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RELEVANT SAMPLING OF BAND-LIMITED FUNCTIONS

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Abstract. We study the random sampling of band-limited functions of several variables. If a band-limited function with bandwidth has its essential support on a cube of volume R^d, then $\mathcal{O}(R^d \log R^d)$ random samples suffice to approximate the function up to a given error with high probability.

1. Introduction

The nonuniform sampling of band-limited functions of several variables remains a challenging problem. Whereas in dimension 1 the density of a set essentially characterizes sets of stable sampling [14], in higher dimensions the density is no longer a decisive property of sets of stable sampling. Only a few strong and explicit sufficient conditions are known, for example, [3], [10], [12].

This difficulty is one of the reasons for taking a probabilistic approach to the sampling problem [2], [20]. At first glance, one would guess that every reasonably homogeneous set of points in $\mathbb{R}^d$ satisfying Landau's necessary density condition will generate a set of stable sampling. This intuition is far from true. To the best of our knowledge, every construction in the literature of sets of random points in $\mathbb{R}^d$ contains either arbitrarily large holes with positive probability or concentrates near the zero manifold of a band-limited function. Both properties are incompatible with a sampling inequality. See [2] for a detailed discussion.

The difficulties with the probabilistic approach lie in the unboundedness of the configuration space $\mathbb{R}^d$ and the infinite dimensionality of the space of band-limited functions. To resolve this issue, we argued in [2] that usually one observes only finitely many samples of a band-limited function and that these

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observations are drawn from a bounded subset of $\mathbb{R}^d$. Moreover, since it does not make sense to sample a given function f in a region where f is small, we proposed to sample f only on its essential support. Since f is sampled only in the relevant region, this method might be called the "relevant sampling of band-limited functions." In this paper, we continue our investigation of the random sampling of band-limited functions and settle a question that was left open in [2], namely how many random samples are required to approximate a band-limited function locally to within a given accuracy?

To fix terms, recall that the space of band-limited functions is defined to be

$$\mathcal{B} = \{ f \in L^2(\mathbb{R}^d) : \operatorname{supp} \hat{f} \subseteq [-1/2, 1/2]^d \},$$

where we have normalized the spectrum to be the unit cube and the Fourier transform is normalized as $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx$. A set $\{x_j : j \in J\} \subseteq \mathbb{R}^d$ is called a set of stable sampling or simply a set of sampling [7], if there exist constants $A, B > 0$, such that a *sampling inequality* holds:

$$A \|f\|_2^2 \le \sum_j |f(x_j)|^2 \le B \|f\|_2^2, \quad \forall f \in \mathcal{B}. \tag{1}$$

Next, we sample only on the essential support of f. Therefore, we let $C_R = [-R/2, R/2]^d$ and define the subset

$$\mathcal{B}(R, \delta) = \left\{ f \in \mathcal{B} : \int_{C_R} |f(x)|^2 \, dx \ge (1 - \delta) \|f\|_2^2 \right\}.$$

As a continuation of [2], we will prove the following sampling theorem.

THEOREM 1. *Let $\{x_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables that are uniformly distributed in C_R. Suppose that $R \ge 2$, that $\delta \in (0, 1)$ and $\nu \in (0, 1/2)$ are small enough, and that $0 < \varepsilon < 1$. There exists a constant κ so that if the number of samples r satisfies*

$$r \ge 2R^d \frac{1 + \nu/3}{\nu^2} \log \frac{2R^d}{\varepsilon}, \tag{2}$$

then the sampling inequality

$$\begin{aligned} \frac{r}{R^d} \left(\frac{1}{2} - \delta - \nu - 12\delta\kappa \right) \|f\|_2^2 \\ \le \sum_{j=1}^r |f(x_j)|^2 \le r \|f\|_2^2 \quad \textit{for all } f \in \mathcal{B}(R, \delta) \end{aligned} \tag{3}$$

holds with probability at least $1 - \varepsilon$. The constant κ can be taken to be $\kappa = e^{d\pi}$.

The formulation of Theorem 1 is similar to [2, Theorem 3.1]. The main point is that only $\mathcal{O}(R^d \log R^d)$ samples are required for a sampling inequality to hold with high probability. In [2], we used a metric entropy argument to show that $\mathcal{O}(R^{2d})$ samples suffice. We expect that the order $\mathcal{O}(R^d \log R^d)$ is optimal. We point out that in addition all constants are now explicit.

Our idea is to replace the sampling of band-limited function in $\mathcal{B}(R,\delta)$ by a finite-dimensional problem, namely the sampling of the corresponding span of prolate spheroidal functions on the cube $[-R/2, R/2]^d$ and then use error estimates. For the probability estimates we use a new tool, namely the powerful matrix Bernstein inequality of Ahlswede and Winter [1] in the optimized version of Tropp [22].

The remainder of the paper contains the analysis of a related finite-dimensional problem for prolate spheroidal functions in Section 2 and transition to the infinite-dimensional problem in $\mathcal{B}(R,\delta)$ with the necessary error estimates in Section 3. The Appendix contains an elementary estimate for the constant κ.

2. Finite-dimensional subspaces of $\mathcal{B}$

We first study a sampling problem in a finite-dimensional subspace related to the set $\mathcal{B}(R,\delta)$.

Prolate spheroidal functions. Let P_R and Q be the projection operators defined by

$$P_R f = \chi_{C_R} f \quad \text{and} \quad Qf = \mathcal{F}^{-1}(\chi_{[-1/2,1/2]^d} \hat{f}), \tag{4}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. The composition of these orthogonal projections

$$A_R = QP_RQ \tag{5}$$

is the operator of time and frequency limiting. This operator arises frequently in the context of band-limited functions and uncertainty principles. The localization operator A_R is a compact positive operator of trace class, and by results of Landau, Slepian, Pollak and Widom [8], [9], [17], [18], [19], [24] the eigenvalue distribution spectrum is precisely known. We summarize the properties of the spectrum that we will need.

Let $A_R^{(1)}$ denote the operator of time-frequency limiting in dimension $d=1$. This operator can be defined explicitly on $L^2(\mathbb{R})$ by the formula

$$\big(A_R^{(1)} f\big)\hat{}\,(\xi) = \int_{-1/2}^{1/2} \frac{\sin \pi R(\xi-\eta)}{\pi(\xi-\eta)} \hat{f}(\eta)\, d\eta \quad \text{for } |\xi| \le 1/2.$$

The eigenfunctions of $A_R^{(1)}$ are the prolate spheroidal functions, and let the corresponding eigenvalues $\mu_k = \mu_k(R)$ be arranged in decreasing order. According to [6], they satisfy

$$0 < \mu_k(R) < 1 \quad \forall k \in \mathbb{N},$$
$$\mu_{[R]+1}(R) \le 1/2 \le \mu_{[R]-1}(R).$$

As a consequence any function with spectrum $[-1/2,1/2]$ and "essential" support on $[-R/2,R/2]$ is close to the span of the first R prolate spheroidal functions. In particular, we may think of $\mathcal{B}(R,\delta)$ as, roughly, almost a subset of a finite-dimensional space of dimension R.

The time-frequency limiting operator A_R on $L^2(\mathbb{R}^d)$ is the d-fold tensor product of $A_R^{(1)}$, $A_R = A_R^{(1)} \otimes \cdots \otimes A_R^{(1)}$. Consequently, $\sigma(A_R)$, the spectrum of A_R, is

$$\sigma(A_R) = \left\{ \lambda \in (0,1) : \lambda = \prod_{j=1}^{d} \mu_{k_j}, \mu_{k_j} \in \sigma\big(A_R^{(1)}\big) \right\}.$$

Since $0 < \mu_k < 1$, A_R possesses at most R^d eigenvalues greater than or equal to $1/2$. Again we arrange the eigenvalues of A_R by magnitude $1 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} \geq \cdots > 0$. Let ϕ_j be the eigenfunction corresponding to λ_j.

We fix R "large" and $\delta \in (0,1)$. Let

$$\mathcal{P}_N = \operatorname{span}\{\phi_j : j = 1, \ldots, N\}$$

be the span of the first N eigenfunctions of the time-frequency limiting operator A_R (one might call functions in $\mathcal{P}_N$ "multivariate prolate polynomials"). For properly chosen N, $\mathcal{P}_N$ consists of functions in $\mathcal{B}(R,\delta)$. See Lemma 5.

By Plancherel's theorem,

$$\langle Qf, g\rangle = \langle \chi_{[-1/2,1/2]^d} \hat{f}, \hat{g}\rangle = \langle \hat{f}, \chi_{[-1/2,1/2]^d} \hat{g}\rangle = \langle f, Qg\rangle.$$

Then for $f \in \mathcal{B}$ we have $Qf = f$, and so

$$\langle A_R f, f\rangle = \langle P_R Q f, Q f\rangle = \langle P_R f, f\rangle = \int_{C_R} \big|f(x)\big|^2 \, dx. \tag{6}$$

We first study random sampling in the finite-dimensional space $\mathcal{P}_N$. In the following $\|f\|_{2,R}$ denotes the normalized L^2-norm of f restricted to the cube $C_R = [-R/2, R/2]^d$:

$$\|f\|_{2,R}^2 = \int_{C_R} \big|f(x)\big|^2 \, dx.$$

PROPOSITION 2. *Let $\{x_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables that are uniformly distributed in $[-R/2, R/2]^d$. Then*

$$\mathbb{P}\left(\inf_{f \in \mathcal{P}_N, \|f\|_2 = 1} \frac{1}{r} \sum_{j=1}^{r} \left(\big|f(x_j)\big|^2 - \frac{1}{R^d} \|f\|_{2,R}^2 \right) \leq -\frac{\nu}{R^d} \right) \leq N \exp\left(-\frac{\nu^2 r/2}{R^d(1+\nu/3)} \right) \tag{7}$$

for $r \in \mathbb{N}$ and $\nu \geq 0$.

Proof. We prove the proposition in several steps. First, since $\mathcal{P}_N$ is finite-dimensional, the sampling inequality for $\mathcal{P}_N$ amounts to a statement about the spectrum of an underlying (random) matrix.

Let $f = \langle c, \phi\rangle = \sum_{k=1}^N c_k\phi_k \in \mathcal{P}_N$, so that $|f(x_j)|^2 = \sum_{k,l=1}^N c_k\overline{c_l}\phi_k(x_j) \times \overline{\phi_l(x_j)}$. Now define the $N \times N$ matrix T_j of rank one by letting the (k,l) entry be

$$(T_j)_{kl} = \phi_k(x_j)\overline{\phi_l(x_j)}. \tag{8}$$

Then $|f(x_j)|^2 = \langle c, T_j c\rangle$. Since each random variable x_j is uniformly distributed over C_R and ϕ_k is the kth eigenfunction of the localization operator A_R, using (6) the expectation of the klth entry is

$$\begin{aligned} \mathbb{E}\big((T_j)_{kl}\big) &= \frac{1}{R^d}\int_{C_R} \phi_k(x)\overline{\phi_l(x)}\,dx \\ &= \frac{1}{R^d}\langle A_R\phi_k, \phi_l\rangle \\ &= \frac{1}{R^d}\lambda_k\delta_{kl}, \quad k,l = 1,\dots,N, \end{aligned} \tag{9}$$

where δ_{kl} is Kronecker's delta. Consequently the expectation of T_j is the diagonal matrix

$$\mathbb{E}(T_j) = \frac{1}{R^d}\operatorname{diag}(\lambda_k) =: \frac{1}{R^d}\Delta. \tag{10}$$

We may now rewrite the expression in (7) as

$$\begin{aligned} &\inf_{f\in\mathcal{P}_N, \|f\|_2=1} \frac{1}{r}\sum_{j=1}^r \left(\big|f(x_j)\big|^2 - \frac{1}{R^d}\|f\|_{2,R}^2 \right) \\ &= \inf_{\|c\|_2=1} \frac{1}{r}\sum_{j=1}^r \big(\langle c, T_j c\rangle - \big\langle c, \mathbb{E}(T_j)c\big\rangle\big) \\ &= \lambda_{\min}\left(\frac{1}{r}\sum_{j=1}^N \big(T_j - \mathbb{E}(T_j)\big) \right), \end{aligned} \tag{11}$$

where we use $\lambda_{\min}(U)$ for the smallest eigenvalue of a self-adjoint matrix U.

Consequently, we have to estimate a probability for the matrix norm of a sum of random matrices. We do this using a matrix Bernstein inequality due to Tropp [22]. Let $\lambda_{\max}(A)$ be the largest singular value of a matrix A so that $\|A\| = \lambda_{\max}(A^*A)^{1/2}$ is the operator norm (with respect to the ℓ^2-norm).

THEOREM 3 (Tropp). *Let X_j be a sequence of independent, random self-adjoint $N \times N$-matrices. Suppose that*

$$\mathbb{E}X_j = 0 \quad \textit{and} \quad \|X_j\| \le B \quad \textit{a.s.}$$

and let

$$\sigma^2 = \left\| \sum_{j=1}^{r} \mathbb{E}\big(X_j^2\big) \right\|.$$

Then for all $t \geq 0$,

$$\mathbb{P}\left(\lambda_{\max}\left(\sum_{j=1}^{r} X_j\right) \geq t\right) \leq N \exp\left(-\frac{t^2/2}{\sigma^2 + Bt/3}\right). \tag{12}$$

To apply the matrix Bernstein inequality, we set $X_j = T_j - \mathbb{E}(T_j)$. We need to calculate $\|X_j\|$ and $\|\sum_j \mathbb{E}(X_j^2)\|$. Clearly $\mathbb{E}(X_j) = 0$.

LEMMA 4. *Under the conditions stated above, we have*

$$\|X_j\| \leq 1,$$
$$\mathbb{E}\big(X_j^2\big) \leq R^{-d}\Delta$$

and

$$\sigma^2 = \left\| \sum_{j=1}^{r} \mathbb{E}\big(X_j^2\big) \right\| \leq \frac{r}{R^d}.$$

Proof. (i) To estimate the matrix norm of X_j, recall that

$$\big|f(x)\big| \leq \|f\|_2 \quad \forall f \in \mathcal{B}. \tag{13}$$

Hence, we obtain

$$\begin{aligned} \|X_j\| &= \sup_{\|f\|_2=1} \Big| \big|f(x_j)\big|^2 - R^{-d}\|f\|_{2,R}^2 \Big| \\ &\leq \|f\|_\infty - R^{-d}\|f\|_{2,R}^2 \leq \|f\|_2 = 1. \end{aligned}$$

(ii) Next, we calculate the matrix $\mathbb{E}(X_j^2)$:

$$\begin{aligned} \mathbb{E}\big(X_j^2\big) &= \mathbb{E}\big(T_j^2\big) - R^{-d}\mathbb{E}(T_j\Delta) - R^{-d}\mathbb{E}(\Delta T_j) + R^{-2d}\Delta^2 \\ &= \mathbb{E}\big(T_j^2\big) - R^{-d}\mathbb{E}(T_j)\Delta - R^{-d}\Delta\mathbb{E}(T_j) + R^{-2d}\Delta^2 \\ &= \mathbb{E}\big(T_j^2\big) - R^{-2d}\Delta^2. \end{aligned}$$

Furthermore, the square of the rank one matrix T_j is the (rank one) matrix

$$\begin{aligned} \big(T_j^2\big)_{km} &= \sum_{l=1}^{N} (T_j)_{kl}(T_j)_{lm} \\ &= \sum_{l} \phi_k(x_j)\overline{\phi_l(x_j)}\phi_l(x_j)\overline{\phi_m(x_j)} \\ &= \left(\sum_{l=1}^{N} \big|\phi_l(x_j)\big|^2\right)(T_j)_{km}. \end{aligned}$$

Writing $m(x)=\sum_{l=1}^N |\phi_l(x)|^2$, we obtain

$$T_j^2 = m(x_j)T_j. \tag{14}$$

Let s be the function whose Fourier transform is given by $\hat{s}=\chi_{[-1/2,1/2]^d}$ and let $T_x f(t)=f(t-x)$ be the translation operator. Then it is well known that $T_x s$ is the reproducing kernel for $\mathcal{B}$, that is,

$$f(x)=\langle f, T_x s\rangle.$$

To see this, by Plancherel's theorem and the inversion formula for the Fourier transform, if $f\in\mathcal{B}$,

$$\begin{aligned}\langle f, T_x s\rangle &= \langle \hat{f}, e^{-2\pi i x\cdot\xi}\hat{s}\rangle = \int_{[-1/2,1/2]^d} e^{2\pi i x\cdot\xi}\hat{f}(\xi)\,d\xi \\ &= \int e^{2\pi i x\cdot\xi}\hat{f}(\xi)\,d\xi = f(x).\end{aligned}$$

Since the eigenfunctions ϕ_l form an orthonormal basis for $\mathcal{B}$, the factor $m(x_j)$ in (14) is majorized by

$$\begin{aligned} m(x_j) &= \sum_{l=1}^N \bigl|\phi_l(x_j)\bigr|^2 = \sum_{l=1}^N \bigl|\langle \phi_l, T_{x_j}s\rangle\bigr|^2 \\ &\le \sum_{l=1}^\infty \bigl|\langle \phi_l, T_{x_j}s\rangle\bigr|^2 = \|T_{x_j}s\|_2^2 = 1.\end{aligned}$$

Since $T_j^2\le T_j$ and the expectation preserves the cone of positive (semi)definite matrices (see, e.g., [22]), we have $\mathbb{E}(T_j^2)\le\mathbb{E}(T_j)=R^{-d}\Delta$, and

$$\mathbb{E}\bigl(X_j^2\bigr)=\mathbb{E}\bigl(T_j^2\bigr)-R^{-2d}\Delta^2\le R^{-d}\Delta.$$

(iii) Now the variance of the sum of positive (semi)definite random matrices is majorized by

$$\sigma^2=\left\|\sum_{j=1}^r \mathbb{E}\bigl(X_j^2\bigr)\right\|\le\left\|\sum_{j=1}^r \mathbb{E}(T_j)\right\|=\frac{r}{R^d}\|\Delta\|\le\frac{r}{R^d}.$$ □

End of the proof of Proposition 2. Now we have all information to finish the proof of Proposition 2. Since $\lambda_{\min}(T)=-\lambda_{\max}(-T)$, we substitute these estimates into the matrix Bernstein inequality with $t=r\nu/R^d$, and obtain that

$$\mathbb{E}\left(\lambda_{\min}\left(\sum_{j=1}^r\bigl(T_j-\mathbb{E}(T_j)\bigr)\right)\le -r\nu/R^d\right)\le N\exp\left(-\frac{r^2\nu^2R^{-2d}/2}{rR^{-d}+r\nu R^{-d}/3}\right).$$

Combined with (11), the proposition is proved. □

Random matrix theory offers several methods to obtain probability estimates for the spectrum of random matrices. In [2], we used the entropy method. We also mention the influential work of Rudelson [15] and the recent papers [11], [16] on random matrices with independent columns. The matrix Bernstein inequality offers a new approach and makes the probabilistic part of the argument almost painless. The matrix Bernstein inequality was first derived in [1] and improved in several subsequent papers, in particular in [13]. The version with the best constants is due to Tropp [22]. Matrix Bernstein inequalities also simplify many probabilistic arguments in compressed sensing; see [4] and [23].

3. From sampling of prolate spheroidal functions to relevant sampling of bandlimited functions

Let α be the value of the Nth eigenvalue of A_R, that is, $\alpha = \lambda_N$, let $E = E_N$ be the orthogonal projections from $\mathcal{B}$ onto $\mathcal{P}_N$, and let $F = F_N = \mathrm{I} - E_N$. Intuitively, since $f \in \mathcal{B}(R, \delta)$ is essentially supported on the cube C_R, it should be close to the span of the largest eigenfunctions of A_R and thus Ff should be small. The following lemma gives a precise estimate. Compare also with the proof of [9, Theorem 3].

Lemma 5. *If $f \in \mathcal{B}(R, \delta)$, then*

$$\|Ef\|_2^2 \geq \left(1 - \frac{\delta}{1-\alpha}\right) \|f\|_2^2,$$

$$\|Ef\|_{2,R}^2 \geq \alpha \left(1 - \frac{\delta}{1-\alpha}\right) \|f\|_2^2,$$

$$\|Ff\|_2^2 \leq \frac{\delta}{1-\alpha} \|f\|_2^2.$$

Proof. Expand $f \in \mathcal{B}$ with respect to the prolate spheroidal functions as $f = \sum_{j=1}^{\infty} c_j \phi_j$. Without loss of generality, we may assume that $\|f\|_2 = \|c\|_2 = 1$. Since $f \in \mathcal{B}(R, \delta)$, we have that

$$1 - \delta \leq \|f\|_{2,R}^2 = \int_{C_R} \left|f(t)\right|^2 dt = \langle A_R f, f \rangle = \sum_{j=1}^{\infty} |c_j|^2 \lambda_j.$$

Set

$$A = \|Ef\|_2^2 = \sum_{j=1}^{N} |c_j|^2$$

and $B = \sum_{j>N} |c_j|^2 = 1 - A = \|Ff\|_2^2$. Since $\lambda_j \le \lambda_N = \alpha$ for $j > N$, we estimate $A = \|Ef\|_2^2$ as follows:

$$\begin{aligned} A &= \sum_{j=1}^{N} |c_j|^2 \ge \sum_{j=1}^{N} |c_j|^2 \lambda_j \\ &= \sum_{j=1}^{\infty} |c_j|^2 \lambda_j - \sum_{j=N+1}^{\infty} |c_j|^2 \lambda_j \\ &\ge 1 - \delta - \lambda_N \sum_{j=N+1}^{\infty} |c_j|^2 \\ &= 1 - \delta - \alpha(1 - A). \end{aligned}$$

The inequality $A \ge 1 - \delta - \alpha(1 - A)$ implies that $\|Ef\|_2^2 = A \ge 1 - \frac{\delta}{1-\alpha}$ and using the orthogonal decomposition $f = Ef + Ff$,

$$B = \|Ff\|_2^2 \le \frac{\delta}{1 - \alpha}.$$

Finally, $\|Ef\|_{2,R}^2 = \sum_{j=1}^{N} \lambda_j |c_j|^2 \ge \alpha A \ge \alpha(1 - \frac{\delta}{1-\alpha})$, as claimed. □

REMARK (due to J.-L. Romero). As mentioned in [2], if $f \in \mathcal{B}(R, \delta)$ and $f(x_j) = 0$ for sufficiently many samples $x_j \in C_R$, then $f \equiv 0$. However, f cannot be completely determined by samples in C_R alone. This is a consequence of the fact that $\mathcal{B}(R, \delta)$ is not a linear space. Given a finite subset $S \subseteq C_R$, consider the finite-dimensional subspace $\mathcal{H}_0$ of $\mathcal{B}$ spanned by the reproducing kernels $T_x s$, $x \in S$. If $\phi \in \mathcal{H}_0^\perp$, then $\phi(x) = \langle \phi, T_x s \rangle = 0$ for $x \in S$. Thus by adding a function in $\mathcal{H}_0^\perp$ of sufficiently small norm to $f \in \mathcal{B}(R, \delta)$, one obtains a different function with the same samples. More precisely, let $f \in \mathcal{B}(R, \delta)$ with $\|f\|_2 = 1$ and $\int_{C_R} |f(x)|^2 \, dx = \gamma > 1 - \delta$ and $\phi \in \mathcal{H}_0^\perp$ with $\|\phi\|_2 = 1$. Then $f(x) + \varepsilon\phi(x) = f(x)$ for $x \in S$ and $f + \varepsilon\phi \in \mathcal{B}(R, \delta)$ for sufficiently small $\varepsilon > 0$.

Despite this non-uniqueness, one can approximate f from the samples up to an accuracy δ, as is shown by the next lemma.

We will require a standard estimate for sampled 2-norms, a so-called Plancherel–Polya–Nikolskij inequality [21]. Assume that $\mathcal{X} = \{x_j\} \subseteq \mathbb{R}^d$ is relatively separated, that is, the "covering index"

$$\max_{k \in \mathbb{Z}^d} \#\mathcal{X} \cap \left(k + [-1/2, 1/2]^d\right) =: N_0 < \infty$$

is finite. Then there exists a constant $\kappa > 0$, such that

$$\sum_{j=1}^{\infty} \left| f(x_j) \right|^2 \le \kappa N_0 \|f\|_2^2 \quad \text{for all } f \in \mathcal{B}. \tag{15}$$

The constant κ can be chosen as $\kappa = e^{d\pi}$. Since the standard proof in [21] uses a maximal inequality with an non-explicit constant, we will give a simple argument using Taylor series in the Appendix.

LEMMA 6. *Let $\{x_j : j = 1, \dots, r\}$ be a finite subset of C_R with covering index N_0. Then the solution to the least square problem*

$$p_{\mathrm{opt}} = \arg\min_{p \in \mathcal{P}_N} \left\{ \sum_{j=1}^{r} \left| f(x_j) - p(x_j) \right|^2 \right\} \tag{16}$$

satisfies the error estimate

$$\sum_{j=1}^{r} \left| f(x_j) - p_{\mathrm{opt}}(x_j) \right|^2 \le N_0 \kappa \frac{\delta}{1-\alpha} \|f\|_2^2 \quad \text{for all } f \in \mathcal{B}(R, \delta). \tag{17}$$

Proof. We combine Lemma 5 with (15).

$$\begin{aligned} \sum_{j=1}^{r} \left| f(x_j) - p_{\mathrm{opt}}(x_j) \right|^2 &\le \sum_{j=1}^{r} \left| f(x_j) - Ef(x_j) \right|^2 \\ &= \sum_{j=1}^{r} \left| Ff(x_j) \right|^2 \le \kappa N_0 \|Ff\|_2^2 \\ &\le \kappa N_0 \frac{\delta}{1-\alpha} \|f\|_2^2. \end{aligned}$$

□

Next, we compare sampling inequalities for the space of prolate polynomials $\mathcal{P}_N$ to sampling inequalities for functions in $\mathcal{B}(R, \delta)$.

LEMMA 7. *Let $\{x_j : j = 1, \dots, r\}$ be a finite subset of C_R with covering index N_0.*

If the inequality

$$\frac{1}{r} \sum_{j=1}^{r} \left(\left| p(x_j) \right|^2 - R^{-d} \|p\|_{2,R}^2 \right) \ge -\frac{\nu}{R^d} \|p\|_2^2 \tag{18}$$

holds for all $p \in \mathcal{P}_N$, then the inequality

$$\sum_{j=1}^{r} \left| f(x_j) \right|^2 \ge A \|f\|_2^2 \tag{19}$$

holds for all $f \in \mathcal{B}(R, \delta)$ with a constant

$$A = \frac{r}{R^d} \left(\alpha - \frac{\alpha\delta}{1-\alpha} - \nu \right) - 2\kappa N_0 \frac{\delta}{1-\alpha}.$$

REMARK. For A to be positive, we need

$$r \ge R^d \frac{2\kappa N_0 \frac{\delta}{1-\alpha}}{\alpha - \frac{\alpha\delta}{1-\alpha} - \nu}.$$

Proof of Lemma 7. Using the triangle inequality and the orthogonal decomposition $f = Ef + Ff$, we estimate

$$\left(\sum_{j=1}^{r}\left|f(x_j)\right|^2\right)^{1/2} \geq \left(\sum_{j=1}^{r}\left|Ef(x_j)\right|^2\right)^{1/2} - \left(\sum_{j=1}^{r}\left|Ff(x_j)\right|^2\right)^{1/2}.$$

Taking squares and using (15) on Ef and Ff in the cross product term, we continue as

$$\begin{aligned}\sum_{j=1}^{r}\left|f(x_j)\right|^2 &\geq \sum_{j=1}^{r}\left|Ef(x_j)\right|^2 - 2\left(\sum_{j=1}^{r}\left|Ef(x_j)\right|^2\right)^{1/2}\left(\sum_{j=1}^{r}\left|Ff(x_j)\right|^2\right)^{1/2} \\ &\quad + \sum_{j=1}^{r}\left|Ff(x_j)\right|^2 \\ &\geq \sum_{j=1}^{r}\left|Ef(x_j)\right|^2 - 2\kappa N_0\|Ef\|_2\|Ff\|_2 \\ &\geq \sum_{j=1}^{r}\left|Ef(x_j)\right|^2 - 2\kappa N_0\frac{\delta}{1-\alpha}\|f\|_2^2,\end{aligned}$$

since by Lemma 5, $\|Ff\|_2^2 \leq \frac{\delta}{1-\alpha}\|f\|_2^2$ and $\|Ef\|_2 \leq \|f\|_2$. Now we make use of hypothesis (18) and Lemma 5 and obtain

$$\begin{aligned}\sum_{j=1}^{r}\left|f(x_j)\right|^2 &\geq \sum_{j=1}^{r}\left|Ef(x_j)\right|^2 - 2\kappa N_0\frac{\delta}{1-\alpha}\|f\|_2^2 \\ &\geq \frac{r}{R^d}\|Ef\|_{2,R}^2 - \frac{\nu r}{R^d}\|Ef\|_2^2 - 2\kappa N_0\frac{\delta}{1-\alpha}\|f\|_2^2 \\ &\geq \frac{\alpha r}{R^d}\left(1 - \frac{\delta}{1-\alpha}\right)\|f\|_2^2 - \frac{\nu r}{R^d}\|f\|_2^2 - 2\kappa N_0\frac{\delta}{1-\alpha}\|f\|_2^2.\end{aligned}$$

So we may choose A to be

$$A = \frac{r}{R^d}\left(\alpha - \frac{\alpha\delta}{1-\alpha} - \nu\right) - 2\kappa N_0\frac{\delta}{1-\alpha}. \qquad \square$$

The final ingredient we need is a deviation inequality for the covering index $N_0 = \max_{k\in\mathbb{Z}^d}\{x_j\} \cap (k + [-1/2, 1/2]^d)$.

LEMMA 8. *Suppose $R \geq 2$ and $\{x_j : j = 1, \ldots, r\}$ are independent and identically distributed random variables that are uniformly distributed over C_R. Let $a > R^{-d}$. Then*

$$\mathbb{P}(N_0 > ar) \leq (R+2)^d \exp\bigl(-r\bigl(a\log\bigl(aR^d\bigr) - \bigl(a - R^{-d}\bigr)\bigr)\bigr).$$

Proof. Let $D_k = k + [-1/2, 1/2]^d$ for $k \in \mathbb{Z}^d$. Note that we need at most $(R+2)^d$ of the D_k's to cover C_R. If $N_0 > ar$, then for at least one k, D_k must contain at least ar of the x_j's. Therefore,

$$\mathbb{P}(N_0 > ar) \le (R+2)^d \max_{k \in \mathbb{Z}^d} \mathbb{P}\big(\#\{x_j\} \cap D_k > ar\big). \tag{20}$$

Fix $k \in \mathbb{Z}^d$. For any $b > 0$, by Chebyshev's inequality

$$\begin{aligned}\mathbb{P}\big(\#\{x_j\} \cap D_k > ar\big) &= \mathbb{P}\left(\sum_{j=1}^{r} \chi_{D_k}(x_j) > ar\right)\\ &= \mathbb{P}\left(\exp\left(b\sum_{j=1}^{r} \chi_{D_k}(x_j)\right) > e^{bar}\right)\\ &\le e^{-bar}\mathbb{E}\exp\left(b\sum_{j=1}^{r} \chi_{D_k}(x_j)\right).\end{aligned}$$

Since the x_j are uniformly distributed over C_R, then $\chi_{D_k}(x_j)$ is equal to 1 with probability at most R^{-d} and otherwise equals zero. Therefore, using the independence,

$$\begin{aligned}\mathbb{P}\big(\#\{x_j\} \cap D_k > ar\big) &\le e^{-bar} \prod_{j=1}^{r} \mathbb{E}e^{b\chi_{D_k}(x_j)}\\ &\le e^{-bar}\big(\big(1 - R^{-d}\big) + e^b R^{-d}\big)^r\\ &= e^{-bar}\big(\big(1 + \big(e^b - 1\big)R^{-d}\big)^r\\ &\le e^{-bar}\big(\exp\big(\big(e^b - 1\big)R^{-d}\big)\big)^r.\end{aligned}$$

With the optimal choice $b = \log(aR^d)$ the last term is then

$$\exp\Big(-r\big(a\log(aR^d) - (a - R^{-d})\big)\Big).$$

Substituting this in (20) proves the lemma. □

By combining the finite-dimensional result of Proposition 2 with the estimates of Lemmas 7 and 8 and the appropriate choice of the free parameters, we obtain the following theorem.

THEOREM 9. *Let $\{x_j : j \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables that are uniformly distributed in C_R. Suppose $R \ge 2$,*

$$\delta < \frac{1}{2(1+12\kappa)}$$

and

$$\nu < \frac{1}{2} - \delta(1+12\kappa),$$

where $\kappa = e^{d\pi}$. *Let*

$$A = \frac{r}{R^d}\left(\frac{1}{2} - \delta - \nu - 12\delta\kappa\right). \tag{21}$$

Then the sampling inequality

$$A\|f\|_2^2 \le \sum_{j=1}^{r} \left|f(x_j)\right|^2 \le r\|f\|_2^2 \quad \textit{for all } f \in \mathcal{B}(R,\delta) \tag{22}$$

holds with probability at least

$$1 - R^d \exp\left(-\frac{\nu^2 r/2}{R^d(1+\nu/3)}\right) - (R+2)^d \exp\left(-\frac{r}{R^d}(3\log 3 - 2)\right). \tag{23}$$

Proof. Since $|f(x)| \le \|f\|_2$ for $f \in \mathcal{B}$, the right-hand inequality in (22) is immediate. We take $\alpha = 1/2$ and $N = R^d$ in Proposition 2 and $a = 3R^{-d}$ in Lemma 8. Let

$$V_1 = \left\{ \inf_{f \in \mathcal{P}_N, \|f\|_2 = 1} \frac{1}{r} \sum_{j=1}^{r} \left(\left|f(x_j)\right|^2 - \frac{1}{R^d}\|f\|_{2,R}^2 \right) \le -\frac{\nu}{R^d} \right\}$$

and let

$$V_2 = \{N_0 > ar\}.$$

By Proposition 2 and Lemma 8, the probability of $(V_1 \cup V_2)^c$ is bounded below by (23). By Lemma 7,

$$\frac{1}{r}\sum_{j=1}^{r} \left|f(x_j)\right|^2 \ge A\|f\|_2^2$$

for all $f \in \mathcal{B}(R,\delta)$ on the set $(V_1 \cup V_2)^c$. With $\alpha = 1/2$ and $N_0 = 3R^{-d}r$ the lower bound A of Lemma 7 simplifies to $A = \frac{r}{R^d}(\frac{1}{2} - \delta - \nu - 12\delta\kappa)$. Our assumptions on δ and ν guarantee that $A > 0$. □

The formulation of Theorem 1 now follows. With $N = R^d$ and $0 < \nu < 1/2 - \delta < 1/2$, if $\varepsilon > 0$ is given and

$$\begin{aligned} r &\ge \max\left(R^d \frac{1+\nu/3}{\nu^2} \log \frac{2R^d}{\varepsilon}, \frac{R^d}{3\log 3 - 2} \log \frac{2(R+2)^d}{\varepsilon} \right) \\ &= R^d \frac{1+\nu/3}{\nu^2} \log \frac{2R^d}{\varepsilon}, \end{aligned} \tag{24}$$

then the probability in (23) will be larger than $1 - \varepsilon$.

REMARK. Observe that the parameters δ and R are not independent. As mentioned in [2, p. 14], for $\mathcal{B}(R,\delta)$ to be non-empty, we need $\delta \ge 2\pi d\sqrt{2R}e^{-\pi R}$ (up to terms of higher order). Thus for small δ as in Theorem 9, we need to choose R of order $R \approx c\log(d/\delta)$.

Appendix: The Plancherel–Polya inequality

We finish by showing that the constant κ in the Plancherel–Polya inequality (15) can be chosen explicitly to be $\kappa = e^{d\pi}$. The argument is simple and well known, see, for example, [5].

LEMMA A.1. *Let* $\{x_j : j \in \mathbb{N}\}$ *be a set in* $\mathbb{R}^d$ *with covering index* N_0. *Then*

$$\sum_{j=1}^{\infty} \left|f(x_j)\right|^2 \le N_0 e^{d\pi} \|f\|_2^2.$$

Proof. Let $k \in \mathbb{Z}^d$ and $x_j \in k + [-1/2, 1/2] =: D_k$. Then $\|x_j - k\|_\infty \le 1/2$. Consider the Taylor expansion of $f(x_j)$ at k (with the usual multi-index notation):

$$\left|f(x_j)\right| = \left|\sum_{\alpha \ge 0} \frac{D^\alpha f(k)}{\alpha!}(x_j - k)^\alpha\right| \le \sum_{\alpha \ge 0} \frac{|D^\alpha f(k)|}{\alpha!}\left(\frac{1}{2}\right)^{|\alpha|}.$$

We now let $\theta \in (0,1)$ and apply Cauchy–Schwarz:

$$\begin{aligned} \left|f(x_j)\right|^2 &\le \sum_{\alpha \ge 0} \frac{1}{\alpha!}\left(\frac{1}{2}\right)^{2\theta|\alpha|} \sum_{\alpha \ge 0} \frac{|D^\alpha f(k)|^2}{\alpha!}\left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|} \\ &= e^{d/4^\theta} \sum_{\alpha \ge 0} \frac{|D^\alpha f(k)|^2}{\alpha!}\left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|}. \end{aligned} \tag{25}$$

If $f \in \mathcal{B}$, then by Shannon's sampling theorem (or because the reproducing kernels $T_k s, k \in \mathbb{Z}^d$, form an orthonormal basis of $\mathcal{B}$) we have

$$\sum_{k \in \mathbb{Z}^d} \left|f(k)\right|^2 = \|f\|_2^2 \quad \forall f \in \mathcal{B}.$$

To estimate the partial derivatives we use Bernstein's inequality $\|D^\alpha f\|_2 \le \pi^{|\alpha|}\|f\|_2$.

We first assume that $N_0 = 1$, that is, each cube D_k contains at most one of the x_j's. Then we obtain, after interchanging the order of summation

$$\begin{aligned} \sum_{j \in \mathbb{N}} \left|f(x_j)\right|^2 &\le e^{d/4^\theta} \sum_{\alpha \ge 0} \sum_{k \in \mathbb{Z}^d} \frac{|D^\alpha f(k)|^2}{\alpha!}\left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|} \\ &= e^{d/4^\theta} \sum_{\alpha \ge 0} \left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|} \frac{\|D^\alpha f\|_2^2}{\alpha!} \\ &\le e^{d/4^\theta} \sum_{\alpha \ge 0} \left(\frac{1}{2}\right)^{2(1-\theta)|\alpha|} \frac{\pi^{2|\alpha|}}{\alpha!}\|f\|_2^2 = e^{d/4^\theta} e^{d\pi^2/4^{1-\theta}} \|f\|_2^2. \end{aligned} \tag{26}$$

The choice $4^\theta = 2/\pi$ yields the constant $\kappa = e^{d/4^\theta} e^{d\pi^2/4^{1-\theta}} = e^{d\pi}$. For arbitrary N_0 we obtain

$$\sum_{j\in\mathbb{N}} \big|f(x_j)\big|^2 = \sum_{k\in\mathbb{Z}^d} \sum_{\{j:x_j\in D_k\}} \big|f(x_j)^2\big| \le N_0 e^{d\pi} \|f\|_2^2,$$

as claimed. □

Possibly the Plancherel–Polya inequality could be improved to a local estimate of the form $\sum_{x_j\in C_R} |f(x_j|^2 \le \tilde{\kappa} N_0 \|f\|_{2,R}^2$, but we did not pursue this question.

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