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THE BEHAVIOR OF THE BOUNDS OF MATRIX-VALUED MAXIMAL INEQUALITY IN $\mathbb{R}^n$ FOR LARGE n

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ABSTRACT. In this paper, we study the behavior of the bounds of matrix-valued maximal inequality in $\mathbb{R}^n$ for large n. The main result of this paper is that the L_p-bounds ($p>1$) can be taken to be independent of n, which is a generalization of Stein and Strömberg's result in the scalar-valued case. We also show that the weak type $(1,1)$ bound has similar behavior as Stein and Stömberg's.

1. Introduction

Let (X,d,μ) be a metric measure space and $\mathcal{B}(\ell_2)$ the matrix algebra of bounded operators on ℓ_2. For a locally integrable $\mathcal{B}(\ell_2)$-valued function f, we define

$$f_r(x)=\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,d\mu(y),$$

where $B(x,r)=\{y\in X: d(x,y)\le r\}$.

We shall study the weak type $(1,1)$ norm of the maximal operator, defined to be the least quantity c_1 such that for all $f\in L_1^+(X;S_1)$, all $\lambda>0$, there exists a projection $e\in\mathcal{P}(L_\infty(X)\bar{\otimes}\mathcal{B}(\ell_2))$ satisfying

$$ef_re\le\lambda,\quad \forall r>0\quad\text{and}\quad \operatorname{tr}\otimes\int e^{\perp}\le\frac{c_1\|f\|_1}{\lambda}. \tag{1.1}$$

Here $L_p(X;S_p)$ denotes the noncommutative L_p spaces associated with von Neumann algebra $\mathcal{A}=L_\infty(X)\bar{\otimes}\mathcal{B}(\ell_2)$, which is the weak closure of the algebra formed by essentially bounded functions $f:X\to\mathcal{B}(\ell_2)$. $L_p^+(\mathcal{A})$ is the positive part of $L_p(\mathcal{A})$. $\mathcal{P}(\mathcal{A})$ denotes the set of all projections in $\mathcal{A}$.

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Analogously to (1.1), the strong (p,p) norm of the maximal operator is defined to be the least quantity c_p such that for all $f \in L_p^+(\mathcal{A})$, there exists $F \in L_p^+(\mathcal{A})$ satisfying

$$f_r \leq F, \quad \forall r > 0 \quad \text{and} \quad \|F\|_p \leq c_p \|f\|_p. \tag{1.2}$$

In the scalar-valued case, that is, replacing $\mathcal{B}(\ell_2)$ by complex numbers $\mathbb{C}$, c_1 and c_p are reduced to be the weak $(1,1)$-boundedness and L_p-boundedness of the Hardy–Littlewood maximal function

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, d\mu(y).$$

This maximal function seems no available for matrix-valued function since we can not compare any two matrices or operators, which is one source of difficulties in the noncommutative analysis. The obstacle has been successfully overcome by the interaction with operator space theory. For instance, Junge in [11] formulated noncommutative Doob's maximal inequality using Pisier's theory of vector-valued noncommutative L_p-space [21]. Later, in [14], Junge and Xu developed a quite involved noncommutative version of Macinkiewiz interpolation theorem. Together with Yeadon's weak type $(1,1)$ maximal ergodic inequality, the interpolation result enable them to establish a noncommutative analogue of the Dunford–Schwartz maximal ergodic inequality. The noncommutative Stein's maximal ergodic inequality has also been obtained in the same paper.

Inspired by the maximal inequalities established in the theory of noncommutative martingale and in the ergodic theory, Mei in [15] considered the operator-valued Hardy–Littlewood maximal inequality in $\mathbb{R}^n$. He made use of the geometric property of $\mathbb{R}^n$ to reduce Hardy–Littlewood maximal inequality to several operator-valued martingale inequalities, which can be viewed as Junge's noncommutative Doob's maximal inequality or Cuculescu's weak type $(1,1)$ inequality for noncommutative martingales. Mei's inequality is exploited by Chen, Xu and Yin in [6] to prove maximal inequalities associated to the integrable rapidly decreasing functions.

The reduction method in Mei's arguments inevitably yields that the constants grow exponentially in n, the dimension of the base space $\mathbb{R}^n$. However, it is well known that the constants c_p when $p > 1$ can be taken to be independent of n in the scalar-valued case. The first result on this topic belongs to Stein [22] (see also the Appendix of [23]), which asserts that when X is the n-dimensional Hilbert space and μ is Lebesgue measure, c_p $(p > 1)$ can be taken to be independent of n. For general n-dimensional normed spaces, Bourgain [1], [2] and Carbery [5] proved that $c_p \leq C(p) < \infty$ provided $p > 3/2$. It is unknown whether or not there is some $1 < p < 3/2$ for which there exist n-dimensional normed spaces X_n such that c_p are unbounded. Bourgain in [3] showed that $c_p \leq C(p,q) < \infty$ for all $p > 1$ when $X = \ell_q^n$ and q is an even integer, which was extended by Müller to $X=\ell_q^n$ for all $1\le q<\infty$. Finally, Bougain [4] proved that $c_p<\infty$ for all $X=\ell_\infty^n$. We refer the readers to the Introduction of [18] for an overall review of the related results.

A dimension independent bound on c_p would mean that the operator-valued Hardy–Littlewood maximal inequality is in essence an infinite dimensional phenomenon. In the scalar-valued case, Stein's dimension independent bound on c_p $(p>1)$ has been exploited by Tišer in [24] to study differentiation of integrals with respect to certain Gaussian measures on Hilbert space. Therefore, it is reasonable for us to expect a similar application of the dimension independent bounds in the operator-valued case. Moreover, even though many operator-valued results are motivated by quantum analysis or probability (see, e.g., [15], [9], [20], [16], [10]), some of them are inversely used to study analysis on some noncommutative structures. For instance, in [6], the authors studied harmonic analysis on quantum torus through operator-valued harmonic analysis by transference technique; Junge Mei and Parcet [12] reduced the analysis on the Fourier multiplier on discret group von Neumann algebras to operator-valued results through Junge's cross product techniques. Hence, there would exist some applications of the dimension independent operator-valued results to the analysis on some noncommutative structures. Last but not least, the dimension free results are particularly of interest in the noncommutative analysis, since our research object is of infinite dimension such as a von Neumann algebra.

In this work, as the first attempt, we restrict us to study the behavior of operator-valued maximal inequality on n-dimensional Hilbert space equipped with Lebesgue measure. An underlying principle is that even though there are many difficulties in transferring classical results to the operator-valued setting (or even noncommutative setting), the metric or geometric properties of the defined spaces may interplay well with the noncommutativity of the range spaces, as happened in [15], [9], [20]. The first result in the paper is on the estimates of c_1.

THEOREM 1.1. *Let $f\in L_1^+(\mathcal{A})$. Then for any $\lambda>0$, there exists a universal constant C and a projection $e\in\mathcal{P}(\mathcal{A})$ such that*

$$ef_re\le\lambda,\quad \forall r>0 \quad \textit{and} \quad \mathrm{tr}\otimes\int e^{\perp}\le\frac{Cn\|f\|_1}{\lambda}.$$

This result is a generalization of the one by Stein and Strömberg. The main ingredient of the proof is Yeadon's noncommutative maximal ergodic theorem [25] (see also below Lemma 2.3). One will find a detailed proof in Section 3.

The main result of this paper is the following dimension independent estimates of c_p for $p>1$.

THEOREM 1.2. *Let $1 < p \leq \infty$ and $f \in L_p^+(\mathcal{A})$. Then there exist a constant C_p which depends only on p but not on n, and a function $F \in L_p(\mathcal{A})$ such that*

$$f_r \leq F, \quad \forall r > 0 \quad \textit{and} \quad \|F\|_p \leq C_p \|f\|_p.$$

This is an matrix-valued analogue of Stein and Strömberg's result. We should point out that the previous two theorems are also true by replacing $B(\ell_2)$ with any von Neumann algebra equipped with a trace. But for simplicity, we only prove them in the matrix-valued case. In Section 4, we prove Theorem 1.2. The main idea is due to Stein and Strömberg, but we should make use of the techniques and tools developped recently in the noncommutative analysis. A key ingredient in Stein's argument is the spherical maximal inequality. In Section 5, we prove an operator-valued version. In a forthcoming paper [8], we prove a noncommutative version of Nevo and Thangavelu's ergodic theorems for radial averages on the Heisenberg Group [19], and this spherical maximal inequality can be viewed as a special case of this kind of maximal ergodic inequalities.

Since this paper depends heavily on noncommutative maximal ergodic inequalities and noncommutative Marcinkiewicz interpolation theorem, and some readers may not be familiar with the main results or its related notations, we shall recall part of them in Section 2. Throughout this paper, C denotes a universal constant, may varying from line to line.

2. Preliminaries

We first recall the definition of the noncommutative maximal norm introduced by Pisier [21] and Junge [11]. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $1 \leq p \leq \infty$. We define $L_p(\mathcal{M}; \ell_\infty)$ to be the space of all sequences $x = (x_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $y = (y_n)$ in $L_\infty(\mathcal{M})$ such that

$$x_n = a y_n b, \quad \forall n \geq 1.$$

The norm of x in $L_p(\mathcal{M}; \ell_\infty)$ is given by

$$\|x\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\Big\{ \|a\|_{2p} \sup_{n\geq 1} \|y_n\|_\infty \|b\|_{2p} \Big\},$$

where the infimum runs over all factorizations of x as above.

We will follow the convention adopted in [14] that $\|x\|_{L_p(\mathcal{M};\ell_\infty)}$ is denoted by $\|\sup_n^+ x_n\|_p$. We should warn the reader that $\|\sup_n^+ x_n\|_p$ is just a notation since $\sup_n x_n$ does not make any sense in the noncommutative setting. We find, however, that $\|\sup_n^+ x_n\|_p$ is more intuitive than $\|x\|_{L_p(\mathcal{M};\ell_\infty)}$. The introduction of this notation is partly justified by the following remark.

REMARK 2.1. Let $x=(x_n)$ be a sequence of selfadjoint operators in $L_p(\mathcal{M})$. Then $x\in L_p(\mathcal{M};\ell_\infty)$ iff there exists a positive element $a\in L_p(\mathcal{M})$ such that $-a\le x_n\le a$ for all $n\ge 1$. In this case, we have

$$\Big\|\sup_{n\ge 1}{}^+ x_n\Big\|_p=\inf\big\{\|a\|_p : a\in L_p(\mathcal{M}), -a\le x_n\le a, \forall n\ge 1\big\}.$$

More generally, if Λ is any index set, we define $L_p(\mathcal{M};\ell_\infty(\Lambda))$ as the space of all $x=(x_\lambda)_{\lambda\in\Lambda}$ in $L_p(\mathcal{M})$ that can be factorized as

$$x_\lambda=ay_\lambda b \quad \text{with } a,b\in L_{2p}(\mathcal{M}), y_\lambda\in L_\infty(\mathcal{M}),\ \sup_\lambda\|y_\lambda\|_\infty<\infty.$$

The norm of $L_p(\mathcal{M};\ell_\infty(\Lambda))$ is defined by

$$\Big\|\sup_{\lambda\in\Lambda}{}^+ x_\lambda\Big\|_p=\inf_{x_\lambda=ay_\lambda b}\Big\{\|a\|_{2p}\sup_{\lambda\in\Lambda}\|y_\lambda\|_\infty\|b\|_{2p}\Big\}.$$

It is shown in [14] that $x\in L_p(\mathcal{M};\ell_\infty(\Lambda))$ iff

$$\sup\Big\{\Big\|\sup_{\lambda\in J}{}^+ x_\lambda\Big\|_p : J\subset\Lambda, J \text{ finite}\Big\}<\infty.$$

In this case, $\|\sup_{\lambda\in\Lambda}{}^+ x_\lambda\|_p$ is equal to the above supremum.

A closely related operator space is $L_p(\mathcal{M};\ell_\infty^c)$ for $p\ge 2$ which is the set of all sequences $(x_n)_n\subset L_p(\mathcal{M})$ such that

$$\Big\|\sup_{n\ge 1}{}^+ |x_n|^2\Big\|_{p/2}^{1/2}<\infty.$$

While $L_p(\mathcal{M};\ell_\infty^r)$ for $p\ge 2$ is the Banach space of all sequences $(x_n)_n\subset L_p(\mathcal{M})$ such that $(x_n^*)_n\in L_p(\mathcal{M};\ell_\infty^c)$. All these spaces fall into the scope of amalgamated L_p spaces intensively studied in [13]. What we need about these spaces is the following interpolation results.

LEMMA 2.2. *Let $2\le p\le\infty$. Then we have*

$$\big(L_p\big(\mathcal{M};\ell_\infty^c\big),L_p\big(\mathcal{M};\ell_\infty^r\big)\big)_{1/2}=L_p(\mathcal{M};\ell_\infty)$$

with equivalent norms.

We refer the reader to [11], [17] and [13] for more properties on these spaces.

Yeadon's weak type $(1,1)$ maximal ergodic inequality for semigroup is stated as follows:

LEMMA 2.3. *Let $(T_t)_{t\ge 0}$ be a semigroup of linear maps on $\mathcal{M}$. Each T_t for $t\ge 0$ satisfies the following properties*:

(i) *T_t is a contraction on $\mathcal{M}: \|Tx\|_\infty\le\|x\|_\infty$ for all $x\in\mathcal{M}$;*
(ii) *T_t is positive*: *$Tx\ge 0$ if $x\ge 0$;*
(iii) *$\tau\circ T\le\tau:\tau(T(x))\le\tau(x)$ for all $x\in L_1(\mathcal{M})\cap\mathcal{M}^+$.*

Let $x \in L_1^+(\mathcal{M})$, then for any $\lambda > 0$, there exists a projection $e \in \mathcal{M}$ such that

$$eM_t(x)e \le \lambda, \quad \forall t > 0 \quad \text{and} \quad \tau\big(e^\perp\big) \le \frac{\|x\|_1}{\lambda},$$

where M_t is defined as

$$M_t = \frac{1}{t}\int_0^t T^s ds, \quad \forall t > 0.$$

In order to extend this result to $p > 1$, Junge and Xu [14] proved the following much involved noncommutative Marcinkiewicz theorem for $L_p(\mathcal{M};\ell_\infty)$.

Lemma 2.4. *Let $1 \le p_0 < p_1 \le \infty$. Let $S = (S_n)_{n\ge 0}$ be a sequence of maps from $L_{p_0}^+(\mathcal{M}) + L_{p_1}^+(\mathcal{M})$ into $L_0^+(\mathcal{M})$. Assume that S is subadditive in the sense that $S_n(x+y) \le S_n(x) + S_n(y)$ for all $n \in \mathbb{N}$. If S is of weak type (p_0,p_0) with constant C_0 and of type C_1, then for any $p_0 < p < p_1$, S is of type (p,p) with constant C_p satisfying*

$$C_p \le C C_0^{1-\theta} C_1^{\theta}\left(\frac{1}{p_0} - \frac{1}{p}\right)^{-2},$$

where θ is determined by $1/p = (1-\theta)/p_0 + \theta/p_1$ and C is a universal constant.

With this interpolation result, they proved that there exists a constant C_p such that

$$\Big\|{\sup_{t>0}}^+ M_t(x)\Big\|_p \le C_p\|x\|_p, \quad \forall x \in L_p(\mathcal{M}). \tag{2.1}$$

Moreover, if additionally each T_t satisfies

(iv) T_t is symmetric relative to τ: $\tau(T(y)^*x) = \tau(y^*T(x))$ for all x, y in the intersection $L_2(\mathcal{M}) \cap \mathcal{M}$,

then

$$\Big\|{\sup_{t>0}}^+ T_t(x)\Big\|_p \le C_p\|x\|_p, \quad \forall x \in L_p(\mathcal{M}), \tag{2.2}$$

with C_p a constant only depending on p.

3. Estimates for c_1

We follow Stein and Strömberg's original argument to prove Theorem 1.1. As we shall see that it is just an application of Yeadon's weak type $(1,1)$ noncommutative maximal ergodic inequality.

Proof of Theorem 1.1. Let $f \in L_1(\mathbb{R}^n; L_1(\mathcal{M}))$. Without loss of generality, we assume f is positive. We then define

$$f_r(x) = \frac{1}{|B(0,r)|}\int_{B(0,r)} f(x-y)\, dy.$$

Recall that the heat-diffusion semigroup on $\mathbb{R}^n$ is given by $T^t g = g * h_t$, $\forall g \in \mathscr{S}(\mathbb{R}^n)$ with

$$h_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

We consider the heat-diffusion semigroup on $L_\infty(\mathbb{R}^n)\,\bar{\otimes}\,\mathcal{M}$ given by $S^t = T^t \otimes id_{\mathcal{M}}$. It is easy to check that $(S^t)_{t\geq 0}$ satisfies (i)–(iii). So by Lemma 2.3, for any $\eta > 0$, there exists a projection $e \in \mathcal{P}(\mathcal{A})$ such that

$$eM_t(f)e \leq \eta, \quad \forall t > 0 \quad \text{and} \quad \mathrm{tr} \otimes \int e^{\perp} \leq \frac{\|f\|_1}{\eta},$$

where

$$M_t(f) = \frac{1}{t}\int_0^t S^s(f)(x)\, ds = \int_{\mathbb{R}^n} \frac{1}{t}\int_0^t h_s(y)\, ds f(x-y)\, dy.$$

As proved in page 265 of [23], for any $r > 0$, there exists some t_r such that

$$\frac{1}{|B(0,r)|}\chi_{B(0,r)(y)} \leq Cn\frac{1}{t_r}\int_0^{t_r} h_s(y)\, ds. \tag{3.1}$$

Hence, obviously we have

$$ef_r e \leq eCnM_{t_r}(f)e \leq Cn\eta.$$

Now for any $\lambda > 0$, take $\eta = \lambda/(Cn)$, we obtain

$$\mathrm{tr} \otimes \int e^{\perp} \leq \frac{Cn\|f\|_1}{\lambda},$$

which finishes the proof. □

Instead of using Yeadon's inequality, but use Junge and Xu's inequality (1.2), in the same spirit, we can deduce that for $1 < p \leq \infty$, there exists an absolute constant $C_p > 0$ such that

$$\Big\| {\sup_{r>0}}^+ f_r \Big\|_p \leq C_p n \|f\|_p, \quad \forall f \in L_p(\mathcal{A}). \tag{3.2}$$

And the constant c_p can be improved to be $O(\sqrt{n})$ by the noncommutative Stein's maximal ergodic inequality (2.2) and the following fundamental estimates [23]: for any $r > 0$, there is $t_r > 0$ such that

$$\frac{1}{|B(0,r)|}\chi_{B(0,r)(y)} \leq Cn^{1/2} h_{t_r}(y). \tag{3.3}$$

4. Estimates for c_p $(p>1)$

We adapt Stein's argument [22] (see also the Appendix of [23]) to the operator-valued setting. The key step of the argument is the following operator-valued spherical maximal inequality. Let $f \in \mathscr{S}(\mathbb{R}^k; S_M)$ (S_M is the set of finite dimension self-adjoint matrix), for any $r>0$, we define

$$f_r^k(x) = \frac{1}{\omega_{k-1}} \int_{S^{k-1}} f\big(x - ry'\big)\, d\sigma\big(y'\big),$$

where $d\sigma$ is the usual measure on S^{k-1}, and ω_{k-1} is its total mass.

Proposition 4.1. *Let $k \geq 3$ and $p > k/(k-1)$, then there exists a constant $A_{k,p}$ such that*

$$\Big\| {\sup_{r>0}}^+ f_r^k \Big\|_p \leq A_{k,p} \|f\|_p, \quad \forall f \in L_p\big(\mathbb{R}^k; S_p\big).$$

We postpone its proof to the next section. The spherical maximal inequality yields the following weighted maximal inequality. Let $f \in \mathscr{S}(\mathbb{R}^k; S_M)$, for any $m \geq 0$ and $r \geq 0$, we define

$$f_r^{k,m}(x) = \left(\int_{|y|\leq r} |y|^m \, dy \right)^{-1} \int_{|y|\leq r} f(x-y)|y|^m \, dy.$$

Proposition 4.2. *Let $k \geq 3$ and $p > k/(k-1)$, then*

$$\Big\| {\sup_{r>0}}^+ f_r^{k,m} \Big\|_p \leq A_{k,p} \|f\|_p, \quad \forall f \in L_p\big(\mathbb{R}^k; S_p\big)$$

with the constant $A_{k,p}$ independent of m.

Proof. Without loss of generality, we assume $f \in \mathscr{S}(\mathbb{R}^k; S_M^+)$. Using polar coordinates, we can write

$$\int_{|y|\leq r} f(x-y)|y|^m \, dy = \int_0^r \int_{S^{k-1}} f\big(x - sy'\big) s^{m+k-1} \, d\sigma\big(y'\big)\, ds. \tag{4.1}$$

By Proposition 4.1, there exists $F \in L_p^+(\mathbb{R}^k; S_p)$ such that

$$f_s^k(x) \leq F(x), \quad \forall s > 0 \quad \text{and} \quad \|F\|_p \leq A_{k,p}\|f\|_p.$$

Hence

$$\text{RHS of (4.1)} \leq F(x)\omega_{k-1} \int_0^r s^{m+k-1}\, ds = F(x)\omega_{k-1} \frac{r^{m+k}}{m+k}.$$

So we have

$$f_r^{k,m} \leq F, \quad \forall r > 0 \quad \text{and} \quad \|F\|_p \leq A_{k,p}\|f\|_p$$

which is the desired result. ☐

We now consider $\mathbb{R}^n$ with $n \geq 3$, and write it as $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ with its points x written by (x_1, x_2). Let ρ denote an arbitrary element of $O(n)$, a rotation of $\mathbb{R}^n$ about the origin. Let $f \in \mathscr{S}(\mathbb{R}^n; S_{\mathcal{M}})$, for each $\rho \in O(n)$, $r > 0$, we define

$$f_r^{k,n-k,\rho}(x) = \left(\int_{|y_1|\leq r} |y_1|^{n-k}\, dy_1\right)^{-1} \int_{|y_1|\leq r} f\big(x - \rho(y_1, 0)\big)|y_1|^{n-k}\, dy_1.$$

Proposition 4.3. *Let $k \geq 3$ and $p > k/(k-1)$, we have*

$$\Big\| {\sup_{r>0}}^+ f_r^{k,n-k,\rho} \Big\|_p \leq A_{k,p}\|f\|_p, \quad \forall f \in L_p(\mathcal{A})$$

with the constant $A_{k,p}$ independent of n.

Proof. Take $f \in \mathscr{S}(\mathbb{R}^n; S_M)$. Again, we assume f is positive. By rotation invariance, it suffices to prove this when ρ is the identity rotation. In this case, we decompose $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, with $x = (x_1, x_2)$. Fix $x_2 \in \mathbb{R}^{n-k}$. By Proposition 4.2, there exist $F_{x_2} \in L_p^+(\mathbb{R}^k; S_p)$ such that

$$f_r^{k,n-k,1}(x_1, x_2) \leq F_{x_2}(x_1), \quad \forall r > 0 \quad \text{and} \quad \|F_{x_2}\|_p \leq A_{k,p}\|f_{x_2}\|_p.$$

Define $F(x_1, x_2) = F_{x_2}(x_1)$ on $\mathbb{R}^n$, then we complete the proof since $f_r^{k,n-k,1} \leq F$ for all $r > 0$ and

$$\begin{aligned}\|F\|_p^p &= \int_{\mathbb{R}^{n-k}} \big\|F(\cdot, x_2)\big\|_p^p\, dx_2 \\ &\leq A_{k,p}^p \int_{\mathbb{R}^{n-k}} \big\|f(\cdot, x_2)\big\|_p^p\, dx_2 = A_{k,p}^p\|f\|_p^p.\end{aligned}$$

□

Let $d\rho$ denote the Haar measure on the group $O(n)$, normalized so that its total measure is 1. Now we are at a position to prove Theorem 1.2.

Proof of Theorem 1.2. The result for $p = \infty$ is trivial. So we only consider the case $1 < p < \infty$. When $n \leq \max(p/(p-1), 2)$, we can use the estimates (3.2). Now, we assume $n > \max(p/(p-1), 2)$. We write $n = k + (n-k)$, where k is the smallest integer greater than $\max(p/(p-1), 2)$. We can assume f is of the form $g \otimes m$ where $g \in \mathscr{S}^+(\mathbb{R}^n)$ and $m \in S_M^+$, since the set of linear combinations of such elements are dense in $L_p(\mathcal{A})$. For such f, we have the following formula

$$\frac{\int_{|y|\leq r} f(y)\, dy}{\int_{|y|\leq r} dy} = \frac{\int_{O(n)} \int_{|y_1|\leq r} f(\rho(y_1, 0))|y_1|^{n-k}\, dy_1\, d\rho}{\int_{|y_1|\leq r} |y_1|^{n-k}\, dy_1}. \tag{4.2}$$

Here $y = (y_1, y_2) \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. To verify (4.2) it suffices to do so for g of the form $g = g_0(|y|)g_1(y')$, where $y' \in S^{n-1}$, and $y = |y|y'$, since linear combination of such functions are dense. Then for such g,

$$\text{LHS of (4.2)} = \int_0^r g_0(t)t^{n-1}\, dt \cdot \int_{S^{n-1}} g_1\big(y'\big)\, d\sigma\big(y'\big) nr^{-n}\omega_{n-1}^{-1} \otimes m.$$

On the other hand, notice that $g(\rho(y_1,0)) = g_0(|y_1|)g_1(\rho(y_1,0))$, so the right-hand side of (4.2) equals

$$\int_0^r g_0(t)t^{n-1}\,dt \cdot \int_{O(n)} \int_{S^{k-1}} g_1\big(\rho\big(y_1',0\big)\big)\,d\sigma\big(y_1'\big)\,d\rho n r^{-n}\omega_{k-1}^{-1} \otimes m.$$

Therefore matters are reduced to check that

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} g_1\big(y'\big)\,d\sigma\big(y'\big) = \frac{1}{\omega_{k-1}} \int_{O(n)} \int_{S^{k-1}} g_1\big(\rho\big(y_1',0\big)\big)\,d\sigma\big(y_1'\big)\,d\rho$$

which is trivial because

$$\int_{O(n)} g_1\big(\rho\big(y_1',0\big)\big)\,d\rho = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} g_1\big(y'\big)\,d\sigma\big(y_1'\big).$$

In (4.2), replace $f(y)$ with $f(x-y)$, we get

$$\frac{\int_{|y|\le r} f(x-y)\,dy}{\int_{|y|\le r} dy} = \frac{\int_{O(n)} \int_{|y_1|\le r} f(x-\rho(y_1,0))|y_1|^{n-k}\,dy_1\,d\rho}{\int_{|y_1|\le r} |y_1|^{n-k}\,dy_1}.$$

By Proposition 4.3, for each $\rho \in O(n)$, there exists $F^\rho \in L_p^+(\mathcal{A})$ such that

$$f_r^{k,n-k,\rho} \le F^\rho, \quad \forall r>0 \quad \text{and} \quad \big\|F^\rho\big\|_p \le A_{k,p}\|f\|_p.$$

Hence, one can very easily deduce that $F(x) = \int_{O(n)} F^\rho(x)\,d\rho$ is in $L_p(\mathcal{A})$ and satisfy

$$f_r^{k,n-k} \le F, \quad \forall r>0 \quad \text{and} \quad \|F\|_p \le A_{k,p}\|f\|_p. \qquad \square$$

5. The proof of Proposition 4.1

In order to simplify the notation, we denote f_t^n by f_t/ω_{n-1}. Hence

$$f_t(x) = \int_{S^{n-1}} f(x-t\theta)\,d\sigma(\theta).$$

We set $m(\xi) = \widehat{d\sigma}(\xi) = 2\pi|\xi|^{(2-n)/2}J_{(n-2)/2}(2\pi|\xi|)$ (see, e.g., Appendix B.4 in [7]). Obviously $m(\xi)$ is an infinitely differential function. We decompose the multiplier $m(\xi)$ into radial pieces as follows: We fix a radial Schwarz function φ_0 in $\mathbb{R}^n$ such that $\varphi_0(\xi) = 1$ when $|\xi| \le 1$ and $\varphi_0(\xi) = 0$ when $|\xi| \ge 2$. For $j \ge 1$, we let

$$\varphi_j(\xi) = \varphi_0\big(2^{-j}\xi\big) - \varphi_0\big(2^{1-j}\xi\big)$$

and we observe that $\varphi_j(\xi)$ is localized near $|\xi| = 2^j$. Then we have

$$\sum_{j\ge 0} \varphi_j = 1.$$

Set $m_j = \varphi_j m$ for all $j \ge 0$. The m_j's are finite supported Schwarz functions that satisfy

$$m = \sum_{j\ge 0} m_j.$$

Hence,

$$f_t(x) = \big(\hat{f}(\cdot)m(t\cdot)\big)^{\vee} = \sum_{j\geq 0}\big(\hat{f}(\cdot)m_j(t\cdot)\big)^{\vee} = \sum_{j\geq 0} f_{t,j}$$

For these $f_{t,j}$, there are the following estimates.

PROPOSITION 5.1. *Let $1 < p \leq \infty$, there exists a constant $C = C(n,p)$ such that*

$$\Big\|{\sup_t}^+ f_{t,0}\Big\|_p \leq C\|f\|_p.$$

More precisely, for $f \in L_p^+(\mathcal{A})$, there exists $F_0 \in L_p(\mathcal{A})$ such that

$$f_{t,0} \leq F_0, \quad \forall t > 0 \quad \text{and} \quad \|F_0\|_p \leq C\|f\|_p. \tag{5.1}$$

PROPOSITION 5.2. *Let $1 < p \leq 2$. There exists a universal constant $C = C(n,p)$ such that for any $j \geq 1$, we have*

$$\Big\|{\sup_t}^+ f_{t,j}\Big\|_p \leq C2^{(n/p-(n-1))j}\|f\|_p, \quad \forall f \in L_p(\mathcal{A}).$$

More precisely, for $f \in L_p^+(\mathcal{A})$, there exists $F_j \in L_p(\mathcal{A})$ such that

$$f_{t,j} \leq F_j, \quad \forall t > 0 \quad \text{and} \quad \|F_j\|_p \leq C2^{(n/p-(n-1))j}\|f\|_p. \tag{5.2}$$

With the two previous estimates, we can finish the proof of Proposition 4.1.

Proof of Proposition 4.1. Let $f \in L_p^+(\mathcal{A})$. When $2 \geq p > n/(n-1)$, by Propositions 5.1 and 5.2, we find F_j's satisfying inequality (5.1) or (5.2). We set

$$F = \sum_{j\geq 0} F_j.$$

Then

$$f_t = \sum_{j\geq 0} f_{t,j} \leq \sum_{j\geq 0} F_j = F, \quad \forall t > 0$$

and

$$\|F\| \leq \sum_{j\geq 0}\|F_j\|_p \leq C\sum_{j\geq 0} 2^{(n/p-(n-1))j}\|f\|_p = C\|f\|_p.$$

When $p \geq 2$, we invoke the noncommutative interpolation theorem, Lemma 2.4 to obtain the estimates. □

The rest of this section is devoted to the proof of the two propositions. Proposition 5.1 is a trivial application of the following Theorem 4.3 of [6].

LEMMA 5.3. *Let ψ be an integrable function on $\mathbb{R}^n$ such that $|\psi|$ is radial and radially decreasing. Let $\psi_t(x) = \frac{1}{t^n}\psi(\frac{x}{t})$ for $x \in \mathbb{R}^n$ and $t > 0$.*

(i) *Let* $f \in L_1(\mathbb{R}^n; S_1)$. *Then for any* $\alpha > 0$ *there exists a projection* $e \in \mathcal{P}(\mathcal{A}) \overline{\otimes} \mathcal{M}$ *such that*

$$\sup_{t>0} \big\| e(\psi_t * f) e \big\|_\infty \leq \alpha \quad \textit{and} \quad \mathrm{tr} \otimes \int e^\perp \leq C_n \|\psi\|_1 \frac{\|f\|_1}{\alpha}.$$

(ii) *Let* $1 < p \leq \infty$. *Then*

$$\Big\| {\sup_{t>0}}^+ \psi_t * f \Big\|_p \leq C_n \|\psi\|_1 \frac{p^2}{(p-1)^2} \|f\|_p, \quad \forall f \in L_p\big(\mathbb{R}^n; S_p\big).$$

On the proof of Proposition 5.2, it suffices to establish the two end-point estimates $p = 2$ and $p = 1$, since a noncommutative version of Marcinkiewicz interpolation theorem is available (see Lemma 2.4).

Lemma 5.4. *There exists a constant* $C = C(n) < \infty$ *such that for any* $j \geq 1$ *we have*

$$\Big\| {\sup_{t>0}}^+ f_{t,j} \Big\|_2 \leq C 2^{(1/2-(n-1)/2)j} \|f\|_2, \quad \forall f \in L_2(\mathcal{A}).$$

Proof. We define a function

$$\tilde{m}_j(\xi) = \xi \cdot \nabla m_j(\xi).$$

Let

$$\tilde{f}_{t,j}(x) = \big(\hat{f}(\cdot) \tilde{m}_j(t\cdot)\big)^\vee(x).$$

And we consider the following two g-functions:

$$G_j(f)(x) = \left(\int_0^\infty \big| f_{t,j}(x) \big|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$\tilde{G}_j(f)(x) = \left(\int_0^\infty \big| \tilde{f}_{t,j}(x) \big|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

For $f \in \mathscr{S}(\mathbb{R}^n, S_{\mathcal{M}}^+)$, the identity

$$s \frac{df_{s,j}}{ds} = \tilde{f}_{s,j}$$

hold for all j and s. By the fundamental theorem of calculus, we deduce that

$$\begin{aligned}
f_{t,j}(x)^2 &= \int_\varepsilon^t \frac{d}{ds} \big(f_{s,j}(x)\big)^2 ds + f_{\varepsilon,j}(x)^2 \\
&= \int_\varepsilon^t s \frac{df^*_{s,j}(x)}{ds} f_{s,j}(x) f^*_{s,j}(x) s \frac{df_{s,j}(x)}{ds} \frac{ds}{s} + f_{\varepsilon,j}(x)^2 \\
&= \int_\varepsilon^t \tilde{f}^*_{s,j}(x) f_{s,j}(x) + f^*_{s,j}(x) \tilde{f}_{s,j}(x) \frac{ds}{s} + f_{\varepsilon,j}(x)^2 \\
&\leq \int_0^\infty \bigg| \tilde{f}^*_{s,j}(x) f_{s,j}(x) \frac{ds}{s} + \int_0^\infty f^*_{s,j}(x) \tilde{f}_{s,j}(x) \bigg| \frac{ds}{s} + f_{\varepsilon,j}(x)^2.
\end{aligned}$$

Hence by triangle inequality and Hölder inequality, we have

$$\begin{aligned}\Big\|\sup_t{}^+|f_{t,j}|^2\Big\|_1^{1/2} &\le \Big\|\int_0^\infty \big|\tilde{f}^*_{s,j}(x)f_{s,j}(x)+f^*_{s,j}(x)\tilde{f}_{s,j}(x)\big|\frac{ds}{s}\Big\|_1^{1/2} \\ &\quad + \big\|f_{\varepsilon,j}(x)^2\big\|_1^{1/2} \\ &\le 2\Big\|\int_0^\infty \tilde{f}^*_{s,j}(x)f_{s,j}(x)\frac{ds}{s}\Big\|_1^{1/2} \\ &\quad + 2\Big\|\int_0^\infty f^*_{s,j}(x)\tilde{f}_{s,j}(x)\frac{ds}{s}\Big\|_1^{1/2} + \big\|f_{\varepsilon,j}(x)^2\big\|_1^{1/2} \\ &\le 4\big\|G_j(f)\big\|_2^{\frac12}\big\|\tilde{G}_j(f)\big\|_2^{\frac12} + \big\|f_{\varepsilon,j}(x)^2\big\|_1^{1/2}. \\ &\le 8\big\|G_j(f)\big\|_2^{\frac12}\big\|\tilde{G}_j(f)\big\|_2^{\frac12}.\end{aligned}$$

The last inequality is due to the fact that $\|f_{\varepsilon,j}(x)^2\|_1^{1/2}$ tends to 0 as ε tends to ∞ by Lebesgue dominated theorem. On the other hand, by the estimates (see, e.g., [7])

$$\big|\hat{d\sigma}(\xi)\big| + \big|\nabla\,\hat{d\sigma}(\xi)\big| \le C_n\big(1+|\xi|\big)^{(1-n)/2},$$

we have

$$\big\|m(\xi)\big\|_\infty \le C2^{-j\frac{n-1}{2}} \quad \text{and} \quad \big\|m(\xi)\big\|_\infty \le C2^{j(1-\frac{n-1}{2})}.$$

Using these elementary estimates and the facts that the functions m_j and $\tilde{m}_j$ are supported in the annuli around $|\xi| = 2^j$, we obtain that these two g-functions are L_2-bounded with norms at most a constant multiple of the quantities $2^{-j\frac{n-1}{2}}$ and $2^{j(1-\frac{n-1}{2})}$ respectively. Hence

$$\Big\|\sup_t{}^+ f_{t,j}\Big\|_2 \le C2^{j(\frac12-\frac{n-1}{2})}\|f\|_2.$$

□

Lemma 5.5. *There exists a constant $C = C(n) < \infty$ such that for all $j > 1$, we have*

$$\Big\|\sup_t{}^+ f_{t,j}\Big\|_{1,\infty} \le C2^j\|f\|_1, \quad \forall f \in L_1(\mathcal{A}).$$

More precisely, for all $\lambda > 0$, there is a projection $e \in \mathcal{P}(\mathcal{A})$ such that

$$\sup_t \|ef_{t,j}e\|_\infty \le \lambda \quad \textit{and} \quad \tau\int\big(e^\perp\big) \le C2^j\|f\|_1.$$

Proof. Let $K_j = (\varphi_j)^\vee * d\sigma = \Phi_{2^{-j}} * d\sigma$, where Φ is a Schwarz function. Setting

$$K_{j,t}(x) = t^{-n}K_j\big(t^{-1}x\big).$$

We have

$$f_{t,j} = K_{j,t} * f.$$

On page 399 of [7], it is shown that for any $M > n$, there exists $C_M < \infty$ such that

$$\left|K_j(x)\right| \leq C_M 2^j \left(1 + |x|\right)^{-M}.$$

Then we complete the proof by Lemma 5.3. □

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