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STRICTLY SINGULAR OPERATORS IN TSIRELSON LIKE SPACES

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Abstract. For each $n \in \mathbb{N}$ a Banach space $\mathfrak{X}^n_{0,1}$ is constructed having the property that every normalized weakly null sequence generates either a c_0 or ℓ_1 spreading models and every subspace has weakly null sequences generating both c_0 and ℓ_1 spreading models. The space $\mathfrak{X}^n_{0,1}$ is also quasiminimal and for every infinite dimensional closed subspace Y of $\mathfrak{X}^n_{0,1}$, for every $S_1, S_2, \dots, S_{n+1}$ strictly singular operators on Y, the operator $S_1 S_2 \cdots S_{n+1}$ is compact. Moreover, for every subspace Y as above, there exist $S_1, S_2, \dots, S_n$ strictly singular operators on Y, such that the operator $S_1 S_2 \cdots S_n$ is non-compact.

Introduction

The strictly singular operators[1] form a two sided ideal which includes the one of the compact operators. In many cases, the two ideal coincide. This happens for the spaces $\ell_p, 1 \leq p < \infty, c_0$, as well as Tsirelson space T (see [13], [25]). On the other hand, in the spaces $L^p[0,1], 1 \leq p < \infty, p \neq 2, C[0,1]$ the two ideals are different. However, a classical result of V. Milman [17], explains that in all the above spaces, the composition of two strictly singular operators is a compact one. The aim of the present paper, is to present examples of spaces where similar properties occur in a hereditary manner. More precisely, we prove the following.

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[1] A bounded linear operator is called strictly singular, if its restriction on any infinite dimensional subspace is not an isomorphism.

THEOREM 0.1. *For every* $n \in \mathbb{N}$ *there exists a reflexive space with a* 1*-unconditional basis, denoted by* $\mathfrak{X}^n_{0,1}$, *such that for every infinite dimensional subspace* Y *of* $\mathfrak{X}^n_{0,1}$ *we have the following.*

(i) *The ideal* $\mathcal{S}(Y)$ *of the strictly singular operators is non-separable.*
(ii) *For every family* $\{S_i\}_{i=1}^{n+1} \subset \mathcal{S}(Y)$, *the composition* $S_1 S_2 \cdots S_{n+1}$ *is a compact operator.*
(iii) *There are* $S_1, \dots, S_n \in S(Y)$, *such that the composition* $S_1 \cdots S_n$ *is non-compact.*

The construction of the spaces $\mathfrak{X}^n_{0,1}$ is based on T. Figiel's and W. B. Johnson's construction of Tsirelson space [13], which is actually the dual of Tsirelson's initial space [25]. Therefore the spaces $\mathfrak{X}^n_{0,1}$ are Tsirelson like spaces and their norm is defined through a saturation with constraints, described by the following implicit formula, which uses the nth Schreier family $\mathcal{S}_n$.

For $x \in c_{00}$

$$\|x\| = \max\left\{\|x\|_0, \sup\left\{\sum_{q=1}^{d} \|E_q x\|_{j_q}\right\}\right\},$$

where the supremum is taken over all $\{E_q\}_{q=1}^d$ which are $\mathcal{S}_n$-admissible successive finite subsets of $\mathbb{N}$, $\{j_q\}_{q=1}^d$ very fast growing (i.e., $2 \le j_1 < \cdots < j_q$ and $j_q > \max E_{q-1}$, for $q > 1$) natural numbers and

$$\|x\|_j = \sup\left\{\frac{1}{j}\sum_{q=1}^{d} \|E_q x\|\right\},$$

where the supremum is taken over all successive finite subsets of the naturals $E_1 < \cdots < E_d, d \le j$.

Saturated norms under constraints were introduced by E. Odell and Th. Schlumprecht [20], [21]. In particular, the space defined in [21] has the property that every bimonotone basis is finitely block represented in every subspace. Recently, in [8], the first and third authors have used these techniques to construct a reflexive hereditarily indecomposable space such that every operator on an infinite dimensional subspace has a non-trivial invariant subspace.

Property (ii) of Theorem 0.1, combined with N. D. Hooker's and G. Sirotkin's real version [14], [23] of V. I. Lomonosov's theorem [16], yields that the strictly singular operators on the subspaces of $\mathfrak{X}^n_{0,1}$ admit non-trivial hyperinvariant subspaces.

Unlike the Tsirelson type spaces, the spaces $\mathfrak{X}^n_{0,1}$ have non-homogeneous asymptotic structure. In particular, every seminormalized weakly null sequence admits either ℓ_1 or c_0 as a spreading model and every subspace Y contains weakly null sequences generating both ℓ_1 and c_0 as spreading models. As a result, the spaces $\mathfrak{X}^n_{0,1}$ do not contain any asymptotic ℓ_p subspace and, as a consequence, the spaces $\mathfrak{X}^n_{0,1}$ do not contain a boundedly distortable subspace [18]. The sequences in $\mathfrak{X}^n_{0,1}$ generating ℓ_1 spreading models admit a further classification in terms of higher order ℓ_1 spreading models. Recall that for $k \in \mathbb{N}$, a bounded sequence $\{x_i\}_{i\in\mathbb{N}}$ generates an ℓ_1^k spreading model if there exists $C > 0$ such that $\|\sum_{i\in F} \lambda_i x_i\| \geq C \sum_{i\in F} |\lambda_i|$ for every $F \in \mathcal{S}_k$. The next proposition provides a precise description of the possible spreading models of $\mathfrak{X}^n_{0,1}$.

PROPOSITION 0.2. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a seminormalized weakly null sequence in $\mathfrak{X}^n_{0,1}$. Then one of the following holds.*

(i) *$\{x_i\}_{i\in\mathbb{N}}$ admits c_0 as a spreading model.*
(ii) *There exists $1 \leq k \leq n$ such that $\{x_i\}_{i\in\mathbb{N}}$ admits an ℓ_1^k spreading model and it does not admit an ℓ_1^{k+1} one.*

The proof of Theorem 0.1(ii) is based on Proposition 0.2 and the following characterization of the non-strictly singular operators on subspaces of $\mathfrak{X}^n_{0,1}$.

PROPOSITION 0.3. *Let Y be an infinite dimensional subspace of $\mathfrak{X}^n_{0,1}$ and $T : Y \to Y$ a bounded linear operator. Then the following are equivalent.*

(i) *The operator T is not a strictly singular operator.*
(ii) *There exists $1 \leq k \leq n$ and a bounded weakly null sequence $\{x_i\}_{i\in\mathbb{N}}$ such that both $\{x_i\}_{i\in\mathbb{N}}$ and $\{Tx_i\}_{i\in\mathbb{N}}$ generate an ℓ_1^k spreading model and do not admit an ℓ_1^{k+1} one.*
(iii) *There exists $\{x_i\}_{i\in\mathbb{N}}$ a bounded weakly null sequence such that both $\{x_i\}_{i\in\mathbb{N}}$ and $\{Tx_i\}_{i\in\mathbb{N}}$ generate a c_0 spreading model.*

A space is called quasi-minimal if any two infinite dimensional subspaces have further subspaces which are isomorphic. A major obstacle in proving the above, is to show that certain normalized block sequences, that can be found in every subspace, are equivalent. This also yields that the space $\mathfrak{X}^n_{0,1}$ is quasi-minimal.

The above proposition combined with the properties of the spreading models of the space $\mathfrak{X}^n_{0,1}$ also allows us to study classes of strictly singular operators on subspaces of the spaces $\mathfrak{X}^n_{0,1}$, which were introduced in [2]. Recall that a bounded linear operator T defined on a Banach space X, is said to be $\mathcal{S}_\xi$-strictly singular (the class is denoted $\mathcal{SS}_\xi(X)$), for $\xi < \omega_1$, if for every Schauder basic sequence $\{x_i\}_i$ in X and $\varepsilon > 0$, there exists a vector x in the linear span of $\{x_i\}_{i\in F}$, where $F \in \mathcal{S}_\xi$ such that $\|Tx\| < \varepsilon\|x\|$. We prove that for $n \in \mathbb{N}$ the space $\mathfrak{X}^n_{0,1}$ satisfies the following:

$$\mathcal{K}(Y) \subsetneq \mathcal{SS}_1(Y) \subsetneq \mathcal{SS}_2(Y) \subsetneq \cdots \subsetneq \mathcal{SS}_n(Y) = \mathcal{S}(Y)$$

and for every $1 \leq k \leq n$, $\mathcal{SS}_k(Y)$ is a two sided ideal. This solves a problem in [24] by being the first example of a space for which the collection $\mathcal{SS}_k(\mathfrak{X}^n_{0,1})$ is a ideal not equal to $\mathcal{K}(\mathfrak{X}^n_{0,1})$ or $\mathcal{SS}(\mathfrak{X}^n_{0,1})$.

The spaces $\mathfrak{X}^n_{0,1}$ can be extended to a transfinite hierarchy $\mathfrak{X}^\xi_{0,1}$ for $1 \leq \xi < \omega_1$. Roughly speaking, the space $\mathfrak{X}^\xi_{0,1}$ is defined with the use of the Schreier family $\mathcal{S}_\xi$ in the place of $\mathcal{S}_n$. In Section 5, we investigate the case the space $\mathfrak{X}^\omega_{0,1}$ and prove results analogous to those in the case of $\mathfrak{X}^1_{0,1}$. We also comment, in passing, that for $\xi = \zeta + (n-1)$ with ζ a limit ordinal satisfying $\eta + \zeta = \zeta$ for $\eta < \zeta$, the strictly singular operators on the space $\mathfrak{X}^\xi_{0,1}$ behave in a similar manner as the spaces $\mathfrak{X}^n_{0,1}$.

The paper is organized into six sections. The first one is devoted to some preliminary concepts and results. In the second section, we introduce the norm of the space $\mathfrak{X}^n_{0,1}$, by defining the norming set W, a subset of c_{00}. The third section includes the study of the spreading models generated by seminormalized sequences of $\mathfrak{X}^n_{0,1}$. Our approach uses tools similar to those in [8]. In particular, to each block sequence $\{x_i\}_{i\in\mathbb{N}}$ of $\mathfrak{X}^n_{0,1}$, we assign a family of indices $\alpha_k(\{x_i\}_i), k = 0, \dots, n-1$ and their behaviour determines the spreading models generated by the subsequences of $\{x_i\}_{i\in\mathbb{N}}$. The fourth section contains the study of equivalent block sequences in $\mathfrak{X}^n_{0,1}$. The proof is rather involved and based on the analysis of the elements of the set W. The equivalence of block sequences is central to our approach and it is critical in the proofs of Proposition 0.3 which, in turn, proves Theorem 0.1. The proofs of the latter results are given in Section 5. In Section 6, we provide the extended hierarchy $\mathfrak{X}^\zeta_{0,1}, 1 \leq \zeta < \omega_1$ and we prove some of the fundamental properties of the spaces.

1. Preliminaries

The Schreier families. The Schreier families is an increasing sequence of families of finite subsets of the naturals, which first appeared in [1], and is inductively defined in the following manner.

Set $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\}$ and $\mathcal{S}_1 = \{F \subset \mathbb{N} : \#F \leq \min F\}$.

Suppose that $\mathcal{S}_n$ has been defined and set $\mathcal{S}_{n+1} = \{F \subset \mathbb{N} : F = \bigcup_{j=1}^k F_j$, where $F_1 < \cdots < F_k \in \mathcal{S}_n$ and $k \leq \min F_1\}$.

If for $n, m \in \mathbb{N}$, we set $\mathcal{S}_n * \mathcal{S}_m = \{F \subset \mathbb{N} : F = \bigcup_{j=1}^k F_j$, where $F_1 < \cdots < F_k \in \mathcal{S}_m$ and $\{\min F_j : j = 1, \dots, k\} \in \mathcal{S}_n\}$, then it is well known [4] and follows easily by induction that $\mathcal{S}_n * \mathcal{S}_m = \mathcal{S}_{n+m}$.

Definition 1.1. Let X be a Banach space, $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in X, $k \in \mathbb{N}$ and $1 \leq p < \infty$. We say that $\{x_i\}_{i\in\mathbb{N}}$ generates an ℓ^k_p spreading model, if there exists a uniform constant $C \geq 1$, such that for any $F \in \mathcal{S}_k$, $\{x_i\}_{i\in F}$ is C-equivalent to the usual basis of $(\mathbb{R}^{\#F}, \|\cdot\|_p)$. The c^k_0 spreading models are defined similarly.

REMARK 1.2. Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. If $\{x_m\}_{m\in\mathbb{N}}$ is a bounded sequence in X such that $\{Tx_m\}_{m\in\mathbb{N}}$ generates an ℓ_1^k spreading model for some $k \in \mathbb{N}$, then $\{x_m\}_{m\in\mathbb{N}}$ generates an ℓ_1^d spreading model, for some $d \geq k$.

DEFINITION 1.3. Let X be a Banach space, $\{x_i\}_{i\in\mathbb{N}}$ be a seminormalized sequence in X and $k \in \mathbb{N}$. We say that $\{x_i\}_{i\in\mathbb{N}}$ generates a strong ℓ_1^k spreading model if there exists a seminormalized sequence $\{x_i^*\}_{i\in\mathbb{N}}$ in X^* which generates a c_0^k spreading model and $\varepsilon > 0$, such that $x_i^*(x_i) > \varepsilon$ for all $i \in \mathbb{N}$ and $x_i^*(x_j) = 0$ for $i \neq j$.

REMARK 1.4. If X is a Banach space, $k \in \mathbb{N}$, $\{x_i\}_i$ is a seminormalized weakly null sequence in X generating a strong ℓ_1^k spreading model and $\{y_i\}_i$ is a sequence in X with $\sum_{i=1}^\infty \|x_i - y_i\| < \infty$, then $\{y_i\}_i$ has a subsequence generating a strong ℓ_1^k spreading model.

The above is easily implied by the following.

LEMMA 1.5. *Let X be a Banach space, $\{x_i\}_{i\in\mathbb{N}}$ be a seminormalized weakly null sequence in X, $\{x_i^*\}_{i\in\mathbb{N}}$ be a seminormalized w^*-null sequence in X^* and $\varepsilon > 0$ such that $x_i^*(x_i) > \varepsilon$ for all $i \in \mathbb{N}$ and $x_i^*(x_j) = 0$ for $i \neq j$. If $\{y_i\}_i$ is a sequence in X with $\sum_{i=1}^\infty \|x_i - y_i\| < \infty$, then there exist a strictly increasing sequence of natural numbers $\{m_i\}_i$ and a seminormalized sequence $\{y_i^*\}_i$ in X^* such that $y_i^*(y_{m_i}) > \varepsilon/2$ for all $i \in \mathbb{N}$, $y_i^*(y_{m_j}) = 0$ for $i \neq j$ and $\sum_{i=1}^\infty \|y_i^* - x_{m_i}^*\| < \infty$.*

Proof. Using the fact that $\{x_i\}_i$ is weakly null, $\{x_i^*\}_i$ is w*-null and $\sum_{i=1}^\infty \|x_i - y_i\| < \infty$, we may pass to appropriate subsequences and relabel such that $\sum_{i\neq j} |x_i^*(y_j)| < \infty$. We may moreover assume that $\{y_i\}_i$ is Schauder basic and set $Y = [\{y_i\}_i]$. For $i \in \mathbb{N}$, define a bounded linear functional $g_i : Y \to \mathbb{R}$ with $g_i(\sum_{j=1}^\infty c_j y_j) = \sum_{j\neq i} c_j x_i^*(y_j)$ and take z_i^* to be a norm preserving extension of g_i to X. Then the $y_i^* = x_i^* - z_i^*$ are the desired functionals. $\square$

REMARK 1.6. If a sequence generates a strong ℓ_1^k spreading model, it generates an ℓ_1^k spreading model. Moreover, the class of strong ℓ_1^k spreading models is strictly smaller than the class of ℓ_1^k spreading models.

Special convex combinations. Next, we recall for $k \in \mathbb{N}$ and $\varepsilon > 0$ the notion of the (k, ε) special convex combinations (see [6], [9]). This is an important tool used throughout the paper.

DEFINITION 1.7. Let $F \subset \mathbb{N}$ and $x = \sum_{i\in F} c_i e_i$ be a vector in c_{00}. Then x is said to be a (k, ε) basic special convex combination (or a (k, ε) basic s.c.c.) if:

(i) $F \in \mathcal{S}_k$, $c_i \geq 0$, for $i \in F$ and $\sum_{i\in F} c_i = 1$.
(ii) For any $G \subset F$, $G \in \mathcal{S}_{k-1}$, we have that $\sum_{i\in G} c_i < \varepsilon$.

DEFINITION 1.8. Let $x_1 < \cdots < x_m$ be vectors in c_{00} and $\psi(k) = \min\operatorname{supp} x_k$, for $k = 1, \ldots, m$. Then $x = \sum_{k=1}^m c_k x_k$ is said to be a (n, ε) special convex combination (or (n, ε) s.c.c.), if $\sum_{k=1}^m c_k e_{\psi(k)}$ is a (n, ε) basic s.c.c.

Repeated averages. For every $k \in \mathbb{N}$ and F a maximal $\mathcal{S}_k$ set we inductively define the repeated average $x_F = \sum_{i \in F} c_i^F e_i$ of F, which is a convex combination of the usual basis of c_{00}.

For $k = 1$ and F a maximal $\mathcal{S}_1$ set, we define $x_F = \frac{1}{\#F} \sum_{i \in F} e_i$.

Let now $k > 1$ and assume that for any F maximal $\mathcal{S}_{k-1}$ set the repeated average x_F has been defined. If F is a maximal $\mathcal{S}_k$ set, then there exist $F_1 < \cdots < F_d$ maximal $\mathcal{S}_{k-1}$ sets such that $F = \bigcup_{q=1}^d F_q$. Set $x_F = \frac{1}{d} \sum_{q=1}^d x_{F_q}$.

The proof of the next proposition can be found in [9, Chapter 2, Proposition 2.3].

PROPOSITION 1.9. *Let $k \in \mathbb{N}$ and F be a maximal $\mathcal{S}_k$ set. Then the repeated average of F $x_F = \sum_{i \in F} c_i e_i$ is a $(k, \frac{3}{\min F})$ basic s.c.c.*

The above proposition yields the following.

PROPOSITION 1.10. *For any infinite subset M of $\mathbb{N}$, $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $F \subset M, \{c_i\}_{i \in F}$, such that $x = \sum_{i \in F} c_i e_i$ is a (k, ε) basic s.c.c.*

2. The space $\mathfrak{X}^n_{0,1}$

Let us fix a natural number n throughout the rest of the paper. We start with the definition of the norm of the space $\mathfrak{X}^n_{0,1}$.

NOTATION. Let $G \subset c_{00}$. If a vector $\alpha \in G$ is of the form $\alpha = \frac{1}{\ell} \sum_{q=1}^d f_q$, for some $f_1 < \cdots < f_d \in G, d \leq \ell$ and $2 \leq \ell$, then α will be called an α-average of size $s(\alpha) = \ell$.

Let $k \in \mathbb{N}$. A finite sequence $\{\alpha_q\}_{q=1}^d$ of α-averages in G will be called $\mathcal{S}_k$ admissible if $\alpha_1 < \cdots < \alpha_d$ and $\{\min\operatorname{supp} \alpha_q : q = 1, \ldots, d\} \in \mathcal{S}_k$.

A sequence $\{\alpha_q\}_q$ of α-averages in G will be called *very fast growing* if $\alpha_1 < \alpha_2 < \cdots$, $s(\alpha_1) < s(\alpha_2) < \cdots$ and $s(\alpha_q) > \max\operatorname{supp} \alpha_{q-i}$ for $1 < q$.

If a vector $g \in G$ is of the form $g = \sum_{q=1}^d \alpha_q$ for an $\mathcal{S}_n$-admissible and very fast growing sequence $\{\alpha_q\}_{q=1}^d \subset G$, then g will be called a Schreier functional.

The norming set. Inductively construct a set $W \subset c_{00}$ in the following manner. Set $W_0 = \{\pm e_i\}_{i \in \mathbb{N}}$. Suppose that $W_0, \ldots, W_m$ have been constructed. Define:

$$W^\alpha_{m+1} = \left\{ \alpha = \frac{1}{\ell} \sum_{q=1}^d f_q : f_1 < \cdots < f_d \in W_m, \ell \geq 2, \ell \geq d \right\},$$

$$W^S_{m+1} = \left\{ g = \sum_{q=1}^d \alpha_q : \{\alpha_q\}_{q=1}^d \subset W_m \ \mathcal{S}_n\text{-admissible and very fast growing} \right\}.$$

Define $W_{m+1} = W^{\alpha}_{m+1} \cup W^{S}_{m+1} \cup W_m$ and $W = \bigcup_{m=0}^{\infty} W_m$.

For $x \in c_{00}$ define $\|x\| = \sup\{f(x) : f \in W\}$ and $\mathfrak{X}^n_{0,1} = \overline{(c_{00}(\mathbb{N}), \|\cdot\|)}$. Evidently $\mathfrak{X}^n_{0,1}$ has a 1-unconditional basis.

One may also describe the norm on $\mathfrak{X}^n_{0,1}$ with an implicit formula. For $j \in \mathbb{N}, j \geq 2, x \in \mathfrak{X}^n_{0,1}$, set $\|x\|_j = \sup\{\frac{1}{j}\sum_{q=1}^{d} \|E_q x\|\}$, where the supremum is taken over all successive finite subsets of the naturals $E_1 < \cdots < E_d, d \leq j$. Then by using standard arguments it is easy to see that

$$\|x\| = \max\left\{ \|x\|_0, \sup\left\{ \sum_{q=1}^{d} \|E_q x\|_{j_q} \right\} \right\},$$

where the supremum is taken over all $\mathcal{S}_n$ admissible finite subsets of the naturals $E_1 < \cdots < E_k$, such that $j_q > \max E_{q-1}$, for $q > 1$.

3. Spreading models of $\mathfrak{X}^n_{0,1}$

In this section, the possible spreading models of block sequences are determined. The method used for this, is based on the α_k indices of block sequences, which are defined below and are similar to the corresponding one in [8]. We show that every subspace of $\mathfrak{X}^n_{0,1}$ admits the same variety of spreading models.

Spreading models of block sequences in $\mathfrak{X}^n_{0,1}$.

Definition 3.1. Let $0 \leq k \leq n-1$, $\{x_i\}_{i\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}^n_{0,1}$ that satisfies the following. For any subsequence $\{x_{i_j}\}_{j\in\mathbb{N}}$ of $\{x_i\}_{i\in\mathbb{N}}$, for any very fast growing sequence of α-averages $\{\alpha_q\}_{q\in\mathbb{N}}$ and any $\{F_j\}_{j\in\mathbb{N}}$ sequence of increasing subsets of the naturals such that $\{\alpha_q\}_{q\in F_j}$ is $\mathcal{S}_k$ admissible we have that $\lim_j \sum_{q\in F_j} |\alpha_q(x_{i_j})| = 0$. Then we say that the α_k-index of $\{x_i\}_{i\in\mathbb{N}}$ is zero and write $\alpha_k(\{x_i\}_i) = 0$. Otherwise we write $\alpha_k(\{x_i\}_i) > 0$.

The next proposition follow straight from the definition.

Proposition 3.2. *Let $0 \leq k \leq n-1$ and $\{x_i\}_{i\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}^n_{0,1}$, then the following statements are equivalent.*

(i) $\alpha_k(\{x_i\}_i) = 0$.
(ii) *For any $\varepsilon > 0$ there exist $j_0, i_0 \in \mathbb{N}$, such that for any $\{\alpha_q\}_{q=1}^{d}$ very fast growing and $\mathcal{S}_k$-admissible sequence of α-average with $s(\alpha_q) \geq j_0$ for $q = 1, \ldots, d$ and for any $i \geq i_0$, we have that $\sum_{q=1}^{d} |\alpha_q(x_i)| < \varepsilon$.*

Lemma 3.3. *Let α be an α-average in W, $\{x_k\}_{k=1}^{m}$ be a normalized block sequence and $\{c_k\}_{k=1}^{m}$ non-negative reals with $\sum_{k=1}^{m} c_k = 1$. Then if $G_\alpha = \{k : \operatorname{ran}\alpha \cap \operatorname{ran} x_k \neq \varnothing\}$, the following holds*:

$$\left| \alpha\left(\sum_{k=1}^{m} c_k x_k \right) \right| < \frac{1}{s(\alpha)} \sum_{i\in G_\alpha} c_i + 2\max\{c_i : i \in G_\alpha\}.$$

Proof. If $\alpha = \frac{1}{p}\sum_{j=1}^{d} f_j$ with $d \le p$. Set

$$\begin{aligned}
E_1 &= \{k \in G_\alpha : \text{there exists at most one } j \text{ with } \operatorname{ran} f_j \cap \operatorname{ran} x_k \neq \varnothing\},\\
E_2 &= \{1,\dots,m\} \setminus E_1,\\
J_k &= \{j : \operatorname{ran} f_j \cap \operatorname{ran} x_k \neq \varnothing\} \quad \text{for } k \in E_2.
\end{aligned}$$

Then it is easy to see that

$$\left|\alpha\left(\sum_{k\in E_1} c_k x_k\right)\right| \le \frac{1}{p}\sum_{k\in G_\alpha} c_k. \tag{1}$$

Moreover,

$$\left|\alpha\left(\sum_{k\in E_2} c_k x_k\right)\right| < 2\max\{c_k : k \in G_\alpha\}. \tag{2}$$

Since $\#E_2 \le 2p$, we have

$$\left|\alpha\left(\sum_{k\in E_2} c_k x_k\right)\right| \le \frac{1}{p}\sum_{k\in E_2} c_k\left(\sum_{j\in J_k}\left|f_j(x_k)\right|\right) < \max\{c_k : k \in G_\alpha\}\frac{2p}{p}.$$

By summing up (1) and (2), the result follows. ☐

LEMMA 3.4. *Let $1 \le k \le n$, $x = \sum_{i=1}^{m} c_i x_i$ be a (k,ε) s.c.c. with $\|x_i\| \le 1$ for $i = 1,\dots,m$. Let also $\{\alpha_q\}_{q=1}^{d}$ be a very fast growing and $\mathcal{S}_{k-1}$-admissible sequence of α-averages. Then the following holds.*

$$\sum_{q=1}^{d}\left|\alpha_q\left(\sum_{i=1}^{m} c_i x_i\right)\right| < \frac{1}{s(\alpha_1)} + 6\varepsilon.$$

Proof. Set

$$\begin{aligned}
G_1 &= \{i : \text{there exists at most one } q \text{ with } \operatorname{ran}\alpha_q \cap \operatorname{ran} x_i \neq \varnothing\},\\
G_2 &= \{i : \text{there exist at least two } q \text{ with } \operatorname{ran}\alpha_q \cap \operatorname{ran} x_i \neq \varnothing\},\\
J &= \{q : \text{there exists } i \in G_1 \text{ with } \operatorname{ran}\alpha_q \cap \operatorname{ran} x_i \neq \varnothing\},\\
G^q &= \{i : \operatorname{ran}\alpha_q \cap \operatorname{ran} x_i \neq \varnothing\} \quad \text{for } q \in J.
\end{aligned}$$

For $q \in J$, by Lemma 3.3 it follows that

$$\left|\alpha_q\left(\sum_{i=1}^{m} c_i x_i\right)\right| < \frac{1}{s(\alpha_q)}\sum_{i\in G^q} c_i + 2\max\{c_i : i \in G^q\}. \tag{3}$$

Choose $i_q \in G^q$ such that $c_{i_q} = \max\{c_i : i \in G^q\}$. Since $\{\alpha_q\}_{q=1}^{d}$ is $\mathcal{S}_{k-1}$-admissible, it follows that $\{\min\operatorname{supp} x_{i_q} : q \in J\}$ is the union of a $\mathcal{S}_{k-1}$ set and a singleton. Therefore, we conclude the following.

$$\sum_{q\in J}\max\{c_i : i \in G^q\} < 2\varepsilon. \tag{4}$$

Hence, combining (3) and (4), we have that

$$\sum_{q=1}^{d}\left|\alpha_q\Big(\sum_{i\in G_1} c_i x_i\Big)\right| < \frac{1}{s(\alpha_1)} + 4\varepsilon. \tag{5}$$

Moreover, it is easy to see that $\{\min\operatorname{supp} x_i : i \in G_2\}$ is the union of a $\mathcal{S}_{k-1}$ set and a singleton and therefore we have the following.

$$\sum_{q=1}^{d}\left|\alpha_q\Big(\sum_{i\in G_2} c_i x_i\Big)\right| \le \Big\|\sum_{i\in G_2} c_i x_i\Big\| \le \sum_{i\in G_2} c_i < 2\varepsilon. \tag{6}$$

Finally, summing up (5) and (6), the desired result follows. □

PROPOSITION 3.5. *Let $0 \le k \le n-1$, $\{x_i\}_{i\in\mathbb{N}} \subset \operatorname{Ba}(\mathfrak{X}^n_{0,1})$ be a normalized block sequence. The following hold*:

(i) *If $\alpha_k(\{x_i\}_i) > 0$, then, by passing to a subsequence, $\{x_i\}_{i\in\mathbb{N}}$ generates a strong ℓ_1^{n-k} spreading model.*

(ii) *If $\alpha_{k'}(\{x_i\}_i) = 0$ for $k' < k$ and $\{w_j\}_{j\in\mathbb{N}}$ is a block sequence of $\{x_i\}_{i\in\mathbb{N}}$ such that $w_j = \sum_{i\in F_j} c_i x_i$ is a $(n-k, \varepsilon_j)$ s.c.c. with $\lim_j \varepsilon_j = 0$, then $\alpha_{n-1}(\{w_j\}_j) = 0$.*

Proof. First, we prove (i). Passing to a subsequence of $\{x_i\}_{i\in\mathbb{N}}$ and relabeling we can find $\varepsilon > 0$, a very fast growing sequence of α-averages $\{\alpha_q\}_{q\in\mathbb{N}}$ and a sequence of successive finite sets $(F_i)_{i=1}^{\infty}$ such that for $i \in \mathbb{N}$ $\{\alpha_q\}_{q\in F_i}$ is $\mathcal{S}_k$ admissible and

$$\sum_{q\in F_i} \alpha_q(x_i) \ge \varepsilon$$

for each $i \in \mathbb{N}$. Passing to a further subsequence and relabeling, we can assume that

$$\max\operatorname{supp}\Big(\sum_{q\in F_i} \alpha_q\Big) < \min\operatorname{supp} x_{i+1}$$

for each $i \in \mathbb{N}$. Set $x_i^* = \sum_{q\in F_i} \alpha_q$. Then $x_i^* \in W$, $x_i^*(x_i) > \varepsilon$ for all $i \in \mathbb{N}$ and $x_j^*(x_i) = 0$ for $i \ne j$. Therefore, $\varepsilon < \|x_i^*\| \le 1$ and all that remains to be shown it that $\{x_i^*\}_{i\in\mathbb{N}}$ generates a c_0^{n-k} spreading model.

Let $F \in \mathcal{S}_{n-k}$. Note that $\{\alpha_q\}_{q\in\bigcup_{i\in F} F_i}$ is $\mathcal{S}_n$ admissible. It follows that $\|\sum_{i\in F} x_i^*\| \le 1$. In other words, $\{x_i^*\}_{i\in\mathbb{N}}$ generates a c_0^{n-k} spreading model.

We now prove (ii). Let $w_j = \sum_{i\in F_j} c_i x_i$ be the $(n-k, \varepsilon_j)$ s.c.c.; we claim that $\alpha_{n-1}(\{w_j\}_j) = 0$. First, pass to a subsequence of $\{w_j\}$ and relabel for simplicity. Now, fix a sequence $\{\alpha_q\}_{q\in\mathbb{N}}$ of very fast growing α-averages and a sequence $(L_j)_{j\in\mathbb{N}}$ of successive finite subsets $\mathbb{N}$ such that $\{\alpha_q\}_{q\in L_j}$ is $\mathcal{S}_{n-1}$ admissible for each $j \in \mathbb{N}$.

Let $\varepsilon > 0$. First, we consider the case $k > 0$. Since $\alpha_{k-1}(\{x_i\}_i) = 0$ and $\{\alpha_q\}_{q\in\mathbb{N}}$ is very fast growing, by Proposition 3.2 we can find $q_0, i_0 \in \mathbb{N}$ such that for each finite set $L \geq q_0$, with $\{\alpha_q\}_{q\in L}$ being $\mathcal{S}_{k-1}$ admissible, and $i \geq i_0$, we have

$$\sum_{q\in L} \big|\alpha_q(x_i)\big| < \varepsilon/3.$$

Find $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$

$$\min L_j \geq q_0, \qquad \min F_j \geq i_0 \quad \text{and} \quad \varepsilon_j < \varepsilon/6. \tag{7}$$

Fix $j \geq j_0$. We claim that

$$\sum_{q\in L_j} \big|\alpha_q(w_j)\big| < \varepsilon.$$

This, of course, implies the $\alpha_{n-1}(\{w_j\}_j) = 0$. To simplify notation, let $L = L_j$ and $F = F_j$. Before passing to the proof we note the following:

For $i \in F$ and $E \subset L$ such that $\{\alpha_q\}_{q\in E}$ is $\mathcal{S}_{k-1}$ admissible, we have

$$\sum_{q\in E} \big|\alpha_q(x_i)\big| < \varepsilon/3. \tag{8}$$

Partition L into the following sets:

$$G_1 = \{i \in F : \text{there is a unique } q \in L \text{ such that } \operatorname{ran}\alpha_q \cap \operatorname{ran} x_i \neq \varnothing\},$$
$$G_2 = \{i \in F : \text{there are at least two } q \in L \text{ such that } \operatorname{ran}\alpha_q \cap \operatorname{ran} x_i \neq \varnothing\}.$$

First, consider the case of G_1. For $q \in L$ let

$$H_q = \{i \in G_1 : \operatorname{ran}\alpha_q \cap \operatorname{ran} x_i \neq \varnothing\}.$$

If $q \neq q'$ then $H_q \cap H_{q'} = \varnothing$; and $\bigcup_{q\in L} H_q \subset F$. Using (8) (for singleton subsets of L) and the convexity of $(c_i)_{i\in F}$, we have

$$\begin{aligned}\sum_{q\in L}\bigg|\alpha_q\bigg(\sum_{i\in G_1} c_i x_i\bigg)\bigg| &= \sum_{q\in L}\bigg|\alpha_q\bigg(\sum_{i\in H_q} c_i x_i\bigg)\bigg| \\ &< \frac{\varepsilon}{3}\sum_{q\in L}\sum_{i\in H_q} c_i \leq \frac{\varepsilon}{3}.\end{aligned}$$

For $i \in G_2$, set

$$\begin{aligned} J_i &= \{q \in L : \operatorname{ran}\alpha_q \cap \operatorname{ran} x_i \neq \varnothing\}, \\ G_2' &= \big\{i \in G_2 : \{\min\operatorname{supp}\alpha_q : q \in J_i\} \notin S_{k-1}\big\}.\end{aligned}$$

This splits the estimates in the following way:

$$\begin{aligned}\sum_{q\in L}\bigg|\alpha_q\bigg(\sum_{i\in G_2} c_i x_i\bigg)\bigg| &\leq \sum_{i\in G_2}\bigg|c_i\bigg(\sum_{q\in J_i}\alpha_q\bigg)(x_i)\bigg| \\ &= \sum_{i\in G_2'} c_i\bigg|\bigg(\sum_{q\in J_i}\alpha_q\bigg)(x_i)\bigg| + \sum_{i\in G_2\setminus G_2'} c_i\bigg|\bigg(\sum_{q\in J_i}\alpha_q\bigg)(x_i)\bigg|.\end{aligned}$$

Since for each $i \in G_2 \setminus G_2'$, $\{\alpha_q\}_{i\in J_i}$ is $\mathcal{S}_{k-1}$ admissible we can apply (8) to conclude that

$$\sum_{i\in G_2\setminus G_2'} c_i \left| \left(\sum_{q\in J_i} \alpha_q \right)(x_i) \right| \le \frac{\varepsilon}{3} \sum_{i\in G_2\setminus G_2'} c_i \le \frac{\varepsilon}{3}.$$

For the final case, we must observe that

$$\big\{\min\operatorname{supp} x_i : i \in G_2'\big\} \in 2S_{n-k-1}. \tag{9}$$

Let $G_2'' = G_2' \setminus \min G_2'$. For each $i \in G_2''$ it is clear that

$$\min\operatorname{supp} x_i \ge \min\operatorname{supp} \alpha_{\min J_{i'}} \quad \text{for } i' < i \text{ and } i' \in G_2'. \tag{10}$$

Find $\ell \in \mathbb{N}$ such that

$$\big\{\min\operatorname{supp} \alpha_{\min J_i} : i \in G_2''\big\} \in S_\ell.$$

Since

$$\{\min\operatorname{supp} \alpha_q : q \in F\} \supset \bigcup_{i\in G_2''} \{\min\operatorname{supp} \alpha_q : q \in J_i\}.$$

The second set is $\mathcal{S}_{n-1}$ admissible. It is clear that for $i \in G_2''$

$$\min\operatorname{supp} \alpha_{\min J_i} = \min\{\min\operatorname{supp} \alpha_q : q \in J_i\}$$

and $\{\min\operatorname{supp} \alpha_q : q \in J_i\} \in S_d$, for some $d \ge k$.

The convolution property of the Schreier sets yields that $\ell + d \le n-1$. Therefore $\ell \le n-d-1 \le n-k-1$. From (10), it follows that

$$\big\{\min\operatorname{supp} x_i : i \in G_2''\big\} \in S_{n-k-1}.$$

Since we are excluding a singleton, (9) follows. Therefore $\sum_{i\in G_2'} c_i < 2\varepsilon_j < \varepsilon/3$, by our choice of j_0 (see (7)). Since $\{x_i\}_i \subset \mathrm{Ba}(\mathfrak{X}^n_{0,1})$

$$\sum_{i\in G_2'} c_i \left(\sum_{q\in J_i} \alpha_q \right)(x_i) \le \sum_{i\in G_2'} c_i < \varepsilon/3.$$

This proves our claim for the case $k > 0$.

Now we consider the case $k = 0$. Find $q_0 \in \mathbb{N}$ such that

$$\frac{1}{s(\alpha_{q_0})} < \varepsilon/2.$$

Now fix $j_0 \in \mathbb{N}$ such that for all $j \ge j_0$

$$\min L_j \ge q_0 \quad \text{and} \quad \varepsilon_j < \varepsilon/8.$$

Fix $j \ge j_0$ and for simplicity let $L = L_j$ and $F = F_j$.

Using Lemma 3.4, we have

$$\sum_{q\in L} \left| \alpha_q \left(\sum_{i\in F} c_i x_i \right) \right| < \frac{\varepsilon}{2} + 4 \cdot \frac{\varepsilon}{8} = \varepsilon.$$

This finishes the proof. $\square$

PROPOSITION 3.6. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \mathrm{Ba}(\mathfrak{X}^n_{0,1})$ be a block sequence such that $\alpha_{n-1}(\{x_i\}_i) = 0$. Then for $\varepsilon > 0$ there is a subsequence $\{x'_i\}_{i\in\mathbb{N}}$ of $\{x_i\}_{i\in\mathbb{N}}$ such that for every $F \in \mathcal{S}_1$*

$$\Big\|\sum_{i\in F} x'_i\Big\| < 1 + \varepsilon.$$

Moreover if $\{x_i\}_{i\in\mathbb{N}}$ is normalized there is a subsequence that generates a spreading model isometric to c_0.

Proof. Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a summable sequence of positive reals, such that $\varepsilon_i > 3\sum_{j>i}\varepsilon_j$ for all $i \in \mathbb{N}$. Using Proposition 3.2, inductively choose a subsequence, again denoted by $\{x_i\}_{i\in\mathbb{N}}$, such that for any $i_0 \geq 2$ and $i \geq i_0$, for any $\{\alpha_q\}_{q=1}^{\ell}$ very fast growing and $\mathcal{S}_{n-1}$-admissible sequence of α-average with $s(\alpha_q) \geq \min\operatorname{supp} x_{i_0}$ for $q = 1,\dots,\ell$, we have that

$$\sum_{q=1}^{\ell}\big|\alpha_q(x_i)\big| < \frac{\varepsilon_{i_0}}{i_0 \max\operatorname{supp} x_{i_0-1}}. \tag{11}$$

We will show that for any $t \leq i_1 < \dots < i_t$, $F \subset \{1,\dots,t\}$ we have

$$\bigg|\alpha\Big(\sum_{j\in F} x_{i_j}\Big)\bigg| < 1 + 2\varepsilon_{i_{\min F}}$$

whenever α is an α-average and

$$\bigg|g\Big(\sum_{j\in F} x_{i_j}\Big)\bigg| < 1 + 3\varepsilon_{i_{\min F}}$$

whenever g is a Schreier functional. This implies the conclusion of the proposition.

For functionals in W_0 the above is clearly true. Assume, for some $m \geq 0$ the above holds for any $t \leq i_1 < \dots < i_t$ and any functional in W_m. In the first case, let $t \leq i_1 < \dots < i_t$ and $\alpha \in W_{m+1}$ with $\alpha = \frac{1}{\ell}\sum_{q=1}^{d} f_q, d \leq \ell$.

Set

$$E_1 = \{q : \text{there exists at most one } j \leq t \text{ such that } \operatorname{ran} f_q \cap \operatorname{ran} x_{i_j} \neq \varnothing\},$$

and $E_2 = \{1,\dots,\ell\} \setminus E_1$. For $q \in E_1$, we have $|f_q(\sum_{j=1}^{n} x_{i_j})| \leq 1$. Therefore $\sum_{q\in E_1} |f_q(\sum_{j=1}^{n} x_{i_j})| \leq \#E_1$.

For q in E_2, let $j_q \in \{1,\dots,t\}$ be minimum such that $\operatorname{ran} x_{i_{j_q}} \cap \operatorname{ran} f_q \neq \varnothing$. If $q < q'$ are in E_2, $j_q < j_{q'}$. By the inductive assumption

$$\begin{aligned}\sum_{q\in E_2}\bigg|f_q\Big(\sum_{j=1}^{t} x_{i_j}\Big)\bigg| &< \sum_{q\in E_2}(1 + 3\varepsilon_{i_{j_q}}) \\ &< \#E_2 + 3\varepsilon_{i_1} + 3\sum_{j>1}\varepsilon_{i_j} < \#E_2 + 4\varepsilon_{i_1}.\end{aligned} \tag{12}$$

Therefore,

$$\left|\alpha\left(\sum_{j=1}^{t} x_{i_j}\right)\right| < \frac{d+4\varepsilon_{i_1}}{\ell} \leq 1+2\varepsilon_{i_1}.$$

Let $g \in W_{m+1}$ with $g = \sum_{q=1}^{d} \alpha_q$ be a Schreier functional. Set

$$j_0 = \min\{j : \operatorname{ran} g \cap \operatorname{ran} x_{i_j} \neq \varnothing\},$$
$$q_0 = \min\{q : \max\operatorname{supp} \alpha_q \geq \min\operatorname{supp} x_{i_{j_0+1}}\}.$$

Decompose $\{q : q > q_0\}$ into successive intervals $\{J_\nu\}_{\nu=1}^{\nu_0}$ such that the following hold:

(i) $\{q : q > q_0\} = \bigcup_{\nu=1}^{\nu_0} J_\nu$ and
(ii) $\{\min\operatorname{supp} \alpha_q : q \in J_\nu\}$ are maximal $\mathcal{S}_{n-1}$ sets (except perhaps the last one).

Since $\{\alpha_q\}_{q=1}^{d}$ is $\mathcal{S}_n$ admissible, $\nu_0 \leq \max\operatorname{supp} x_{i_{j_0}}$. By definition, for $q > q_0$

$$s(\alpha_q) > \max\operatorname{supp} \alpha_{q_0} \geq \min\operatorname{supp} x_{i_{j_0+1}}.$$

Therefore, we can apply (11) to conclude that

$$\sum_{q>q_0}\left|\alpha_q\left(\sum_{j=1}^{t} x_{i_j}\right)\right| = \sum_{\nu=1}^{\nu_0}\sum_{q\in J_\nu}\left|\alpha_q\left(\sum_{j>j_0} x_{i_j}\right)\right| < \nu_0 \cdot \frac{\varepsilon_{i_{j_0+1}}}{i_{j_0+1}\max\operatorname{supp} x_{i_{j_0}}} \cdot t < \varepsilon_{i_{j_0}}. \tag{13}$$

For the other part of the functional, we consider two cases.

Case 1. Assume that for $q < q_0$, $\alpha_q(\sum_{j=1}^{t} x_{i_j}) = 0$. In this case we simply apply the inductive assumption to conclude that $\alpha_{q_0}(\sum_{j=1}^{t} x_{i_j}) < 1+2\varepsilon_{i_{j_0}}$. Combining this with (13) finishes the proof.

Case 2. If the first case does not hold, we have that $s(\alpha_{q_0}) \geq \min\operatorname{supp} x_{i_{j_0}}$. Using (11), we have

$$\sum_{q<q_0}\left|\alpha_q\left(\sum_{j=1}^{t} x_{i_j}\right)\right| + \left|\alpha_{q_0}\left(\sum_{j=1}^{t} x_{i_j}\right)\right| = \sum_{q<q_0}\left|\alpha_q(x_{i_{j_0}})\right| + \left|\alpha_{q_0}\left(\sum_{j=j_0}^{t} x_{i_j}\right)\right| < 1+\varepsilon_{i_{j_0}}. \tag{14}$$

Combining this with (13) gives the desired result. $\square$

PROPOSITION 3.7. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a seminormalized block sequence in $\mathfrak{X}_{0,1}^n$ and $0 \leq k \leq n-1$. The following assertions are equivalent.*

(i) *$\alpha_{k'}(\{x_i\}_i) = 0$ for $k' < k$.*
(ii) *$\{x_i\}_{i\in\mathbb{N}}$ has no subsequence generating an ℓ_1^{n-k+1} spreading model.*

Proof. First, assume that (i) holds. Towards a contradiction, assume that passing, if necessary, to a subsequence, $\{x_i\}_{i\in\mathbb{N}}$ generates an ℓ_1^{n-k+1} spreading model, with a lower constant $\theta > 0$.

We may choose $\{F_j\}_{j\in\mathbb{N}}$ increasing $\mathcal{S}_{n-k}$ sets with $F_j \geq j$ for all $j\in\mathbb{N}$, $\{\varepsilon_j\}_{j\in\mathbb{N}}$ positive reals with $\lim_j \varepsilon_j = 0$ and $\{c_i\}_{\in F_j}$ positive reals, such that $w_j = \sum_{i\in F_j} c_i x_i$ is a $(n-k,\varepsilon_j)$ s.c.c. for all $j\in\mathbb{N}$.

If $M = \sup\{\|w_j\| : j\in\mathbb{N}\}$, it follows that $\theta < \|w_j\| \leq M$ for all $j\in\mathbb{N}$.

For any $t \leq j_1 < \cdots < j_t$, $\bigcup_{q=1}^t F_{j_q}$ is a $\mathcal{S}_{n-k+1}$ set, therefore

$$\left\|\sum_{q=1}^t w_{j_q}\right\| > \theta\cdot t. \tag{15}$$

Propositions 3.5(ii) and 3.6, yield that passing, if necessary, to subsequence, for any $t \leq j_1 < \cdots < j_t$ the following holds.

$$\left\|\sum_{q=1}^t w_{j_q}\right\| < 2M. \tag{16}$$

For t appropriately large, (15) and (16) together yield a contradiction.

Now assume that (ii) holds. Let $0 \leq k' \leq n-1$ such that $\alpha_{k'}(\{x_i\}_i) > 0$. Proposition 3.5(i) yields that passing, if necessary, to a subsequence, $\{x_i\}_{i\in\mathbb{N}}$ generates an $\ell_1^{n-k'}$ spreading model. Since (ii) holds, we have that $n-k' < n-k+1$, therefore $k \leq k'$ and this completes the proof. $\square$

PROPOSITION 3.8. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a seminormalized block sequence in $\mathfrak{X}^n_{0,1}$ and $0 \leq k \leq n-1$. The following assertions are equivalent.*

(i) $\alpha_k(\{x_i\}_i) > 0$.

(ii) *$\{x_i\}_{i\in\mathbb{N}}$ has a subsequence generating a strong ℓ_1^{n-k} spreading model.*

Proof. If (i) holds, then by Proposition 3.5 so does (ii).

Assume now that (ii) is holds. Pass to a subsequence of $\{x_i\}_{i\in\mathbb{N}}$ generating an ℓ_1^{n-k} spreading model and relabel for simplicity. Towards a contradiction assume that $\alpha_k(\{x_i\}_i) = 0$.

Consider first the case $k = n-1$. Then by Proposition 3.6, $\{x_i\}_{i\in\mathbb{N}}$ has a subsequence generating a c_0 spreading model, which is absurd.

Otherwise, if $k < n-1$, then evidently we have that $\alpha_{k'}(\{x_i\}_i) = 0$ for $k' < k+1$. Proposition 3.7 yields a contradiction. $\square$

Combining Propositions 3.6, 3.7 and 3.8, we conclude the following.

COROLLARY 3.9. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}^n_{0,1}$. Then the following assertions are equivalent.*

(i) *Any subsequence of $\{x_i\}_{i\in\mathbb{N}}$ has a further subsequence generating an isometric c_0 spreading model.*

(ii) $\alpha_{n-1}(\{x_i\}_i) = 0$.

REMARK 3.10. Every normalized weakly null sequence generating a c_0 spreading model satisfies $\alpha_{n-1}(\{x_i\}_i)=0$. The above yields that c_0 spreading models generated by normalized weakly null sequences are always isometric to the usual basis of c_0.

COROLLARY 3.11. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}^n_{0,1}$ and $0\le k\le n-1$. Then the following assertions are equivalent.*

(i) *$\alpha_k(\{x_i\}_i)>0$ and $\alpha_{k'}(\{x_i\}_i)=0$ for $k'<k$.*

(ii) *$\{x_i\}_{i\in\mathbb{N}}$ has a subsequence generating a strong ℓ_1^{n-k} spreading model and no subsequence of $\{x_i\}_{i\in\mathbb{N}}$ generates a strong ℓ_1^{n-k+1} spreading model.*

(iii) *$\{x_i\}_{i\in\mathbb{N}}$ has a subsequence generating an ℓ_1^{n-k} spreading model and no subsequence of $\{x_i\}_{i\in\mathbb{N}}$ generates an ℓ_1^{n-k+1} spreading model.*

Proof. Assume first that (i) holds. Propositions 3.7 and 3.8 yield that (ii) also holds.

Assume now that (ii) is true. To prove that (iii) is true as well, all that needs to be shown is that no subsequence of $\{x_i\}_{i\in\mathbb{N}}$ generates an ℓ_1^{n-k+1} spreading model. Towards a contradiction, assume that this is not the case. Proposition 3.7 yields that there exists $k'<k$ such that $\alpha_{k'}(\{x_i\}_i)>0$. In turn, Proposition 3.8 yields that $\{x_i\}_{i\in\mathbb{N}}$ has a subsequence that generates a strong $\ell_1^{n-k'}$ spreading model. The fact that $k'<k$ and no subsequence $\{x_i\}_{i\in\mathbb{N}}$ generates a strong ℓ_1^{n-k+1} spreading model yields a contradiction.

For the last part, assume that (iii) holds. We will show that so does (i). Proposition 3.7 yields that $\alpha_{k'}(\{x_i\}_i)=0$ for $k'<k$. Towards a contradiction, assume that $\alpha_k(\{x_i\}_i)=0$.

If $k=n-1$, Corollary 3.9 yields that any subsequence of $\{x_i\}_{i\in\mathbb{N}}$ has a further subsequence generating a c_0 spreading model, which is absurd.

Otherwise, if $k<n-1$, then $\alpha_{k'}(\{x_i\}_i)=0$ for $k'<k+1$. Once more, Proposition 3.7 yields that no subsequence of $\{x_i\}_{i\in\mathbb{N}}$ generates an ℓ_1^{n-k} spreading model, a contradiction which completes the proof. ☐

Corollaries 3.9 and 3.11 easily yield the following.

COROLLARY 3.12. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a normalized weakly null sequence in $\mathfrak{X}^n_{0,1}$. Then passing, if necessary, to a subsequence, exactly one of the following holds.*

(i) *$\{x_i\}_{i\in\mathbb{N}}$ generates an isometric c_0 spreading model.*

(ii) *There exists $0\le k\le n-1$ such that $\{x_i\}_{i\in\mathbb{N}}$ generates a strong ℓ_1^{n-k} spreading model and no subsequence of it generates an ℓ_1^{n-k+1} spreading model.*

REMARK 3.13. Corollaries 3.11 and 3.12 yield that whenever a normalized weakly null sequence generates an ℓ_1^{n-k} spreading model, for some $0\le k\le n$, then passing, if necessary, to a subsequence, it generates a strong ℓ_1^{n-k} spreading model.

As we will show in Proposition 3.18, any block subspace of $\mathfrak{X}^n_{0,1}$, hence any subspace of $\mathfrak{X}^n_{0,1}$, contains a normalized weakly null sequence generating a c_0 spreading model and for any $0 \le k \le n-1$, it contains a normalized weakly null sequence generating an ℓ_1^{n-k} spreading model having no subsequence generating an ℓ_1^{n-k+1} spreading model.

Although in the usual sense of spreading models, any subspace of $\mathfrak{X}^n_{0,1}$ admits exactly two types of them, in the sense of higher order spreading models, any subspace of $\mathfrak{X}^n_{0,1}$ admits exactly $n+1$ types.

It is an interesting question, whether for given $n \in \mathbb{N}$ there exists a Banach space X, such that any subspace of it admits exactly $n+1$ types of spreading models, in the usual sense.

Spreading models of subspaces of $\mathfrak{X}^n_{0,1}$.

PROPOSITION 3.14. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}^n_{0,1}$ that generates a spreading model isometric to c_0, $\{F_j\}_{j\in\mathbb{N}}$ be a sequence of successive subsets of the naturals, such that $\#F_j \le \min F_j$, for all $j \in \mathbb{N}$ and $\lim_j \#F_j = \infty$. Then if $y_j = \sum_{i\in F_j} x_i$, there exists a subsequence of $\{y_j\}_{j\in\mathbb{N}}$ generating an ℓ_1^n spreading model.*

Proof. Since $\{x_i\}_{i\in\mathbb{N}}$ generates a spreading model isometric to c_0, it follows that $\|y_j\| \to 1$. By Proposition 3.5, it suffices to choose $\{y_{j_m}\}_{m\in\mathbb{N}}$ a subsequence of $\{y_j\}_{j\in\mathbb{N}}$, such that $\alpha_0(\{y_{j_m}\}_m) > 0$. Set $j_1 = 1$ and assume that $j_1, \dots, j_{m-1}$ have been chosen. Set $d = \max\{\max\operatorname{supp} y_{j_{m-1}}, \#F_{j_{m-1}}\}$ and choose $j_m > j_{m-1}$ such that $\#F_{j_m} > d$.

To see that $\{y_{j_m}\}_{m\in\mathbb{N}}$ generates an ℓ_1^n spreading model, notice that for $m > 1$, there exists an α-average α_m with $\operatorname{ran}\alpha_m \subset \operatorname{ran} y_{j_m}$ and $s(\alpha_m) = \#F_{j_m} > \max\{\max\operatorname{supp}\alpha_{m-1}, s(\alpha_{m-1})\}$ such that $\alpha_m(y_{j_m}) \to 1$. Therefore $\alpha_0(\{y_{j_m}\}_m) > 0$. □

COROLLARY 3.15. *The space $\mathfrak{X}^n_{0,1}$ does not contain seminormalized weakly null sequences generating c_0^2 or ℓ_1^{n+1} spreading models.*

Proof. Assume that there exists a seminormalized weakly null sequence $\{x_i\}_{i\in\mathbb{N}}$ generating a c_0^2 spreading model. We may therefore assume that it is a block sequence. By Proposition 3.14, it follows that there exist $\{F_j\}_{j\in\mathbb{N}}$ increasing, Schreier admissible subsets of the naturals and $\theta > 0$, such that $\|\sum_{q=1}^m \sum_{i\in F_{j_q}} x_i\| > \theta \cdot m$ for any $m \le j_1 < \cdots < j_m$. Since for any such $F_{j_1} < \cdots < F_{j_m}$ we have that $\bigcup_{q=1}^m F_{j_q} \in \mathcal{S}_2$, it follows that $\{x_i\}_{i\in\mathbb{N}}$ does not generate a c_0^2 spreading model.

The fact that $\mathfrak{X}^n_{0,1}$ does not contain seminormalized weakly null sequences generating ℓ_1^{n+1} spreading models follows from Corollary 3.12. □

PROPOSITION 3.16. *Let $0 \le k \le n-1$ and $\{x_i\}_{i\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}^n_{0,1}$ that generates an ℓ_1^{n-k} spreading model and no subsequence* *of it generates an ℓ_1^{n-k+1} spreading model. Then there exists $\{F_j\}_{j\in\mathbb{N}}$ an increasing sequence of subsets of the naturals and $\{c_i\}_{i\in F_j}$ non-negative reals with $\sum_{i\in F_j} c_i = 1$, satisfying the following. If we set $w_j = \sum_{i\in F_j} c_i x_i$, then $\{w_j\}_{j\in\mathbb{N}}$ is seminormalized and generates a c_0 spreading model.*

Proof. By Corollary 3.11, it follows that $\alpha_k(\{x_i\}_i) > 0$ and $\alpha_{k'}(\{x_i\}_i) = 0$ for $k' < k$. Choose $\{F_j\}_{j\in\mathbb{N}}$ and increasing sequence of $\mathcal{S}_{n-k}$ subsets of the naturals and $\{c_i\}_{i\in F_j}$ non-negative reals such that $w_j = \sum_{i\in F_j} c_i x_i$ is a $(n-k,\varepsilon_j)$ s.c.c. with $\lim_j \varepsilon_j = 0$.

Since $\{x_i\}_{i\in\mathbb{N}}$ generates an ℓ_1^{n-k} spreading model, it follows that $\{w_j\}_{j\in\mathbb{N}}$ is seminormalized. Moreover, Proposition 3.5(ii) yields that $\alpha_{n-1}(\{w_j\}_j) = 0$. Applying Proposition 3.6 we conclude the desired result. □

PROPOSITION 3.17. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}_{0,1}^n$ generating an ℓ_1^n spreading model and $1 \le k \le n-1$. Then there exists $\{F_j\}_{j\in\mathbb{N}}$ an increasing sequence of subsets of the naturals and $\{c_i\}_{i\in F_j}$ non-negative reals with $\sum_{i\in F_j} c_i = 1$, satisfying the following. If we set $w_j = \sum_{i\in F_j} c_i x_i$, then $\{w_j\}_{j\in\mathbb{N}}$ is seminormalized, generates an ℓ_1^{n-k} spreading model and no subsequence of it generates an ℓ_1^{n-k+1} spreading model.*

Proof. By Corollary 3.11 and passing, if necessary to a subsequence, there exists $\{\alpha_i\}_{i\in\mathbb{N}}$ a very fast growing sequence of α-averages, such that $\operatorname{ran}\alpha_i \subset \operatorname{ran} x_i$ and $\theta > 0$ such that $\alpha_i(x_i) > \theta$ for all $i \in \mathbb{N}$. Choose $\{F_j\}_{j\in\mathbb{N}}$ and increasing sequence of $\mathcal{S}_k$ subsets of the naturals and $\{c_i\}_{i\in F_j}$ non-negative reals such that $w_j = \sum_{i\in F_j} c_i x_i$ is a (k,ε_j) s.c.c. with $\lim_j \varepsilon_j = 0$.

Since $\{x_i\}_{i\in\mathbb{N}}$ generates an ℓ_1^n spreading model and $k < n$, we have that $\{w_j\}_{j\in\mathbb{N}}$ is seminormalized.

To see that $\{w_j\}_{j\in\mathbb{N}}$ has a subsequence generating an ℓ_1^{n-k} spreading model, by Corollary 3.11 it is enough to show that $\alpha_k(\{w_j\}_j) > 0$. It is straightforward to check that the sequences $\{\alpha_i\}_{i\in\mathbb{N}}$ and $\{F_j\}_{j\in\mathbb{N}}$ previously chosen, witness this fact.

It remains to be shown that no subsequence of $\{w_j\}_{j\in\mathbb{N}}$ generates an ℓ_1^{n-k+1} spreading model. Once more, by Corollary 3.11 it is enough to check that $\alpha_{k-1}(\{x_i\}_i) = 0$.

Pass to a subsequence of $\{w_j\}_{j\in\mathbb{N}}$, relabel for simplicity, let $\{\alpha'_i\}_{i\in\mathbb{N}}$ be a very fast growing sequence of α-averages and $\{G_j\}_{j\in\mathbb{N}}$ be an increasing sequence of subsets of the naturals such that $\{\alpha'_i\}_{i\in G_j}$ is $\mathcal{S}_{k-1}$ admissible for all $j \in \mathbb{N}$. Lemma 3.4 yields the following.

$$\lim_{j\to\infty} \sum_{i\in G_j} \left|\alpha'_i(w_j)\right| \le \lim_{j\to\infty} \left(\frac{1}{s(\alpha'_{\min G_j})} + 6\varepsilon_j\right) = 0.$$

By definition, this means that $\alpha_{k-1}(\{x_i\}_i) = 0$ and this completes the proof. □

PROPOSITION 3.18. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$. Then there exists a normalized weakly null sequence in Y generating an isometric c_0 spreading model. Moreover, for $0 \le k \le n-1$ there exists a sequence in Y that generates an ℓ_1^{n-k} spreading model and no subsequence of it generates an ℓ_1^{n-k+1} one.*

Proof. Assume first that Y is a block subspace. We first show that Y admits an isometric c_0 spreading model. Let $\{x_i\}_{i\in\mathbb{N}}$ be a normalized block sequence in Y. If $\{x_i\}_{i\in\mathbb{N}}$ has a subsequence generating a c_0 spreading model, then by Remark 3.10 there is nothing to prove.

If this is not the case, by Corollary 3.9 we conclude that $\alpha_{n-1}(\{x_i\}_i) > 0$. Set $k_0 = \min\{k' : \alpha_{k'}(\{x_i\}_i) > 0\}$. Corollary 3.11 yields that passing, if necessary, to a subsequence, $\{x_i\}_{i\in\mathbb{N}}$ generates an $\ell_1^{n-k_0}$ spreading model and no further subsequence of it generates an $\ell_1^{n-k_0+1}$ one. Proposition 3.16 yields that $\{x_i\}_{i\in\mathbb{N}}$ has a further seminormalized block sequence $\{w_j\}_{j\in\mathbb{N}}$ generating a c_0 spreading model. If we set $y_j = \frac{w_j}{\|w_j\|}$, then by Remark 3.10 $\{y_j\}_{j\in\mathbb{N}}$ is the desired sequence.

We now prove that Y admits an ℓ_1^n spreading model. Take $\{x_i\}_{i\in\mathbb{N}}$ a normalized block sequence in Y generating an isometric c_0 spreading model. By Proposition 3.14 there exists $\{w_j\}_{j\in\mathbb{N}}$ a further block sequence of $\{x_i\}_{i\in\mathbb{N}}$ generating an ℓ_1^n spreading model. By Corollary 3.15 $\{w_j\}_{j\in\mathbb{N}}$ is the desired sequence.

Let $1 \le k \le n-1$. We show that there exists a sequence in Y that generates an ℓ_1^{n-k} spreading model and no subsequence of it generates an ℓ_1^{n-k+1} one. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in Y generating an ℓ_1^n spreading model. Simply apply Proposition 3.17 to find the desired sequence.

Therefore the statement is true for block subspaces. The fact that any subspace of $\mathfrak{X}^n_{0,1}$ contains a sequence arbitrarily close to a block sequence completes the proof. $\square$

From this, it follows that $\mathfrak{X}^n_{0,1}$ cannot contain c_0 or ℓ_1, therefore from James' theorem for spaces with an unconditional basis [15], the next result follows.

COROLLARY 3.19. *The space $\mathfrak{X}^n_{0,1}$ is reflexive.*

COROLLARY 3.20. *Let Y be an infinite dimensional, closed subspace of $\mathfrak{X}^n_{0,1}$. Then Y^* admits a spreading model isometric to ℓ_1. Moreover, for $0 \le k \le n-1$ there exists a sequence in Y^* generating a c_0^{n-k} spreading model, such that no subsequence of it generates a c_0^{n-k+1} one.*

Proof. Since Y contains a sequence $\{x_i\}_{i\in\mathbb{N}}$ generating a spreading model isometric to c_0, which we may assume is unconditional Schauder basic, such that $\{x_i\}_{i\ge j}$ has an unconditional basic constant $c_j \to 1$, as $j \to \infty$, then for any normalized $\{x_i^*\}_{i\in\mathbb{N}} \subset Y^*$, such that $x_i^*(x_i) = 1$, we have that $\{x_i^*\}_{i\in\mathbb{N}}$ generates a spreading model isometric to ℓ_1.

Let now $0 \le k \le n-1$. Use Proposition 3.18 to choose $\{x_i\}_{i\in\mathbb{N}}$ a normalized weakly null sequence in Y, generating an ℓ_1^{n-k} spreading model, such that no subsequence of it generates an ℓ_1^{n-k+1} one.

By Remark 3.13 and passing if necessary to a subsequence, there exist $\varepsilon > 0$ and $\{x_i^*\}_{i\in\mathbb{N}}$ a seminormalized sequence in X^* generating a c_0^{n-k} spreading model satisfying the following. $x_i^*(x_i) > \varepsilon$ for all $i \in \mathbb{N}$ and $\sum_{i\neq j}|x_i^*(x_j)| < \infty$. Since $\{x_i\}_{i\in\mathbb{N}}$ has no subsequence generating an ℓ_1^{n-k+1} spreading model, it follows that $\{x_i^*\}_{i\in\mathbb{N}}$ has no subsequence generating a c_0^{n-k+1} one.

Let $I^* : \mathfrak{X}^{n*}_{0,1} \to Y^*$ be the dual operator of $I : Y \to \mathfrak{X}^n_{0,1}$. Then, since $\|I^*\| = 1$, to see that this generates a c_0^{n-k} spreading model, all that needs to be shown is that $\{I^*x_i^*\}_{i\in\mathbb{N}}$ is bounded from below. Indeed, $\|I^*x_i^*\| \ge (I^*x_i^*)(x_i) = x_i^*(x_i) > \varepsilon$.

It remains to be shown $\{I^*x_i^*\}_{i\in\mathbb{N}}$ has no subsequence generating a c_0^{n-k+1} spreading model. Since it is seminormalized, $(I^*x_i^*)(x_i) = x_i^*(x_i) > \varepsilon$ for all $i \in \mathbb{N}$, and $\sum_{i\neq j}|(I^*x_i^*)(x_j)| = \sum_{i\neq j}|x_i^*(x_j)| < \infty$ and $\{x_i\}_{i\in\mathbb{N}}$ has no subsequence generating an ℓ_1^{n-k+1} spreading model, the result easily follows. $\square$

4. Equivalent block sequences in $\mathfrak{X}^n_{0,1}$

In this section, we prove that the space $\mathfrak{X}^n_{0,1}$ is quasi minimal by showing that every two block subspaces have further block sequences which are equivalent. Our method is based on the analysis of the functionals of the norming set W and we use some techniques first appeared in [3].

In Tsirelson space, whenever two seminormalized block sequences $\{x_m\}_{m\in\mathbb{N}}$, $\{y_m\}_{m\in\mathbb{N}}$ satisfy $x_m < y_{m+1}$ and $y_m < x_{m+1}$ for all $m \in \mathbb{N}$, then they are equivalent (see [11]). In the space $\mathfrak{X}^n_{0,1}$ this is false, since seminormalized sequences satisfying this condition may be constructed generating different spreading models, therefore they cannot be equivalent.

Even in the case for sequences satisfying the above condition, which moreover generate the same spreading model, we are unable to prove that they have equivalent subsequences, not even if they only consist of elements of the basis. The reason for this is the fact that when constructing Schreier functionals in the norming set W, unlike the norming set of Tsirelson space, very fast growing sequences of α-averages need to be taken.

In order to compensate for this fact, the following is done. Let $\{x_m\}_{m\in\mathbb{N}}$, $\{y_m\}_{m\in\mathbb{N}}$ be normalized block sequences, both generating ℓ_1^n spreading models, such that $x_m < y_{m+1}$ and $y_m < x_{m+1}$ for all $m \in \mathbb{N}$. we show that by appropriately blocking both sequences in the same manner, we obtain sequences which are equivalent. More precisely, we prove the following.

Proposition 4.1. *Let $\{x_m\}_{m\in\mathbb{N}}, \{y_m\}_{m\in\mathbb{N}}$ be normalized block sequences in $\mathfrak{X}^n_{0,1}$, both generating ℓ_1^n spreading models, such that $x_m < y_{m+1}$ and $y_m < x_{m+1}$ for all $m \in \mathbb{N}$. Then there exist $\{F_m\}_{m\in\mathbb{N}}$ successive subsets* *of the naturals and* $\{c_i\}_{i\in F_m}$ *non-negative reals, for all* $m\in\mathbb{N}$, *such that if* $z_m=\sum_{i\in F_m} c_i x_i$ *and* $w_m=\sum_{i\in F_m} c_i y_i$, *then* $\{z_m\}_{m\in\mathbb{N}}$ *and* $\{w_m\}_{m\in\mathbb{N}}$ *are seminormalized and equivalent.*

Our method for showing the equivalence of $\{z_m\}_{m\in\mathbb{N}}$ and $\{w_m\}_{m\in\mathbb{N}}$ is based on the following. For every f in W there exist g^1,g^2,g^3 in W such that $\theta f(z_m)<g^1(w_m)+g^2(w_m)+g^3(w_m)+\varepsilon_m$, for some fixed constant θ and $\{\varepsilon_m\}_{m\in\mathbb{N}}$ a summable sequence of positive reals. The choice of the g^i uses the tree analysis of f given below. Clearly the roles of $\{z_m\}_{m\in\mathbb{N}}$ and $\{w_m\}_{m\in\mathbb{N}}$ can be reversed and this yields the equivalence of the two sequences.

The tree analysis of a functional $f\in W$. Let $f\in W$. We construct a finite, single rooted tree Λ and choose $\{f_\lambda\}_{\lambda\in\Lambda}\subset W$, which will be called a tree analysis of f.

Set $f_\varnothing=f$, where $\varnothing$ denotes the root of the tree to be constructed. Choose $m\in\mathbb{N}$, such that $f\in W_m$. If $m=0$, then the tree analysis of f is $\{f_\varnothing\}$. Otherwise, if f is a Schreier functional, $f=\sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d\subset W_{m-1}$ is a very fast growing and $\mathcal{S}_n$-admissible sequence of α-averages, set $\{f_j\}_{j=1}^d$ to be the immediate successors of $f_\varnothing$. If f is an α-average, $f=\frac{1}{n}\sum_{j=1}^d f_j$, where $\{f_1<\cdots<f_d\}\subset W_{m-1}$, set $\{f_j\}_{j=1}^d$ to be the immediate successors of $f_\varnothing$.

Suppose that the nodes of the tree and the corresponding functionals have been chosen up to a height $\ell<m$ such that $f_\lambda\in W_{m-h(\lambda)}$. Let λ be such that $h(\lambda)=\ell$. If $f_\lambda\in W_0$, then don't extend any further and λ is a terminal node of the tree. If f_λ is a Schreier functional, $f_\lambda=\sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d\subset W_{m-\ell-1}$ is a very fast growing and $\mathcal{S}_n$-admissible sequence of α-averages, set $\{f_j\}_{j=1}^d$ to be the immediate successors of f_λ.

If f_λ is an α-average, $f_\lambda=\frac{1}{n}\sum_{j=1}^d f_j$, where $\{f_1<\cdots<f_d\}\subset W_{m-\ell-1}$, set $\{f_j\}_{j=1}^d$ to be the immediate successors of f_λ.

Remark. If f_{λ^-} is a Schreier functional, $f_{\lambda^-}=\sum_{j=1}^d f_j$ and there exists $j>1$ such that $f_\lambda=f_j$, then f_λ is of the form $f_\lambda=\frac{1}{m}\sum_{j=1}^\ell g_j$, where $m>\max\operatorname{supp} f_{j-1}$. In this case, set $\{g_j\}_{j=1}^\ell$ to be the immediate successors of f_λ.

It is clear that the procedure ends in at most $m+1$ steps.

Definition 4.2. Let $x\in\mathfrak{X}^n_{0,1}$, $f\in W$ such that $\operatorname{supp} f\cap\operatorname{supp} x\neq\varnothing$, $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f.

(i) We say that f_μ covers x, with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$, for some $\mu\in\Lambda$, if $\operatorname{supp} f_\mu\cap\operatorname{supp} x=\operatorname{supp} f\cap\operatorname{supp} x$.

(ii) We say that f_μ covers x for the first time, with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$, for some $\mu\in\Lambda$, if $\mu=\max\{\lambda\in\Lambda : f_\lambda \text{ covers } x\}$.

Definition 4.3. Let $x\in\mathfrak{X}^n_{0,1}$, $f\in W, \{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f, $\lambda\in\Lambda$ be the node of Λ such that f_λ covers x for the first time, with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$. If $\{\mu_j\}_{j=1}^d$ are the immediate successors of λ in Λ, $j_1=\min\{j:\ \operatorname{ran} f_{\mu_j}\cap\operatorname{ran} x\neq\varnothing\}, j_2=\max\{j:\ \operatorname{ran} f_{\mu_j}\cap\operatorname{ran} x\neq\varnothing\}$, set $x^1=x|_{[1,\dots,\max\operatorname{supp} f_{\mu_{j_1}}]}$, $x^3=x|_{[\min\operatorname{supp} f_{\mu_{j_2}},+\infty)}$, $x^2=x-x^1-x^3$. Then x^1,x^2,x^3 are called the initial, the middle and the final part of x respectively, with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$.

REMARK 4.4. If $\operatorname{supp} f\cap\operatorname{supp} x$ is not a singleton, then x_1 and x_3 are not zero and $x_1<x_3$. However x_2 might be zero.

LEMMA 4.5. *Let $\{x_m\}_{m\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}^n_{0,1}$, $f\in W$, $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f and $G=\{m\in\mathbb{N}:\ \operatorname{supp} f\cap\operatorname{supp} x_m\neq\varnothing\}$. For $m\in G$ set λ_m,λ_m^1 to be the nodes of Λ that cover x_m,x_m^1 for the first time, respectively and assume that $\#\{\operatorname{supp} f_{\lambda_m}\cap\operatorname{supp} x_m\}>1$, for all $m\in G$. Then:*

(i) $\lambda_m^1>\lambda_m$ *and* $\max\operatorname{supp} f_{\lambda_m^1}<\max\operatorname{supp} x_m$, *for all $m\in G$.*
(ii) *For any $m\in G$ and $\lambda\geq\lambda_m^1$ such that* $\operatorname{ran} f_\lambda\cap\operatorname{ran} x_m^1\neq\varnothing$ *and* $\operatorname{ran} f_\lambda\cap\operatorname{ran} x_\ell^1\neq\varnothing$, *for some $\ell\neq m$, we have that $\ell<m$ and $\lambda_\ell^1>\lambda$.*
(iii) *The map $m\to\lambda_m^1$ is one to one.*

Proof. Let $m\in G$. Evidently $\lambda_m^1\geq\lambda_m$. Suppose that $\lambda_m^1=\lambda_m$. This means that x_m^1 and x_m are covered for the first time simultaneously, which can only be the case if $\#\{\operatorname{supp} f_{\lambda_m}\cap\operatorname{supp} x_m\}=1$. Moreover, $\max\operatorname{supp} f_{\lambda_m^1}\leq\max\operatorname{supp} x_m^1$ and by Remark 4.4, we have that $\max\operatorname{supp} x_m^1<\max\operatorname{supp} x_m^3=\max\operatorname{supp} x_m$.

For the second statement, notice that since $\lambda\geq\lambda_m^1$, it follows that $\max\operatorname{supp} f_\lambda\leq\max\operatorname{supp} f_{\lambda_m^1}<\max\operatorname{supp} x_m$, therefore $\ell<m$. Moreover, since $\operatorname{supp} f_\lambda\cap\operatorname{supp} x_\ell\neq\varnothing$, λ is comparable to λ_ℓ^1. If $\lambda_\ell^1\leq\lambda$, then $\max\operatorname{supp} f_\lambda\leq\max\operatorname{supp} f_{\lambda_\ell^1}<\max\operatorname{supp} x_\ell$, which contradicts the fact that $\operatorname{ran} f_\lambda\cap\operatorname{ran} x_m^1\neq\varnothing$.

The third statement follows from the second one. ☐

The next lemma is proved in exactly the same way.

LEMMA 4.6. *Let $\{x_m\}_{m\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}^n_{0,1}$, $f\in W$, $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f and $G=\{m\in\mathbb{N}:\ \operatorname{supp} f\cap\operatorname{supp} x_m\neq\varnothing\}$. For $m\in G$ set λ_m,λ_m^3 to be the nodes of Λ that cover x_m,x_m^3 for the first time respectively and assume that $\#\{\operatorname{supp} f_{\lambda_m}\cap\operatorname{supp} x_m\}>1$, for all $m\in G$. Then:*

(i) $\lambda_m^3>\lambda_m$ *and* $\min\operatorname{supp} f_{\lambda_m^3}>\min\operatorname{supp} x_m$, *for all $m\in G$.*
(ii) *For any $m\in G$ and $\lambda\geq\lambda_m^3$ such that* $\operatorname{ran} f_\lambda\cap\operatorname{ran} x_m^3\neq\varnothing$ *and* $\operatorname{ran} f_\lambda\cap\operatorname{ran} x_\ell^3\neq\varnothing$, *for some $\ell\neq m$, we have that $\ell>m$ and $\lambda_\ell^3>\lambda$.*
(iii) *The map $m\to\lambda_m^3$ is one to one.*

The proof of the next lemma is even simpler and therefore it is omitted.

LEMMA 4.7. *Let $\{x_m\}_{m\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}^n_{0,1}$, $f\in W$, $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f and $G=\{m\in\mathbb{N}:\ \operatorname{supp} f\cap\operatorname{supp} x_m\neq\varnothing\}$. For $m\in G$ set*

λ_m, λ_m^2 to be the nodes of Λ that cover x_m, x_m^2 for the first time respectively and assume that $\#\{\operatorname{supp} f_{\lambda_m} \cap \operatorname{supp} x_m\} > 1$, for all $m \in G$. Then, for any $m \in G$ with $x_m^2 \neq 0$, for any $\lambda \leq \lambda_m^2$ such that $\operatorname{supp} f_\lambda \cap \operatorname{supp} x_\ell^2 \neq \varnothing$, for some $\ell \neq m$, it follows that $\lambda \leq \lambda_\ell^2$.

LEMMA 4.8. *Let $x_1, \dots, x_m$, $y_1, \dots, y_m$ be finite normalized block sequences such that $x_i < y_{i+1}$ and $y_i < x_{i+1}$ for $i = 1, \dots, m-1$. Assume moreover that $\{x_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ are both equivalent to the usual basis of $(\mathbb{R}^m, \|\cdot\|_1)$, with a lower constant $\theta > 0$. Let $\{c_i\}_{i=1}^m$ be non-negative reals with $\sum_{i=1}^m c_i = 1$ and set $z = \sum_{i=1}^m c_i x_i$ and $w = \sum_{i=1}^m c_i y_i$. Then:*

(i) *If $f \in W$ is an α-average of size $s(f) = p$, then there exists $g \in W$ such that $\operatorname{ran} g \subset \operatorname{ran} f \cap \operatorname{ran} w$ and $\frac{1}{p} g(w) > \theta f(z) - 3\max\{c_i : i = 1, \dots, m\}$.*

(ii) *Let $f \in W$. Then there exists $g \in W$ with $\operatorname{ran} g \subset \operatorname{ran} f \cap \operatorname{ran} w$, such that $g(w) > \theta f(z) - 2\max\{c_i : i = 1, \dots, m\}$.*

Proof. For the proof of the first statement, set $i_1 = \min\{i : \operatorname{ran} f \cap \operatorname{ran} x_i \neq \varnothing\}, i_2 = \max\{i : \operatorname{ran} f \cap \operatorname{ran} x_i \neq \varnothing\}$. By Lemma 3.3, we conclude that

$$f(z) < \frac{1}{p} \sum_{i=i_1}^{i_2} c_i + 2\max\{c_i : i = 1, \dots, m\}. \tag{17}$$

Since $\|\sum_{i=i_1+1}^{i_2-1} c_i y_i\| > \theta \sum_{i=i_1}^{i_2} c_i - 2\max\{c_i : i = 1, \dots, m\}$, we may choose $g \in W$ such that

$$g\left(\sum_{i=i_1+1}^{i_2-1} c_i y_i\right) > \theta \sum_{i=i_1}^{i_2} c_i - 2\max\{c_i : i = 1, \dots, m\}. \tag{18}$$

We may clearly assume that $\operatorname{ran} g \subset \operatorname{ran}\{\bigcup_{i=i_1+1}^{i_2-1} \operatorname{ran} y_i\} \subset \operatorname{ran} f \cap \operatorname{ran} w$. Finally, combining (17) and (18), and doing some easy calculations we conclude that g is the desired functional.

To prove the second statement, define i_1, i_2 as before. Then, one evidently has that $\|\sum_{i=i_1+1}^{i_2-1} c_i y_i\| > \theta \sum_{i=i_1}^{i_2} c_i - 2\max\{c_i : i = 1, \dots, m\}$, therefore there exists $g \in W$ such that

$$g\left(\sum_{i=i_1+1}^{i_2-1} c_i y_i\right) > \theta \sum_{i=i_1}^{i_2} c_i - 2\max\{c_i : i = 1, \dots, m\}. \tag{19}$$

It is also clear that

$$f(z) \leq \sum_{i=i_1}^{i_2} c_i. \tag{20}$$

As previously, we may assume that $\operatorname{ran} g \subset \operatorname{ran} f \cap \operatorname{ran} w$. Combining (19) and (20), we conclude the desired result. $\square$

For $\{x_m\}_{m\in\mathbb{N}}, \{y_m\}_{m\in\mathbb{N}}$ normalized block sequences in $\mathfrak{X}^n_{0,1}$ both generating ℓ_1^n spreading models, we appropriately block both sequences in the same manner to obtain further seminormalized block sequences $\{z_m\}_{m\in\mathbb{N}}$ and $\{w_m\}_{m\in\mathbb{N}}$. For f a given functional in W, we decompose z_m into z^1_m, z^2_m, z^3_m its initial, middle and final part, as previously described. Next, we proceed to construct g^1, g^2 g^3 functionals in W, such that each g^i acting on w_m, pointwise dominates f acting on z^i_m, for $i=1,2,3$. The choice of the functionals g^i, $i=1,2,3$ is presented in the following three lemmas.

Lemma 4.9. *Let $\{x_m\}_{m\in\mathbb{N}}, \{y_m\}_{m\in\mathbb{N}}$ be normalized block sequences in $\mathfrak{X}^n_{0,1}$, both generating ℓ_1^n spreading models, with a lower constant $\theta>0$, such that $x_m < y_{m+1}$ and $y_m < x_{m+1}$ for all $m\in\mathbb{N}$. Let $\{F_m\}_{m\in\mathbb{N}}$ be successive subsets of the naturals, $\{c_i\}_{i\in\mathbb{N}}$ be non-negative reals and $\{\varepsilon_m\}_{m\in\mathbb{N}}, \{\delta_m\}_{m\in\mathbb{N}}$ be positive reals satisfying the following*:

(i) $F_m \in \mathcal{S}_n$ *and* $z_m = \sum_{i\in F_m} c_i x_i, w_m = \sum_{i\in F_m} c_i y_i$ *are both* (n,ε_m) *s.c.c. for all* $m\in\mathbb{N}$.

(ii) $\max\operatorname{supp} z_m\big(\frac{1}{\min\operatorname{supp} z_{m+1}} + 6\varepsilon_{m+1}\big) < \frac{\delta_{m+1}}{4}$, *for all* $m\in\mathbb{N}$.

Let also $f\in W$, with a tree analysis $\{f_\lambda\}_{\lambda\in\Lambda}$ and z^1_m be the initial part of z_m with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$, for all $m\in\mathbb{N}$. Then there exists $g^1\in W$, such that

$$g^1(w_m) > 2\theta f\big(z^1_m\big) - 5\delta_m, \quad \textit{for all } m\in\mathbb{N}.$$

Proof. Let $f\in W$. We may assume that $f(e_j)\geq 0$, for all $j\in\mathbb{N}$, that $\operatorname{supp} f \subset \bigcup_{m\in\mathbb{N}}\operatorname{supp} z_m$ and that $e^*_j(z_m)\geq 0, e^*_j(w_m)\geq 0$ for all $j,k\in\mathbb{N}$. Set $G=\{m\in\mathbb{N}:\ \operatorname{supp} f\cap\operatorname{supp} x_m\neq\varnothing\}$.

We may assume that for any $m\in G$, $\operatorname{supp} f\cap\operatorname{supp} z_m$ is not a singleton. Otherwise there exists $f'\in W$ that satisfies this condition for $G'=\{m\in\mathbb{N}:\ \operatorname{supp} f'\cap\operatorname{supp} z_m\neq\varnothing\}$ and $f'(z_m)\geq f(z_m)-\varepsilon_m$, for all $m\in\mathbb{N}$.

Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f. Denote by z^1_m the initial part of z_m and λ^1_m the node of Λ that cover z^1_m for the first time, for all $m\in G$, all with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$.

We proceed to the construction of g^1. Set

$$\begin{aligned}\mathcal{C}^1_m = &\big\{\lambda\in\Lambda: \lambda\geq\lambda^1_m, \min\big\{\operatorname{supp} f_\lambda\cap\operatorname{supp} z^1_m\big\} = \min\big\{\operatorname{supp} f_{\lambda^1_m}\cap\operatorname{supp} z^1_m\big\}\big\}\\ &\cup\big\{\lambda\in\Lambda: \lambda\leq\lambda^1_m\big\}.\end{aligned}$$

Notice that $\mathcal{C}^1_m$ is a maximal chain in Λ. Set

$$\begin{aligned}\nu^1_m &= \max\big\{\lambda\in\mathcal{C}^1_m:\ \operatorname{ran} f_\lambda\cap\operatorname{ran} z^1_\ell\neq\varnothing,\ \text{for some } \ell\neq m\big\},\\ \mu_m &= \min\big\{\lambda\in\mathcal{C}^1_m: \lambda\geq\lambda^1_m, f_\lambda \text{ is an } \alpha\text{-average and there exists } \beta\in\operatorname{succ}(\lambda)\\ &\qquad \text{such that } \operatorname{ran} f_\beta\cap\operatorname{ran} z^1_m\neq\varnothing \text{ and} \operatorname{ran} f_\beta\cap\operatorname{ran} z^1_\ell=\varnothing \text{ for } \ell\neq m\big\},\end{aligned}$$

where $\operatorname{succ}(\lambda)$ are the immediate successors of λ in Λ.

Claim. *If for some $m\in G$ we have that $\lambda^1_m\leq\nu^1_m<\mu_m$, then $\nu^1_m=\mu^-_m$.*

Proof. First, notice that in this case $f_{\nu_m^1}$ must be a Schreier functional. If $f_{\nu_m^1}$ were an α-average, then if we denote its immediate successor in $\mathcal{C}_m^1$ by β, then $\operatorname{ran} f_\beta \cap \operatorname{ran} z_m^1 \neq \varnothing$ and $\operatorname{ran} f_\beta \cap \operatorname{ran} z_\ell^1 = \varnothing$ for $\ell \neq m$, therefore μ_m^1 would not be the minimal element satisfying this condition, since we assumed that $\nu_m^1 < \mu_m$, a contradiction. Since $f_{\nu_m^1}$ is a Schreier functional, it follows that if we denote its immediate successor in $\mathcal{C}_m^1$ by β, then f_β is an α-average, such that $\operatorname{ran} f_\beta \cap \operatorname{ran} z_m^1 \neq \varnothing$ and $\operatorname{ran} f_\beta \cap \operatorname{ran} z_\ell^1 = \varnothing$ for $\ell \neq m$. Since $\nu_m^1 < \mu_m$, it follows that $\beta = \mu_m$. □

Set

$$\Lambda_1 = \left\{\lambda \in \Lambda : \text{there exists } m \in G \text{ such that } \lambda \leq \lambda_m^1\right\} \\ \cup \left\{\lambda \in \Lambda : \text{there exists } m \in G \text{ such that } \lambda \leq \mu_m \text{ and } \nu_m^1 \geq \lambda_m^1\right\}.$$

For $\lambda \in \Lambda_1$, set

$$G_\lambda = \left\{m \in G : \lambda \leq \lambda_m^1\right\} \cup \left\{m \in G : \lambda \leq \mu_m \text{ and } \nu_m^1 \geq \lambda_m^1\right\}.$$

For every $\lambda \in \Lambda_1$, we will inductively construct $g_\lambda^1 \in W$ satisfying the following.

(i) $g_\lambda^1(w_m) > \theta f_\lambda(z_m^1) - 4\delta_m$, for all $m \in G_\lambda$.
(ii) $\operatorname{ran} g_\lambda^1 \subset \operatorname{ran} f_\lambda \cap \operatorname{ran}\{\cup\{\operatorname{ran} w_m : m \in G_\lambda\}\}$.
(iii) If f_λ is an α-average, then so is g_λ^1 and $s(g_\lambda^1) = s(f_\lambda)$.

Before proceeding to the construction, we would like to stress out that Lemma 4.5 assures us that whenever a functional f_λ, $\lambda \in \Lambda_1$ acts on more than one vectors z_k^1, then all vectors except for the rightmost one, have been covered for the first time in a previous step. Therefore in this case, we are free to focus the inductive step on one vector. In particular, if $\lambda \in \Lambda, \lambda \geq \lambda_m^1$ for some $m \in G$, such that $\operatorname{ran} f_\lambda \cap \operatorname{ran} z_m^1 \neq \varnothing$ and $\operatorname{ran} f_\lambda \cap \operatorname{ran} z_\ell^1 \neq \varnothing$ for $\ell \neq m$, then besides the fact that $\ell < m$ and $\lambda_\ell^1 \geq \lambda$, it also follows that $\lambda \in \mathcal{C}_m^1$ (as well as $\lambda \in \mathcal{C}_\ell^1$).

Let $\lambda \in \Lambda_1$. We distinguish six cases. The first inductive step falls under the first two.

Case 1. There exists $m \in G$ such that $\lambda = \lambda_m^1 = \mu_m$ and $\nu_m < \lambda_m^1$.

In this case f_λ is an α-average, $f_\lambda = \frac{1}{p}\sum_{j=1}^d f_{\beta_j}$, where $\{\beta_j\}_{j=1}^d$ are the immediate successors of λ. By Lemma 4.8, there exists $g \in G$, such that $\operatorname{ran} g \subset \operatorname{ran} f_\lambda \cap \operatorname{ran} w_m$ and $\frac{1}{p}g(w_m) > \theta f_\lambda(z_m^1) - 3\max\{c_i : i \in F_m\}$. Set $g_\lambda^1 = \frac{1}{p}g$. Since $\max\{c_i : i \in F_m\} < \varepsilon_m < \delta_m$, we conclude that g_λ^1 satisfies the inductive assumption.

Case 2. There exists $m \in G$ such that $\lambda = \lambda_m^1 < \mu_m$ and $\nu_m < \lambda_m^1$.

Then f_λ is a Schreier functional, $f_\lambda = \sum_{j=1}^d f_{\beta_j}$. Then again by Lemma 4.8, there exists $g \in W$ such that $\operatorname{ran} g \subset \operatorname{ran} f_\lambda \cap \operatorname{ran} w_m$ and $g(w_m) > \theta f(z_m^1) - 3\max\{c_i : i \in F_m\}$. Set $g_\lambda^1 = g$. As in the previous case, we conclude that g_λ^1 satisfies the inductive assumption.

Case 3. For any $m \in G$ such that $\operatorname{ran} f_\lambda \cap \operatorname{ran} z_m^1 \neq \varnothing$, we have that $\lambda < \lambda_m^1$.

Then if $f_\lambda = \sum_{j=1}^d f_{\beta_j}$ (or $f_\lambda = \frac{1}{p}\sum_{j=1}^d f_{\beta_j}$), for $j = 1, \dots, d$ there exist $g_{\beta_j}^1$, already satisfying the inductive assumption. Then it is easy to see that $g_\lambda^1 = \sum_{j=1}^d g_{\beta_j}^1 \in W$ (or $g_\lambda^1 = \frac{1}{p}\sum_{j=1}^d g_{\beta_j}^1 \in W$) and is the desired functional.

Case 4. There exists $m \in G$ such that $\lambda > \mu_m$.

Since $\lambda \in \Lambda_1$, there exists at least one $\ell < m$ in G, such that $\lambda < \lambda_\ell^1$. If $f_\lambda = \sum_{j=1}^d f_{\beta_j}$ (or $f_\lambda = \frac{1}{p}\sum_{j=1}^d f_{\beta_j}$), set $j_0 = \max\{j :$ there exists $\ell < k$ such that $\operatorname{ran} f_{\beta_j} \cap \operatorname{ran} z_\ell^1 \neq \varnothing\}$. Then it is easy to see that $g_\lambda^1 = \sum_{j=1}^{j_0} g_{\beta_j}^1 \in W$ (or $g_\lambda^1 = \frac{1}{p}\sum_{j=1}^{j_0} g_{\beta_j}^1 \in W$) and satisfies the inductive assumption.

Case 5. There exists $m \in G$ such that $\lambda = \mu_m$ and $\lambda_m^1 \leq \nu_m^1$.

This both covers the case when $\mu_m = \lambda_m^1$ and when $\mu_m > \lambda_m^1$. The claim yields that in either case $\nu_m^1 \geq \mu_m^-$.

If $\nu_m^1 = \mu_m^-$, simply repeat what was done in case 1. Otherwise, $\nu_m^1 \geq \mu_m$ and there exist at least one $\ell < m$ in G, such that $\operatorname{ran} f_\lambda \cap \operatorname{ran} z_\ell^1 \neq \varnothing$. If $f_\lambda = \frac{1}{p}\sum_{j=1}^d f_{\beta_j}$, set $j_0 = \max\{j :$ there exists $\ell < m$ such that $\operatorname{ran} f_{\beta_j} \cap \operatorname{ran} z_\ell^1 \neq \varnothing\}$. Since $\lambda = \mu_m$, we have that $j_0 < d$. Apply Lemma 4.8 and find $g \in W, \operatorname{ran} g \subset \operatorname{ran} f_\lambda \cap \operatorname{ran} w_m$ such that $\frac{1}{p}g(w_k) > \theta f_\lambda(z_k^1) - 3\max\{c_i : i \in F_m\}$. Set $g_\lambda^1 = \frac{1}{p}\sum_{j=1}^{j_0} g_{\beta_j}^1 + \frac{1}{p}g$. Then $g_\lambda^1 \in W$ and satisfies the inductive assumption. In particular, note that $g_\lambda^1(w_m) > \theta f_\lambda(z_m^1) - 3\delta_m$.

Case 6. There exists $m \in G$, such that $\lambda_m^1 \leq \lambda < \mu_m$ and $\nu_m^1 \geq \lambda_m^1$.

We will prove by induction on $q = |\lambda| - |\mu_m|$ that there exists $g_\lambda^1 \in W$ satisfying conditions (i), (ii) and (iii) from our initial inductive assumption and moreover a stronger version of condition (i). In particular:

If f_λ is an α-average, then $g_\lambda^1(w_m) > \theta f_\lambda(z_m^1) - 3\delta_m - \frac{\delta_m}{4}$.

If f_λ is a Schreier functional, then $g_\lambda^1(w_m) > \theta f_\lambda(z_m^1) - 3\delta_m - \frac{\delta_m}{2}$.

For convenience start the induction for $q = 0$, i.e. $\lambda = \mu_m$. As we have noted in this case $g_\lambda^1(w_m) > \theta f_\lambda(z_m^1) - 3\delta_m$.

Assume that it is true for some $q < |\lambda_m^1| - |\mu_m|$. Then for λ such that $|\lambda| - |\mu_m| = q + 1$, the claim yields that $\nu_m^1 \geq \lambda$.

If f_λ is an α-average, $f_\lambda = \frac{1}{p}\sum_{j=1}^d f_{\beta_j}$, since $\lambda \leq \nu_m^1$, $\lambda < \mu_m$, we have that $\operatorname{ran} f_{\beta_d} \cap \operatorname{ran} z_\ell^1 \neq \varnothing$, for some $\ell < m$. Therefore $\operatorname{ran} f_{\beta_j} \cap \operatorname{ran} z_m^1 = \varnothing$ for $j < d$ and there exists $g_{\beta_d}^1$ satisfying the stronger inductive assumption. Set $g_\lambda^1 = \frac{1}{p}\sum_{j=1}^d g_{\beta_j}^1$. As always $g_\lambda^1 \in W$ and it satisfies the initial inductive assumption. It also satisfies the stronger one. Indeed, $g_\lambda^1(w_m) = \frac{1}{p}g_{\beta_d}^1(w_m) > \frac{1}{p}(\theta f_{\beta_d}(z_m^1) - 3\delta_m - \frac{\delta_m}{2}) = \frac{1}{p}(p\theta f_\lambda(z_m^1) - 3\delta_m - \frac{\delta_m}{2}) = \theta f_\lambda(z_m^1) - \frac{3\delta_m}{p} - \frac{\delta_m}{2p} > \theta f_\lambda(z_m^1) - 3\delta_m - \frac{\delta_m}{4}$.

If f_λ is a Schreier functional, $f_\lambda = \sum_{j=1}^d f_{\beta_j}$, since $\lambda \leq \nu_m^1$ we have that $\operatorname{ran} f_{\beta_j} \cap \operatorname{ran} z_\ell^1 \neq \varnothing$, for some $\ell < m$ and some $j \leq d$. Set $j_0 = \min\{j :\ \operatorname{ran} f_{\beta_j} \cap$

$\operatorname{ran} z_m^1 \neq \varnothing\}$. Therefore $\operatorname{ran} f_{\beta_j} \cap \operatorname{ran} z_m^1 = \varnothing$ for $j < j_0$ and there exists an α-average $g_{\beta_{j_0}}^1$ satisfying the stronger inductive assumption.

Choose $\{J_r\}_{r=1}^{r_0}$ successive subsets of the naturals satisfying the following.

(i) $\bigcup_{r=1}^{r_0} J_r = \{j : j_0 < j \leq d\}$.

(ii) $\{\min\operatorname{supp} f_j : j \in J_r\}$ is a maximal $\mathcal{S}_{n-1}$ set for $r < r_0$ and $\{\min\operatorname{supp} f_j : j \in J_{r_0}\} \in \mathcal{S}_{n-1}$.

We conclude that $r_0 \leq \max\operatorname{supp} z_{m-1}$. Moreover, Lemma 3.4 yields that for $r \leq r_0$

$$\sum_{j\in J_r} f_j\left(z_m^1\right) < \frac{1}{\min\operatorname{supp} z_m} + 6\varepsilon_m.$$

Assumption (ii) of the proposition yields that $\sum_{j>j_0} f_{\beta_j}(z_m^1) < \frac{\delta_m}{4}$.

Set $g_\lambda^1 = \sum_{j=1}^{j_0} g_{\beta_j}^1$. Then $g_\lambda^1(w_m) = g_{\beta_{j_0}}^1(w_m) > \theta f_{\beta_{j_0}}(z_m^1) - 3\delta_m - \frac{\delta_m}{4} = \theta\sum_{j=1}^{j_0} f_{\beta_j}(z_m^1) - 3\delta_m - \frac{\delta_m}{4} > \theta f_\lambda(z_m^1) - \frac{\delta_m}{4} - 3\delta_m - \frac{\delta_m}{4} = \theta f_\lambda(z_m^1) - 3\delta_m - \frac{\delta_m}{2}$. This ends the inductive step in case 6 and also the initial induction.

Set $g^1 = g_\varnothing^1$. Then:

$$g^1(w_m) > \theta f\left(z_m^1\right) - 4\delta_m, \quad \text{for all } m \in G.$$

Lifting the restriction that for any $m \in G$, $\operatorname{supp} f \cap \operatorname{supp} z_m$ is not a singleton, in the general case we conclude that $g^1(w_m) > \theta f(z_m^1) - 5\delta_m$, for all $m \in G$. ☐

LEMMA 4.10. *Let $\{x_m\}_{m\in\mathbb{N}}, \{y_m\}_{m\in\mathbb{N}}$ be normalized block sequences in $\mathfrak{X}_{0,1}^n$, both generating ℓ_1^n spreading models, with a lower constant $\theta > 0$, such that $x_m < y_{m+1}$ and $y_m < x_{m+1}$ for all $m \in \mathbb{N}$. Let $\{F_m\}_{m\in\mathbb{N}}$ be successive subsets of the naturals, $\{c_i\}_{i\in\mathbb{N}}$ be non-negative reals and $\{\varepsilon_m\}_{m\in\mathbb{N}}, \{\delta_m\}_{m\in\mathbb{N}}$ be positive reals satisfying the following*:

(i) *$F_m \in \mathcal{S}_n$ and $z_m = \sum_{i\in F_m} c_i x_i, w_m = \sum_{i\in F_m} c_i y_i$ are both (n, ε_m) s.c.c. for all $m \in \mathbb{N}$.*

(ii) *$\max\operatorname{supp} z_m(\frac{1}{\min\operatorname{supp} z_{m+1}} + 6\varepsilon_{m+1}) < \frac{\delta_{m+1}}{4}$, for all $m \in \mathbb{N}$.*

Let also $f \in W$, with a tree analysis $\{f_\lambda\}_{\lambda\in\Lambda}$ and z_m^3 be the final part of z_m with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$, for all $m \in \mathbb{N}$. Then there exists $g^3 \in W$, such that

$$g^3(w_m) > \frac{\theta}{2} f\left(z_m^3\right) - 3\delta_m, \quad \textit{for all } m \in \mathbb{N}.$$

Proof. Let $f \in W$. As in the previous proof, assume that $f(e_j) \geq 0$, for all $j \in \mathbb{N}$, that $\operatorname{supp} f \subset \bigcup_{m\in\mathbb{N}} \operatorname{supp} z_m$ and that $e_j^*(z_m) \geq 0, e_j^*(w_m) \geq 0$ for all $j, k \in \mathbb{N}$. Set $G = \{m \in \mathbb{N} : \operatorname{supp} f \cap \operatorname{supp} x_m \neq \varnothing\}$.

Assume again that for any $m \in G$, $\operatorname{supp} f \cap \operatorname{supp} z_m$ is not a singleton. Otherwise there exists $f' \in W$ that satisfies this condition for $G' = \{m \in \mathbb{N} : \operatorname{supp} f' \cap \operatorname{supp} z_m \neq \varnothing\}$ and $f'(z_m) \geq f(z_m) - \varepsilon_m$, for all $m \in \mathbb{N}$.

Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f. Denote by z_m^3 the final part of z_m and λ_m^3 the node of Λ that cover z_m^3 for the first time, for all $m\in G$, all with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$.

Set

$$\begin{aligned}\mathcal{C}_m^3 &= \big\{\lambda\in\Lambda : \lambda\geq\lambda_m^3, \max\big\{\operatorname{supp} f_\lambda\cap\operatorname{supp} z_m^3\big\} = \max\big\{\operatorname{supp} f_{\lambda_m^3}\cap\operatorname{supp} z_m^3\big\}\big\}\\ &\quad \cup\big\{\lambda\in\Lambda : \lambda\leq\lambda_m^3\big\},\\ \nu_m^3 &= \max\big\{\lambda\in\mathcal{C}_m^3 : \operatorname{ran} f_\lambda\cap\operatorname{ran} z_\ell^3\neq\varnothing, \text{for some } \ell\neq m\big\}.\end{aligned}$$

Set

$$\Lambda_3 = \big\{\lambda\in\Lambda : \text{there exists } m\in G \text{ such that } \lambda\leq\lambda_m^3\big\}.$$

For every $\lambda\in\Lambda_3$, we will inductively construct $g_\lambda^3\in W$ satisfying the following.

(i) $g_\lambda^3(w_m) > \frac{\theta}{2} f_\lambda(z_m^3) - 2\delta_m$, for all $m\in G$ such that $\lambda_m^3\geq\lambda$.
(ii) $\operatorname{ran} g_\lambda^3\subset\operatorname{ran} f_\lambda\cap\operatorname{ran}\{\cup\{\operatorname{ran} w_m : \lambda_m^3\geq\lambda\}\}$.
(iii) If f_λ is an α-average, then so is g_λ^3 and $s(g_\lambda^3) = s(f_\lambda)$.

Just as in the construction of g^1, Lemma 4.6 assures us that whenever a functional f_λ, $\lambda\in\Lambda_2$ acts on more than one vectors z_m^3, then all vectors except for the leftmost one, have been covered for the first time in a previous step.

Let $\lambda\in\Lambda_3$. We distinguish 4 cases, the first inductive step falls under the first case.

Case 1. There exists $m\in G$, such that $\lambda=\lambda_m^3$ and $\nu_m^3<\lambda_m^3$.

If f_λ is an α-average, $f_\lambda=\frac{1}{p}\sum_{j=1}^d f_{\beta_j}$, by Lemma 4.8 there exists $g\in W$ such that $\operatorname{ran} g\subset\operatorname{ran} f_\lambda\cap\operatorname{ran} w_m$ and $\frac{1}{p}g(w_m) > \theta f_\lambda(z_m^3) - 3\max\{c_i : i\in F_m\}$. Set $g_\lambda^3 = \frac{1}{p}g$.

If f_λ is a Schreier functional, then by Lemma 4.8 there exists $g\in W$ such that $\operatorname{ran} g\subset\operatorname{ran} f_\lambda\cap\operatorname{ran} w_m$ and $g(w_m) > f_\lambda(z_m^3) - 2\max\{c_i : i\in F_m\}$. Set $g_\lambda^3 = g$.

Case 2. For any $m\in G$ such that $\operatorname{ran} f_\lambda\cap\operatorname{ran} z_m^3\neq\varnothing$, we have that $\lambda<\lambda_m^3$. If $f_\lambda=\sum_{j=1}^d f_{\beta_j}$ (or $f_\lambda=\frac{1}{p}\sum_{j=1}^d f_{\beta_j}$), set $g_\lambda^3=\sum_{j=1}^d g_{\beta_j}^3$ (or $g_\lambda^3=\frac{1}{p}\sum_{j=1}^d g_{\beta_j}^3$).

Case 3. There exists $m\in G$, such that $\lambda>\lambda_m^3$.

Since $\lambda\in\Lambda_3$, there exists at least one $\ell>m$ such that $\lambda_\ell^3>\lambda$. If $f_\lambda=\sum_{j=1}^d f_{\beta_j}$ (or $f_\lambda=\frac{1}{p}\sum_{j=1}^d f_{\beta_j}$), set $j_0=\min\{j : \operatorname{ran} f_{\beta_j}\cap\operatorname{ran} z_\ell^3\neq\varnothing, \text{for some } \ell>k\}$. Set $g_\lambda^3=\sum_{j=j_0}^d g_{\beta_j}^3$ (or $g_\lambda^3=\frac{1}{p}\sum_{j=j_0}^d g_{\beta_j}^3$).

Case 4. There exists $m\in G$, such that $\lambda=\lambda_m^3$ and $\nu_m^3\geq\lambda_m^3$.

If f_λ is an α-average, $f_\lambda=\frac{1}{p}\sum_{j=1}^d f_{\beta_j}$, set $j_0=\min\{j : \operatorname{ran} f_{\beta_j}\cap\operatorname{ran} z_\ell^3\neq\varnothing$ for some $\ell>m\}$. Then $j_0>1$, otherwise z_m^3 would have been covered for the first time in a previous step. By Lemma 4.8, there exists $g\in W$ such that $\operatorname{ran} g \subset \operatorname{ran} f_\lambda \cap \operatorname{ran} w_m$ and $\frac{1}{p} g(w_m) > \theta f_\lambda(z_m^3) - 3\max\{c_i : i \in F_m\}$. Set $g_\lambda^3 = \frac{1}{p} g + \frac{1}{p}\sum_{j=j_0}^d g_{\beta_j}^3$.

If f_λ is a Schreier functional, $f_\lambda = \sum_{j=1}^d f_{\beta_j}$, set again $j_0 = \min\{j : \operatorname{ran} f_{\beta_j} \cap \operatorname{ran} z_\ell^3 \neq \varnothing$ for some $\ell > m\}$ and as before $j_0 > 1$. By Lemma 4.8, there exists $g \in W$ such that $\operatorname{ran} g \subset \operatorname{ran}(\sum_{j<j_0} f_{\beta_j}) \cap \operatorname{ran} w_m$ and $g(w_m) > \theta \sum_{j<j_0} f_{\beta_j}(z_m^3) - 2\max\{c_i : i \in F_m\}$. Since $j_0 > 1$, it follows that $s(f_{\beta_{j_0}}) > \min\operatorname{supp} z_m$. By Lemma 3.3 and assumption (ii) of the lemma we are proving, we conclude the following.

$$\begin{aligned} f_{\beta_{j_0}}\left(z_m^3\right) &< \frac{1}{\min\operatorname{supp} z_m} + 2\max\{c_i : i \in F_m\} \\ &< \frac{1}{\min\operatorname{supp} z_m} + 2\varepsilon_m < \frac{\delta_m}{4}. \end{aligned}$$

Set $g_\lambda^3 = \frac{1}{2} g + \sum_{j=j_0}^d g_{\beta_j}^3$. Then $g_\lambda(w_m) > \frac{\theta}{2} f_\lambda(z_m^3) - 2\delta_m$.

This ends induction. Set $g^3 = g_\varnothing^3$. Then:

$$g^3(w_m) > \frac{\theta}{2} f\left(z_m^3\right) - 2\delta_m, \quad \text{for all } k \in G.$$

Lifting the restriction that for any $m \in G$, $\operatorname{supp} f \cap \operatorname{supp} z_m$ is not a singleton, in the general case we conclude that $g^3(w_m) > \frac{\theta}{2} f(z_m^3) - 3\delta_m$, for all $m \in G$. ☐

LEMMA 4.11. *Let $\{x_m\}_{m\in\mathbb{N}}, \{y_m\}_{m\in\mathbb{N}}$ be normalized block sequences in $\mathfrak{X}_{0,1}^n$, both generating ℓ_1^n spreading models, with a lower constant $\theta > 0$, such that $x_m < y_{m+1}$ and $y_m < x_{m+1}$ for all $m \in \mathbb{N}$. Let $\{F_m\}_{m\in\mathbb{N}}$ be successive subsets of the naturals, $\{c_i\}_{i\in\mathbb{N}}$ be non-negative reals and $\{\varepsilon_m\}_{m\in\mathbb{N}}, \{\delta_m\}_{m\in\mathbb{N}}$ be positive reals satisfying the following*:

(i) *$F_m \in \mathcal{S}_n$ and $z_m = \sum_{i\in F_m} c_i x_i, w_m = \sum_{i\in F_m} c_i y_i$ are both (n, ε_m) s.c.c. for all $m \in \mathbb{N}$.*
(ii) $\max\operatorname{supp} z_m(\frac{1}{\min\operatorname{supp} z_{m+1}} + 6\varepsilon_{m+1}) < \frac{\delta_{m+1}}{4}$, *for all $m \in \mathbb{N}$.*

Let also $f \in W$, with a tree analysis $\{f_\lambda\}_{\lambda\in\Lambda}$ and z_m^2 be the middle part of z_m with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$, for all $m \in \mathbb{N}$. Then there exists $g^2 \in W$, such that

$$g^2(w_m) > \frac{\theta}{2} f\left(z_m^2\right) - 5\delta_m, \quad \textit{for all } m \in \mathbb{N}.$$

Proof. Let $f \in W$. As usually, assume that $f(e_j) \geq 0$, for all $j \in \mathbb{N}$, that $\operatorname{supp} f \subset \bigcup_{m\in\mathbb{N}} \operatorname{supp} z_m$ and that $e_j^*(z_m) \geq 0, e_j^*(w_m) \geq 0$ for all $j, k \in \mathbb{N}$. Set $G = \{m \in \mathbb{N} : \operatorname{supp} f \cap \operatorname{supp} x_m \neq \varnothing\}$.

Assume again that for any $m \in G$, $\operatorname{supp} f \cap \operatorname{supp} z_m$ is not a singleton. Otherwise there exists $f' \in W$ that satisfies this condition for $G' = \{m \in \mathbb{N} : \operatorname{supp} f' \cap \operatorname{supp} z_m \neq \varnothing\}$ and $f'(z_m) \geq f(z_m) - \varepsilon_m$, for all $m \in \mathbb{N}$.

Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a tree analysis of f. Denote by z_m^2 the middle part of z_m and λ_m^2 the node of Λ that cover z_m^2 for the first time, for all $m\in G$, all with respect to $\{f_\lambda\}_{\lambda\in\Lambda}$.

Set

$$\Lambda_2=\big\{\lambda\in\Lambda: \text{there exists } m\in G \text{ such that } z_m^2\neq 0 \text{ and } \lambda\leq\lambda_m^2\big\}.$$

For every $\lambda\in\Lambda_2$, we will inductively construct $g_\lambda^2\in W$ such that:

(i) $g_\lambda^2(w_m)>\frac{\theta}{2}f_\lambda(z_m^2)-4\delta_m$, for all $m\in G$.

(ii) $\operatorname{ran} g_\lambda^2\subset\operatorname{ran} f_\lambda$.

(iii) If f_λ is an α-average, then so is g_λ^2 and $s(g_\lambda^2)=s(f_\lambda)$.

By Lemma 4.7, it follows that whenever $\lambda\in\Lambda_2$ such that $f_\lambda\cap\operatorname{ran} z_m^2$, for some m, then $\lambda\leq\lambda_m^2$. Therefore, although it might be the case that f_λ covers many z_m^2 for the first time simultaneously, it cannot act on any z_m^2 without covering it.

The first inductive step it similar to the general one, therefore let $\lambda\in\Lambda_2$ and assume that the inductive assumption holds for any $\mu>\lambda$.

Case 1. f_λ is an α-average.

Set

$$D=\big\{m\in G:\lambda=\lambda_m^2\big\},\qquad E=\big\{m\in G:\lambda<\lambda_m^2\big\}.$$

If $f_\lambda=\frac{1}{p}\sum_{j=1}^d f_{\beta_j}$, set

$$H=\big\{j:\ \operatorname{ran} f_{\beta_j}\cap\operatorname{ran} z_m^2\neq\varnothing \text{ for some } m\in E\big\}.$$

As we have noted, $\operatorname{ran} f_{\beta_j}\cap\operatorname{ran} z_m^2=\varnothing$, for any $j\in H, m\in D$.

For $m\in D$, since $\lambda=\lambda_m^2$, there exists at least one j_m, such that $\operatorname{ran} f_{\beta_{j_m}}\cap \operatorname{ran} z_\ell^2=\varnothing$ for any $\ell\neq m$, in fact there exist $j_{m_1}<j_{m_2}$ such that $\operatorname{ran} f_{\beta_{j_{m_i}}}\subset \operatorname{ran} z_m^2$, for $i=1,2$. Therefore, $\#H<p-\#D$.

For $m\in D$ apply Lemma 4.8 and find $g_m\in W$, such that $\operatorname{ran} g_m\subset\operatorname{ran} f_\lambda\cap \operatorname{ran} w_m$ and $\frac{1}{p}g(w_m)\geq\theta f_\lambda(z_2^m)-3\max\{c_i: i\in F_m\}$. We may assume that $\operatorname{ran} g\subset\operatorname{ran} z_m^2$ (to see this restrict f_λ to the range of z_m^2).

Set $g_\lambda^2=\frac{1}{p}\sum_{m\in D}g_m+\frac{1}{p}\sum_{j\in H}g_{\beta_j}^2$. By the above it follows that $g_\lambda^2\in W$ and that it satisfies the inductive assumption.

Case 2. f_λ is a Schreier functional.

$$D=\big\{m\in G:\lambda=\lambda_m^2\big\},\qquad E=\big\{m\in G:\lambda<\lambda_m^2\big\}.$$

If $f_\lambda=\sum_{j=1}^d f_{\beta_j}$, set

$$H=\big\{j:\ \operatorname{ran} f_{\beta_j}\cap\operatorname{ran} z_m^2\neq\varnothing \text{ for some } m\in E\big\}.$$

Again, $\operatorname{ran} f_{\beta_j}\cap\operatorname{ran} z_m^2=\varnothing$, for any $j\in H, m\in D$.

Set $m_1=\min\{m:\ \operatorname{ran} f_\lambda\cap\operatorname{ran} z_m^2\neq\varnothing\}$. Let $m\in D, m>m_1$. Set $j_m=\min\{j:\ \operatorname{ran} f_{\beta_j}\cap\operatorname{ran} z_m^2\neq\varnothing\}$. Then $\operatorname{ran} f_{\beta_{j_m}}\subset\operatorname{ran} z_m^2$.

By applying Lemma 4.8 find $g_m\in W$ an α-functional of size $s(g_m)=s(f_{\beta_{j_m}})$ such that $\operatorname{ran} g_m\subset\operatorname{ran} f_{\beta_{j_m}}\cap\operatorname{ran} w_m$ and $g_m>\theta f_{\beta_{j_m}}(z_m^2)-$

$3\max\{c_i : i \in F_m\}$. By the fact that $\{f_{\beta_j}\}_{j=1}^d$ is admissible and very fast growing, just as in case 6 of the proof of Lemma 4.9, it follows that $\sum_{j>j_m} f_{\beta_j}(z_m^2) < \frac{\delta_m}{4}$.

If $\min D > m_1$, set $g_\lambda^2 = \sum_{j\in H} g_{\beta_j}^2 + \sum_{m\in D} g_m$.

If $\min D = m_1$, set $j_0 = \max\{j : \operatorname{ran} f_{\beta_j} \cap \operatorname{ran} z_{m_1}^2 \neq \varnothing\}$. Just as in case 4 of the proof of Lemma 4.10, find $g_{m_1} \in W$, such that $\operatorname{ran} g_{m_1} \subset \operatorname{ran}(\sum_{j<j_0} f_{\beta_j}) \cap \operatorname{ran} w_{m_1}$ and $g_{m_1}(w_{m_1}) > \theta \sum_{j<j_0} f_{\beta_j}(z_{m_1}^2) - 2\max\{c_i : i \in F_m\}$. Again we have that $f_{\beta_{j_0}}(z_{m_1}^2) < \frac{\delta_m}{4}$. Set $g_\lambda^2 = \frac{1}{2} g_{m_1} + \sum_{j\in H} g_{\beta_j}^2 + \sum_{m\in D\setminus\{m_1\}} g_m$.

The inductive construction is complete. Set $g^2 = g_\varnothing^2$. Then:

$$g^2(w_m) > \frac{\theta}{2} f\left(z_m^2\right) - 4\delta_m, \quad \text{for all } m \in G.$$

Lifting the restriction that for any $m \in G$, $\operatorname{supp} f \cap \operatorname{supp} z_m$ is not a singleton, in the general case we conclude that $g^2(w_m) > \frac{\theta}{2} f(z_m^2) - 5\delta_m$, for all $m \in G$. □

We are now ready to prove the main result of this section.

Proof of Proposition 4.1. Fix $\theta > 0$ such that both $\{x_m\}_{m\in\mathbb{N}}$ and $\{y_m\}_{m\in\mathbb{N}}$ generate ℓ_1^n spreading models with a lower constant θ. Fix $\{\delta_m\}_{m\in\mathbb{N}}$ a sequence of positive reals, such that $\sum_{m=1}^\infty \delta_m < \frac{\theta^2}{13}$. Inductively choose $\{F_m\}_{m\in\mathbb{N}}$ successive subsets of the naturals and $\{c_i\}_{i\in F_m}$ non-negative reals, satisfying the following:

(i) $F_m \in \mathcal{S}_n$ and $z_m = \sum_{i\in F_m} c_i x_i, w_m = \sum_{i\in F_m} c_i y_i$ are both (n, ε_m) s.c.c. for all $m \in \mathbb{N}$.

(ii) If we set

$$\begin{aligned} M_m &= \max\{\max\operatorname{supp} z_m, \max\operatorname{supp} w_m\},\\ N_m &= \min\{\min\operatorname{supp} z_m, \min\operatorname{supp} w_m\}, \end{aligned}$$

then $M_m(\frac{1}{N_{m+1}} + 6\varepsilon_{m+1}) < \frac{\delta_{m+1}}{4}$, for all $m \in \mathbb{N}$.

We will show that for any $\{r_m\}_{m=1}^d \subset \mathbb{R}$, we have that $\|\sum_{m=1}^d r_m w_m\| > \frac{\theta^2}{3}\|\sum_{m=1}^d c_m z_m\|$.

Let $f \in W$. As always may assume that $1 \geq r_m \geq 0, e_j^*(z_m) \geq 0, e_j^*(w_m) \geq 0, f(e_j) \geq 0$, for all $m, j \in \mathbb{N}$. We may also assume that $1 \geq \|\sum_{m=1}^n r_k z_m\| > \theta$, therefore we may assume that $1 \geq f(\sum_{m=1}^d c_m z_m) > \theta$. By Lemmas 4.9, 4.10 and 4.11, there exist $g^1, g^2, g^3 \in W$, such that $(g^1 + g^2 + g^3)(\sum_{m=1}^d r_m w_m) > 2\theta f(\sum_{m=1}^d r_m z_m) - 13\sum_{m=1}^\infty \delta_m > 2\theta^2 - \theta^2 = \theta^2$. Hence $\|\sum_{m=1}^d r_m w_m\| > \frac{\theta^2}{3}$ and this means that $\|\sum_{m=1}^d r_m w_m\| > \frac{\theta^2}{3}\|\sum_{m=1}^d r_m z_m\|$.

By symmetricity of the arguments it follows that $\{z_m\}_{m\in\mathbb{N}}$ also dominates $\{w_m\}_{m\in\mathbb{N}}$, therefore $\{z_m\}_{m\in\mathbb{N}}$ and $\{w_m\}_{m\in\mathbb{N}}$ are equivalent. □

Corollary 4.12. *The space $\mathfrak{X}_{0,1}^n$ is quasi-minimal.*

Proof. If X,Y are block subspaces of $\mathfrak{X}^n_{0,1}$, choose $\{x_k\}_{k\in\mathbb{N}}$ in X and $\{y_k\}_{k\in\mathbb{N}}$ in Y normalized block sequences both generating ℓ_1^n spreading models. Then obviously one may pass to subsequences satisfying the assumption of Proposition 4.1, therefore X,Y contain further subspaces that are isomorphic. Since any subspace contains an isomorph of a block subspace, the result follows. □

5. Strictly singular operators

In this section, we provide necessary and sufficient conditions for a bounded operator defined on a subspace of $\mathfrak{X}^n_{0,1}$, to be non-strictly singular. The proof of this is based on results from the previous section and yields the following. For any Y subspace of $\mathfrak{X}^n_{0,1}$ and $S_1,S_2,\ldots,S_{n+1}$ strictly singular operators on Y, the composition $S_1S_2\cdots S_{n+1}$ is a compact operator. We show that the strictly singular operators on the subspaces of $\mathfrak{X}^n_{0,1}$ admit non-trivial hyperinvariant subspaces. Next, we provide a method for constructing strictly singular operators on subspaces of $\mathfrak{X}^n_{0,1}$, which is used to prove the non-separability of $\mathcal{S}(Y)$ and also to build $S_1,\ldots,S_n$ in $\mathcal{S}(Y)$, such that the composition $S_1\cdots S_n$ is non-compact. We close this section by combining the above results with the properties of the α-indices to show that $\{\mathcal{SS}_k(Y)\}_{k=1}^n$ is a strictly increasing family of two sided ideals.

Theorem 5.1. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$ and $T:Y\to\mathfrak{X}^n_{0,1}$ be a bounded linear operator. Then the following assertions are equivalent.*

(i) *T is not strictly singular.*
(ii) *There exists a sequence $\{x_m\}_{m\in\mathbb{N}}$ in Y generating a c_0 spreading model, such that $\{Tx_m\}_{m\in\mathbb{N}}$ generates a c_0 spreading model.*
(iii) *There exists $1\le k\le n$ and a sequence $\{x_m\}_{m\in\mathbb{N}}$ in Y, such that both $\{x_m\}_{m\in\mathbb{N}}$ and $\{Tx_m\}_{m\in\mathbb{N}}$ generate an ℓ_1^k spreading model but no subsequences of $\{x_m\}_{m\in\mathbb{N}}$ and $\{Tx_m\}_{m\in\mathbb{N}}$ generate an ℓ_1^{k+1} one.*

Proof. Assume that there exists $1\le k\le n$ and a sequence $\{x_m\}_{m\in\mathbb{N}}$ in Y, such that both $\{x_m\}_{m\in\mathbb{N}}$ and $\{Tx_m\}_{m\in\mathbb{N}}$ generate an ℓ_1^k spreading model but no subsequences of $\{x_m\}_{m\in\mathbb{N}}$ and $\{Tx_m\}_{m\in\mathbb{N}}$ generate an ℓ_1^{k+1} one.

If $\{x_m\}_{m\in\mathbb{N}}$ converges weakly to a non-zero element x, then $\{x_m-x\}_{m\in\mathbb{N}}$, as well as $\{Tx_m-Tx\}_{m\in\mathbb{N}}$ generate ℓ_1^k spreading models and no subsequences of them generate an ℓ_1^{k+1} one. Therefore, we may assume that they are both normalized block sequences. Set $I_m=\operatorname{ran}(\operatorname{ran}x_m\cup\operatorname{ran}Tx_m)$ and passing, if necessary, to a subsequence of $\{x_m\}_{m\in\mathbb{N}}$, $\{I_m\}_{m\in\mathbb{N}}$ are increasing subsets of the naturals.

Corollary 3.11 yields that $\alpha_{n-k}(\{x_m\}_m)>0,\alpha_{n-k}(\{Tx_m\}_m)>0$ as well as $\alpha_{k'}(\{x_m\}_m)=0,\alpha_{k'}(\{Tx_m\}_m)=0$, for $k'<n-k$.

Choose $\{F_m\}_{m\in\mathbb{N}}$ increasing subsets of the naturals $\{c_i\}_{i\in F_m}$ non-negative reals for all $m\in\mathbb{N}$ such that the following are satisfied.

(i) $\sum_{i\in F_m} c_i x_i$ as well as $\sum_{i\in F_m} c_i T x_i$ are (k,ε_m) s.c.c. with $\lim_m \varepsilon_m = 0$.
(ii) $F_m \in \mathcal{S}_k$.

Since $F_m \in \mathcal{S}_k$ and $\{x_m\}_{m\in\mathbb{N}}$, $\{Tx_m\}_{m\in\mathbb{N}}$ generate ℓ_1^k spreading models, we conclude that, if $z_m = \sum_{i\in F_m} c_i x_i$ for all $m\in\mathbb{N}$, then $\{z_m\}_{m\in\mathbb{N}}$, as well as $\{Tz_m\}_{m\in\mathbb{N}}$ are seminormalized. Moreover, since $\alpha_{k'}(\{x_m\}_m) = 0, \alpha_{k'}(\{Tx_m\}_m) = 0$, for $k' < n-k$, by Proposition 3.5(ii) we conclude that $\alpha_{n-1}(\{z_m\}_m) = 0$ as well as $\alpha_{n-1}(\{Tz_m\}_m) = 0$. By Proposition 3.6, we conclude that passing, if necessary to a subsequence, both $\{z_m\}_{m\in\mathbb{N}}$ and $\{Tz_m\}_{m\in\mathbb{N}}$ generate c_0 spreading models.

Assume now that there exists a sequence $\{x_m\}_{m\in\mathbb{N}}$ in Y generating a c_0 spreading model, such that $\{T(x_m)\}_{m\in\mathbb{N}}$ generates a c_0 spreading model. This means that $\{x_m\}_{m\in\mathbb{N}}$, as well as $\{Tx_m\}_{m\in\mathbb{N}}$ are weakly null, we may therefore assume that they are both normalized block sequences. Apply Proposition 3.14 and find $\{F_m\}_{m\in\mathbb{N}}$ increasing subsets of the naturals, such that if $y_m = \sum_{i\in F_m} y_i$, then both $\{y_m\}_{m\in\mathbb{N}}$ and $\{Ty_m\}_{m\in\mathbb{N}}$ generate ℓ_1^n spreading models. Set $I_m = \operatorname{ran}(\operatorname{ran} y_m \cup \operatorname{ran} Ty_m)$ and passing, if necessary, to a subsequence of $\{y_m\}_{m\in\mathbb{N}}$, $\{I_m\}_{m\in\mathbb{N}}$ are increasing subsets of the naturals. This means that the assumption of Proposition 4.1 is satisfied. Hence, there exists a further block sequence $\{w_m\}_{m\in\mathbb{N}}$ of $\{y_m\}_{m\in\mathbb{N}}$, such that $\{w_m\}_{m\in\mathbb{N}}$ is equivalent to $\{Tw_m\}_{m\in\mathbb{N}}$. We conclude that T is not strictly singular.

Assume now, that T is not strictly singular and let $1\le k\le n$. Then there exists Z a subspace of Y, such that $T|_Z$ is an isomorphism. Proposition 3.18 yields that any subspace of $\mathfrak{X}^n_{0,1}$ contains a sequence generating an ℓ_1^k spreading model, such that no subsequence of it generates an ℓ_1^{k+1} one, thus so does Z. Since $T|_Z$ is an isomorphism, the third assertion must be true. □

The following definition is from [2].

Definition 5.2. Let X be a Banach space and k be a natural number. We denote by $\mathcal{SS}_k(X)$ the set of all bounded linear operators $T : X\to X$ satisfying the following: for every Schauder basic sequence $\{x_i\}_i$ in X and $\varepsilon>0$, there exists $F\in\mathcal{S}_k$ and a vector x in the linear span of $\{x_i\}_{i\in F}$ such that $\|Tx\| < \varepsilon\|x\|$.

Proposition 5.3. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$, $T : Y\to Y$ be a bounded linear operator and $1\le k\le n$. The following assertions are equivalent.*

(i) *The operator T is in $\mathcal{SS}_k(Y)$.*
(ii) *For every seminormalized weakly null sequence $\{x_i\}_i$ in Y, $\{Tx_i\}_i$ does not admit an ℓ_1^k spreading model.*

Proof. The implication (i) ⇒ (ii) follows easily using Remark 1.2 and therefore we omit it. Let us assume that (ii) holds, and towards a contradiction suppose that T is not in $\mathcal{SS}_k(Y)$, that is, there exist a normalized weakly null sequence $\{x_i\}_i$ in Y and $\varepsilon > 0$ satisfying the following: for every $F \in \mathcal{S}_k$ and real numbers $\{c_i\}_{i\in F}$ we have that

$$\Big\|T\Big(\sum_{i\in F} c_i x_i\Big)\Big\| \geq \varepsilon\Big\|\sum_{i\in F} c_i x_i\Big\|. \tag{21}$$

Let us first notice that T is strictly singular. Indeed, if not then there exists a closed infinite dimensional subspace Z of Y such that $T|_Z$ is an isomorphism. Proposition 3.18 yields that there exists a normalized weakly null sequence $\{z_i\}_i$ in Z generating an ℓ_1^k spreading model. Since $T|_Z$ is an isomorphism, $\{Tz_i\}_i$ generates an ℓ_1^k spreading model as well, which contradicts (ii).

We shall now show that $\{Tx_i\}_i$ does not admit a c_0 spreading model. Assume that this is not the case, pass to a subsequence of $\{x_i\}_i$ and relabel so that $\{Tx_i\}_i$ generates a c_0 spreading model. Applying Theorem 5.1 and Corollary 3.12, we may assume that $\{x_i\}_i$ generates an ℓ_1 spreading model. This implies that there exists $F \in \mathcal{S}_1$ such that $\|T(\frac{1}{\#F}\sum_{i\in F} x_i)\| < \varepsilon\|\frac{1}{\#F}\sum_{i\in F} x_i\|$, which contradicts (21).

Corollary 3.12 and Remark 1.2 imply that there exist natural numbers $1 \leq d \leq m \leq n$ and a subsequence of $\{x_i\}_i$, again denoted by $\{x_i\}_i$, such that $\{Tx_i\}_i$ generates an ℓ_1^d spreading model and does not admit an ℓ_1^{d+1} one, while $\{x_i\}_i$ generates an ℓ_1^m spreading model and does not admit an ℓ_1^{m+1} one. Theorem 5.1 implies that $d+1 \leq m$. Combining the above it is easy to see that there exists $F \in \mathcal{S}_{d+1}$ and real numbers $\{c_i\}_{i\in F}$ such that $\|T(\sum_{i\in F} c_i x_i)\| < \varepsilon\|\sum_{i\in F} c_i x_i\|$. However, (ii) yields that $d+1 \leq k$ and hence $F \in \mathcal{S}_k$ which contradicts (21). □

PROPOSITION 5.4. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$, and $\{x_m\}_{m\in\mathbb{N}}$ be a seminormalized weakly null sequence in Y. Then for every $1 \leq k \leq n$ and $S_1, S_2, \ldots, S_k : Y \to Y$ strictly singular operators, $\{S_1S_2\cdots S_k x_m\}_{m\in\mathbb{N}}$ has no subsequence generating an ℓ_1^{n+1-k} spreading model. In particular, $S_1S_2\cdots S_k$ is in $\mathcal{SS}_{n+1-k}(Y)$.*

Proof. The second assertion of this proposition evidently follows from the first one and Proposition 5.3. We prove the first assertion by induction on k. For $k=1$ and $S : Y \to Y$ a strictly singular operator, assume that $\{Sx_m\}_m$ generates an ℓ_1^n spreading model. The boundedness of S yields that $\{x_m\}_m$ must also generate an ℓ_1^n spreading model, while by Corollary 3.15 neither $\{x_m\}_m$ nor $\{Sx_m\}_m$ admit an ℓ_1^{n+1} spreading model. Theorem 5.1 yields that S cannot be strictly singular which is absurd.

Assume now that the statement holds for some $1 \leq k < n$ and let $S_1, \ldots, S_{k+1} : Y \to Y$ be strictly singular operators. If $\{S_1S_2\cdots S_{k+1}x_m\}_m$ generates an ℓ_1^{n-k} spreading model, then the boundedness of the operators yields that $\{S_2 \cdots S_{k+1} x_m\}_m$ generates an ℓ_1^{n-k} spreading model as well. By the inductive assumption it follows that neither of the sequences $\{S_1 S_2 \cdots S_{k+1} x_m\}_m$, $\{S_2 \cdots S_{k+1} x_m\}_m$ admits an ℓ_1^{n+1-k} spreading model. Once more, Theorem 5.1 yields that S_1 cannot be strictly singular, a contradiction which completes the proof. □

PROPOSITION 5.5. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$ and $S_1, S_2, \ldots, S_{n+1} : Y \to Y$ be strictly singular operators. Then $S_1 S_2 \cdots S_{n+1}$ is compact.*

Proof. Since $\mathfrak{X}^n_{0,1}$ is reflexive, it is enough to show that for any weakly null sequence $\{x_m\}_{m\in\mathbb{N}}$, we have that $\{S_1 S_2 \cdots S_{n+1} x_m\}_{m\in\mathbb{N}}$ norm converges to zero. By Proposition 5.4, the sequence $\{S_2 \cdots S_{n+1} x_m\}_m$ does not admit an ℓ_1 spreading model and hence, by Corollary 3.12 it is either norm null or it has some subsequence generating a c_0 spreading model.

If it is norm null, then there is nothing to prove. If, on the other hand, $\{S_2 \cdots S_{n+1} x_m\}_m$ generates a c_0 spreading model, then Theorem 5.1 and the fact that S_1 is strictly singular yield that $\{S_1 S_2 \cdots S_{n+1} x_m\}_{m\in\mathbb{N}}$ norm converges to zero. □

COROLLARY 5.6. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$ and $S : Y \to Y$ be a non-zero strictly singular operator. Then S has a non-trivial closed hyperinvariant subspace.*

Proof. Assume first that $S^{n+1} = 0$. Then it is straightforward to check that $\ker S$ is a non-trivial closed hyperinvariant subspace of S.

Otherwise, if $S^{n+1} \neq 0$, then Corollary 5.5 yields that S^{n+1} is compact and non-zero. Since S commutes with S^{n+1}, by Theorem 2.1 from [23], it is enough to check that for any $\alpha, \beta \in \mathbb{R}$ such that $\beta \neq 0$, we have that $(\alpha I - S)^2 + \beta^2 I \neq 0$ (see also [14, Theorem 2]). Since S is strictly singular, it is easy to see that this condition is satisfied. □

REMARK 5.7. The space $\mathfrak{X}^n_{0,1}$ is also defined over the complex field, satisfying all the above and following properties. For the complex $\mathfrak{X}^n_{0,1}$ Corollary 5.6 is an immediate consequence of the classical Lomonosov theorem [16].

REMARK 5.8. A well-known result due to M. Aronszajn and K. T. Smith [10], asserts that compact operators always admit non-trivial invariant subspaces. As it is shown by C. J. Read in [22], there do exist strictly singular operators on Banach spaces, not admitting any non-trivial invariant subspaces. Therefore, one may not hope to extend M. Aronszajn's and K. T. Smith's result to strictly singular operators. In [7] a hereditarily indecomposable Banach space $\mathfrak{X}_K$ is presented satisfying the scalar plus compact property. It follows that any operator acting on this space, admits a non-trivial closed invariant subspace. Moreover, in [8] a reflexive hereditarily indecomposable

Banach space $\mathfrak{X}_{\mathrm{ISP}}$ is constructed such that any operator acting on a subspace of $\mathfrak{X}_{\mathrm{ISP}}$, admits a non-trivial closed invariant subspace.

The next corollary is an immediate consequence of the previous one.

COROLLARY 5.9. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$ and $T : Y \to Y$ be a linear operator that commutes with a non-zero strictly singular operator. Then T admits a non-trivial closed invariant subspace.*

Before stating the next theorem, we need the following lemma concerning sequences that do not have a subsequence generating an ℓ_1^{k+1} spreading model.

LEMMA 5.10. *Let $0 \le k \le n-1$ and $\{x_i\}_{i\in\mathbb{N}} \subset \mathrm{Ba}(\mathfrak{X}^n_{0,1})$ be a block sequence such that no subsequence of it, generates an ℓ_1^{k+1} spreading model. Then for every $m \in \mathbb{N}$ there exists $L \in [\mathbb{N}]^\infty$ such that for any $m \le F_1 < \cdots < F_m$ maximal $\mathcal{S}_k$ subsets of L the following holds.*

$$\left\| \sum_{j=1}^m \sum_{i\in F_j} c_i^{F_j} x_i \right\| < 2.$$

Proof. Fix $m \in \mathbb{N}$ and let $\mathcal{G}$ to be the collection of finite sets F satisfying $F = \bigcup_{j=1}^m F_j$, where $m \le F_1 < \cdots < F_m$ are maximal $\mathcal{S}_k$ sets for all $i \in \{1, \ldots, m\}$ and

$$\left\| \sum_{i=1}^m \sum_{i\in F_j} c_i^{F_j} x_i \right\| \ge 2.$$

Assume the conclusion of the lemma is false. Then, by definition, the collection $\mathcal{G}$ is large in the $\mathbb{N}$. A theorem of Nash-Williams [19] gives us an $L \in [\mathbb{N}]$ such that $\mathcal{G}$ for all $M \in [L]$ and initial segment of M is in $\mathcal{G}$ (i.e., $\mathcal{G}$ is very large in L).

Therefore for any $F_1 < \cdots < F_m$ (assume $\min L \ge m$) maximal $\mathcal{S}_k$ subsets of L, we have

$$\left\| \sum_{i=1}^m \sum_{i\in F_j} c_i^{F_j} x_i \right\| \ge 2. \tag{22}$$

We show this yields a contradiction. Let $(F_j)_j$ be an increasing sequence of maximal $\mathcal{S}_k$ subset of L and define $y_j = \sum_{i\in F_j} c_i^{F_j} x_i$. By Proposition 3.7, $\alpha_{n-k-1}(\{x_i\}_i) = 0$. Since each $\{y_j\}_j \subset \mathrm{Ba}(\mathfrak{X}^n_{0,1})$ and each y_j is a $(k, 3/\min F_j)$ s.c.c. Proposition 3.5(2), implies that $\alpha_{n-1}(\{y_j\}_j) = 0$. By Proposition 3.6, there is a subsequence of $\{y'_j\}_{j\in\mathbb{N}}$ of $\{y_j\}_{j\in\mathbb{N}}$ such that for $m \le k_1 < \cdots < k_m$ we have

$$\left\| \sum_{j=1}^m \sum_{i\in F_{k_j}} c_i^{F_{k_j}} x_i \right\| = \left\| \sum_{j=1}^m y'_{k_j} \right\| < 2.$$

This contradicts (22). $\square$

The next proposition is an intermediate step towards showing that for any Y infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$, there exist $S_1,\dots,S_n : Y\to Y$ strictly singular operators, such that $S_1\cdots S_n$ is non-compact.

PROPOSITION 5.11. *Let $0\le k\le n-1$ and Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$. Let also $\{x_i^*\}_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{X}^{n*}_{0,1}$ generating a c_0^{k+1} spreading model and $\{x_i\}_{i\in\mathbb{N}}$ be a seminormalized weakly null sequence in Y, such that no subsequence of it generates an ℓ_1^{k+1} spreading model. Then passing, if necessary, to subsequences of $\{x_i^*\}_{i\in\mathbb{N}}$ and $\{x_i\}_{i\in\mathbb{N}}$, the map $T:Y\to Y$ with $Tx=\sum_{1=1}^\infty x_i^*(x)x_i$ is bounded, strictly singular and non-compact.*

Proof. Passing, if necessary, to a subsequence, we may assume that $\{x_i\}_{i\in\mathbb{N}}$ is a normalized block sequence.

It follows from Lemma 5.10 and a standard diagonal argument that there is an $L\in[\mathbb{N}]$ such for all $m\in\mathbb{N}$ and $m\le F_1<\cdots<F_m$ maximal $\mathcal{S}_k$ sets in L

$$\left\|\sum_{j=1}^m\sum_{i\in F_j}c_i^{F_j}x_i\right\|<2. \tag{23}$$

Choose a subsequence $(i_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that $i_j\ge 2^{j+3}+1$ for all $j\in\mathbb{N}$. We claim that the map

$$Tx=\sum_{j\in L}x^*_{i_j}(x)x_j$$

is the desired one.

Let $x\in Y,\|x\|=1$ and $x^*\in Y^*,\|x^*\|=1$. We may assume that $x^*(x_j)\ge 0$ for all $j\in L$. We partition L in the following way: For $q=0,1,\dots$ set

$$\begin{aligned}
B_q&=\left\{j\in L:\frac{1}{2^{q+1}}<x^*(x_j)\le\frac{1}{2^q}\right\},\\
C_q&=\{j\in B_q: j\ge q+1\},\\
D_q&=\{j\in B_q: j\le q\}.
\end{aligned}$$

Evidently we have

$$\left|\sum_{j\in D_q}x^*(x_j)x^*_{i_j}(x)\right|\le\frac{q}{2^q}. \tag{24}$$

Decompose C_q into successive subsets $\{C_q^\ell\}_{\ell=0}^{p_q}$ of L such that the following are satisfied:

(i) $C_q=\bigcup_{\ell=0}^{p_q}C_q^\ell$.
(ii) $C_q^0=C_q\cap\{q+1,\dots,2^{q+1}\}$ and for $\ell>0$ C_q^ℓ is a maximal $\mathcal{S}_k$ set (except perhaps the last one).

We claim that $p_q < 2^{q+3}$. Let $I_q \subset \{1,\dots,p_q\}$ be an $\mathcal{S}_1$ set such that $\#I_q \geq p_q/2$. From (23) and the definition of B_q we have

$$2 > \Big\|\sum_{\ell\in I_q}\sum_{j\in C_q^\ell} c_j^{C_q^\ell} x_j\Big\| \geq \sum_{\ell\in I_q}\sum_{j\in C_q^\ell} c_j^{C_q^\ell} x^*(x_j) \geq \frac{p_q}{2^{q+2}}.$$

Therefore, $p_q < 2^{q+3}$.

Now set

$$G_q^\ell = \{i_j : j \in C_q^\ell\} \quad \text{for } \ell = 0,\dots,p_q.$$

Then it is easy to check the following.

(i) $G_q^0 \in \mathcal{S}_1$ and $\min G_q^0 > 2^{q+3}$.

(ii) $G_q^\ell \in \mathcal{S}_k$ for $\ell > 0$.

Since $p_q < 2^{q+3}$, the set $G_q = \bigcup_{\ell=0}^{p_q} G_q^\ell \in \mathcal{S}_{k+1}$. Since $\{x_i^*\}_{i\in\mathbb{N}}$ generates a c_0^{k+1} spreading model, we conclude the following:

$$\Big|\sum_{j\in C_q} x^*(x_j)x_{i_j}^*(x)\Big| < 2\max\{|x^*(x_j)| : j \in C_q\}. \tag{25}$$

Summing up (24) and (25), we conclude that $\|T\| \leq 2\sum_{q=0}^\infty \frac{1+q}{2^q}$.

To see that T is non-compact, consider the biorthogonal functionals $\{f_k\}_{k\in L}$ of $\{x_{i_j}^*\}_{j\in L}$. Since $\{f_k\}_{k\in L}$ is a seminormalized sequence we have

$$\|T(f_k - f_m)\| = \|x_k - x_m\|$$

for $m \neq k$ in L. Therefore $\{T(f_k)\}_{k\in\mathbb{N}}$ has no norm convergent subsequence.

To prove that S is strictly singular, first notice that for $x \in Y, \|x\| = 1$, $x^* \in Y^*, \|x^*\| = 1, j_0 \in \mathbb{N}$, we have that

$$\begin{aligned}
x^*(Tx) &\leq \sum_{q=0}^{q_0-1}\Big|\sum_{j\in B_q} x_{i_j}^*(x)x^*(x_j)\Big| + 2\sum_{q=q_0}^\infty \frac{(q+1)}{2^q}\\
&\leq \sum_{q=0}^{q_0-1}\Big(\Big|\sum_{j\in D_q} x_{i_j}^*(x)x^*(x_j)\Big| + \Big|\sum_{j\in C_q} x_{i_j}^*(x)x^*(x_j)\Big|\Big) + 2\sum_{q=q_0}^\infty \frac{(q+1)}{2^q}\\
&< \sum_{q=0}^{q_0-1}(q+2)\sup\{|x_{i_j}^*(x)| : j \in \mathbb{N}\} + 2\sum_{q=q_0}^\infty \frac{(q+1)}{2^q}.
\end{aligned}$$

Therefore, $\|Tx\| \leq \frac{q_0^2+3q_0}{2}\sup\{|x_{i_j}^*(x)| : j \in \mathbb{N}\} + 2\sum_{q=q_0}^\infty \frac{(q+1)}{2^q}$.

Let Z be an infinite dimensional closed subspace of Y and $\varepsilon > 0$. Since Z does not contain c_0, it follows that for any $\delta > 0$ there exists $x \in Z, \|x\| = 1$, such that $\sup\{|x_{i_j}^*(x)| : j \in \mathbb{N}\} < \delta$. For appropriate choices of q_0 and δ, it follows that there exists $x \in X, \|x\| = 1$ such that $\|Tx\| < \varepsilon$, thus T is strictly singular.

The proof of the boundedness is based on the proof of Proposition 3.1 from [5] and the proof of the strict singularity of T originated from an unpublished result due to A. Pelczar-Barwacz. $\square$

Remark 5.12. The proof of the above proposition actually yields, that for L, M infinite subsets of the naturals, the map $T_{L,M} = \sum_{i=1}^{\infty} x^*_{L(i)}(x)x_{M(i)}$ remains bounded, strictly singular and non-compact.

Corollary 5.13. *For any infinite dimensional closed subspace Y of $\mathfrak{X}^n_{0,1}$, the ideal $\mathcal{S}(Y)$ of strictly singular operators is non-separable.*

Proof. Choose $\{x^*_i\}_{i\in\mathbb{N}}$ a seminormalized sequence in $\mathfrak{X}^{n*}_{0,1}$ generating a c^n_0 spreading model and $\{x_i\}_{i\in\mathbb{N}}$ a seminormalized weakly null sequence in Y not having a subsequence generating an ℓ^n_1 spreading model, such that the map $T : Y \to Y$ with $Tx = \sum_{i=1}^{\infty} x^*_i(x)x_i$ is bounded, strictly singular and non-compact. By Remark 5.12, for any L infinite subset of the naturals, the operator $T_L : Y \to Y$ with $T_L x = \sum_{i=1}^{\infty} x_L(i)^*(x)x_i$ is bounded, strictly singular and non-compact. Therefore $\mathcal{S}(Y)$ contains an uncountable ε-separated subset, hence it is non-separable. $\square$

Proposition 5.14. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$. Then there exist $S_1, \ldots, S_n : Y \to Y$ strictly singular operators, such that for $0 \le k \le n-2$ the composition $S_{n-k} \cdots S_n$ is in $\mathcal{SS}_{n-k}(Y)$ and not in $\mathcal{SS}_{n-k-1}(Y)$ and $S_1 \cdots S_n$ is in $\mathcal{SS}_1(Y)$ and it is not compact.*

Proof. Using Proposition 3.18, Remark 3.13, Proposition 5.11 and Remark 5.12, for $k = 1, \ldots, n$ choose $\{x_{k,i}\}_{i\in\mathbb{N}}$ normalized weakly null sequences in Y and $\{x^*_{k,i}\}_{i\in\mathbb{N}}$ normalized weakly null sequences in $\mathfrak{X}^{n*}_{0,1}$ satisfying the following.

- (i) $\{x_{k,i}\}_{i\in\mathbb{N}}$ generates an ℓ^{k-1}_1 spreading model and no subsequence of it generates an ℓ^k_1 one for $k = 2, \ldots, n$, while $\{x_{1,i}\}_{i\in\mathbb{N}}$ generates a c_0 spreading model.
- (ii) $\{x^*_{k,i}\}_{i\in\mathbb{N}}$ generates a c^k_0 spreading model for $k = 1, \ldots, n$.
- (iii) There exists $\varepsilon_k > 0$ such that $x^*_{k+1,i}(x_{k,i}) > \varepsilon_k$ for all $i \in \mathbb{N}$ and $x^*_{k+1,i}(x_{k,j}) = 0$ for $i \neq j$, $k = 1, \ldots, n-1$.
- (iv) The map $S_k : Y \to Y$ with $S_k(x) = \sum_{i=1}^{\infty} x^*_{k,i}(x)x_{k,i}$ is bounded strictly singular and non-compact.

We shall inductively prove the following. For $k = 0, \ldots, n-1$ there exists a sequence of seminormalized positive real numbers $\{c_{k,i}\}_{i\in\mathbb{N}}$ such that

$$S_{n-k} \cdots S_{n-1} S_n x = \sum_{i=1}^{\infty} c_{k,i} x^*_{n,i}(x) x_{n-k,i}.$$

For $k=0$, the assumption holds, for $c_{0,i}=1$ for all $i\in\mathbb{N}$. Assume that it holds for some $k<n-1$. Then, by the inductive assumption

$$S_{n-k-1}\cdots S_n x=\sum_{i=1}^{\infty} x^*_{n-k-1,i}\left(\sum_{j=1}^{\infty} c_{k,j}x^*_{n,j}(x)x_{n-k,j}\right)x_{k,i}$$
$$=\sum_{i=1}^{\infty} c_{k,i}x^*_{n-k-1,i}(x_{n-k,i})x^*_{n,i}(x)x_{k,i}.$$

Set $c_{k+1,i}=c_{k,i}x^*_{n-k-1,i}(x_{n-k,i})$ for all $i\in\mathbb{N}$. Then $c_{k+1,i}>c_{k,i}\varepsilon_{n-k}$, for all $i\in\mathbb{N}$, therefore $\{c_{k+1,i}\}_{i\in\mathbb{N}}$ is seminormalized. The induction is complete.

Let now $0\le k\le n-2$. Proposition 5.4 yields that $S_{n-k}\cdots S_n$ is in $\mathcal{SS}_{n-k}(Y)$. Moreover, if we consider $\{y_i\}_i$ to be a seminormalized sequence in Y, biorthogonal to $\{x^*_{n,i}\}_{i\in\mathbb{N}}$, then $S_{n-k}\cdots S_n y_i=c_{k,i}x_{n-k,i}$ and therefore by (i) $\{S_n y_i\}_i$ generates an ℓ_1^{n-k-1} spreading model. Proposition 5.3 yields that $S_{n-k}\cdots S_n$ is not in $\mathcal{SS}_{n-k-1}(Y)$

The fact that $S_1\cdots S_n$ is in $\mathcal{SS}_1(Y)$ and it is not compact is proved similarly. □

Proposition 5.15. *Let Y be an infinite dimensional closed subspace of $\mathfrak{X}^n_{0,1}$. Then $\mathcal{K}(Y)\subsetneq\mathcal{SS}_1(Y)\subsetneq\mathcal{SS}_2(Y)\subsetneq\cdots\subsetneq\mathcal{SS}_n(Y)=\mathcal{S}(Y)$ and for every $1\le k\le n$, $\mathcal{SS}_k(Y)$ is a two sided ideal.*

Proof. The fact that $\mathcal{SS}_n(Y)=\mathcal{S}(Y)$ follows from Proposition 5.4 while the fact that $\mathcal{K}(Y)\subsetneq\mathcal{SS}_1(Y)\subsetneq\mathcal{SS}_2(Y)\subsetneq\cdots\subsetneq\mathcal{SS}_n(Y)$ follows from Proposition 5.14. Fix $1\le k\le n$. We will show that $\mathcal{SS}_k(Y)$ is a two sided ideal and for that it is enough to show that whenever S,T are in $\mathcal{SS}_k(Y)$, then so is $S+T$. The other properties of an ideal were verified in [2] and hold for any space.

We shall show that for every seminormalized weakly null sequence $\{x_i\}_i$ in Y, $\{(S+T)x_i\}_i$ does not admit an ℓ_1^k spreading model and by Proposition 5.3 we will be done.

We may assume that $\{Sx_i\}_i,\{Tx_i\}_i$ and $\{(S+T)x_i\}_i$ are all seminormalized block sequences. Since S and T are both in $\mathcal{SS}_k(Y)$, by Proposition 5.3 neither $\{Sx_i\}_i$ nor $\{Tx_i\}_i$ admits an ℓ_1^k spreading model. Proposition 3.7 yields that $\alpha_{k'}(\{Sx_i\}_i)=0$ as well as $\alpha_{k'}(\{Tx_i\}_i)=0$ for $k'<n-k+1$. It immediately follows from the definition of the α-index that $\alpha_{k'}(\{(S+T)x_i\}_i)=0$ for $k'<n-k+1$. Once more, Proposition 3.7 yields that $\{(S+T)x_i\}_i$ does not admit an ℓ_1^k spreading model. □

6. The space $\mathfrak{X}^{\omega}_{0,1}$

Recall that

$$\mathcal{S}_\omega=\{F\subset\mathbb{N}: n\le F \text{ and } F\in\mathcal{S}_n \text{ for some } n\in\mathbb{N}\}.$$

The space $\mathfrak{X}^{\omega}_{0,1}$ is defined in the natural way allowing $\mathcal{S}_\omega$-admissible successive subsets of $\mathbb{N}$. In this section let W denote the norming set of $\mathfrak{X}^{\omega}_{0,1}$. For this space, we have the following proposition.

Proposition 6.1. *The following hold for* $\mathfrak{X}^{\omega}_{0,1}$.

(i) *Every normalized weakly null sequence has a subsequence generating a* c_0 *or* ℓ_1^ω *spreading model.*
(ii) *Every non-trivial spreading model of* $\mathfrak{X}^{\omega}_{0,1}$ *is either isomorphic to* c_0 *or* ℓ_1.
(iii) *Every subspace of* $\mathfrak{X}^{\omega}_{0,1}$ *admits a spreading model isometric to* c_0 *and a spreading model isometric to* ℓ_1.
(iv) *Let* Y *be an infinite dimensional subspace of* $\mathfrak{X}^{\omega}_{0,1}$. *The following are equivalent.*
   (a) $T: Y \to Y$ *is not a strictly singular.*
   (b) *There is a weakly null sequence* $\{x_i\}_{i\in\mathbb{N}}$ *such that both* $\{x_i\}_{i\in\mathbb{N}}$ *and* $\{Tx_i\}_{i\in\mathbb{N}}$ *generate a* ℓ_1^ω *spreading model.*
   (c) *There is a weakly null sequence* $\{y_i\}_{i\in\mathbb{N}}$ *such that both* $\{y_i\}_{i\in\mathbb{N}}$ *and* $\{Ty_i\}_{i\in\mathbb{N}}$ *generate a* c_0 *spreading model.*

Since the proof of (ii) and (iv) are almost identical to the finite order case, we omit them. Below we include the sketches of the proofs of (i) and (iii). These are also similar to the corresponding proofs for $\mathfrak{X}^{n}_{0,1}$, however, there are some technical differences that are worth pointing out.

Clearly for each $1 \le \xi < \omega_1$ the space $\mathfrak{X}^{\xi}_{0,1}$ can be defined using the Schreier family $\mathcal{S}_\xi$ where appropriate. See [1] for the definition of $\mathcal{S}_\xi$. Whenever ξ is a countable limit ordinal satisfying $\eta + \xi = \xi$ for all $\eta < \xi$, we claim that the above proposition holds replacing ω with ξ. If ξ is of the form $\xi = \zeta + (n-1)$, where ζ is a limit ordinal satisfying the above condition and $n \in \mathbb{N}$, we have observed that the spreading models in this space behave analogously to those in $\mathfrak{X}^{n}_{0,1}$. The technical difficulty in including the proofs of these results is that they require us to introduce the higher order repeated averages and modify the proofs to accommodate more complicated nature of the Schreier sets of transfinite order. However, there does not seem to be any non-technical obstruction to proceeding in this direction.

It is worth pointing out that for countable ordinal numbers ξ failing the condition $\eta + \xi = \xi$ for all $\eta < \xi$, the space $\mathfrak{X}^{\xi}_{0,1}$ fails to satisfy (i). For example, in the space $\mathfrak{X}^{\omega\cdot 2}_{0,1}$ every seminormalized weakly null sequence admits either c_0 as a spreading model, or ℓ_1^ζ, for $\omega \le \zeta \le \omega\cdot 2$.

The following definition is found in [8, Definition 3.1].

Definition 6.2. Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}^{\omega}_{0,1}$.

We write $\alpha_{<\omega}(\{x_i\}_{i\in\mathbb{N}})=0$ if for any $n\in\mathbb{N}$, any fast growing sequence $\{\alpha_q\}_{q\in\mathbb{N}}$ of α-averages in W and for any $\{F_k\}_{k\in\mathbb{N}}$ increasing sequence of subsets of $\mathbb{N}$, such that $\{\alpha_q\}_{q\in F_k}$ is $\mathcal{S}_n$, the following holds: For any subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ we have $\lim_k \sum_{q\in F_k}|\alpha_q(x_{n_k})|=0$. If this is not the case, we write $\alpha_{<\omega}(\{x_i\}_{i\in\mathbb{N}})>0$.

Notice that for any limit ordinal $\xi<\omega_1$ it is easy to define the corresponding index $\alpha_{<\xi}$ using the sequence or ordinals increasing up to ξ. The next proposition is proved in [8, Proposition 3.3]. We note that in contrast with the finite order case, the argument is not completely trivial; however, for the sake of brevity we omit it.

Proposition 6.3. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}^{\omega}_{0,1}$. The following are equivalent.*

(i) $\alpha_{<\omega}(\{x_k\})=0$.
(ii) *For any $\varepsilon>0$ there exists $j_0\in\mathbb{N}$ such that for any $j\geq j_0$ there is an $k_j\in\mathbb{N}$ such that for any $k\geq k_j$, and for any $\{\alpha_q\}_{q=1}^d$ $\mathcal{S}_j$-admissible and very fast growing sequence of α-averages such that $s(\alpha_q)>j_0$ for $q=1,\dots,d$, we have that $\sum_{q=1}^d|\alpha_q(x_k)|<\varepsilon$.*

As in the finite case, we need to use the index to establish existence of the spreading models.

Proposition 6.4. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}^{\omega}_{0,1}$. Then the following hold*:

(i) *If $\alpha_{<\omega}(\{x_i\})>0$, then, by passing to a subsequence, $\{x_i\}_{i\in\mathbb{N}}$ generates a strong ℓ_1^ω spreading model.*
(ii) *If $\alpha_{<\omega}(\{x_i\})=0$ then there is a sequence of $\{x_i\}$ that generates a c_0 spreading model.*

Proof. First we prove (i). By Definition 6.2, there is an $d\in\mathbb{N}$, a very fast growing sequence of α-averages $\{\alpha_q\}_{q\in\mathbb{N}}$ in W, and sequence $\{F_i\}_{i\in\mathbb{N}}$ of successive finite subsets such that $\{\alpha_q\}_{q\in F_i}$ is $\mathcal{S}_d$ for each $i\in\mathbb{N}$ and

$$\sum_{q\in F_i}\left|\alpha_q(x_i)\right|>\varepsilon.$$

Relabeling so that $F_1\geq d$ we have that $(F_i)_{i\in\mathbb{N}}$ that for $G\in\mathcal{S}_\omega$, we have $\bigcup_{i\in G}F_i\in S_\omega$. Pass to a further subsequence such that of $\{x_i\}_{i\in\mathbb{N}}$ such that

$$\max\operatorname{supp}\left(\sum_{q\in F_i}\alpha_q\right)<\min\operatorname{supp}x_{i+1}.$$

Let $x_i^*=\sum_{q\in F_i}\alpha_q$. Note that $\varepsilon<\|x_i^*\|\leq 1$. If $G\in S_\xi$ the above argument yields that $\sum_{i\in G}x_i^*$ is a Schreier functional. Therefore, $\|\sum_{i\in G}x_i^*\|\leq 1$. This implies $\{x_i^*\}_{i\in\mathbb{N}}$ generates a c_0^ξ spreading model, as desired.

The proof has the same structure as the proof of Proposition 3.6 and so we will sketch some of the details. Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a summable sequence of positive reals such that $\varepsilon_i > 3\sum_{j>i}\varepsilon_j$ for all $i\in\mathbb{N}$. Using Proposition 6.3, inductively choose a subsequence, again denoted by $\{x_i\}_{i\in\mathbb{N}}$, such that for $i_0\geq 2$ and $j_0=\max\operatorname{supp} x_{i_0-1}$ if $\{\alpha_q\}_{q=1}^{\ell}$ is $\mathcal{S}_{j_0}$-admissible $s(\alpha_q)\geq \min\operatorname{supp} x_{i_0}$ then for all $i\geq i_0$

$$\sum_{q=1}^{\ell}\big|\alpha_q(x_i)\big| < \frac{\varepsilon_{i_0}}{i_0\max\operatorname{supp} x_{i_0-1}}.$$

As before, we will show that for any $t\leq i_1<\cdots<i_t$, $F\subset\{1,\dots,t\}$ we have

$$\left|\alpha\left(\sum_{j\in F}x_{i_j}\right)\right| < 1+2\varepsilon_{i_{\min F}}$$

whenever α is an α-average and

$$\left|g\left(\sum_{j\in F}x_{i_j}\right)\right| < 1+3\varepsilon_{i_{\min F}}$$

whenever g is Schreier functional. This implies the proposition.

For functionals in W_0 the above is clearly true. Assume for $m\geq 0$ that above holds for $t\leq i_1<\cdots<i_t$ and any functional in W_m. In the first case, let $t\leq i_1<\cdots<i_t$ and $\alpha\in W_{m+1}$. In this case, we refer the reader to the analogous step in the proof of Proposition 3.6.

Let $g\in W_{m+1}$ such that $g=\sum_{q=1}^{d}\alpha_q$ be a Schreier functional. We assume without loss of generality that

$$\operatorname{ran} g\cap\operatorname{ran} x_{i_j}\neq\varnothing \quad \text{for all } j=1,\dots,t. \tag{26}$$

Set

$$q_0=\min\{q:\ \max\operatorname{supp}\alpha_q\geq\min\operatorname{supp} x_{i_2}\}.$$

By definition of $\mathcal{S}_\omega$, $\{\alpha_q\}_{q=1}^{d}$ is $S_{\min\operatorname{supp}\alpha_1}$-admissible. Also, by definition, for $q>q_0$

$$s(\alpha_q)>\max\operatorname{supp}\alpha_{q_0}\geq\min\operatorname{supp} x_{i_2}.$$

Using (26)

$$\min\operatorname{supp}\alpha_1\leq\max\operatorname{supp} x_{i_1}.$$

These facts together allow us to use or initial assumption on the sequence $\{x_i\}_{i\in\mathbb{N}}$ (for $i_0=i_2$) and conclude that for $j\geq 2$

$$\sum_{q>q_0}\big|\alpha_q(x_{i_j})\big| < \frac{\varepsilon_{i_2}}{i_2\max\operatorname{supp} x_{i_1}}. \tag{27}$$

Using the fact that $i_2\geq t$, it follows that

$$\sum_{q>q_0}\left|\alpha_q\left(\sum_{j=1}^{t}x_{i_j}\right)\right| < \varepsilon_{i_1}.$$

As before, we consider two more cases.

Case 1. Assume that for $q < q_0$, $\alpha_q(\sum_{j=1}^t x_{i_j}) = 0$. In this case apply the induction for α_{q_0}.

Case 2. Alternatively, assume $s(\alpha_{q_0}) \geq \min \operatorname{supp} x_{i_1}$. In this case, since the singleton α_{q_0} is $\mathcal{S}_0$ admissible, we can apply our initial assume to conclude that $|\alpha_{q_0}(\sum_{j=1}^t x_{i_j})| < \varepsilon_{j_1}$. Combining previous estimates gives the desired result. □

The next proposition implies item (iii) of Proposition 6.1.

PROPOSITION 6.5. *Let $\{x_k\}_{k\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}^{\omega}_{_{0,1}}$ and $\{F_k\}$ be an sequence of successive subsets of naturals such that* $\lim_{k\to\infty} \#F_k = \infty$.

(i) *If $\{x_k\}_{k\in\mathbb{N}}$ generates a spreading model equivalent to c_0, $F_k \in \mathcal{S}_1$ for $k \in \mathbb{N}$ and $y_k = \sum_{i\in F_k} x_i$, then a subsequence of $\{y_k\}_{k\in\mathbb{N}}$ generates an ℓ_1^{ω} spreading model.*

(ii) *Suppose $\{x_k\}_{k\in\mathbb{N}}$ generates an ℓ_1^{ω} spreading model, $F_k \in \mathcal{S}_\omega$ and F_k is maximal $\mathcal{S}_\omega$ for each $k \in \mathbb{N}$ (i.e., maximal in $\mathcal{S}_{\min F_k}$). Let $w_k = \sum_{j\in F_k} c_j x_i$ where w_k is $(\min F_k, 3/\min F_k)$ s.c.c. Then a subsequence of $\{w_k\}_{k\in\mathbb{N}}$ generates a c_0 spreading model.*

Proof. The proof of (i) is identical to that of Proposition 3.14.

To prove (ii) it suffices to show $\alpha_{<\omega}(\{w_k\}) = 0$. We use Proposition 6.3. Let $\varepsilon > 0$. Find $j_0 > 2/\varepsilon$. Let $j \geq j_0$ and let $k_j \in \mathbb{N}$ such that $36/\min F_{k_j} < \varepsilon$. Let $k \geq k_j$, $\{\alpha_q\}_{q=1}^d$ be $\mathcal{S}_j$-admissible and very fast growing sequence of α-averages such that $s(\alpha_q) > j_0$ for $q = 1, \dots, d$. Clearly, $j < F_{k_j}$. Using Lemma 3.4

$$\sum_{q=1}^{d} \left| \alpha_q \left(\sum_{j\in F_k} c_j^{F_k} x_j \right) \right| < \frac{1}{s(\alpha_1)} + 6\frac{3}{\min F_k} < \varepsilon.$$

□

Problems and questions

There are some questions and problems concerning the structure of $\mathfrak{X}^{n}_{_{0,1}}$ and its dual which are open for us.

PROBLEM 1. (i) Is $\mathfrak{X}^{n}_{_{0,1}}$ minimal?

(ii) Does any sequence generating a c_0 spreading model have a subsequence equivalent to some subsequence of the basis?

If this is true, then Proposition 3.18 yields that $\mathfrak{X}^{n}_{_{0,1}}$ is sequentially minimal.

In particular, it is open to us whether two subsequences $\{e_{i_m}\}_{m\in\mathbb{N}}$, $\{e_{j_m}\}_{m\in\mathbb{N}}$ of the basis, such that $i_m < j_{m+1}$ and $j_m < i_{m+1}$ for all $m \in \mathbb{N}$, are equivalent.

Moreover, we do not know which class of Banach spaces in the classification appearing in [12] the subspaces of $\mathfrak{X}^{n}_{_{0,1}}$ belong to.

The next problem concerns the structure of $\mathfrak{X}^{n*}_{0,1}$ and its strictly singular operators.

Problem 2. (i) Does any block sequence in $\mathfrak{X}^{n*}_{0,1}$ contain a subsequence generating a c_0^k, $k=1,\dots,n$ or ℓ_1 spreading model?

(ii) Does any subspace of $\mathfrak{X}^{n*}_{0,1}$ admit $c_0^k, k=1,\dots,n$ and ℓ_1 spreading models?

The latter is equivalent to the corresponding problem for quotients of $\mathfrak{X}^{n}_{0,1}$, namely if every quotient of $\mathfrak{X}^{n}_{0,1}$ admits c_0 and ℓ_1^k, $k=1,\dots,n$ spreading models. Note that Corollary 3.20 yields that the same question for quotients of $\mathfrak{X}^{n*}_{0,1}$ has an affirmative answer.

(iii) Does $\mathfrak{X}^{n*}_{0,1}$ satisfy that whenever $S_1,\dots,S_{n+1}:\mathfrak{X}^{n*}_{0,1}\to\mathfrak{X}^{n*}_{0,1}$ are strictly singular, then the composition $S_1\cdots S_{n+1}$ is compact, as in $\mathfrak{X}^{n}_{0,1}$?

A way of answering this affirmatively is to show that any subspace Y of $\mathfrak{X}^{n}_{0,1}$, contains a further subspace which is complemented in $\mathfrak{X}^{n}_{0,1}$, which seems possible.

As it was pointed out to us by Anna Pelczar-Barwacz, since c_0 and ℓ_1 are both block finitely representable in every subspace of $\mathfrak{X}^{n}_{0,1}$, it follows that $\mathfrak{X}^{n}_{0,1}$ is arbitrarily distortable.

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