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SOME RESULTS ON LOCAL HOMOLOGY AND LOCAL COHOMOLOGY MODULES

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ABSTRACT. In this paper, we obtain some results about the local homology modules of Artinian modules, and by Matlis duality we obtain some results about the local cohomology modules of finitely generated modules.

1. Introduction

Throughout this paper, $(R,\mathfrak{m})$ is a commutative Noetherian local ring, $\mathfrak{a}$ is an ideal of R. Let M be an R-module. In [3], N. T. Cuong and T. T. Nam defined the local homology modules $\mathrm{H}_i^{\mathfrak{a}}(M)$ with respect to $\mathfrak{a}$ by

$$\mathrm{H}_i^{\mathfrak{a}}(M) = \varprojlim_n \mathrm{Tor}_i^R\big(R/\mathfrak{a}^n, M\big).$$

This definition is dual to Grothendieck's definition of local cohomology modules and coincides with the definition of Greenless and May [5] when M is an Artinian R-module. For each $i \geq 0$, the ith local cohomology module of M with respect to an ideal $\mathfrak{a}$ is defined as $\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_n \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M)$. Also, the cohomological dimension of M with respect to $\mathfrak{a}$, denoted by $\mathrm{cd}(\mathfrak{a}, M)$, is defined as $\mathrm{cd}(\mathfrak{a}, M) := \sup\{i : \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}$. For basic results about local homology, we refer the reader to [3], [4] and [11]; for local cohomology refer to [2]. In this article, we obtain some results for the Artinianness of local homology modules and by Matlis duality we extend some results for the finiteness of local cohomology modules. In [3], it is shown that for an Artinian R module M,

$$\inf\big\{i \in \mathbb{N} : \mathrm{H}_i^{\mathfrak{a}}(M) \text{ is not Artinian}\big\} = \inf\Big\{i \in \mathbb{N} : \mathfrak{a} \nsubseteq \sqrt{\big(0 : \mathrm{H}_i^{\mathfrak{a}}(M)\big)}\Big\}.$$

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Also, it is well known that for a finitely generated R-module M,

$$\begin{aligned}&\inf\{i \in \mathbb{N} : \mathrm{H}^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\}\\ &= \inf\Big\{i \in \mathbb{N} : \mathfrak{a} \nsubseteq \sqrt{(0 : \mathrm{H}^i_{\mathfrak{a}}(M))}\Big\}\end{aligned}$$

(see [2, 9.1.2]).

In this paper, among other things, we generalize the above results. In fact, we show that for any Artinian R-module M we have:

$$\inf\{i \in \mathbb{N} : \mathrm{H}_i^{\mathfrak{a}}(M) \text{ is not representable}\} = \inf\Big\{i \in \mathbb{N} : \mathfrak{a} \nsubseteq \sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M))}\Big\}$$

and for any finitely generated R-module M we have:

$$\inf\{i \in \mathbb{N} : \mathrm{H}^i_{\mathfrak{a}}(M) \text{ is not good}\} = \inf\Big\{i \in \mathbb{N} : \mathfrak{a} \nsubseteq \sqrt{(0 : \mathrm{H}^i_{\mathfrak{a}}(M))}\Big\}.$$

Also we obtain a result about the top local cohomology modules of finitely generated modules. In this result, we prove that if M is a nonzero finitely generated R-module and $\mathfrak{a}$ is an ideal of R then Matlis dual of $\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}(\mathfrak{a},M)}(M)$ is not representable.

Throughout the paper, $\mathrm{D}(\cdot)$ denotes the Matlis duality functor $\mathrm{Hom}_R(\cdot, \mathrm{E}(R/\mathfrak{m}))$.

2. The results

A nonzero R-module M is called secondary if its multiplication map by any element a of R is either surjective or nilpotent. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. As the sum of the empty family of submodules of an R-module is zero, we shall regard a zero R-module as representable. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation, $M = S_1 + S_2 + \cdots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i} : i = 1, \ldots, n\}$ (see [7]).

Theorem 2.1. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$, M an R-module and i an integer. Assume that $\mathrm{H}_i^{\mathfrak{a}}(M)$ is nonzero and representable. Then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Att}_R \mathrm{H}_i^{\mathfrak{a}}(M)$.*

Proof. Let $\mathrm{H}_i^{\mathfrak{a}}(M) = S_1 + S_2 + \cdots + S_n$ be a minimal secondary representation of $\mathrm{H}_i^{\mathfrak{a}}(M)$ where S_j is $\mathfrak{p}_j$-secondary for $j = 1, 2, \ldots, n$. If $\mathfrak{a} \nsubseteq \mathfrak{p}_j$ for some $j \in \{1, \ldots, n\}$, then there exists $t \in \mathfrak{a} \setminus \mathfrak{p}_j$. But $S_j \neq 0$ and so there exists $0 \neq \alpha = (\alpha_k) \in S_j \leq \mathrm{H}_i^{\mathfrak{a}}(M) = \varprojlim_n \mathrm{Tor}_i^R(R/\mathfrak{a}^n, M)$.

Let α_k be the first nonzero component of α. Since $t \notin \mathfrak{p}_j$, we have $tS_j = S_j$. Thus $S_j = t^k S_j \subseteq t^k \mathrm{H}_i^{\mathfrak{a}}(M)$ and so $\alpha \in t^k \mathrm{H}_i^{\mathfrak{a}}(M)$. On the other hand $t^k \mathrm{Tor}_i^R(R/\mathfrak{a}^k, M) = 0$, it follows that the kth component of each element of

$t^k \mathrm{H}_i^{\mathfrak{a}}(M)$ is zero. But $\alpha \in t^k \mathrm{H}_i^{\mathfrak{a}}(M)$ and the kth component of α is nonzero, which is a contradiction. Thus $\mathfrak{a} \subseteq \mathfrak{p}_j$ for $j = 1, 2, \ldots, n$. This completes the proof. □

COROLLARY 2.2. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *an* R*-module and* i *an integer. If* $\mathrm{H}_i^{\mathfrak{a}}(M)$ *is representable, then* $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M))}$. *In particular,* $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M))}$ *if* $\mathrm{H}_i^{\mathfrak{a}}(M)$ *is Artinian.*

Proof. By [2, 7.2.11], $\sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M))} = \bigcap_{\mathfrak{p} \in \mathrm{Att}\, \mathrm{H}_i^{\mathfrak{a}}(M)} \mathfrak{p}$ and by Theorem 2.1, $\mathfrak{a} \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Att}\, \mathrm{H}_i^{\mathfrak{a}}(M)} \mathfrak{p}$. Thus $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M))}$. □

THEOREM 2.3. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *an Artinian* R*-module, and let* $t \in \mathbb{N}$. *Then the following statements are equivalent*:

(i) $\mathrm{H}_i^{\mathfrak{a}}(M)$ *is Artinian for all* $i < t$.
(ii) $\mathrm{H}_i^{\mathfrak{a}}(M)$ *is representable for all* $i < t$.
(iii) $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(M))}$ *for all* $i < t$.

Proof. (i) ⇒ (ii) by [8, Theorem 6.1], every Artinian R-module is representable.

(ii) ⇒ (iii) by Corollary 2.2.

(iii) ⇒ (i) by [3, Proposition 4.7]. □

THEOREM 2.4. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module, and let* i *be an integer. If* $\mathrm{D}(\mathrm{H}^i_{\mathfrak{a}}(M))$ *is a representable* R*-module, then* $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}^i_{\mathfrak{a}}(M))}$.

Proof. By [3, Proposition 3.3], $\mathrm{D}(\mathrm{H}^i_{\mathfrak{a}}(M)) \simeq \mathrm{H}_i^{\mathfrak{a}}(\mathrm{D}(M))$ and so our hypothesis implies that $\mathrm{H}_i^{\mathfrak{a}}(\mathrm{D}(M))$ is a representable R-module. It follows from Corollary 2.2 that $\mathfrak{a}^k \mathrm{H}_i^{\mathfrak{a}}(\mathrm{D}(M)) \simeq \mathfrak{a}^k \mathrm{D}(\mathrm{H}^i_{\mathfrak{a}}(M)) = 0$ for some $k \in \mathbb{N}$. Now the result follows by this fact that $\mathfrak{a}^k \mathrm{D}(\mathrm{H}^i_{\mathfrak{a}}(M)) = 0$ is equivalent to $\mathfrak{a}^k \mathrm{H}^i_{\mathfrak{a}}(M) = 0$. □

An R-module M is called good if its zero submodule has a primary decomposition.

THEOREM 2.5. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module, and let* $t \in \mathbb{N}$. *Then the following statements are equivalent*:

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is finitely generated for all* $i < t$.
(ii) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is good for all* $i < t$.
(iii) $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}^i_{\mathfrak{a}}(M))}$ *for all* $i < t$.

Proof. (i) ⇒ (ii). By [8, Theorem 6.8], every finitely generated R-module is a good R-module.

(ii) ⇒ (iii) Since Matlis dual of a good module is representable (see [1, Corollary 3.2]) the result follows by Theorem 2.4.

(iii) ⇒ (i) by [2, 9.1.2]. □

THEOREM 2.6. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module, and let $t\in\mathbb{N}$. If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is flat for all $i<t$, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is finitely generated for all $i<t$.*

Proof. It is well known that Matlis dual of a flat R-module is injective R-module. But by [2, 7.2.10], every injective R-module is representable and so Matlis dual of a flat R-module is representable. Hence by Theorem 2.4, $\mathfrak{a}\subseteq\sqrt{(0:\mathrm{H}^i_{\mathfrak{a}}(M))}$ for all $i<t$. Thus, Theorem 2.5 completes the proof. □

Let M be an Artinian R-module. The Notherian dimension of M, $\mathrm{Ndim}_R(M)$, is defined by induction. If $M=0$, we put $\mathrm{Ndim}_R(M)=-1$. For any integer $t\geq 0$, if $\mathrm{Ndim}_R(M)<t$ is false and whenever $M_1\subseteq M_2\subseteq\cdots$ is an ascending chain of submodules of M then there exists an integer m_0 such that $\mathrm{Ndim}_R(M_{m+1}/M_m)<t$ for all $m\geq m_0$, then we put $\mathrm{Ndim}_R(M)=t$. In case M is an Aartinian module, $\mathrm{Ndim}_R(M)<\infty$. (See [10] and [6].)

LEMMA 2.7 ([3, Corollary 4.5]). *Let M be an Artinian module and $\mathfrak{a}$ an ideal of R. Then $\mathrm{H}^{\mathfrak{a}}_i(\bigcap_{n>0}\mathfrak{a}^nM)\simeq\mathrm{H}^{\mathfrak{a}}_i(M)$ for all $i>0$ and $\mathrm{H}^{\mathfrak{a}}_0(\bigcap_{n>0}\mathfrak{a}^nM)=0$.*

THEOREM 2.8. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a nonzero Artinian R-module and t an integer. Then the following statements are equivalent:*

(i) $\mathrm{H}^{\mathfrak{a}}_i(M)$ *is Artinian for all $i>t$.*
(ii) $\mathrm{Ass}_R(\mathrm{H}^{\mathfrak{a}}_i(M))\subseteq\{\mathfrak{m}\}$ *for all $i>t$.*
(iii) $\mathrm{H}^{\mathfrak{a}}_i(M)=0$ *for all $i>t$.*
(iv) $\mathrm{H}^{\mathfrak{a}}_i(M)$ *is representable for all $i>t$.*
(v) $\mathfrak{a}\subseteq\sqrt{(0:\mathrm{H}^{\mathfrak{a}}_i(M))}$ *for all $i>t$.*

Proof. (i) ⇒ (ii): It is clear.

(ii) ⇒ (iii): We proceed by induction on $n:=\mathrm{Ndim}_R M$. Let $n=0$. Since $\mathrm{H}^{\mathfrak{a}}_i(M)=0$ for all $i>0$, by [3, Proposition 4.8], the result follows in this case. Now suppose, inductively that $n>0$ and the result is true for $n-1$. By Lemma 2.7, we can replace M by $\bigcap_{n>0}\mathfrak{a}^nM$. But $\bigcap_{n>0}\mathfrak{a}^nM=\mathfrak{a}^kM$ for some $k\in\mathbb{N}$ and so we may assume that $\mathfrak{a}M=M$. Since M is Artinian, $xM=M$ for some $x\in\mathfrak{a}$. Thus for all $i>t$, the exact sequence

$$0\to(0:_M x)\to M\xrightarrow{x}M\to 0$$

implies that

$$\to\mathrm{H}^{\mathfrak{a}}_i(0:_M x)\to\mathrm{H}^{\mathfrak{a}}_i(M)\xrightarrow{x}\mathrm{H}^{\mathfrak{a}}_i(M)\to.$$

By the above exact sequence $\mathrm{Ass}_R(\mathrm{H}^{\mathfrak{a}}_i(0:_M x))\subseteq\{\mathfrak{m}\}$ for all $i>t$. Since $\mathrm{Ndim}_R(0:_M x)\leq n-1$ (see [4, Lemma 4.7]), induction hypothesis implies that $\mathrm{H}^{\mathfrak{a}}_i(0:_M x)=0$ for all $i>t$ and so we have the injection $0\to\mathrm{H}^{\mathfrak{a}}_i(M)\xrightarrow{x}\mathrm{H}^{\mathfrak{a}}_i(M)$ for all $i>t$. If $\mathrm{H}^{\mathfrak{a}}_i(M)\neq 0$ for some $i>t$, then $\mathrm{Ass}_R(\mathrm{H}^{\mathfrak{a}}_i(M))=\{\mathfrak{m}\}$. Thus if $0\neq y\in\mathrm{H}^{\mathfrak{a}}_i(M)$, then there exists a positive integer ν such that $\mathfrak{m}^\nu y=0$. But

$x \in \mathfrak{m}$ and so $x^\nu y = 0$. Now from the above injection we conclude that $y = 0$, which is a contradiction. Hence, $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i > t$.

(iii) $\Rightarrow$ (iv): It is clear.

(iv) $\Rightarrow$ (v): By Corollary 2.2.

(v) $\Rightarrow$ (i): We proceed by induction on $n := \mathrm{Ndim}_R M$. Let $n = 0$. By [3, Proposition 4.8], $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$ and so the result follows in this case. So, let $n > 0$. As we did in the proof of (ii) $\Rightarrow$ (iii), there exists the following exact sequence

$$\to \mathrm{H}_i^{\mathfrak{a}}(0 :_M x) \to \mathrm{H}_i^{\mathfrak{a}}(M) \xrightarrow{x} \mathrm{H}_i^{\mathfrak{a}}(M) \to,$$

where $x \in \mathfrak{a}$. It follows from the above exact sequence and the hypothesis that $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}_i^{\mathfrak{a}}(0 :_M x))}$ for all $i > t$ (see [2, Lemma 9.1.1]). Since $\mathrm{Ndim}_R(0 :_M x) \leq n - 1$, we have $\mathrm{H}_i^{\mathfrak{a}}(0 :_M x)$ is Artinian for all $i > t$, by induction hypothesis and so $(0 :_{\mathrm{H}_i^{\mathfrak{a}}(M)} x)$ is Artinian for all $i > t$. On the other hand, since $x \in \mathfrak{a}$, there exists $k \in \mathbb{N}$ such that $x^k \mathrm{H}_i^{\mathfrak{a}}(M) = 0$ by hypothesis. Thus, $\mathrm{H}_i^{\mathfrak{a}}(M)$ is Rx-torsion for all $i > t$. Therefore $\mathrm{H}_i^{\mathfrak{a}}(M)$ is Artinian by [9, Theorem 1.3], for all $i > t$. This completes the proof. □

An R-module L is called cocyclic if it is a submodule of $\mathrm{E}(R/\mathfrak{m})$. A prime ideal p is called coassociated to a nonzero R-module M if there is a cocyclic homomorphic image T of M with $\mathfrak{p} = \mathrm{Ann}(T)$. The set of coassociated primes of M is denoted by $\mathrm{Coass}_R(M)$. (See [12].)

It is well known that the top local cohomology module of a nonzero finitely generated module is not finitely generated. In the remainder of this paper, we show that these modules are not good and are not flat.

Theorem 2.9. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module, and let $t \in \mathbb{N}$. Then the following statements are equivalent*:

(i) $\mathrm{H}_{\mathfrak{a}}^i(M) = 0$ *for all* $i > t$.
(ii) $\mathrm{H}_{\mathfrak{a}}^i(M)$ *is finitely generated for all* $i > t$.
(iii) $\mathrm{Coass}_R(\mathrm{H}_{\mathfrak{a}}^i(M)) \subseteq \{\mathfrak{m}\}$ *for all* $i > t$.
(iv) $\mathrm{D}(\mathrm{H}_{\mathfrak{a}}^i(M))$ *is representable for all* $i > t$.
(v) $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{H}_{\mathfrak{a}}^i(M))}$ *for all* $i > t$.

Proof. Clearly, we can assume that $M \neq 0$.

(i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): It is clear by [12, 2.10].

(iii) $\Rightarrow$ (iv): By [12, 1.7], $\mathrm{Coass}_R(\mathrm{H}_{\mathfrak{a}}^i(M)) = \mathrm{Ass}_R \mathrm{D}(\mathrm{H}_{\mathfrak{a}}^i(M))$. Thus, $\mathrm{Ass}_R(\mathrm{H}_i^{\mathfrak{a}}(\mathrm{D}(M))) \subseteq \{\mathfrak{m}\}$ for all $i > t$. Hence by Theorem 2.8, $\mathrm{H}_i^{\mathfrak{a}}(\mathrm{D}(M)) \simeq \mathrm{D}(\mathrm{H}_{\mathfrak{a}}^i(M))$ is representable for all $i > t$.

(iv) $\Rightarrow$ (v): By Theorem 2.4.

(v) $\Rightarrow$ (i): Hypothesis implies that $\mathfrak{a} \subseteq \sqrt{(0 : \mathrm{D}(\mathrm{H}_{\mathfrak{a}}^i(M)))}$ for all $i > t$. But $\mathrm{D}(\mathrm{H}_{\mathfrak{a}}^i(M)) \simeq \mathrm{H}_i^{\mathfrak{a}}(\mathrm{D}(M))$ and so by Theorem 2.8, $\mathrm{H}_i^{\mathfrak{a}}(\mathrm{D}(M)) = 0$ for all $i > t$.

Thus $\mathrm{D}(\mathrm{H}^i_{\mathfrak{a}}(M)) = 0$ for all $i > t$. Therefore, we conclude that $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i > t$. □

Corollary 2.10. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module, and let* $t \in \mathbb{N}$. *Then the following statements are equivalent*:

(i) $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ *for all* $i > t$.
(ii) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is good for all* $i > t$.
(iii) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is flat for all* $i > t$.

Proof. Matlis dual of a good R-module or a flat R-module is representable and so the result follows by Theorem 2.9. □

Corollary 2.11. *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a nonzero finitely generated* R*-module. Then* $\mathrm{D}(\mathrm{H}^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M))$ *is not representable. Therefore* $\mathrm{H}^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$ *is not good and is not flat.*

Proof. Since $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i > \mathrm{cd}(\mathfrak{a}, M)$ and $\mathrm{H}^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M) \neq 0$, the result follows by Theorem 2.9 and Corollary 2.10. □

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