z : \mathcal{B}_0(\triangle, E) \to E$ the evaluation map $\delta_z(f) = f(z)$.

We now show that all such functionals are extreme points of $\mathcal{Y}_1^*$. We observe that the smoothness and reflexivity assumption on E implies that E^* is strictly convex and then every norm 1 functional in E^* is an extreme point of E_1^*. Furthermore, the smoothness and the reflexivity of E implies that for every unit vector v in E, there exists a unique functional v^* in E_1^*, such that $v^*(v) = 1$.

Lemma 2.1. *A functional τ is an extreme point of $\mathcal{Y}_1^*$ if and only if $\tau = e^*\delta_z$, with $e^* \in \operatorname{ext}(E_1^*)$ and $z \in \triangle$.*

Proof. We refer the reader to Corollary 2.3.6 in [19] which states that $\operatorname{ext}(\mathcal{Y}_1^*) \subset \{e^*\delta_z : e^* \in \operatorname{ext}(E_1^*),\ \text{and } z \in \triangle\}$. Given $z_0 \in \triangle$ and $e^* \in \operatorname{ext}(E_1^*)$ we show that $e^*\delta_{z_0}$ is an extreme point of $\mathcal{Y}_1^*$. We assume otherwise, then

$$e^*\delta_{z_0} = \frac{\varphi_1 + \varphi_2}{2}, \tag{1}$$

for φ_1 and φ_2 in $\mathcal{Y}_1^*$.

Since $\mathcal{Y}$ is a closed subspace of $\mathcal{C}_0(\triangle, E)$, the Hahn–Banach theorem implies the existence of extensions of φ_1 and φ_2, to $\mathcal{C}_0(\triangle, E)$. These functionals are written as

$$\tilde{\varphi_1}(F) = \int_\triangle F\, d\nu^* \quad \text{and} \quad \tilde{\varphi_2}(F) = \int_\triangle F\, d\mu^*,$$

with ν^* and μ^* representing regular vector valued Borel measures on $\triangle$ with values on E^*.

We consider the function in $\mathcal{B}_0(\triangle, E)$

$$f_0(z) = \frac{(1 - |z_0|^2)z}{1 - \overline{z_0}z} e,$$

with $e \in E$ such that $e^*(e) = 1$. Furthermore, $\sup_{|z|<1}(1 - |z|^2)\|f_0'(z)\| = (1 - |z_0|^2)\|f_0'(z_0)\|$ and, for all $z \in \triangle \setminus \{z_0\}$,

$$\big(1 - |z|^2\big)\big\|f_0'(z)\big\| < \big(1 - |z_0|^2\big)\big\|f_0'(z_0)\big\| = 1.$$

We apply (1) to the function $F_0(z) = (1 - |z|^2)f_0'(z)$ to conclude that $\varphi_1(F_0) = \varphi_2(F_0) = 1$. If $|\nu^*|(\triangle \setminus \{z_0\}) > 0$, then there exists a compact subset K of $\triangle \setminus \{z_0\}$ such that $|\nu^*|(K) > 0$. Clearly,

$$\sup_{z\in K}\big\|F_0(z)\big\| = \sup_{z\in K}\big(1 - |z|^2\big)\big\|f_0'(z)\big\| = \alpha < 1.$$

Hence,

$$\begin{aligned}1 = \tilde{\varphi_1}(F_0) = &\left|\int_{\triangle} F_0\, d\nu^*\right| = \left|\int_{\{z_0\}} F_0\, d\nu^* + \int_K F_0\, d\nu^* + \int_{(\triangle\setminus\{z_0\})\setminus K} F_0\, d\nu^*\right|\\ &\le |\nu^*|\big(\{z_0\}\big) + \alpha|\nu^*|(K) + |\nu^*|\big((\triangle \setminus \{z_0\}) \setminus K\big)\\ &< |\nu^*|(\triangle) = 1.\end{aligned}$$

This leads to an absurdity and shows that $|\nu^*|(\triangle \setminus \{z_0\}) = 0$ and $\nu^*(\triangle \setminus \{z_0\}) = 0$. This also implies that $\nu^*\{z_0\}$ is a norm one functional. A similar reasoning applies to μ^*. Given $F \in \mathcal{Y}$, we have

$$\begin{aligned}e^*\delta_{z_0}(F) =& \big(1-|z_0|^2\big)e^*\big(f'(z_0)\big) = \frac{\tilde{\varphi_1}(F)+\tilde{\varphi_2}(F)}{2}\\ =& \frac{1}{2}\left(\int_{\{z_0\}} F\, d\nu^* + \int_{\{z_0\}} F\, d\mu^*\right)\\ =& \frac{1}{2}\big[\nu^*(z_0)\big(1-|z_0|^2\big)f'(z_0) + \mu^*(z_0)\big(1-|z_0|^2\big)f'(z_0)\big].\end{aligned}$$

Therefore,

$$e^*\big(f'(z_0)\big) = \frac{\nu^*(z_0)(f'(z_0)) + \mu^*(z_0)(f'(z_0))}{2}.$$

The strict convexity of the scalar field implies that $e^*(f'(z_0)) = \nu^*(z_0) \times (f'(z_0)) = \mu^*(z_0)(f'(z_0))$. From the smoothness of E we have that $e^* = \nu^* = \mu^*$ and $\varphi_1 = \varphi_2$. This completes the proof. □

The next corollary gives a description of the extreme points of $\mathcal{B}_0(\triangle, E)_1^*$.

Corollary 2.2. *A functional $\tau \in \mathcal{B}_0(\triangle, E)_1^*$ is an extreme point if and only if $\tau(f) = e^*(\Phi(f)(z))$, with $z \in \triangle$ and $e^* \in \text{ext}(E_1^*)$.*

Proof. The isometry Φ induces the isometry $\Phi^* : \mathcal{Y}^* \to \mathcal{B}_0(\triangle, E)^*$, which defines a bijection between the corresponding sets of extreme points, consequently we have that $\Phi^*(e^*\delta_z) \in \text{ext}(\mathcal{B}_0(\triangle, E)_1^*)$, with $e^*\delta_z \in \text{ext}(\mathcal{Y}_1^*)$. Therefore,

$$\Phi^*\big(e^*\delta_z\big)(f) = e^*\big(\Phi(f)(z)\big).$$

This completes the proof. □

Remark 2.3. We observe that the function $f \to (f(0), f - f(0))$ defines a surjective isometry from $\mathcal{B}_*(\triangle, E)$ onto $E \oplus_1 \mathcal{B}_0(\triangle, E)$.

It is well known (cf. [19]) that $\text{ext}(\mathcal{B}_*(\triangle, E)_1^*) = \text{ext}(E_1^* \oplus_\infty (\mathcal{B}_0(\triangle, E)_1^*))$. Therefore, $\text{ext}(\mathcal{B}_*(\triangle, E)_1^*) = \{(v^*, \tau) : v^* \in \text{ext}(E_1^*), \tau \in \text{ext}(\mathcal{B}_0(\triangle, E)_1^*)$ with $(v^*, \tau)(f) = v^*(f(0)) + \tau(f - f(0))\}$.

We recall that the assumptions on E imply that every norm one functional in E^* is an extreme point of E_1^*.

3. A characterization of the surjective isometries on $\mathcal{B}_0(\triangle, E)$

In this section, we show that surjective linear isometries on $\mathcal{B}_0(\triangle, E)$ are translations of weighted composition operators.

We consider a surjective linear isometry $T: \mathcal{B}_0(\triangle, E) \to \mathcal{B}_0(\triangle, E)$ and define $S: \mathcal{Y} \to \mathcal{Y}$ such that $S \circ \Phi = \Phi \circ T$. Hence, $S^*: \mathcal{Y}^* \to \mathcal{Y}^*$ induces a permutation of $\operatorname{ext}(\mathcal{Y}_1^*)$. Therefore, for every $u^* \in \operatorname{ext}(E_1^*)$ and $z \in \triangle$, there exist $v^* \in \operatorname{ext}(E_1^*)$ and $w \in \triangle$ such that

$$S^*\big(u^*\delta_z\big) = v^*\delta_w,$$

equivalently we write

$$(2)\quad \big(1-|z|^2\big)u^*\big((Tf)'(z)\big) = \big(1-|w|^2\big)v^*\big(f'(w)\big), \quad \text{for every } f \in \mathcal{B}_0(\triangle, E).$$

Conceivably v^* and w depend on the choice of u^* and z, this determines the following two maps:

$$\begin{array}{rl} \sigma: & \triangle \times E_1^* \to \triangle, \\ & (z, u^*) \ \to w, \end{array} \quad \text{and} \quad \begin{array}{rl} \Gamma: & \triangle \times E_1^* \to E_1^*, \\ & (z, u^*) \ \to v^*. \end{array}$$

In the next two lemmas, we show that σ is independent of the second coordinate and Γ is independent of the first.

LEMMA 3.1. *Let $z_0 \in \triangle$ and $u_0^* \in E_1^*$. Then σ restricted to the set $\{(z_0, u^*) : u^* \in E_1^*\}$ is constant and it induces a disc automorphism, also denoted by σ, defined by $\sigma(z) = \sigma(z, u_0^*)$.*

Proof. We consider two distinct functionals in E_1^*, u^* and u_1^*, then we write

$$(3)\qquad \big(1-|z_0|^2\big)u^*\big((Tf)'(z_0)\big) = \big(1-|w|^2\big)v^*\big(f'(w)\big)$$

and

$$(4)\qquad \big(1-|z_0|^2\big)u_1^*\big((Tf)'(z_0)\big) = \big(1-|w_1|^2\big)v_1^*\big(f'(w_1)\big).$$

We consider $f_0 \in \mathcal{B}_0(\triangle, E)$, given by $f_0(z) = z \cdot v$, with $v \in E$ such that $v^*(v) = 1$. Applying equation (3) to f_0 we obtain $u^*((Tf_0)'(z_0)) = \frac{1-|w|^2}{1-|z_0|^2} \leq 1$ and $|w| \geq |z_0|$. Since T is surjective there exists $f \in \mathcal{B}_0(\triangle, E)$ such that $(Tf)(z) = z \cdot u$, then equation (3) applied to this function f yields $(1-|z_0|^2) = (1-|w|^2)v^*(f'(w))$. Hence, $(1-|z_0|^2) \leq (1-|w|^2)$ or $|w| \leq |z_0|$. Therefore, $|w| = |z_0|$. A similar argument using (4) implies that $|w_1| = |z_0|$ and $|w| = |w_1|$. If $w \neq w_1$, then we select a norm 1 function f_1 such that $f_1'(w) = v$ and $f_1'(w_1) = v_1$. The equations in (3) and (4) applied to f_1 yield

$$u^*\big[(Tf_1)'(z_0)\big] = u_1^*\big[(Tf_1)'(z_0)\big] = 1.$$

It follows from the smoothness of E_1^* that $u^* = u_1^*$. Therefore,

$$(5)\qquad v^*\big(f'(w)\big) = v_1^*\big(f'(w_1)\big), \quad \text{for every } f \in \mathcal{B}_0(\triangle, E).$$

This implies that $v^* = v_1^*$ and $f'(w) = f'(w_1)$, for every $f \in \mathcal{B}_0(\triangle, E)$. This contradiction implies that σ only depends on the value of the first coordinate.

Thus it induces a map (also denoted by σ) on the open disc. Since T is a surjective isometry the same reasoning applied to the inverse implies that σ is bijective.

We now show that σ is analytic. We apply the equation (2) to the functions $f_0(z) = \frac{z^2}{2}v$ and $f_1(z) = zv$ to obtain the following:

$$\big(1-|z|^2\big)u^*\big[(Tf_0)'(z)\big] = \big(1-\big|\sigma(z)\big|^2\big)v^*\big(f_0'\big(\sigma(z)\big)\big)$$

and

$$\big(1-|z|^2\big)u^*\big[(Tf_1)'(z)\big] = \big(1-\big|\sigma(z)\big|^2\big).$$

For every $z \in \triangle$, we have $u^*[(Tf_1)'(z)] \neq 0$. Therefore

$$\sigma(z) = \frac{u^*[(Tf_0)'(z)]}{u^*[(Tf_1)'(z)]}.$$

This shows that σ is analytic and then a disc automorphism. □

A disc automorphism σ is a bijective and analytic map on the open disc. It is of the form $\sigma(z) = \lambda\frac{z-z_0}{1-\overline{z_0}z}$, with λ a modulus one complex number and $z_0 \in \triangle$. The derivative $\sigma'(z) = \lambda\frac{1-|z_0|^2}{(1-\overline{z_0}z)^2}$. It is a straightforward calculation to check that $|\sigma'(z)| = \frac{1-|\sigma(z)|^2}{1-|z|^2}$.

Lemma 3.2. *If $u^* \in E_1^*$, then Γ restricted to the set $\{(z,u^*) : z \in \triangle\}$ is constant.*

Proof. The equation displayed in (2) is rewritten as

$$\begin{aligned}&\big(1-|z|^2\big)u^*\big[(Tf)'(z)\big]\\ &\quad= \big(1-\big|\sigma(z)\big|^2\big)\Gamma\big(u^*,z\big)\big[f'\big(\sigma(z)\big)\big], \quad \forall f \in \mathcal{B}_0(\triangle,E) \text{ and } z \in \triangle.\end{aligned}$$

Therefore, we get

$$u^*\big[(Tf)'(z)\big] = \frac{|\sigma'(z)|}{\sigma'(z)}\Gamma\big(u^*,z\big)\big[(f\circ\sigma)'(z)\big], \quad \forall f \in \mathcal{B}_0(\triangle,E),$$

since $\frac{1-|\sigma(z)|^2}{1-|z|^2}\sigma'(z) = |\sigma'(z)|$.

Equivalently, we write

$$\frac{u^*[(Tf)'(z)]}{\Gamma(u^*,z)[(f\circ\sigma)'(z)]} = \frac{|\sigma'(z)|}{\sigma'(z)}.$$

Thus the left-hand side is independent of the choice of u^* and f. Further, $\frac{|\sigma'(z)|}{\sigma'(z)}$ is analytic on the open disc because $z \to \frac{u^*[(Tf)'(z)]}{\Gamma(u^*,z)[(f\circ\sigma)'(z)]}$ is analytic. An application of the Maximum Modulus Principle asserts that $\frac{|\sigma'(z)|}{\sigma'(z)}$ is constant, i.e. $\frac{|\sigma'(z)|}{\sigma'(z)} = e^{i\alpha}$, for every z in the disc.

Then

$$u^*\big[(Tf)'(z)\big] = e^{i\alpha}\Gamma\big(u^*,z\big)\big[(f\circ\sigma)'(z)\big], \quad \forall z \in \triangle. \tag{6}$$

We set $v_z^* = \Gamma(u^*, z)$, for every $z \in \triangle$. Since T is surjective, let f be such that $(Tf)(z) = e^{i\alpha} zu$, then $(f \circ \sigma)'(z) = v_z$. The map $z \to (f \circ \sigma)'(z)$ is analytic, this means for every bounded functional, τ in E^*, $z \to \tau((f \circ \sigma)'(z))$ is analytic. In particular, given $z_0 \in \triangle$, $z \to v_{z_0}^*((f \circ \sigma)'(z))$ is analytic and attains a maximum value at z_0. This implies that $\Gamma(u^*, z)$ is constant.

Thus, Γ restricted to $\{(z, u^*) : z \in \triangle\}$ is constant. □

REMARK 3.3. The previous lemma implies that Γ induces a mapping from E_1^* onto E_1^*, which for simplicity it will also be denoted by Γ.

We collect some useful properties of Γ. First $\Gamma(\lambda u^*) = \lambda \Gamma(u^*)$, with λ a modulus 1 complex number. Then, for every scalar λ, we set $\Gamma(\lambda u^*) = \lambda \Gamma(u^*)$. If we set $v_1^* = \Gamma(u_1^*)$, $v_2^* = \Gamma(u_2^*)$ and $v^* = \Gamma(\frac{u_1^* + u_2^*}{\|u_1^* + u_2^*\|})$, then for every $f \in \mathcal{B}_0(\triangle, E)$ and $z \in \triangle$,

$$\begin{aligned}(1 - |z|^2) \frac{u_1^* + u_2^*}{\|u_1^* + u_2^*\|} \left[(Tf)'(z)\right] &= \left(1 - \left|\sigma(z)\right|^2\right) v^* \left[f'\left(\sigma(z)\right)\right] \\ &= \frac{1}{\|u_1^* + u_2^*\|} \left(1 - \left|\sigma(z)\right|^2\right) \left[v_1^* + v_2^*\right] \left[f'\left(\sigma(z)\right)\right].\end{aligned}$$

This implies that $v^* = \frac{v_1^* + v_2^*}{\|u_1^* + u_2^*\|}$, or equivalently

$$\Gamma\left(\frac{u_1^* + u_2^*}{\|u_1^* + u_2^*\|}\right) = \frac{1}{\|u_1^* + u_2^*\|} \left(\Gamma\left(u_1^*\right) + \Gamma\left(u_2^*\right)\right).$$

Hence, we extend Γ to a linear map $\Gamma : E^* \to E^*$. We notice that given two distinct functionals u_1^* and u_2^* we set $\Gamma(\frac{u_1^* - u_2^*}{\|u_1^* - u_2^*\|}) = w^*$. Therefore, $\Gamma(u_1^*) - \Gamma(u_2^*) = \|u_1^* - u_2^*\| w^*$ and

$$\left\|\Gamma\left(u_1^*\right) - \Gamma\left(u_2^*\right)\right\| \le \left\|u_1^* - u_2^*\right\|.$$

As in [12] (see p. 60) we employ the following result due to G. Ding from [16], see also [15].

THEOREM 3.4. *Let E and F be two real Banach spaces. Suppose V_0 is a Lipschitz mapping from E_1 into F_1 (the respective unit spheres) with Lipschitz constant equal to 1, that is $\|V_0(x) - V_0(y)\| \le \|x - y\|$, for every x, y in E_1. Assume also that V_0 is a surjective mapping such that for any $x, y \in E_1$ and $r > 0$, we have*

$$\left\|V_0(x) - rV_0(y)\right\| \wedge \left\|V_0(x) + rV_0(-y)\right\| \le \|x - ry\|$$

and $\|V_0(x) - V_0(-x)\| = 2$. Then V_0 can be extended to be a real linear isometry from E onto F.

Since Γ satisfies the conditions set in the Theorem 3.4, this assures the existence of a surjective real linear isometry from $E^* \to E^*$ that extends Γ. For simplicity of notation, we denote this extension also by Γ. We observe that the complex linearity of the isometry T implies that of Γ. Since E is reflexive then the adjoint of Γ induces a surjective linear isometry on E, we call this isometry V, therefore we have

$$u^*\big((Tf)'(z)\big) = u^*\big(V(f\circ\sigma)'(z)\big),$$

for every $u^* \in E^*$, $f \in \mathcal{B}_0(\triangle, E)$ and $z \in \triangle$. This implies that $(Tf)'(z) = V(f\circ\sigma)'(z)$. A straightforward integration yields

$$Tf(z) = V\big[(f\circ\sigma)(z) - (f\circ\sigma)(0)\big], \quad \forall f \in \mathcal{B}_0(\triangle, E), \text{ and } z \in \triangle.$$

We summarize these considerations in the following theorem.

Theorem 3.5. *Let E be a smooth, strictly convex and reflexive complex Banach space. Then $T : \mathcal{B}_0(\triangle, E) \to \mathcal{B}_0(\triangle, E)$ is a surjective linear isometry if and only if there exist a surjective linear isometry $V : E \to E$ and a disc automorphism σ such that for every $f \in \mathcal{B}_0(\triangle, E)$ and $z \in \triangle$,*

$$Tf(z) = V\big[(f\circ\sigma)(z) - (f\circ\sigma)(0)\big].$$

Proof. The necessity follows from previous considerations. We now show the sufficiency, that is, any mapping of the form described in the theorem is indeed a surjective isometry. Such an operator is bijective, with inverse $T^{-1}f(z) = V^{-1}[f(\sigma^{-1}(z)) - f(\sigma^{-1}(0))]$. We now show that $Tf(x) = V[(f\circ\sigma)(x) - (f\circ\sigma)(0)]$, with σ a disc automorphism and V a surjective isometry on E, is an isometry. We have

$$\begin{aligned}\|Tf\|_{\mathcal{B}_0(\triangle,E)} &= \sup_{z\in\triangle}\big(1-|z|^2\big)\big\|\sigma'(z)V\big(f'\big(\sigma(z)\big)\big)\big\| \\ &= \sup_{z\in\triangle}\big(1-|z|^2\big)\big|\sigma'(z)\big|\big\|f'\big(\sigma(z)\big)\big\|.\end{aligned}$$

We set $w = \sigma(z)$, then if $\sigma(z) = \lambda\frac{z-a}{1-\overline{a}z}$ we have $\sigma^{-1}(w) = \frac{\lambda a + w}{\lambda + \overline{a}w}$. Therefore

$$\begin{aligned}\big(1-|z|^2\big)\big|\sigma'(z)\big| &= \frac{(1-|a|^2)}{|1-\overline{a}\frac{w+\lambda a}{\lambda+\overline{a}w}|^2}\left(1-\left|\frac{w+\lambda a}{\lambda+\overline{a}w}\right|^2\right) \\ &= \big(1-|w|^2\big).\end{aligned}$$

This implies that $\|Tf\|_{\mathcal{B}_0(\triangle,E)} = \|f\|_{\mathcal{B}_0(\triangle,E)}$ and completes the proof. $\square$

4. Hermitian operators

In this section, we use the form of the surjective isometries to derive information about the hermitian operators on $\mathcal{B}_0(\triangle, E)$, see [2] and [3]. An operator A is hermitian if and only if iA is the generator of a strongly continuous one-parameter group of surjective isometries, see [17]. We recall that bounded hermitian operators give rise to uniformly continuous one-parameter groups of surjective isometries.

We consider one-parameter group of surjective isometries on $\mathcal{B}_0(\triangle, E)$, Theorem 3.5 implies that each isometry determines both a disc automorphism and a surjective isometry on E. The next proposition states that the group properties of the underlying group of isometries transfer to the defining families.

Proposition 4.1. *Let E be a smooth, strictly convex and reflexive complex Banach space, then $\{T_t\}_{t\in\mathbb{R}}$ is a one parameter group of surjective isometries on $\mathcal{B}_0(\triangle, E)$ if and only if there exist a one parameter group of disc automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ and one parameter group of surjective isometries on E, $\{V_t\}_{t\in\mathbb{R}}$ such that*

$$T_t(f)(z) = V_t\big[f\big(\sigma_t(z)\big) - f\big(\sigma_t(0)\big)\big], \quad \forall f \in \mathcal{B}_0(\triangle, E).$$

Proof. Let $\{T_t\}_{t\in\mathbb{R}}$ be a one parameter group of surjective isometries on $\mathcal{B}_0(\triangle, E)$. If $T_0 = I$ we have

$$V_0\big[f\circ\sigma_0 - f\big(\sigma_0(0)\big)\big] = f, \quad \forall f \in \mathcal{B}_0(\triangle, E).$$

For $f_1(z) = zv$ and $f_2(z) = z^2 v$, with v a unit vector in E, we obtain

$$\big[\sigma_0(z) - \sigma_0(0)\big]V_0(v) = zv,$$
$$\big[\sigma_0(z)^2 - \sigma_0(0)^2\big]V_0(v) = z^2 v.$$

This implies that $[\sigma_0(z) + \sigma_0(0)]zv = z^2 v$ and $\sigma_0(z) + \sigma_0(0) = z$, for every $z \in \triangle \setminus \{0\}$. The continuity of σ_0 implies that $\sigma_0(z) + \sigma_0(0) = z$, for every $z \in \triangle$. If $z = 0$ then $\sigma_0(0) = 0$ and $\sigma_0(z) = z$. Given t and s in $\mathbb{R}$, we have $T_{t+s}(f) = T_t[T_s(f)]$, then

$$\begin{aligned} T_t\big[T_s(f)\big] &= V_t\big[T_s(f)\circ\sigma_t - T_s(f)\big(\sigma_t(0)\big)\big] \\ &= V_t\big\{V_s\big[f(\sigma_s\circ\sigma_t) - f\big(\sigma_s(0)\big)\big] - V_s\big[f(\sigma_s\circ\sigma_t)(0) - f\big(\sigma_s(0)\big)\big]\big\} \\ &= V_t V_s\big(f(\sigma_s\circ\sigma_t) - f\big(\sigma_s\big(\sigma_t(0)\big)\big)\big). \end{aligned}$$

On the other hand, $T_{t+s}(f) = V_{t+s}[f\circ\sigma_{t+s} - f(\sigma_{t+s}(0))]$. Hence,

$$(*) \qquad \begin{aligned} &V_{t+s}\big[f\circ\sigma_{t+s} - f\big(\sigma_{t+s}(0)\big)\big] \\ &\quad = V_t V_s\big(f(\sigma_s\circ\sigma_t) - f\big(\sigma_s\big(\sigma_t(0)\big)\big)\big), \quad \forall f \in \mathcal{B}_0(\triangle, E). \end{aligned}$$

In particular, for f_1 and f_2 defined above, we have

$$[V_t V_s v]\big[(\sigma_s\circ\sigma_t)(z) - (\sigma_s\circ\sigma_t)(0)\big] = V_{t+s}v\big[\sigma_{s+t}(z) - \sigma_{t+s}(0)\big],$$
$$[V_t V_s v]\big[(\sigma_s\circ\sigma_t)(z)^2 - (\sigma_s\circ\sigma_t)(0)^2\big] = V_{t+s}v\big[\sigma_{s+t}(z)^2 - \sigma_{t+s}(0)^2\big].$$

Therefore,

$$\big[(\sigma_s\circ\sigma_t)(z) + (\sigma_s\circ\sigma_t)(0)\big]\big[\sigma_{s+t}(z) - \sigma_{t+s}(0)\big] = \sigma_{s+t}(z)^2 - \sigma_{t+s}(0)^2.$$

For $z \neq 0$, we have that

$$(\sigma_s\circ\sigma_t)(z) + (\sigma_s\circ\sigma_t)(0) = \sigma_{s+t}(z) + \sigma_{t+s}(0).$$

Since all functions are continuous

$$(\sigma_s\circ\sigma_t)(z) + (\sigma_s\circ\sigma_t)(0) = \sigma_{s+t}(z) + \sigma_{t+s}(0), \quad \forall z \in \triangle.$$

For $z=0$, we have $(\sigma_s \circ \sigma_t)(0) = \sigma_{t+s}(0)$. Then $\sigma_s \circ \sigma_t = \sigma_{s+t}$ and from $(*)$ we conclude that $V_t V_s = V_{t+s}$. The converse implication follows from straightforward calculations. This concludes the proof. □

The next result addresses the question of whether the strong continuity of a one-parameter group of surjective isometries $\{T_t\}_{t\in\mathbb{R}}$ also transfers to the defining symbols.

PROPOSITION 4.2. *Let E be a smooth, strictly convex and reflexive complex Banach space. If $\{T_t\}_{t\in\mathbb{R}}$ is a strongly continuous one parameter group of surjective isometries on $\mathcal{B}_0(\triangle, E)$, then there exist a strongly continuous one parameter group of surjective isometries on E, $\{V_t\}_{t\in\mathbb{R}}$ and a continuous one parameter group of disc automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ such that*

$$T_t(f)(z) = V_t\big(f\big(\sigma_t(z)\big) - f\big(\sigma_t(0)\big)\big), \quad \forall f \in \mathcal{B}_0(\triangle, E)\ \forall z \in \triangle.$$

Proof. Proposition 4.1 implies the existence of one parameter groups of surjective isometries on E and disc automorphisms, $\{S_t\}$ and $\{\sigma_t\}$ respectively, such that

$$T_t(f)(z) = V_t\big(f\big(\sigma_t(z)\big) - f\big(\sigma_t(0)\big)\big), \quad \forall f \in \mathcal{B}_0(\triangle, E)\ \forall z \in \triangle.$$

Since $\{T_t\}_{t\in\mathbb{R}}$ is strongly continuous, in particular for $f_1(z) = z\mathbf{v}$, $f_2(z) = z^2\mathbf{v}$ and $f_3(z) = z^3\mathbf{v}$ ($\mathbf{v} \in E_1$, $z \in \triangle$ and $i = 1, 2$, or 3) we have

$$\big\|\big[\sigma_t(z)^i - \sigma_t(0)^i\big]V_t(\mathbf{v}) - z^i\mathbf{v}\big\| \to 0 \quad \text{as } t \to 0.$$

Given $z_0 \neq 0$, and $\varphi \in E_1^*$ such that $\varphi(\mathbf{v}) = 1$,

$$\lim_{t\to 0}\big[\sigma_t(z_0) - \sigma_t(0)\big]\varphi\big(V_t(\mathbf{v})\big) = z_0 \quad \text{and}$$
$$\lim_{t\to 0}\big[\sigma_t(z_0)^2 - \sigma_t(0)^2\big]\varphi\big(V_t(\mathbf{v})\big) = z_0^2,$$

implies that

$$\lim_{t\to 0}\big(\sigma_t(z_0) + \sigma_t(0)\big) = z_0. \tag{7}$$

Also

$$\lim_{t\to 0}\big[\sigma_t(z_0) - \sigma_t(0)\big]\varphi\big(V_t(\mathbf{v})\big) = z_0 \quad \text{and}$$
$$\lim_{t\to 0}\big[\sigma_t(z_0)^3 - \sigma_t(0)^3\big]\varphi\big(V_t(\mathbf{v})\big) = z_0^3,$$

implies

$$\lim_{t\to 0}\big(\sigma_t(z_0)^2 + \sigma_t(z_0)\sigma_t(0) + \sigma_t(0)^2\big) = z_0^2. \tag{8}$$

It follows from (7) and (8) that $\lim_{t\to 0}\sigma_t(z_0)\sigma_t(0) = 0$. This implies that $\lim_{t\to 0}\sigma_t(0) = 0$, otherwise there exists a sequence $\{t_n\}$ such that $\sigma_{t_n}(0)$ would converges to some complex number $w(\neq 0)$ in the closed disc. Hence, for every $z_0 \neq 0$ $\{\sigma_{t_n}(z_0)\}_n$ converges to zero and $w = z_0$. This leads to an absurdity and proves that $\lim_{t\to 0} \sigma_t(0) = 0$ and $\lim_{t\to 0} \sigma_t(z_0) = z_0$. This establishes the continuity of $\{\sigma_t\}$. For $z_0 \neq 0$,

$$\lim_{t\to 0} \frac{[\sigma_t(z_0) - \sigma_t(0)]V_t(\mathbf{v})}{\sigma_t(z_0) - \sigma_t(0)} = \frac{z_0 \mathbf{v}}{z_0} = \mathbf{v},$$

which completes the proof. □

COROLLARY 4.3. *Let E be a smooth, strictly convex and reflexive complex Banach space. If A is a (not necessarily bounded) hermitian operator on $\mathcal{B}_0(\triangle, E)$, then there exist a hermitian operator (not necessarily bounded) V on E and a continuous group of disc automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ such that*

$$A(f)(z) = V\big[f(z)\big] + \big[\partial_t \sigma_t(z)\big]_{t=0} f'(z).$$

If A is bounded then $\{\sigma_t\}_{t\in\mathbb{R}}$ is the trivial group and $A(f)(z) = V[f(z)]$, with V bounded.

Nontrivial disc automorphisms can be extended to conformal maps on the plane and as such, they are characterized according to their fixed points. More precisely, they fall into three types: an elliptic automorphism has a single fixed point in the disc and another one in the interior of its complement; a hyperbolic automorphism has two distinct fixed points on the boundary of the disc and a parabolic has a single fixed point on the boundary of the disc, cf. [27] and [29].

It has been shown that all disc automorphisms of a nontrivial one-parameter group family of disc automorphisms share the same fixed points, cf. [5] and also [6]. Thus, we consider the following three cases:

(i) Elliptic.
$$\varphi_t(z) = \frac{(e^{ict} - |\tau|^2)z - \tau(e^{ict} - 1)}{1 - |\tau|^2 e^{ict} - \overline{\tau}(1 - e^{ict})z},$$
with $c \in \mathbb{R} \setminus \{0\}$, $\tau \in \mathbb{C}$ such that $|\tau| < 1$.

(ii) Hyperbolic.
$$\varphi_t(z) = \frac{(\beta e^{ct} - \alpha)z + \alpha\beta(1 - e^{ct})}{(e^{ct} - 1)z + (\beta - \alpha e^{ct})},$$
with c a positive real number, $|\alpha| = |\beta| = 1$ and $\alpha \neq \beta$.

(iii) Parabolic.
$$\varphi_t(z) = \frac{(1 - ict)z + ict\alpha}{-ic\overline{\alpha}tz + 1 + ict},$$
with $c \in R \setminus \{0\}$ and $|\alpha| = 1$.

In [4], Berkson, Kaufman and Porta show the existence of an invariant polynomial associated with one parameter group of disc automorphisms

$$\varphi_t(z) = a(t)\frac{z - b(t)}{1 - \overline{b(t)}z},$$

with $|a(t)| = 1$ and $|b(t)| < 1$. This polynomial is given by

$$P(z) = \overline{b'(0)}z^2 + a'(0)z - b'(0).$$

It is a straightforward computation to check that

$$\partial_t \varphi_t(z)|_{t=0} = P(z) \quad \text{and} \quad \partial_t \varphi_t'(z)|_{t=0} = P'(z).$$

The invariant polynomial for each of the three types of nontrivial disc automorphisms is given by:

(i) Elliptic. $P(z) = -\frac{ic}{1-|\tau|^2}\{(\overline{\tau}z - 1)(z-\tau)\}$ ($|\tau| < 1$).
(ii) Hyperbolic. $P(z) = -\frac{c}{\beta-\alpha}\{z^2 - (\alpha+\beta)z + \alpha\beta\}$ ($|\alpha| = |\beta| = 1$ and $\alpha \neq \beta$).
(iii) Parabolic. $P(z) = i\overline{\alpha}c(z-\alpha)^2$ ($c \in R \setminus \{0\}$ and $|\alpha| = 1$).

Since hermitian operators are generators of strongly continuous one-parameter groups of surjective isometries we derive a representation for the $\mathcal{B}_0(\triangle, E)$ setting.

PROPOSITION 4.4. *Let E be a smooth, strictly convex and reflexive complex Banach space. If a closed operator A with domain $\mathcal{D}(A)$, a dense subset of $\mathcal{B}_0(\triangle, E)$ is hermitian then there exists a closed and densely defined hermitian operator V on E and a nonzero real number c, and complex numbers τ, α and β such that $|\tau| < 1$ and $|\alpha| = |\beta| = 1$ and one of the following holds*:

(1) $A(f)(z) = V(f(z)), f \in \mathcal{B}_0(\triangle, E)$ *and* $z \in \triangle$.
(2) $A(f)(z) = V(f(z)) + \frac{c}{1-|\tau|^2}\{(\overline{\tau}z-1)(z-\tau)\}f'(z), f \in \mathcal{D}(A)$ *and* $z \in \triangle$.
(3) $A(f)(z) = V(f(z)) - i\frac{|c|}{\beta-\alpha}\{z^2 - (\alpha+\beta)z + \alpha\beta\}f'(z), f \in \mathcal{D}(A)$ *and* $z \in \triangle$.
(4) $A(f)(z) = V(f(z)) - \overline{\alpha}c(z-\alpha)^2 f'(z), f \in \mathcal{D}(A)$ *and* $z \in \triangle$.

Proof. Given a hermitian operator A satisfying the conditions stated, then $\{e^{-itA}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of surjective isometries on $\mathcal{B}_0(\triangle, E)$. Theorem 3.5 applies to assert the existence of a strongly continuous one-parameter group of surjective isometries on E, $\{V_t\}_{t\in\mathbb{R}}$ and a continuous group of disc automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ such that

$$e^{-itA}(f)(z) = V_t\big(f\big(\sigma_t(z)\big) - f\big(\sigma_t(0)\big)\big), \quad \forall f \in \mathcal{D}(A).$$

We denote by V the generator of $\{V_t\}_{t\in\mathbb{R}}$ then

$$A(f)(z) = V\big(f(z)\big) - i\partial_t\big(\sigma_t'(z)\big)|_{t=0} f'(z), \quad \forall f \in \mathcal{D}(A).$$

The considerations in the preamble to the proposition justify the three last cases listed. If $\sigma_t(z) = z$ for all t, then $\partial_t(\sigma_t'(z)) = 0$ and $A(f)(z) = V(f(z)), f \in \mathcal{B}_0(\triangle, E)$ and $z \in \triangle$. This completes the proof. □

REMARK 4.5. In the scalar case, $\mathcal{B}(\triangle)$ is known be a Grothendieck space with the Dunford–Pettis property (see [28]). As a consequence of this fact Blasco et. al. in [7] (see also [8]) showed that all strongly continuous groups on $\mathcal{B}(\triangle)$ are uniformly continuous. Therefore only the trivial group of disc automorphisms is permissible (i.e., $\{\sigma_t\} = \{\text{id}\}$) and the hermitian operators are just real multiples of the identity. This is in contrast with our case because of the following example. Suppose $E = \ell_2$, $\sigma_t(z) = z$ and set

$$T_t(f)(z) = \big(e^{it} f_1(z), e^{2it} f_2(z), \dots\big).$$

This is a family of strongly continuous surjective isometries but not uniformly continuous. The generator of this group is given by

$$Af(z) = \big(f_1(z), 2f_2(z), 3f_3(z), \dots\big)$$

which is clearly an unbounded operator.

We also have the following characterization for bounded hermitian operators on $\mathcal{B}_0(\triangle, E)$.

Corollary 4.6. *Let E be a smooth, strictly convex and reflexive complex Banach space. If A is a bounded hermitian operator on $\mathcal{B}_0(\triangle, E)$ then there exists a bounded hermitian operator V on E such that*

$$A(f)(z) = V\big(f(z)\big), \quad \forall f \in \mathcal{B}_0(\triangle, E) \text{ and } z \in \triangle.$$

Proof. The operator A is of one of the forms listed in the Proposition 4.4, the sequence of functions $f_n(z) = z^n \mathbf{v}$, with $\mathbf{v}$ a unit vector in E, are in $\mathcal{B}_0(\triangle, E)$. Thus, the respective sequence of norms is uniformly bounded and $\|Af\|$ is unbounded. This implies that $\sigma_t'(z)|_{t=0} = 0$ and $\sigma_t(z) = z$. This completes the proof. □

Remark 4.7. It is a known fact that Banach spaces with the Grothendieck property and the Dunford–Pettits property only support bounded hermitian operators, see [7], [28]. The little Bloch scalar valued space, $\mathcal{B}_0(\triangle)$ has these two properties (cf. [28]) and thus every hermitian operator on $\mathcal{B}(\triangle)$ is bounded. This implies that if a hermitian operator A on $\mathcal{B}_0(\triangle, E)$ with an eigenspace containing one dimensional subspace $\{h(z)v : h \in \mathcal{B}(\triangle), v \in E_1\}$ then A is of the form $A(f)(z) = Vf(z)$.

Corollary 4.6 allows us to extend our characterization to surjective isometries of $\mathcal{B}_*(\triangle, E)$. As pointed out in Remark 2.3, $\mathcal{B}_*(\triangle, E)$ is isometrically isomorphic to the ℓ_1-sum of E with $\mathcal{B}_0(\triangle, E)$. Moreover, if E does not admit L_1-projections (i.e. a bounded hermitian operator P on E such that $P^2 = P$ and for every $v \in E$, $\|v\|_E = \|Pv\|_E + \|(I - P)v\|_E$) then also $\mathcal{B}_0(\triangle, E)$ does not admit L_1-projections. In fact, assuming P represents a L_1-projection on $\mathcal{B}_0(\triangle, E)$, Corollary 4.6 implies that $P(f)(z) = V(f(z))$, with V a bounded hermitian projection on E. Therefore $P(h\mathbf{v})(z) = h(z)V\mathbf{v}$, for $h \in \mathcal{B}_0(\triangle)$. In particular for $h(z) = z$, $\|\mathbf{v}\| = \|V\mathbf{v}\| + \|(I - V)\mathbf{v}\|$ which implies that E supports L_1-projections.

We employ Proposition 4.3 in [24], a surjective isometry on $\mathcal{B}_*(\triangle, E)$ can be written as a direct sum of a surjective isometry on E and a surjective isometry on $\mathcal{B}_0(\triangle, E)$. Therefore, a surjective isometry T on $\mathcal{B}_*(\triangle, E)$ is given by

$$T(f)(z) = Uf(0) + V\big[(f \circ \sigma)(x) - (f \circ \sigma)(0)\big],$$

with σ a disc automorphism, U and V surjective isometries on E. We summarize these considerations in the next result.

THEOREM 4.8. *Let E be a smooth, strictly convex and reflexive complex Banach space. Then $T : \mathcal{B}_*(\triangle, E) \to \mathcal{B}_*(\triangle, E)$ is a surjective linear isometry if and only if there exist surjective linear isometries on E, U and V, and a disc automorphism σ such that, for every $f \in \mathcal{B}_*(\triangle, E)$ and $z \in \triangle$,*

$$Tf(z) = U\big[f(0)\big] + V\big[f\big(\sigma(z)\big) - f\big(\sigma(0)\big)\big].$$

The next corollary extends the results stated in Propositions 4.1 and 4.2 to $\mathcal{B}_*(\triangle, E)$.

COROLLARY 4.9. *Let E be a smooth, strictly convex and reflexive complex Banach space. Then $\{T_t\}_{t\in\mathbb{R}}$ is a strongly continuous one parameter group of surjective isometries on $\mathcal{B}_*(\triangle, E)$ if and only if there exist a continuous one parameter group of disc automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ and strongly continuous one parameter groups of surjective isometries on E, $\{U_t\}_{t\in\mathbb{R}}$ and $\{V_t\}_{t\in\mathbb{R}}$ such that*

$$T_t(f)(z) = U_t\big(f(0)\big) + V_t\big(f\big(\sigma_t(z)\big) - f\big(\sigma_t(0)\big)\big), \quad \forall f \in \mathcal{B}_0(\triangle, E)\ \forall z \in \triangle.$$

Proof. Since E is a smooth and strictly convex complex Banach space, it does not support L_1-projections, Theorem 4.8 applies and for each $t \in \mathbb{R}$,

$$T_t(f)(z) = U_t\big(f(0)\big) + V_t\big(f\big(\sigma_t(z)\big) - f\big(\sigma_t(0)\big)\big), \quad \forall f \in \mathcal{B}_0(\triangle, E)\ \forall z \in \triangle.$$

The proof given for Proposition 4.2 shows that $\{\sigma_t\}_{t\in\mathbb{R}}$ is a one parameter group of disc automorphisms and $\{S_t\}_{t\in\mathbb{R}}$ is a strongly continuous one parameter group of surjective isometries on E. Then by considering constant functions we also derive that $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous one parameter group of surjective isometries on E. The converse implies follows from straightforward computations. ☐

COROLLARY 4.10. *Let E be a Hilbert space. If A is a (not necessarily bounded) hermitian operator on $\mathcal{B}_*(\triangle, E)$, then there exist hermitian operators (not necessarily bounded) U and V on E and a continuous group of disc automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ such that*

$$A(f)(z) = U\big[f(0)\big] + V\big[f(z)\big] + \big[\partial_t\sigma_t(z)\big]_{t=0} f'(z).$$

If A is bounded then $A(f)(z) = U[f(0)] + V[f(z)]$, with U and V bounded.

5. Generalized bi-circular projections

In this section, we characterize the generalized bi-circular projections on $\mathcal{B}_0(\triangle, E)$. We recall that a generalized bi-circular projection P satisfies $P^2 = P$ and $P + \lambda(I - P) = T$ with T a surjective isometry and λ a modulus 1 complex number different from 1, [20]. We refer the reader to the following papers for additional information about this type of projections, [10], [11], [20] and [26].

A straightforward computation yields the following algebraic equation $T^2 - (\lambda+1)T + \lambda I = 0$.

THEOREM 5.1. *Let E be a smooth and strictly convex complex Banach space. Then P is a generalized bi-circular projection on $\mathcal{B}_0(\triangle, E)$ if and only if there exists an isometric reflection T (i.e. $T^2 = I$) such that*

$$P = \frac{I+T}{2}.$$

Proof. If P is a generalized bi-circular projection, then $P + \lambda(I-P) = T$ with $\lambda \in \mathbb{T} \setminus \{1\}$ and T a surjective isometry. An application of Theorem 3.5 implies that there exist a surjective linear isometry $V : E \to E$ and a disc automorphism σ such that for every $f \in \mathcal{B}_0(\triangle, E)$ and $z \in \triangle$

$$Tf(z) = V\big[(f\circ\sigma)(z) - (f\circ\sigma)(0)\big].$$

The automorphism σ is of the form $\sigma(z) = \mu\frac{z-\alpha}{1-\overline{\alpha}z}$ with $\mu \in \mathbb{T}$ and $|\alpha| < 1$. The condition $P^2 = P$ implies that $T^2 - (\lambda+1)T + \lambda I = 0$. Therefore, we have

$$\begin{aligned} V^2\big[f\big((\sigma\circ\sigma)(z)\big) - f\big((\sigma\circ\sigma)(0)\big)\big] \\ -(\lambda+1)V\big[f\big((\sigma)(z)\big) - f\big((\sigma)(0)\big)\big] + \lambda f(z) = 0, \end{aligned} \tag{9}$$

for every $f \in \mathcal{B}_0(\triangle, E)$ and $z \in \triangle$. By differentiating (9), we obtain

$$\begin{aligned} V^2\big[f'\big((\sigma\circ\sigma)(z)\big)\sigma'\big(\sigma(z)\big)\sigma'(z)\big] \\ -(\lambda+1)V\big[f'\big((\sigma)(z)\big)\sigma'(z)\big] + \lambda f'(z) = 0. \end{aligned} \tag{10}$$

The equation displayed in (10) applied to $f(z) = \frac{z^2}{2}\mathbf{v}$ (with $\mathbf{v}$ a vector in E of norm 1) and with $z = \alpha$ yields

$$V^2\mathbf{v} = \frac{\lambda}{\mu^3}\mathbf{v}.$$

Applying (10) to $f(z) = \frac{z^2}{2}\mathbf{v}$ and setting $z = 0$, we obtain

$$\big(V^2\mathbf{v}\big)\mu^3\frac{-\mu\alpha-\alpha}{1+\mu|\alpha|^2}\frac{1-|\alpha|^2}{(1+\mu|\alpha|^2)^2}\big(1-|\alpha|^2\big) - (V\mathbf{v})(\lambda+1)(-\mu\alpha)\mu\big(1-|\alpha|^2\big) = 0.$$

We assume that $\lambda \neq -1$, then straightforward calculations show that

$$V = \frac{\lambda(\mu+1)(1-|\alpha|^2)}{(\lambda+1)\mu^2(1+\mu|\alpha|^2)^3}I. \tag{11}$$

This last equation implies that $\mu \neq -1$. Once more, applying equation (10) to $f(z) = z\mathbf{v}$ and setting $z = \alpha$ we obtain

$$V = \frac{\lambda(\mu+1)(1-|\alpha|^2)}{\mu^2(\lambda+1)}I. \tag{12}$$

From (11) and (12), we derive $(1+\mu|\alpha|^2)^3 = 1$. This leads to $1+\mu|\alpha|^2 = 1$, $1+\mu|\alpha|^2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ or $1+\mu|\alpha|^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. It is easy to show that only the first equation leads to the solution $\alpha = 0$. Therefore, $V = \frac{\lambda(\mu+1)}{(\lambda+1)\mu^2} I$ and $\sigma(z) = \mu z$. Since V is an isometry the $|\mu+1| = |\lambda+1|$, and thus $\mu = \lambda$ or $\lambda = \overline{\mu}$.

We consider two cases.

1. If $\lambda = \mu$, then $V = \overline{\lambda} I$ and equation (9) applied to $f(z) = z\mathbf{v}$ implies

$$\lambda^4 - \lambda(\lambda+1) + \lambda = 0$$

and thus $\lambda = 1$. This is impossible.

2. If $\lambda = \overline{\mu}$, then $V = \overline{\mu}^2 I$. We differentiate equation (9) and applied to $f(z) = z^3\mathbf{v}$ to obtain

$$\mu^4 - (\mu+1)\mu^2 + \mu = 0.$$

This equation has solutions ± 1. Either case leads to a contradiction since we have assumed that $\lambda \neq -1$.

This contradiction shows that $\lambda = -1$ and (10) reduces to

$$V^2\big[f'\big((\sigma\circ\sigma)(z)\big)\sigma'\big(\sigma(z)\big)\sigma'(z)\big] = f'(z), \tag{13}$$

which applied to $f(z) = \frac{z^2}{2}\mathbf{v}$ with $z = \alpha$ yields

$$\big(V^2\mathbf{v}\big)\big(-\mu^3\alpha\big) = \alpha\mathbf{v}.$$

Therefore, $V^2 = -\overline{\mu}^3 I$.

The equation (13) applied to $f(z) = z\mathbf{v}$ and $z = \alpha$ gives

$$-\overline{\mu}^3\sigma'(0)\sigma'(\alpha) = 1.$$

Therefore, $\mu = -1$ and $V^2 = I$. We also have $\sigma\circ\sigma(z) = z$. Therefore, $T^2 = I$ and proves that P is the average of the identity operator with a reflection. The reverse implication is clear. □

A generalized bi-circular projection P on $\mathcal{B}_*(\triangle, E)$ is given

$$P = \frac{1}{1-\lambda}(T - \lambda I)$$

with T a surjective isometry on $\mathcal{B}_*(\triangle, E)$ and λ a modulus 1 scalar different from 1. Theorem 4.8 implies the existence of surjective isometries on E, U and V, also a disc automorphism σ such that $T(f)(z) = U(f(0)) + V[f(\sigma(z)) - f(\sigma(0))]$.

The form for the surjective isometries on $\mathcal{B}_*(\triangle, E)$ implies that P leaves invariant the subspace of all constant functions and also $\mathcal{B}_0(\triangle, E)$. Applying Theorem 5.1, we conclude that the restriction of P to $\mathcal{B}_0(\triangle, E)$ is the average of I with an isometric reflection on $\mathcal{B}_0(\triangle, E)$, thus $V^2 = I$ and $\sigma^2 = \mathrm{id}_\triangle$. Therefore, P is the average of the identity on $\mathcal{B}_*(\triangle, E)$ with a surjective isometry T. Since $T = 2P - I$ is such that $T^2 = I$, then generalized bi-circular projections on $\mathcal{B}_*(\triangle, E)$ are the average of the identity operator with an isometric reflection.

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References

[1] E. Behrends and P. Greim, *A note on the paper "Hermitian operators on $C(X,E)$ and the Banach–Stone theorem" by R. Fleming and J. Jamison*, Math. Z. **175** (1980), 299. MR 0602642

[2] E. Berkson and H. Porta, *Hermitian operators and one parameter groups in Hardy spaces*, Trans. Amer. Math. Soc. **185** (1973), 373–391. MR 0338833

[3] E. Berkson and A. Sourour, *The Hermitian operators on some Banach spaces*, Studia Math. **52** (1974), 33–41. MR 0355668

[4] E. Berkson, R. Kaufman and H. Porta, *Möbius transformations of the disc and one-parameter groups of isometries of H^p*, Trans. Amer. Math. Soc. **199** (1974), 223–239. MR 0361923

[5] E. Berkson and H. Porta, *One-parameter groups of isometries on Hardy spaces of the torus*, Trans. Amer. Math. Soc. **220** (1976), 373–391. MR 0417855

[6] E. Berkson and H. Porta, *One-parameter groups of isometries on Hardy spaces of the torus: Spectral theory*, Trans. Amer. Math. Soc. **227** (1977), 357–370. MR 0451035

[7] O. Blasco, M. Contreras, S. Diaz-Madrigal, J. Martinez and A. Siskakis, *Semigroups of composition operators in BMOA and the extension of a theorem of Sarason*, Integral Equations and Operator Theory **61** (2008), 45–62. MR 2414439

[8] O. Blasco, M. Contreras, S. Diaz-Madrigal, J. Martinez, M. Papadimitrakis and A. Siskakis, *Semigroups of composition operators and integral operators in spaces of analytic functions*, Ann. Acad. Sci. Fenn. Math. **38** (2013), no. 1, 67–89. MR 3076799

[9] F. Botelho, R. Fleming and J. Jamison, *Extreme points and isometries on vector-valued Lipschitz spaces*, J. Math. Anal. Appl. **381** (2011), 821–832. MR 2802117

[10] F. Botelho and J. Jamison, *Generalized bi-circular projections on spaces of analytic functions*, Acta Sci. Math. (Szeged) **75** (2009), no. 3–4, 527–546. MR 2590348

[11] F. Botelho and J. Jamison, *Generalized bi-circular projections on $\mathcal{C}(\Omega, X)$*, Rocky Mountain J. Math. **40** (2010), no. 1, 77–83. MR 2607109

[12] F. Botelho, J. Jamison and B. Zheng, *Isometries on spaces of vector-valued Lipschitz functions*, Positivity **17** (2013), no. 1, 47–65. MR 3027645

[13] B. Brosowski and F. Deustch, *Geometric properties of sums*, J. Approx. Theory **10** (1974), 245–267. MR 0358180

[14] J. Cima and W. Wogen, *On isometries of the Bloch space*, Illinois J. Math. **24** (1980), no. 2, 313–316. MR 0575069

[15] G. Ding, *The 1-Lipschitz mappings between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space*, Sci. China Ser. A **45** (2001), no. 4, 479–483. MR 1912120

[16] L. Li and R. Liu, *On the isometric extension problem: A survey*, Tamsui Oxf. J. Inf. Math. Sci. **24** (2008), no. 4, 339–354. MR 2503701

[17] K.-J. Engel and R. Nagel, *A short course on operator semigroups*, Universitext, Springer, New York, 2006. MR 2229872

[18] R. Fleming and J. Jamison, *Hermitian operators on $C(X,E)$ and the Banach–Stone theorem*, Math. Z. **170** (1980), no. 1, 77–84. MR 0558889

[19] R. Fleming and J. Jamison, *Isometries on Banach spaces: Function spaces*, Chapman & Hall/CRC, Boca Raton, FL, 2003. MR 1957004

[20] M. Fosner, D. Ilisevic and C. Li, *G-invariant norms and bicircular projections*, Linear Algebra Appl. **420** (2007), no. 2–3, 596–608. MR 2278235

[21] K. Hoffman, *Banach spaces of analytic functions*, Dover Publ, Inc., New York, 1962. MR 0133008

[22] W. Hornor and J. E. Jamison, *Isometries of some Banach spaces of analytic functions*, Integral Equations Operator Theory **41** (2001), 410–425. MR 1857800

[23] J. E. Jamison, *Bicircular projections on some Banach spaces*, Linear Algebra Appl. **420** (2007), 29–33. MR 2277626

[24] J. E. Jamison, *Isometric equivalence of certain operators on Banach spaces*, Integral Equations Operator Theory **56** (2006), 469–482. MR 2284711

[25] K. Jarosz, *Isometries in semisimple, commutative Banach algebras*, Proc. Amer. Math. Soc. **94** (1985), 65–71. MR 0781058

[26] R. King, *Generalized bi-circular projections on certain Hardy spaces*, J. Math. Anal. Appl. **408** (2013), no. 1, 35–39. MR 3079944

[27] S. Krantz, *Complex analysis: The geometric viewpoint*, The Carus Mathematical Monographs, vol. 23, Math. Assoc. Amer., Washington, DC, 1990. MR 1074176

[28] H. P. Lotz, *Uniform convergence of operators on L^∞ and similar spaces*, Math. Z. **190** (1985), 207–220. MR 0797538

[29] J. E. Marsden, *Basic complex analysis*, W. H. Freeman and Company, San Francisco, CA, 1973. MR 0352419

[30] L. L. Stachó and B. Zalar, *Bicircular projections on some matrix and operator spaces*, Linear Algebra Appl. **384** (2004), 21–42. MR 2055340

[31] I. Vidav, *Eine metrische Kennzeichnung der selb-adjungierten operatoren*, Math. Z. **66** (1956), 121–128. MR 0084733

[32] I. Vidav, *Spectra of perturbed semigroups with applications to transport theory*, J. Math. Anal. Appl. **30** (1970), 264–279. MR 0259662

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