

ONE AND TWO WEIGHT NORM INEQUALITIES FOR RIESZ POTENTIALS

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ABSTRACT. We consider weighted norm inequalities for the Riesz potentials I_α , also referred to as fractional integral operators. First, we prove mixed A_p - A_∞ type estimates in the spirit of (*Indiana Univ. Math. J.* **61** (2012) 2041–2052, *Anal. PDE* **6** (2013) 777–818, *Houston J. Math.* **38** (2012) 799–814). Then we prove strong and weak type inequalities in the case $p < q$ using the so-called log bump conditions. These results complement the strong type inequalities of Pérez (*Indiana Univ. Math. J.* **43** (1994) 663–683) and answer a conjecture from (Weights, extrapolation and the theory of Rubio de Francia (2011) Birkhäuser). For both sets of results, our main tool is a corona decomposition adapted to fractional averages.

1. Introduction

In this paper, we prove one and two weight norm inequalities for the Riesz potentials (also referred to as fractional integral operators):

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n.$$

Each of our results is analogous to a corresponding result for Calderón–Zygmund operators, and so our work parallels recent development in the study of sharp one and two weight norm inequalities for these operators. Our results are linked by a common technique that also originated in the study of singular integrals: a corona decomposition adapted to the fractional case.

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The natural scaling of the operator I_α shows that if $I_\alpha : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$, then p and q must satisfy the Sobolev relationship

$$(1) \quad \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n},$$

and so this is a natural condition to assume when studying one weight inequalities. Muckenhoupt and Wheeden [28] proved the following result.

THEOREM 1.1. *Given α , $0 < \alpha < n$ and p , $1 < p < n/\alpha$, define q by (1). Then the following are equivalent:*

(1) $w \in A_{p,q}$:

$$[w]_{A_{p,q}} = \sup_Q \left(\int_Q w(x)^q dx \right)^{1/q} \left(\int_Q w(x)^{-p'} dx \right)^{1/p'} < \infty;$$

(2) I_α satisfies the weak type inequality

$$\sup_t t \|w \chi_{\{x: |I_\alpha f(x)| > t\}}\|_q \leq C \|fw\|_p;$$

(3) I_α satisfies the strong type inequality

$$\|(I_\alpha f)w\|_q \leq C \|fw\|_p.$$

More recently, the second author with Lacey, Pérez and Torres [20] proved sharp bounds for these inequalities in terms of the $[w]_{A_{p,q}}$ constant. This question was motivated by the corresponding problem for Calderón–Zygmund singular integrals, which has been studied intensively for more than a decade, and was recently solved in full generality by Hytönen [12]. For the complete history of the problem we also refer the reader to [2], [18], [22], [24], [32] and the references they contain.

The problem of sharp constants for Riesz potentials is more tractable if we reformulate Theorem 1.1 in terms of the Muckenhoupt A_p weights. Recall that for $1 < p < \infty$, $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \left(\int_Q w(x) dx \right) \left(\int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty.$$

Let $u = w^q$ and $\sigma = w^{-p'}$ and define the function $s(\cdot)$ by

$$s(p) := 1 + \frac{q}{p'} = q \left(1 - \frac{\alpha}{n} \right) = p \left(\frac{n - \alpha}{n - \alpha p} \right).$$

Note that it follows at once from this that $s(p)' = s(q')$. Then it is straightforward to show that the following are equivalent: $w \in A_{p,q}$, $u \in A_{s(p)}$ and $\sigma \in A_{s(q')}$, and

$$[w]_{A_{p,q}} = [u]_{A_{s(p)}}^{1/q} = [\sigma]_{A_{s(q')}}^{1/p'}.$$

Moreover, by a change of variables we can restate the weak and strong type inequalities in terms of u and σ :

$$(2) \quad \|I_\alpha(f\sigma)\|_{L^{q,\infty}(u)} \lesssim \|f\|_{L^p(\sigma)}$$

and

$$(3) \quad \|I_\alpha(f\sigma)\|_{L^q(u)} \lesssim \|f\|_{L^p(\sigma)}.$$

(This formulation has the advantage that it makes the connection between the one and two weight inequalities more natural: see below.) It was shown in [20] that

$$(4) \quad \|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\rightarrow L^{q,\infty}(u)} \lesssim [u]_{A_{s(p)}}^{1-\frac{\alpha}{n}}$$

and

$$(5) \quad \|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\rightarrow L^q(u)} \lesssim [u]_{A_{s(p)}}^{1-\frac{\alpha}{n}} + [\sigma]_{A_{s(q')}}^{1-\frac{\alpha}{n}} \simeq [u]_{A_{s(p)}}^{(1-\frac{\alpha}{n})\max(1,\frac{p'}{q})}.$$

Our first result is an improvement of these inequalities. It is again motivated by the corresponding problem for Calderón–Zygmund operators: see [13], [15], [17]. There, a precise bound involving the A_p constant and the smaller A_∞ constant was given. Recall that $w \in A_\infty$ if

$$[w]_{A_\infty} = \sup_Q \exp\left(\int_Q -\log(w(x)) dx\right) \left(\int_Q w(x) dx\right) < \infty.$$

We have that $w \in A_\infty$ if and only if $w \in A_p$ for some $p > 1$, and

$$[w]_{A_\infty} = \lim_{p \rightarrow \infty} [w]_{A_p}.$$

(This limit was proved by Sbordone and Wik [38].) There are several equivalent definitions of the A_∞ condition (see [10]). One in particular has been shown to be very useful in the study of sharp constant problems. We say that a weight $w \in A'_\infty$ if

$$[w]_{A'_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(\chi_Q w)(x) dx < \infty,$$

where M is the Hardy–Littlewood maximal operator. Independently, Fujii [8] and Wilson [41], [42] (also see [43]) showed that $w \in A_\infty$ if and only if $w \in A'_\infty$. Hytönen and Pérez [15] showed that $[w]_{A'_\infty} \lesssim [w]_{A_\infty}$, and in fact $[w]_{A'_\infty}$ can be substantially smaller. Using this definition, we can state our result.

THEOREM 1.2. *Given α , $0 < \alpha < n$, and p , $1 < p < n/\alpha$, define q by (1). Let $w \in A_{p,q}$ and set $u = w^q$ and $\sigma = w^{-p'}$. Then*

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\rightarrow L^{q,\infty}(u)} \lesssim [u]_{A_{s(p)}}^{\frac{1}{q}} [u]_{A'_\infty}^{\frac{1}{p'}}$$

and

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma)\rightarrow L^q(u)} \lesssim [u]_{A_{s(p)}}^{\frac{1}{q}} ([u]_{A'_\infty}^{\frac{1}{p'}} + [\sigma]_{A'_\infty}^{\frac{1}{q}}).$$

REMARK 1.3. Recently, Lerner [23] introduced a different approach to improving the sharp A_p estimates for singular integrals using a mixed A_p - A_r condition, $1 < r < \infty$. He showed that this condition is not readily comparable to the A_p - A_∞ condition we are using. It is an open question whether the corresponding conditions can be proved for Riesz potentials.

We will actually prove Theorem 1.2 as a special case of a two weight result. Note that while we have assumed in inequalities (2) and (3) that u and σ are linked via the weight $w \in A_{p,q}$, we do not a priori have to assume this. We cannot completely decouple the weights u and σ but we can weaken their connection. In this context, it is natural to generalize the A_p condition to hold for a pair of weights: we say $(u, \sigma) \in A_p$ if

$$[u, \sigma]_{A_p} = \sup_Q \left(\int_Q u(x) dx \right) \left(\int_Q \sigma(x) dx \right)^{p-1} < \infty.$$

It is well known that this condition is necessary for many two weight inequalities, but not sufficient. For example, $(u, \sigma) \in A_p$ is necessary for (2). However, if we assume that u and/or σ are in A_∞ , then it is sufficient, and we can generalize Theorem 1.2 as follows.

THEOREM 1.4. *Given α , $0 < \alpha < n$, and p , $1 < p < n/\alpha$, define q by (1). Suppose (u, σ) is a pair of weights with $[u, \sigma]_{A_{s(p)}} < \infty$. If $u \in A_\infty$, then*

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^{q,\infty}(u)} \lesssim [u, \sigma]_{A_{s(p)}}^{\frac{1}{q}} [u]_{A'_\infty}^{\frac{1}{p'}}.$$

Moreover, if both u and σ belong to A_∞ , then

$$\|I_\alpha(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^q(u)} \lesssim [u, \sigma]_{A_{s(p)}}^{\frac{1}{q}} \left([u]_{A'_\infty}^{\frac{1}{p'}} + [\sigma]_{A'_\infty}^{\frac{1}{q}} \right).$$

REMARK 1.5. To see that Theorem 1.4 does indeed generalize Theorem 1.2, set $u = w^q$ and $\sigma = w^{-p'}$. Then $[u, \sigma]_{A_{s(p)}}^{\frac{1}{q}} = [\sigma, u]_{A_{s(q')}}^{\frac{1}{p'}}$ and $\frac{1}{q} + \frac{1}{p'} = 1 - \frac{\alpha}{n}$.

A nonquantitative version of this result was implicit in Pérez [30]. In the study of two weight norm inequalities for singular integrals, it has long been part of the folklore that assuming $(u, \sigma) \in A_p$ with the additional hypothesis that u and σ are in A_∞ is a sufficient condition. This was implicit in Neugebauer [29] and was the motivation for results by Fujii [9], Leckband [21], and Rakotondratsimba [33]. The sharp analog of Theorem 1.4 for singular integrals is due to Hytönen and Lacey [13].

If we drop the assumption that u and σ are A_∞ weights, we need to assume a stronger condition than two weight A_p for norm inequalities to hold. However, when working in this generality we no longer have to assume that p and q satisfy the Sobolev relationship (1). Instead, we only assume that $p \leq q$. (The case $q < p$ is much more difficult; see, for instance, Verbitsky [40].) In this case, the weights for the weak and strong type inequalities were characterized by Sawyer [35], [36].

THEOREM 1.6. *Given α , $0 < \alpha < n$, and p, q , $1 < p \leq q < \infty$, the weak type inequality (2) holds if and only if for every cube Q ,*

$$(6) \quad \left(\int_Q I_\alpha(\chi_Q u)(x)^{p'} \sigma(x) dx \right)^{1/p'} \lesssim \left(\int_Q u(x) dx \right)^{1/q'}.$$

The strong type inequality (3) holds if and only if for every cube Q , inequality (6) holds and

$$(7) \quad \left(\int_Q I_\alpha(\chi_Q \sigma)(x)^q u(x) dx \right)^{1/q} \lesssim \left(\int_Q \sigma(x) dx \right)^{1/p}.$$

While necessary and sufficient, the so-called testing conditions in Theorem 1.6 have the drawback that they involve the Riesz potential itself. Another approach is to find sharp sufficient conditions that resemble the $A_{p,q}$ condition of Muckenhoupt and Wheeden. This approach was introduced by Pérez [30], [31] and involves replacing the local L^p norm with a larger norm in the scale of Orlicz spaces.

To state these results, we need to make some preliminary definitions. A Young function is a function $\Phi : [0, \infty) \rightarrow [0, \infty)$ that is continuous, convex and strictly increasing, $\Phi(0) = 0$ and $\Phi(t)/t \rightarrow \infty$ as $t \rightarrow \infty$. Given a cube Q we define the localized Luxemburg norm by

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \int_Q \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

When $\Phi(t) = t^p$, $1 < p < \infty$, this becomes the L^p norm and we write

$$\|f\|_{p,Q} = \left(\int_Q |f(x)|^p dx \right)^{1/p}.$$

The associate function of Φ is the Young function

$$\bar{\Phi}(t) = \sup_{s>0} \{st - \Phi(s)\}.$$

Note that $\bar{\bar{\Phi}} = \Phi$. A Young function Φ satisfies the B_p condition if for some $c > 0$,

$$\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty.$$

Important examples of such functions are $\Phi(t) = t^{sp}$, $s > 1$, whose associate function is $\bar{A}(t) = t^{(sp)'}$, and $\Phi(t) = t^p \log(e + t)^{-1-\varepsilon}$, $\varepsilon > 0$, which have associate functions $\bar{\Phi}(t) \approx t^{p'} \log(e + t)^{p'-1+\delta}$, $\delta > 0$. We refer to these associate functions as power bumps and log bumps.

Pérez proved the following strong type inequality.

THEOREM 1.7. *Given α , $0 < \alpha < n$, and p, q , $1 < p \leq q < \infty$, the strong type inequality (3) holds for every pair of weights (u, σ) that satisfies*

$$(8) \quad \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, Q} \|\sigma^{\frac{1}{p'}}\|_{\Psi, Q} < \infty,$$

where Φ, Ψ are Young functions such that $\bar{\Phi} \in B_{q'}$ and $\bar{\Psi} \in B_p$.

The corresponding two weight result for singular integrals (with $p = q$ and $\alpha = 0$) was a long-standing conjecture motivated by Theorem 1.7. It was recently proved by Lerner [22]. For a detailed history of this problem, see [2], [3] and the references they contain.

Much less is known about two weight, weak type inequalities for the Riesz potential. It has long been known that for singular integrals, a sufficient condition for the weak (p, p) inequality is that the weights satisfy

$$\sup_Q \|u^{\frac{1}{p}}\|_{\Phi, Q} \|\sigma^{\frac{1}{p'}}\|_{p', Q} < \infty,$$

where Φ is the log bump $\Phi(t) = t^p \log(e + t)^{p-1+\delta}$ (see [5]). It is conjectured that it suffices to take $\bar{\Phi} \in B_{p'}$ (see [3]). Moreover, it was conjectured that the corresponding result holds for Riesz potentials.

CONJECTURE 1.8. *Given α , $0 < \alpha < n$, and p, q , $1 < p \leq q < \infty$, then the weak type inequality (2) holds for every pair of weights (u, σ) that satisfies*

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, Q} \|\sigma^{\frac{1}{p'}}\|_{p', Q} < \infty,$$

where Φ is a Young function such that $\bar{\Phi} \in B_{q'}$.

Until now, Conjecture 1.8 was only known when Φ is power bump (see [1], [6]) or a log bump of the form $\Phi(t) = t^q \log(e + t)^{2q-1+\delta}$. (This is proved in [3] when $p = q$, but the same proof works in the case $q > p$.) In the scale of log bumps, the conjecture should hold with the smaller exponent $q - 1 + \delta$. Our first result is a proof of this for a limited range of values of p and q .

THEOREM 1.9. *Given α , $0 < \alpha < n$, and p, q , $1 < p \leq q < \infty$, suppose*

$$(9) \quad \frac{p'}{q'} \left(1 - \frac{\alpha}{n}\right) \geq 1.$$

Then the weak type inequality (2) holds for every pair of weights (u, σ) that satisfies

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, Q} \|\sigma^{\frac{1}{p'}}\|_{p', Q} < \infty,$$

where $\Phi(t) = t^q \log(e + t)^{q-1+\delta}$, $\delta > 0$.

The restriction (9) holds when p and q satisfy the Sobolev relationship (1), and it also holds for p and q close to these values. It does not hold, however, when $p = q$. This condition appears to be intrinsic to our proof and a new approach will be necessary to prove Theorem 1.9 for the full range of p and q .

REMARK 1.10. The proof of the corresponding result for singular integrals is much simpler than the proof of Theorem 1.9: it follows by extrapolation from a two weight, weak (1, 1) inequality for singular integrals. It is conjectured that a similar inequality holds for Riesz potentials, and this would yield a simpler proof of Theorem 1.9. See [3] for complete details.

REMARK 1.11. The weak type results for singular integral operators are sharp in the sense that they are false if we take $\delta = 0$ in the definition of Φ (see [5]). Though it has not appeared explicitly in the literature, the same is true for Riesz potentials. For an example involving commutators of Riesz potentials, see [4].

By a small modification of our proof of Theorem 1.9, we can extend this result to a class of Young functions referred to as loglog bumps (cf. [3]). Our proof builds upon the recent work in [7], where a weak type inequality for singular integrals involving loglog bumps was proved.

THEOREM 1.12. *With the same hypotheses as before, the conclusion of Theorem 1.9 remains true if $\Phi(t) = t^q \log(e + t)^{q-1} \log \log(e^e + t)^{q-1+\delta}$ for $\delta > 0$ sufficiently large.*

In the scale of loglog bumps, Conjecture 1.8 holds for loglog bumps if we take any $\delta > 0$. But again this restriction on δ seems to be intrinsic to our proof.

Our second result in this vein is an improvement of Theorem 1.7 in the scale of log bumps. We believe that the single condition (8) with a bump on each term can be replaced by two conditions, each with a single bump condition. This is referred to as a separated bump condition. More precisely, we make the following conjecture.

CONJECTURE 1.13. *Given α , $0 < \alpha < n$, and p, q , $1 < p \leq q < \infty$, then the strong type inequality (3) holds for every pair of weights (u, σ) that satisfies*

$$\begin{aligned} \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, Q} \|\sigma^{\frac{1}{p'}}\|_{p', Q} < \infty, \\ \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{q, Q} \|\sigma^{\frac{1}{p'}}\|_{\Psi, Q} < \infty, \end{aligned}$$

where Φ, Ψ are Young functions such that $\bar{\Phi} \in B_{q'}$ and $\bar{\Psi} \in B_p$.

The motivation for this conjecture is recent work on two weight norm inequalities for singular integrals. The corresponding conjecture for singular integrals has been implicit in the literature, as it is closely connected to a long-standing conjecture of Muckenhoupt and Wheeden, now known to be false. It was recently made explicit in [7]; this paper also discusses its connection with the Muckenhoupt–Wheeden conjecture. Moreover, the authors also proved the conjecture in the special case of log bumps and certain loglog bumps. We can prove these kinds of result for Riesz potentials.

THEOREM 1.14. *Given α , $0 < \alpha < n$, and p, q , $1 < p \leq q < \infty$, suppose*

$$(10) \quad \min\left(\frac{q}{p}, \frac{p'}{q'}\right) \left(1 - \frac{\alpha}{n}\right) \geq 1.$$

Then the strong type inequality (3) holds for every pair of weights (u, σ) that satisfies

$$\begin{aligned} \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, Q} \|\sigma^{\frac{1}{p'}}\|_{p', Q} < \infty, \\ \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{q, Q} \|\sigma^{\frac{1}{p'}}\|_{\Psi, Q} < \infty, \end{aligned}$$

where $\Phi(t) = t^q \log(e + t)^{q-1+\delta}$ and $\Psi(t) = t^{p'} \log(e + t)^{p'-1+\delta}$.

THEOREM 1.15. *With the same hypotheses as before, the conclusion of Theorem 1.14 remains true if*

$$\begin{aligned} \Phi(t) &= t^q \log(e + t)^{q-1} \log \log(e^e + t)^{q-1+\delta}, \\ \Psi(t) &= t^{p'} \log(e + t)^{p'-1} \log \log(e^e + t)^{p'-1+\delta} \end{aligned}$$

for $\delta > 0$ sufficiently large.

Similar to the restriction in Theorem 1.9, (10) includes p and q that satisfy the Sobolev relationship (1) but does not extend to include the case $p = q$.

Finally, as an application of our weak type results we can prove a two weight, Sobolev inequality. Such inequalities follow immediately from our strong type results and the well-known inequality

$$|f(x)| \lesssim I_1(|\nabla f|)(x).$$

However, by the truncation method of Maz'ya [26] (see also [11], [25]), a strong type inequality for the gradient can be deduced from a weak type inequality for the Riesz potential. The following corollary to Theorem 1.9 can be proved exactly as [20, Theorem 2.7]. (See also [3, Lemma 4.31].)

COROLLARY 1.16. *Given p, q , $1 < p \leq q < \infty$, suppose $\frac{p'}{q'} \geq n'$. Then for all smooth functions f with compact support,*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u(x) dx\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p v(x) dx\right)^{1/p}$$

for all pairs of weights (u, v) that satisfy

$$\sup_Q |Q|^{\frac{1}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, Q} \|v^{-1/p}\|_{p', Q} < \infty,$$

where $\Phi(t) = t^q \log(e + t)^{q-1+\delta}$.

Organization. The remainder of this paper is organized as follows. In Section 2, we gather some results about dyadic operators that are used in our proofs. In particular, we state a sharp dyadic version of Theorem 1.6. In Section 3, we prove Theorem 1.4, and in Section 4 we prove Theorems 1.9, 1.12, 1.14 and 1.15.

Throughout the paper, all of the notation we will use will be standard or defined as needed. All cubes in \mathbb{R}^n will assume to be half open with sides parallel to the axes. Given a cube, Q , $\ell(Q)$ will denote its side length. Given a set $E \subseteq \mathbb{R}^n$, $|E|$ will denote the Lebesgue measure of E , $w(E) = \int_E w \, dx$ the weighted measure of E , and $\bar{f}_E w \, dx = |E|^{-1} \int_E w \, dx = w(E)/|E|$ the average of w over E . In proving inequalities, if we write $A \lesssim B$, we mean that $A \leq CB$, where the constant C can depend on α, p and n , but does not depend on the weights u or σ , nor on the function. If we write $A \simeq B$, then $A \lesssim B$ and $B \lesssim A$.

2. Dyadic Riesz potentials

In this section, we define two dyadic versions of the Riesz potential, and show how these can be used to approximate I_α . We begin by defining special collections of cubes, known as *dyadic grids* or *filtrations*. A dyadic grid \mathcal{D} is a countable collection of cubes that has the following properties:

- (1) $Q \in \mathcal{D} \Rightarrow \ell(Q) = 2^k$ for some $k \in \mathbb{Z}$,
- (2) $Q, P \in \mathcal{D} \Rightarrow Q \cap P \in \{\emptyset, P, Q\}$,
- (3) and for each $k \in \mathbb{Z}$ the set $\mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^k\}$ forms a partition of \mathbb{R}^n .

The collection of dyadic cubes used to form the well-known Calderón–Zygmund decomposition are a dyadic grid, as are all of the translates of these cubes. Below we will make extensive use of the dyadic grids

$$\mathcal{D}^t = \{2^{-k}([0, 1]^n + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}, \quad t \in \{0, 1/3\}^n.$$

The importance of these grids is shown by the following proposition; a proof can be found in [22].

PROPOSITION 2.1. *Given any cube Q in \mathbb{R}^n there exists a $t \in \{0, 1/3\}^n$ and a cube $Q_t \in \mathcal{D}^t$ such that $Q \subseteq Q_t$ and $\ell(Q_t) \leq 6\ell(Q)$.*

Given a dyadic grid, \mathcal{D} , define the dyadic Riesz potential operator

$$(11) \quad I_\alpha^\mathcal{D} f(x) = \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}} \int_Q f(y) \, dy \cdot \chi_Q(x).$$

Dyadic Riesz potentials were first introduced by Sawyer and Wheeden [30] (see also [37]). They proved (essentially) that the Riesz potential lies in the convex hull of all the dyadic Riesz potentials. Here we prove a sharper version of this result.

PROPOSITION 2.2. *Given $0 < \alpha < n$ and a nonnegative function f , then for any dyadic grid \mathcal{D} ,*

$$I_\alpha^{\mathcal{D}} f(x) \lesssim I_\alpha f(x).$$

Conversely, we have that

$$I_\alpha f(x) \lesssim \max_{t \in \{0,1/3\}^n} I_\alpha^{\mathcal{D}^t} f(x).$$

Note that as a corollary to Proposition 2.2 we have that $I_\alpha f$ is pointwise equivalent to a linear combination of dyadic Riesz potentials:

$$I_\alpha f(x) \simeq \sum_{t \in \{0,1/3\}^n} I_\alpha^{\mathcal{D}^t} f(x).$$

Proof of Proposition 2.2. Fix a nonnegative function f , $x \in \mathbb{R}^n$, and a dyadic grid \mathcal{D} . Let $\{Q_k\}_{k \in \mathbb{Z}}$ be the unique sequence of dyadic cubes in \mathcal{D} such that $\ell(Q_k) = 2^k$ and $x \in Q_k$. Fix $N \geq 1$; then

$$\begin{aligned} & \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 2^N}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f(y) dy \cdot \chi_Q(x) \\ &= \sum_{k=-\infty}^N \frac{1}{|Q_k|^{1-\frac{\alpha}{n}}} \int_{Q_k} f(y) dy \\ &= \sum_{k=-\infty}^N \frac{1}{|Q_k|^{1-\frac{\alpha}{n}}} \int_{Q_k \setminus Q_{k-1}} f(y) dy + \sum_{k=-\infty}^N \frac{1}{|Q_k|^{1-\frac{\alpha}{n}}} \int_{Q_{k-1}} f(y) dy \\ &\lesssim \sum_{k=-\infty}^N \int_{Q_k \setminus Q_{k-1}} \frac{f(y)}{|x-y|^{n-\alpha}} dy + 2^{\alpha-n} \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 2^N}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f(y) dy \cdot \chi_Q(x) \\ &= \int_{Q_N} \frac{f(y)}{|x-y|^{n-\alpha}} dy + 2^{\alpha-n} \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 2^N}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f(y) dy \cdot \chi_Q(x). \end{aligned}$$

Since $\alpha < n$ we can rearrange terms and take the limit as $N \rightarrow \infty$ to get

$$I_\alpha^{\mathcal{D}} f(x) \lesssim I_\alpha f(x).$$

To prove the second inequality, let $Q(x, r)$ be the cube of side-length $2r$ centered at x . By standard estimates (see, for example, [20]),

$$I_\alpha f(x) \leq 2^{n-\alpha} \sum_{k \in \mathbb{Z}} (2^{-k})^{n-\alpha} \int_{Q(x, 2^k)} f(y) dy.$$

By Proposition 2.1, for each $k \in \mathbb{Z}$ there exists $t \in \{0, 1/3\}^n$ and $Q_t \in \mathcal{D}^t$ such that $Q(x, 2^k) \subset Q_t$ and

$$2^{k+1} = \ell(Q) \leq \ell(Q_t) \leq 6\ell(Q(x, 2^k)) = 12 \cdot 2^k.$$

Since $\ell(Q_t) = 2^j$ for some j , we must have that $2^{k+1} \leq \ell(Q_t) \leq 2^{k+3}$. Hence,

$$\begin{aligned} I_\alpha f(x) &\leq 2^{n-\alpha} \sum_{k \in \mathbb{Z}} (2^{-k})^{n-\alpha} \int_{Q(x, 2^k)} f(y) dy \\ &\lesssim \sum_{k \in \mathbb{Z}} \sum_{t \in \{0, 1/3\}^n} \sum_{\substack{Q \in \mathcal{D}^t \\ 2^{k+1} \leq \ell(Q) \leq 2^{k+3}}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f(y) dy \cdot \chi_Q(x) \\ &\lesssim \sum_{t \in \{0, 1/3\}^n} \sum_{Q \in \mathcal{D}^t} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f(y) dy \cdot \chi_Q(x) \\ &\lesssim \max_{t \in \{0, 1/3\}^n} I_\alpha^{\mathcal{D}^t} f(x). \end{aligned} \quad \square$$

We now show that in the definition of $I_\alpha^{\mathcal{D}}$ we can replace the summation over \mathcal{D} by a summation over a subset of the dyadic grid whose members have good intersection properties. We call such a subset a *sparse family* (cf. [14], [22]). Given a dyadic grid \mathcal{D} , a subset $\mathcal{S} \subseteq \mathcal{D}$ is a sparse family of dyadic cubes if for every $Q \in \mathcal{S}$,

$$(12) \quad \left| \bigcup_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q}} Q' \right| \leq \frac{1}{2} |Q|.$$

If \mathcal{S} is a sparse family and we define the sets

$$E(Q) = Q \setminus \left(\bigcup_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q}} Q' \right), \quad Q \in \mathcal{S},$$

then the collection $\{E(Q)\}_{Q \in \mathcal{S}}$ is pairwise disjoint and for each Q ,

$$(13) \quad |E(Q)| \leq |Q| \leq 2|E(Q)|.$$

Though the terminology is recent, particular sparse families have long played a role in the applications of Calderón–Zygmund theory. See, for example, [10, Chapter 4, Lemma 2.5] or [3, Appendix A].

Given α , $0 < \alpha < n$, and a sparse family $\mathcal{S} \subseteq \mathcal{D}$, define the sparse dyadic Riesz potential

$$I_\alpha^{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} |Q|^{\frac{\alpha}{n}} \int_Q f dy \cdot \chi_Q(x).$$

The connection between dyadic Riesz potentials and their sparse counterparts is given by the following result. The ideas underlying the proof are not new: they are implicit in [20], [30], [37].

PROPOSITION 2.3. *Given a bounded, nonnegative function f with compact support and a dyadic grid \mathcal{D} , there exists a sparse family \mathcal{S} such that for all*

$\alpha, 0 < \alpha < n,$

$$I_\alpha^\mathcal{D} f(x) \lesssim I_\alpha^S f(x).$$

Proof. Let $a = 2^{n+1}$. For each $k \in \mathbb{Z}$ define

$$\mathcal{Q}^k = \left\{ P \in \mathcal{D} : a^k < \int_P f dy \leq a^{k+1} \right\}.$$

Then for every $P \in \mathcal{D}$ such that $\int_P f dy \neq 0$, there exists a unique k such that $P \in \mathcal{Q}^k$. Therefore,

$$\begin{aligned} I_\alpha^\mathcal{D} f(x) &= \sum_{P \in \mathcal{D}} \frac{1}{|P|^{1-\frac{\alpha}{n}}} \int_P f dy \cdot \chi_P(x) \\ &= \sum_k \sum_{P \in \mathcal{Q}^k} \frac{1}{|P|^{1-\frac{\alpha}{n}}} \int_P f dy \cdot \chi_P(x) \leq \sum_k a^{k+1} \sum_{P \in \mathcal{Q}^k} |P|^{\frac{\alpha}{n}} \cdot \chi_P(x). \end{aligned}$$

Now let \mathcal{S}_k be the collection of disjoint, maximal cubes $Q \in \mathcal{D}$ such that

$$\int_Q f dx > a^k.$$

(Such a collection exists since \mathcal{D} is a dyadic grid and f is bounded and has compact support.) Let $\mathcal{S} = \bigcup_k \mathcal{S}_k$. Then for every $P \in \mathcal{Q}^k$ there exists $Q \in \mathcal{S}_k$ such that $Q \supseteq P$. Hence, we have that

$$I_\alpha^\mathcal{D} f(x) \leq a \sum_k a^k \sum_{Q \in \mathcal{S}_k} \sum_{\substack{P \in \mathcal{D} \\ P \subseteq Q}} |P|^{\frac{\alpha}{n}} \cdot \chi_P(x).$$

The inner sum can be evaluated:

$$\sum_{\substack{P \in \mathcal{D} \\ P \subseteq Q}} |P|^{\frac{\alpha}{n}} \cdot \chi_P(x) = \sum_{r=0}^\infty \sum_{\substack{P \in \mathcal{D} : P \subseteq Q \\ \ell(P)=2^{-r}\ell(Q)}} |P|^{\frac{\alpha}{n}} \cdot \chi_P(x) = \frac{1}{1-2^{-\alpha}} |Q|^{\frac{\alpha}{n}} \cdot \chi_Q(x).$$

Moreover, since $a^k < \int_Q f dy$ if $Q \in \mathcal{S}_k$, we have that

$$I_\alpha^\mathcal{D} f(x) \lesssim I_\alpha^S f(x).$$

Finally, we show that \mathcal{S} is sparse. If $Q \in \mathcal{S}$, then $Q \in \mathcal{S}_k$ for some $k \in \mathbb{Z}$; hence, by the maximality of the cubes in \mathcal{S} ,

$$\left| \bigcup_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q}} Q' \right| = \sum_{\substack{Q' \in \mathcal{S}^{k+1} \\ Q' \subseteq Q}} |Q'| < \frac{1}{a^k} \sum_{\substack{Q' \in \mathcal{S}^{k+1} \\ Q' \subseteq Q}} \int_{Q'} f dx \leq \frac{1}{a^k} \int_Q f dx \leq \frac{2^n}{a} |Q| = \frac{1}{2} |Q|. \quad \square$$

As a consequence of Propositions 2.2 and 2.3, to prove our main results it will suffice to work with a general dyadic grid \mathcal{D} and a sparse Riesz potential I_α^S . To prove bounds for I_α^S we will use a dyadic version of Theorem 1.6 due to Lacey, Sawyer, and Uriarte-Tuero [19] that gives precise bounds in terms of

testing conditions. To state their result, we need a definition. Given a dyadic grid \mathcal{D} and $R \in \mathcal{D}$, let

$$I_\alpha^{\mathcal{S}(R)} f(x) = \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq R}} |Q|^{\frac{\alpha}{n}} \int_Q f \, dx \cdot \chi_Q(x).$$

For $1 < p \leq q < \infty$ and a pair of weights (u, σ) define

$$[u, \sigma]_{(I_\alpha^{\mathcal{S}})^{p,q}} = \sup_{R \in \mathcal{D}} \sigma(R)^{-1/p} \left(\int_R I_\alpha^{\mathcal{S}(R)} (\chi_R \sigma)^q u \, dx \right)^{1/q}.$$

PROPOSITION 2.4. Fix α , $0 < \alpha < n$, and p, q , $1 < p \leq q < \infty$. Let \mathcal{D} be a dyadic grid and let \mathcal{S} be a sparse subset of \mathcal{D} . Given any pair of weights (u, σ) , the following equivalences hold:

$$\begin{aligned} \|I_\alpha^{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^{q,\infty}(u)} &\simeq [\sigma, u]_{(I_\alpha^{\mathcal{S}})^{q',p'}}, \\ \|I_\alpha^{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^q(u)} &\simeq [u, \sigma]_{(I_\alpha^{\mathcal{S}})^{p,q}} + [\sigma, u]_{(I_\alpha^{\mathcal{S}})^{q',p'}}. \end{aligned}$$

3. Proof of Theorem 1.4

Our main result in this section is the following.

THEOREM 3.1. Given α , $0 < \alpha < n$, and p , $1 < p < n/\alpha$, define q by (1). Suppose (u, σ) is a pair of weights with $[u, \sigma]_{A_{s(p)}} < \infty$, \mathcal{D} is a dyadic grid with sparse subset \mathcal{S} . If $u \in A_\infty$, then

$$(14) \quad [\sigma, u]_{(I_\alpha^{\mathcal{S}})^{q',p'}} \lesssim [u, \sigma]_{A_{s(p)}}^{\frac{1}{q}} [u]_{A'_\infty}^{\frac{1}{p'}}.$$

The constant in (14) is independent of \mathcal{D} and \mathcal{S} .

The operator $I_\alpha^{\mathcal{S}}$ is self adjoint; hence, by symmetry we also have the dual testing condition

$$[u, \sigma]_{(I_\alpha^{\mathcal{S}})^{p,q}} \lesssim [\sigma, u]_{A_{s(q')}}^{\frac{1}{p'}} [\sigma]_{A'_\infty}^{\frac{1}{q}}$$

provided $\sigma \in A_\infty$. By Propositions 2.1, 2.3 and 2.4, Theorem 1.4 follows at once from Theorem 3.1.

The proof of Theorem 3.1 requires three lemmas. To state the first, we define the fractional maximal operator with respect to a dyadic grid \mathcal{D} . Given α , $0 < \alpha < n$, and a nonnegative measure μ on \mathbb{R}^n define

$$M_{\alpha,\mu}^{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{D}} \frac{1}{\mu(Q)^{1-\frac{\alpha}{n}}} \int_Q |f| \, d\mu \cdot \chi_Q(x).$$

LEMMA 3.2. Given α , $0 < \alpha < n$, and p , $1 < p < n/\alpha$, define q by (1). If the measure μ is such that $\mu(\mathbb{R}^n) = \infty$, then $M_{\alpha,\mu}^{\mathcal{D}} : L^p(\mu) \rightarrow L^q(\mu)$. If $p = 1$, then $M_{\alpha,\mu}^{\mathcal{D}} : L^1(\mu) \rightarrow L^{q,\infty}(\mu)$.

The proof of Lemma 3.2 is standard: see [39] for $\alpha = 0$ and [27] for $0 < \alpha < n$.

The second lemma is a fractional Carleson embedding theorem. We do not believe that this result is new; however, we give the short proof because we were unable to find it in the literature.

LEMMA 3.3. *Given a dyadic grid \mathcal{D} and a nonnegative measure μ such that $\mu(\mathbb{R}^n) = \infty$, suppose $\{c_Q\}_{Q \in \mathcal{D}}$ is a sequence of nonnegative numbers satisfying*

$$\sum_{Q \subseteq R} c_Q \leq A\mu(R), \quad R \in \mathcal{D}.$$

Given α , $0 < \alpha < n$, and p , $1 < p < n/\alpha$, define q by (1). Then for all nonnegative functions f ,

$$\left(\sum_{Q \in \mathcal{D}} c_Q \cdot \left(\frac{1}{\mu(Q)^{1-\frac{\alpha}{n}}} \int_Q f d\mu \right)^q \right)^{1/q} \leq A^{1/q} \|M_{\alpha,\mu}^{\mathcal{D}} f\|_{L^q(\mu)} \lesssim A^{1/q} \|f\|_{L^p(\mu)}.$$

Proof. The second inequality follows at once from Lemma 3.2. To prove the first, without loss of generality we may assume that f is bounded and has compact support. Let (\mathcal{D}, ν) be the measure space with $\nu(Q) = c_Q$, and define

$$a_{\alpha,\mu}(f, Q) = \frac{1}{\mu(Q)^{1-\frac{\alpha}{n}}} \int_Q f d\mu.$$

Then

$$\sum_{Q \in \mathcal{D}} c_Q \cdot (a_{\alpha,\mu}(f, Q))^q = q \int_0^\infty \lambda^{q-1} \nu(\{Q \in \mathcal{D} : a_{\alpha,\mu}(f, Q) > \lambda\}) d\lambda.$$

Let $\Omega_\lambda = \{Q \in \mathcal{D} : a_{\alpha,\mu}(f, Q) > \lambda\}$ and Ω_λ^* be the set of all maximal (with respect to inclusion) dyadic cubes R such that $a_{\alpha,\mu}(f, R) > \lambda$. Then the cubes in Ω_λ^* are pairwise disjoint, each $Q \in \Omega_\lambda$ is contained in some $R \in \Omega_\lambda^*$, and

$$\bigcup_{R \in \Omega_\lambda^*} R = \{M_{\alpha,\mu}^{\mathcal{D}} f > \lambda\}.$$

Hence,

$$\nu(\Omega_\lambda) = \sum_{Q \in \Omega_\lambda} c_Q \leq \sum_{R \in \Omega_\lambda^*} \sum_{Q \subseteq R} c_Q \leq A \sum_{R \in \Omega_\lambda^*} \mu(R) = A\mu(\{M_{\alpha,\mu}^{\mathcal{D}} f > \lambda\}),$$

and so

$$\begin{aligned} \sum_{Q \in \mathcal{D}} c_Q \cdot (a_{\alpha,\mu}(f, Q))^q &\leq Aq \int_0^\infty \lambda^{q-1} \mu(\{M_{\alpha,\mu}^{\mathcal{D}} f > \lambda\}) d\lambda \\ &= A \|M_{\alpha,\mu}^{\mathcal{D}} f\|_{L^q(\mu)}^q. \end{aligned} \quad \square$$

The last lemma is a crucial exponential decay estimate in the spirit of the John–Nirenberg inequality for BMO functions. Similar estimates can be found in [16, Lemma 5.5] and [18, Lemma 3.15]. Our proof is simplified because we are able to take advantage of the sparse family of cubes.

LEMMA 3.4. *Let \mathcal{S} be a sparse family of cubes. For any cube R_0 and every $k \geq 1$,*

$$(15) \quad \left| \left\{ x \in R_0 : \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq R_0}} \chi_Q(x) > k \right\} \right| \leq 2^{-k} |R_0|.$$

Proof. Given R_0 , set $\mathcal{S}(R_0) = \{Q \in \mathcal{S} : Q \subseteq R_0\}$. Let $\mathcal{P}_1(R_0)$ be the collection of all maximal cubes in $\mathcal{S}(R_0)$. Define $\mathcal{P}_{k+1}(R_0)$ inductively to be the collection of all $Q \in \mathcal{S}(R_0)$ that are maximal with respect to inclusion and such that there exists $Q' \in \mathcal{P}_k(R_0)$ with $Q \subsetneq Q'$. In other words, $\mathcal{P}_{k+1}(R_0)$ is the collection of maximal cubes that are properly contained in the members of $\mathcal{P}_k(R_0)$. We will refer to the members of \mathcal{P}_k as “cubes at the k th level down.” Let

$$\Omega_k = \bigcup_{Q \in \mathcal{P}_k(R_0)} Q.$$

We claim that

$$\left\{ x \in R_0 : \sum_{Q \in \mathcal{S}(R_0)} \chi_Q(x) > k \right\} = \Omega_{k+1}.$$

Notice that the function

$$f_{R_0}(x) = \sum_{Q \in \mathcal{S}(R_0)} \chi_Q(x)$$

is an integer valued function that counts the number of cubes in $\mathcal{S}(R_0)$ that contain x . With this in mind, it is easy to see that

$$\left\{ x \in R_0 : \sum_{Q \in \mathcal{S}(R_0)} \chi_Q(x) > k \right\} \supseteq \Omega_{k+1}.$$

To see the reverse inclusion, note that if

$$x \in \left\{ x \in R_0 : \sum_{Q \in \mathcal{S}(R_0)} \chi_Q(x) > k \right\},$$

then x belongs to at least $k + 1$ cubes of $\mathcal{S}(R_0)$, so x must belong to a cube in $\mathcal{P}_{k+1}(R_0)$. Finally, by the sparsity condition on the family $\mathcal{S}(R_0)$ and the disjointness of the families $\mathcal{P}_k(R_0)$ we have

$$|\Omega_{k+1}| \leq \frac{1}{2} |\Omega_k| \leq \frac{1}{4} |\Omega_{k-1}| \leq \dots \leq \frac{1}{2^k} |\Omega_1| \leq \frac{1}{2^k} |R_0|. \quad \square$$

REMARK 3.5. We note one identity from the proof of Lemma 3.4 that we will use below:

$$\left\{ x \in R_0 : \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq R_0}} \chi_Q(x) > k \right\} = \bigcup_{Q \in \mathcal{P}_{k+1}(R_0)} Q,$$

where $\mathcal{P}_{k+1}(R_0)$ is the collection of maximal cubes in \mathcal{S} contained in R_0 at the $(k + 1)$ th level down.

Proof of Theorem 3.1. To prove (14), fix $R \in \mathcal{D}$ and let $\mathcal{S}(R) = \{Q \in \mathcal{S} : Q \subseteq R\}$. It will suffice to show that

$$(16) \quad \left(\int_R I_\alpha^{S(R)} (\chi_R u)^{p'} \sigma dx \right)^{1/p'} \lesssim [u, \sigma]_{A_{s(p)}}^{1/q} [u]_{A_\infty}^{1/p'} u(R)^{1/q'}.$$

To estimate the operator $I_\alpha^{S(R)}$ we need to decompose the family $\mathcal{S}(R)$ into a collection of smaller sets. The first step allows us to “freeze” (i.e., gain local control of) the $A_{s(p)}$ constant of u . For each $a \in \mathbb{Z}$, define

$$\mathcal{Q}^a := \left\{ Q \in \mathcal{S}(R) : 2^a < \left(\int_Q u dx \right)^{\frac{1}{q}} \left(\int_Q \sigma dx \right)^{\frac{1}{p'}} \leq 2^{a+1} \right\}.$$

The set \mathcal{Q}^a is empty if $2^a > [u, \sigma]_{A_{s(p)}}^{1/q}$, so we may assume that

$$-\infty < a \leq \log_2 [u, \sigma]_{A_{s(p)}}^{1/q} = \Gamma(u).$$

In particular, we have that

$$(17) \quad \sum_{a=-\infty}^{\Gamma(u)} 2^a \lesssim [u, \sigma]_{A_{s(p)}}^{1/q}.$$

Our next step is to perform a corona decomposition of $\mathcal{S}(R)$ similar to that in [18]. Given a , let C_0^a be the set of maximal cubes in \mathcal{Q}^a . For each $k \geq 1$, define the set C_k^a by induction to be the (possibly empty) collection of cubes $Q \in \mathcal{Q}^a$ such that following three criteria are satisfied:

- (1) there exists $P \in C_{k-1}^a$ containing Q ,
- (2) the inequality

$$(18) \quad |Q|^{\frac{\alpha}{n}} \int_Q u dx > 2|P|^{\frac{\alpha}{n}} \int_P u dx$$

holds,

- (3) and Q is maximal with respect to inclusion in \mathcal{Q}^a .

Set $\mathcal{C}^a = \bigcup_k C_k^a$; we refer this set as the collection of stopping cubes for the corona decomposition of \mathcal{Q}^a . \mathcal{C}^a can be thought of as the collection cubes in \mathcal{Q}^a whose fractional average increases by a factor of two when passing from parent to child in \mathcal{C}^a .

By the maximality of the stopping cubes, given any $Q \in \mathcal{Q}^a$ there exists a smallest $P \in \mathcal{C}^a$ such that $P \supseteq Q$ and the reverse of inequality (18),

$$(19) \quad |Q|^{\frac{\alpha}{n}} \int_Q u \, dx \leq 2|P|^{\frac{\alpha}{n}} \int_P u \, dx,$$

holds. Denote this cube P by $\Pi^a(Q)$. For each $P \in \mathcal{C}^a$ let

$$\mathcal{Q}^a(P) = \{Q \in \mathcal{Q}^a : \Pi^a(Q) = P\}.$$

Then inequality (19) holds for all $Q \in \mathcal{Q}^a(P)$.

Finally, we want to control one more value: for every integer $b \geq 0$ and $P \in \mathcal{C}^a$, let $\mathcal{Q}_b^a(P)$ be the set of $Q \in \mathcal{Q}^a(P)$ such that

$$(20) \quad 2^{-b}|P|^{\frac{\alpha}{n}} \int_P u \, dx < |Q|^{\frac{\alpha}{n}} \int_Q u \, dx \leq 2^{-b+1}|P|^{\frac{\alpha}{n}} \int_P u \, dx.$$

By the above definitions, we have that

$$\mathcal{S}(R) = \bigcup_{a=-\infty}^{\Gamma(u)} \mathcal{Q}^a, \quad \mathcal{Q}^a = \bigcup_{P \in \mathcal{C}^a} \mathcal{Q}^a(P), \quad \mathcal{Q}^a(P) = \bigcup_{b=0}^{\infty} \mathcal{Q}_b^a(P),$$

and each of these unions is disjoint. Therefore, we can decompose the operator as follows:

$$\begin{aligned} I_{\alpha}^{\mathcal{S}(R)} u &= \sum_{a=-\infty}^{\Gamma(u)} \sum_{P \in \mathcal{C}^a} \sum_{b=0}^{\infty} \sum_{Q \in \mathcal{Q}_b^a(P)} |Q|^{\frac{\alpha}{n}} \int_Q u \, dx \cdot \chi_Q \\ &\leq 2 \sum_{a=-\infty}^{\Gamma(u)} \sum_{b=0}^{\infty} 2^{-b} \sum_{P \in \mathcal{C}^a} |P|^{\frac{\alpha}{n}} \int_P u \, dx \sum_{Q \in \mathcal{Q}_b^a(P)} \chi_Q. \end{aligned}$$

For each $k \geq 0$, define

$$E_b^a(k, P) = \left\{ x \in P : k < \sum_{Q \in \mathcal{Q}_b^a(P)} \chi_Q(x) \leq k + 1 \right\}$$

and

$$F_b^a(k, P) = \left\{ x \in P : \sum_{Q \in \mathcal{Q}_b^a(P)} \chi_Q(x) > k \right\}.$$

Then we have that

$$\begin{aligned} I_{\alpha}^{\mathcal{S}(R)} u(x) &\lesssim \sum_{a=-\infty}^{\Gamma(u)} \sum_{b=0}^{\infty} 2^{-b} \sum_{k=0}^{\infty} (k+1) \sum_{P \in \mathcal{C}^a} |P|^{\frac{\alpha}{n}} \int_P u \, dx \cdot \chi_{E_b^a(k, P)}(x) \\ &\leq \sum_{a=-\infty}^{\Gamma(u)} \sum_{b=0}^{\infty} 2^{-b} \sum_{k=0}^{\infty} (k+1) \sum_{P \in \mathcal{C}^a} |P|^{\frac{\alpha}{n}} \int_P u \, dx \cdot \chi_{F_b^a(k, P)}(x). \end{aligned}$$

Hence, by Minkowski's inequality,

$$\begin{aligned}
 (21) \quad & \left(\int_R (I_\alpha^{S(R)} u)^{p'} \sigma dx \right)^{1/p'} \\
 & \lesssim \sum_{a=-\infty}^{\Gamma(u)} \sum_{b=0}^{\infty} 2^{-b} \sum_{k=0}^{\infty} (k+1) \\
 & \quad \times \left(\int_R \left(\sum_{P \in \mathcal{C}^a} |P|^{\frac{\alpha}{n}} \int_P u dx \cdot \chi_{F_b^a(k,P)} \right)^{p'} \sigma dx \right)^{1/p'}.
 \end{aligned}$$

To estimate the last term, we will first show that for each a, b and k ,

$$\begin{aligned}
 (22) \quad & \left(\int_R \left(\sum_{P \in \mathcal{C}^a} |P|^{\frac{\alpha}{n}} \int_P u dx \cdot \chi_{F_b^a(k,P)} \right)^{p'} \sigma dx \right)^{1/p'} \\
 & \lesssim \left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u dx \right)^{p'} \cdot \sigma(F_b^a(k,P)) \right)^{1/p'}.
 \end{aligned}$$

To prove this, note that since the cubes in \mathcal{C}^a are stopping cubes, the set of $x \in R$ that belongs to infinitely many $P \in \mathcal{C}^a$ has measure zero. Fix $x \in R$ not in this set, and let $\{P_i\}_{i=0}^m$ be the stopping cubes such that $P_0 \subset P_1 \subset \dots \subset P_m \subset R$ and $x \in F_b^a(k, P_i)$. By the definition of the stopping cubes, we have that

$$|P_i|^{\frac{\alpha}{n}} \int_{P_i} u dx < 2^{-i} |P_0|^{\frac{\alpha}{n}} \int_{P_0} u dx.$$

Therefore,

$$\begin{aligned}
 & \left(\sum_{P \in \mathcal{C}^a} |P|^{\frac{\alpha}{n}} \int_P u dx \cdot \chi_{F_b^a(k,P)}(x) \right)^{p'} \\
 & = \left(\sum_{i=0}^m |P_i|^{\frac{\alpha}{n}} \int_{P_i} u dx \right)^{p'} < \left(\sum_{i=0}^m 2^{-i} \right)^{p'} \left(|P_0|^{\frac{\alpha}{n}} \int_{P_0} u dx \right)^{p'} \\
 & < 2^{p'} \sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u dx \right)^{p'} \chi_{F_b^a(k,P)}(x).
 \end{aligned}$$

If we integrate this quantity over R with respect to σdx , we get inequality (22).

To continue, suppose for a moment that we have the exponential decay estimate

$$(23) \quad \sigma(F_b^a(k,P)) \lesssim 2^{-ck} \sigma(P).$$

Then by inequalities (22) and (23), we have that

$$\begin{aligned}
 (24) \quad & \left(\int_R (I_\alpha^{S(R)} u)^{p'} \sigma \, dx \right)^{1/p'} \\
 & \lesssim \sum_{a=-\infty}^{\Gamma(u)} \sum_{b=0}^{\infty} 2^{-b} \sum_{k=0}^{\infty} 2^{-ck} (k+1) \left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(P) \right)^{1/p'} \\
 & \lesssim \sum_{a=-\infty}^{\Gamma(u)} \left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(P) \right)^{1/p'}.
 \end{aligned}$$

To estimate the final sum, note first that by the definition of \mathcal{Q}^a , if $P \in \mathcal{C}^a$,

$$\left(\int_P u \, dx \right)^{s(q')-1} \left(\int_P \sigma \, dx \right) \lesssim 2^{ap'}.$$

Therefore, we have that

$$\begin{aligned}
 (25) \quad & \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(P) \\
 & = \left(\frac{1}{u(P)^{1-\frac{\alpha}{n}}} \int_P \chi_R \cdot u \, dx \right)^{p'} \frac{u(P)^{s(q')} \sigma(P)}{|P|^{s(q')}} \\
 & \lesssim 2^{ap'} \left(\frac{1}{u(P)^{1-\frac{\alpha}{n}}} \int_P \chi_R \cdot u \, dx \right)^{p'} u(P).
 \end{aligned}$$

For cubes $Q \in \mathcal{D}$, define the sequence $\{c_Q\}$ by

$$c_Q = \begin{cases} u(Q), & Q \in \mathcal{C}^a, \\ 0, & Q \notin \mathcal{C}^a. \end{cases}$$

We claim this is a Carleson sequence and

$$(26) \quad \sum_{Q \subseteq P} c_Q \lesssim [u]_{A'_\infty} u(P).$$

Fix a cube P ; since $\mathcal{C}^a \subset \mathcal{S}(R)$,

$$\begin{aligned}
 \sum_{Q \subseteq P} c_Q & \leq \sum_{\substack{Q \in \mathcal{S}(R) \\ Q \subseteq P}} u(Q) \\
 & \lesssim \sum_{\substack{Q \in \mathcal{S}(R) \\ Q \subseteq P}} \frac{u(Q)}{|Q|} |E(Q)| \\
 & \leq \int_P M(\chi_P u) \, dx \leq [u]_{A'_\infty} u(P).
 \end{aligned}$$

Therefore, if we combine inequalities (24) and (25), then by Lemmas 3.2 and 3.3 and inequality (17) we have that

$$\begin{aligned} & \left(\int_R (I_\alpha^{S(R)} u)^{p'} \sigma \, dx \right)^{1/p'} \\ & \lesssim \sum_{a=-\infty}^{\Gamma(u)} 2^a \left(\sum_{P \in \mathcal{C}^a} \left(\frac{1}{u(P)^{1-\frac{\alpha}{n}}} \int_P \chi_R \cdot u \, dx \right)^{p'} u(P) \right)^{1/p'} \\ & \lesssim [u]_{A'_\infty}^{1/p'} \left(\sum_{a=-\infty}^{\Gamma(u)} 2^a \right) \left(\int_{\mathbb{R}^n} M_{\alpha,u}^{\mathcal{D}} (\chi_R u)^{p'} u \, dx \right)^{1/p'} \\ & \lesssim [u, \sigma]_{A_s(p)}^{1/q} [u]_{A'_\infty}^{1/p'} u(R)^{1/q'}. \end{aligned}$$

To complete the proof it remains to prove inequality (23): for a, b, k , fixed and $P \in \mathcal{C}^a$,

$$\sigma(F_b^a(k, P)) = \sigma \left(\left\{ x \in P : \sum_{Q \in \mathcal{Q}_b^a(P)} \chi_Q(x) > k \right\} \right) \lesssim 2^{-ck} \sigma(P).$$

If $x \in F_b^a(k, P)$, then clearly $x \in Q$ for some $Q \in \mathcal{Q}_b^a(P)$. Therefore, if we let \mathcal{M} be the collection of maximal, disjoint cubes $Q \in \mathcal{Q}_b^a(P)$ contained in P , we have that

$$\begin{aligned} (27) \quad & \sigma \left(\left\{ x \in P : \sum_{Q \in \mathcal{Q}_b^a(P)} \chi_Q(x) > k \right\} \right) \\ & = \sum_{M \in \mathcal{M}} \sigma \left(\left\{ x \in M : \sum_{\substack{Q \in \mathcal{Q}_b^a(P) \\ Q \subseteq M}} \chi_Q(x) > k \right\} \right). \end{aligned}$$

Fix $M \in \mathcal{M}$ and notice that the family of cubes $Q \in \mathcal{Q}_b^a(P)$ is a sparse family of cubes contained in P . For each $M \in \mathcal{M}$, as in Lemma 3.4 (see Remark 3.5) we may write

$$\left\{ x \in M : \sum_{\substack{Q \in \mathcal{Q}_b^a(P) \\ Q \subseteq M}} \chi_Q(x) > k \right\} = \bigcup_{L \in \mathcal{P}_{k+1}(M)} L,$$

where the union is made up of maximal cubes in contained in M at the $(k + 1)$ th level down.

For any cube, $Q \in \mathcal{Q}_b^a(P)$ (and in particular if $Q = L \in \mathcal{P}_{k+1}(M)$ or $M \in \mathcal{M}$) by the definition of $\mathcal{Q}^a(P)$ and (20) we have that

$$(28) \quad \sigma(Q) \simeq 2^{ap'} 2^{b\frac{p'}{q}} \left(\frac{|P|^{1-\frac{\alpha}{n}}}{u(P)} \right)^{\frac{p'}{q}} |Q|^{\frac{p'}{q} \frac{\alpha}{n} + 1}.$$

Therefore, we can estimate as follows: by one side of inequality (28), with $Q = L$

$$\begin{aligned} & \sigma\left(\left\{x \in M : \sum_{\substack{Q \in \mathcal{Q}_b^\alpha(P) \\ Q \subseteq M}} \chi_Q(x) > k\right\}\right) \\ &= \sum_{L \in \mathcal{P}_{k+1}(M)} \sigma(L) \\ &\lesssim 2^{ap'} 2^{b\frac{p'}{q}} \left(\frac{|P|^{1-\frac{\alpha}{n}}}{u(P)}\right)^{\frac{p'}{q}} \sum_{L \in \mathcal{P}_{k+1}(M)} |L|^{\frac{p'}{q}\frac{\alpha}{n}+1}; \end{aligned}$$

since $1 + \frac{\alpha}{n} \frac{p'}{q} \geq 1$,

$$\begin{aligned} &\leq 2^{ap'} 2^{b\frac{p'}{q}} \left(\frac{|P|^{1-\frac{\alpha}{n}}}{u(P)}\right)^{\frac{p'}{q}} \left(\sum_{L \in \mathcal{P}_{k+1}(M)} |L|\right)^{\frac{p'}{q}\frac{\alpha}{n}+1} \\ &\leq 2^{ap'} 2^{b\frac{p'}{q}} \left(\frac{|P|^{1-\frac{\alpha}{n}}}{u(P)}\right)^{\frac{p'}{q}} \left|\left\{x \in M : \sum_{\substack{Q \in \mathcal{Q}_b^\alpha(P) \\ Q \subseteq M}} \chi_Q(x) > k\right\}\right|^{\frac{p'}{q}\frac{\alpha}{n}+1}; \end{aligned}$$

by inequality (15) and the other half of inequality (28) with $Q = M$,

$$\begin{aligned} &\leq 2^{ap'} 2^{b\frac{p'}{q}} \left(\frac{|P|^{1-\frac{\alpha}{n}}}{u(P)}\right)^{\frac{p'}{q}} 2^{-k(\frac{p'}{q}\frac{\alpha}{n}+1)} |M|^{\frac{p'}{q}\frac{\alpha}{n}+1} \\ &\lesssim 2^{-k(\frac{p'}{q}\frac{\alpha}{n}+1)} \sigma(M). \end{aligned}$$

If we combine this inequality with (27), we get

$$\begin{aligned} &\sigma\left(\left\{x \in P : \sum_{Q \in \mathcal{Q}_b^\alpha(P)} \chi_Q(x) > k\right\}\right) \\ &\leq \sum_{M \in \mathcal{M}} \sigma\left(\left\{x \in M : \sum_{\substack{Q \in \mathcal{Q}_b^\alpha(P) \\ Q \subseteq M}} \chi_Q(x) > k\right\}\right) \\ &\lesssim 2^{-ck} \sum_{M \in \mathcal{M}} \sigma(M) \\ &\leq 2^{-ck} \sigma(P) \end{aligned}$$

as desired. □

4. Logarithmic bump conditions

In this section, we prove Theorems 1.9, 1.12, 1.14 and 1.15. We first consider the results for log bumps.

THEOREM 4.1. Fix α , $0 < \alpha < n$, and $1 < p < q < \infty$ such that $\frac{p'}{q'}(1 - \frac{\alpha}{n}) \geq 1$. Suppose $\Phi(t) = t^q \log(e + t)^{q-1+\delta}$ for some $\delta > 0$ and (u, σ) is a pair of weights that satisfies

$$K = \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, Q} \|\sigma^{\frac{1}{p'}}\|_{p', Q} < \infty.$$

Then for every dyadic grid \mathcal{D} with sparse subset \mathcal{S} ,

$$(29) \quad [\sigma, u]_{(I_{\mathcal{S}}^{\mathcal{D}})^{q', p'}} \lesssim K.$$

Similarly, if $\frac{q}{p}(1 - \frac{\alpha}{n}) \geq 1$, $\Psi(t) = t^{p'} \log(e + t)^{p'-1+\delta}$ and the pair (u, σ) satisfies

$$K = \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{p, Q} \|\sigma^{\frac{1}{p'}}\|_{\Psi, Q} < \infty,$$

then for every dyadic grid \mathcal{D} with sparse subset \mathcal{S} ,

$$(30) \quad [u, \sigma]_{(I_{\mathcal{S}}^{\mathcal{D}})^{p, q}} \lesssim K.$$

As in the previous section, Theorems 1.9 and 1.14 follow immediately from Theorem 4.1 and the results in Section 2.

For the proof of Theorem 4.1, we need three lemmas. The first is classical, see [3], [34].

LEMMA 4.2. Given a Young function Φ , for every cube Q and functions f and g ,

$$\int |fg| dx \lesssim \|f\|_{\Phi, Q} \|g\|_{\bar{\Phi}, Q}.$$

To state the second, we need a definition. Given a Young function Φ define the corresponding maximal function,

$$M_{\Phi} f(x) = \sup_{Q \ni x} \|f\|_{\Phi, Q}.$$

The following result is due to Pérez [31] (also see [3]).

LEMMA 4.3. Given a Young function Φ and any p , $1 < p < \infty$,

$$\|M_{\Phi} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $\Phi \in B_p$.

The third lemma is from [7].

LEMMA 4.4. Given q , $1 < q < \infty$, let $\Phi(t) = t^q \log(e + t)^{q-1+\delta}$ and $\bar{\Phi}(t) = t^q \log(e + t)^{q-1+\delta/2}$. Then there exists γ , $0 < \gamma < 1$, such that for every cube Q ,

$$(31) \quad \|u^{\frac{1}{q}}\|_{\Phi_0, Q} \lesssim \|u^{\frac{1}{q}}\|_{\Phi, Q}^{1-\gamma} \|u^{\frac{1}{q}}\|_{q, Q}^{\gamma}.$$

Proof of Theorem 4.1. The proof is very similar to the proof of Theorem 3.1 and we will sketch briefly those parts that are the same. As before, we will only prove (29); the proof of (30) is the same after making the obvious changes. Fix $R \in \mathcal{D}$. Then it will suffice to prove that

$$(32) \quad \left(\int_R (I_\alpha^{S(R)} u)^{p'} \sigma \, dx \right)^{1/p'} \lesssim K u(R)^{1/q'}.$$

We decompose the family $\mathcal{S}(R)$; however, in the first step there is a significant difference. For $a \in \mathbb{Z}$ define

$$\mathcal{Q}^a := \left\{ Q \in \mathcal{S}(R) : 2^a < |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\int_Q u \, dx \right)^{\frac{1}{q}} \left(\int_Q \sigma \, dx \right)^{\frac{1}{p'}} \leq 2^{a+1} \right\}.$$

Since $\|u^{\frac{1}{q}}\|_{q,Q} \leq \|u^{\frac{1}{q}}\|_{\Phi,Q}$, by our assumption on (u, σ) , the set \mathcal{Q}^a is empty if $a > \log_2 K$. Therefore, we will sum over a contained in the set

$$\Omega(K) = \mathbb{Z} \cap (-\infty, \log_2 K].$$

With this definition of \mathcal{Q}^a , for $a \in \Omega(K)$, define \mathcal{C}^a , $\mathcal{Q}^a(P)$ and $\mathcal{Q}_b^a(P)$ exactly as before. Then the same argument shows that

$$(33) \quad \begin{aligned} & \left(\int_R (I_\alpha^{S(R)} u)^{p'} \sigma \, dx \right)^{1/p'} \\ & \lesssim \sum_{a \in \Omega(K)} \sum_{b=0}^\infty 2^{-b} \sum_{k=0}^\infty (k+1) \\ & \quad \times \left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(F_b^a(k, P)) \right)^{1/p'}. \end{aligned}$$

As before we have that

$$\sigma(F_b^a(k, P)) \lesssim 2^{-ck} \sigma(P);$$

the proof is essentially the same as the proof of (23): the key difference is that for $Q \in \mathcal{Q}_b^a(P)$ we now have

$$\sigma(Q) \simeq 2^{ap'} 2^{b \frac{p'}{q}} \left(\frac{|P|^{1-\frac{\alpha}{n}}}{u(P)} \right)^{\frac{p'}{q}} |Q|^{\frac{p'}{q'}(1-\frac{\alpha}{n})}.$$

Moreover, we note that it is in this part of the proof that we use the assumption that $\frac{p'}{q}(1-\frac{\alpha}{n}) \geq 1$ in order to pull this power out of the sum. (Cf. the argument immediately following (28).) If we substitute this into (33), we can now sum

in b and k to get

$$\begin{aligned}
 (34) \quad & \left(\int_R (I_\alpha^{S(R)} u)^{p'} \sigma \, dx \right)^{1/p'} \\
 & \lesssim \sum_{a \in \Omega(K)} \sum_{b=0}^\infty \sum_{k=0}^\infty 2^{-ck} (k+1) \left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(P) \right)^{1/p'} \\
 & \lesssim \sum_{a \in \Omega(K)} \left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(P) \right)^{1/p'}.
 \end{aligned}$$

To evaluate the inner sum, we apply Lemma 4.4: since $\sigma(P) = \|\sigma^{\frac{1}{p'}}\|_{p',P}^{p'} |P|$,

$$\begin{aligned}
 & \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \sigma(P) \\
 & = |P|^{\frac{\alpha}{n} p'} \left(\int_P u \, dx \right)^{p'} \sigma(P) \\
 & \lesssim |P|^{\frac{\alpha}{n} p'} \|u^{\frac{1}{q}}\|_{\Phi_{0,P}}^{p'} \|u^{\frac{1}{q'}}\|_{\Phi_{0,P}}^{p'} \sigma(P) \\
 & \lesssim |P|^{\frac{\alpha}{n} p'} \|u^{\frac{1}{q}}\|_{\Phi,P}^{(1-\gamma)p'} \|u^{\frac{1}{q}}\|_{q,P}^{p'\gamma} \|u^{\frac{1}{q'}}\|_{\Phi_{0,P}}^{p'} \sigma(P) \\
 & \lesssim |P|^{\frac{\alpha}{n} p' + \frac{p'}{q} - \frac{p'}{p}} \|u^{\frac{1}{q}}\|_{\Phi,P}^{(1-\gamma)p'} \|\sigma^{\frac{1}{p}}\|_{p,P}^{(1-\gamma)p'} \|u^{\frac{1}{q}}\|_{q,P}^{p'\gamma} \\
 & \quad \times \|\sigma^{\frac{1}{p}}\|_{p,P}^{\gamma p'} \|u^{\frac{1}{q'}}\|_{\Phi_{0,P}}^{p'} |P|^{1 + \frac{p'}{p} - \frac{p'}{q}} \\
 & \lesssim K^{(1-\gamma)p'} 2^{\alpha p' \gamma} \|u^{\frac{1}{q'}}\|_{\Phi_{0,P}}^{p'} |P|^{\frac{p'}{q'}}.
 \end{aligned}$$

Therefore, the inner sum in (34) becomes

$$\left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \sigma(P) \right)^{1/p'} \lesssim K^{1-\gamma} 2^{\gamma a} \left(\sum_{P \in \mathcal{C}^a} \|u^{\frac{1}{q'}}\|_{\Phi_{0,P}}^{p'} |P|^{\frac{p'}{q'}} \right)^{1/p'};$$

since $q'/p' \leq 1$ we may pull this power into the sum, and since the cubes in \mathcal{C}^a are a sparse family we can apply inequality (13) to get

$$\begin{aligned}
 & \lesssim K^{1-\gamma} 2^{\gamma a} \left(\sum_{P \in \mathcal{C}^a} \|u^{\frac{1}{q'}}\|_{\Phi_{0,P}}^{q'} |E(P)| \right)^{1/q'} \\
 & \leq K^{1-\gamma} 2^{\gamma a} \left(\sum_{P \in \mathcal{C}^a} \int_{E(P)} M_{\Phi_0} \left(u^{\frac{1}{q'}} \chi_R \right)^{q'} \, dx \right)^{1/q'} \\
 & \leq K^{1-\gamma} 2^{\gamma a} \left(\int_{\mathbb{R}^n} M_{\Phi_0} \left(u^{\frac{1}{q'}} \chi_R \right)^{q'} \, dx \right)^{1/q'} \\
 & \lesssim K^{1-\gamma} 2^{\gamma a} u(R)^{1/q'}.
 \end{aligned}$$

In the final inequality, we used Lemma 4.3; we can do this because

$$\bar{\Phi}_0(t) \approx t^{q'} \log(e+t)^{-1-\frac{\delta}{2(q-1)}}$$

satisfies the $B_{q'}$ condition.

Finally, given the factor $2^{\gamma a}$, if we plug this estimate into (34), the final sum in a converges and is bounded by K^γ . We therefore get the desired inequality and this completes the proof. \square

REMARK 4.5. In the proof of Theorem 4.1, we actually get a sharper “mixed” estimate. If we define

$$[u, \sigma]_{A_{p,q}^\alpha} := \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\int_Q u \, dx \right)^{\frac{1}{q}} \left(\int_Q \sigma \, dx \right)^{\frac{1}{p'}}$$

then a careful analysis of the constants in the proof shows that we actually get the sharper bound

$$[\sigma, u]_{(I_S^{\alpha})_{q',p'}}^{\mathcal{D}} \lesssim K^{1-\gamma} [u, \sigma]_{A_{p,q}^\alpha}^\gamma \leq K.$$

Moreover, if we modify the definition of Φ_0 by replacing $\delta/2$ by a suitable constant, we can prove that for any ε , $0 < \varepsilon < 1$, we can get the bound

$$[\sigma, u]_{(I_S^{\alpha})_{q',p'}}^{\mathcal{D}} \leq C(\varepsilon) K^{1-\varepsilon} [u, \sigma]_{A_{p,q}^\alpha}^\varepsilon,$$

where $C(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 1$. Details are left to the interested reader.

We now prove Theorems 1.12 and 1.15. To do so, we need to extend Theorem 4.1 to the scale of loglog bumps.

THEOREM 4.6. *The conclusions of Theorem 4.1 remain true with the same hypotheses if we replace the Young functions Φ and Ψ with*

$$\begin{aligned} \Phi(t) &= t^q \log(e+t)^{q-1} \log \log(e^e+t)^{q-1+\delta}, \\ \Psi(t) &= t^{p'} \log(e+t)^{p'-1} \log \log(e^e+t)^{p'-1+\delta}, \end{aligned}$$

where $\delta > 0$ is taken sufficiently large.

We will briefly sketch the proof of Theorem 4.6, as it is very similar to the proof of Theorem 4.1. The main difference is that we must replace Lemma 4.4 with the following result which was also proved in [7].

LEMMA 4.7. *Given q , $1 < q < \infty$, let*

$$\Phi(t) = t^q \log(e+t)^{q-1} \log \log(e^e+t)^{q-1+\delta}$$

and

$$\Phi_0(t) = t^q \log(e+t)^{q-1} \log \log(e^e+t)^{q-1+\delta/2}.$$

Then

$$\|u^{\frac{1}{q}}\|_{\Phi_0, Q} \lesssim \|u^{\frac{1}{q}}\|_{\Phi, Q} \cdot \phi\left(\frac{\|u^{\frac{1}{q}}\|_{q, Q}}{\|u^{\frac{1}{q}}\|_{\Phi, Q}}\right),$$

where $\phi(t) = \log(C/t)^{-\kappa}$ with κ, C constants. Moreover if $\delta > 0$ is sufficiently large, then $\kappa > 1$.

Sketch of the proof of Theorem 4.6. We use the same notation as in Theorem 4.1. The proof is identical until estimate (34):

$$\begin{aligned} & \left(\int_R (I_\alpha^{S(R)} u)^{p'} \sigma \, dx \right)^{1/p'} \\ & \lesssim \sum_{a \in \Omega(K)} \sum_{b=0}^\infty 2^{-b} \sum_{k=0}^\infty 2^{-ck} (k+1) \left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(P) \right)^{1/p'}. \end{aligned}$$

Once again we can sum the series in b and k , and the problem is to sum the series in a , where

$$a \in \Omega(K) = \mathbb{Z} \cap (-\infty, \log_2 K].$$

At this stage, we use Lemma 4.7 to estimate the last sum. We have that

$$\begin{aligned} & \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \sigma(P) \\ & = |P|^{\frac{\alpha}{n} p'} \left(\int_P u \, dx \right)^{p'} \sigma(P) \lesssim |P|^{\frac{\alpha}{n} p'} \|u^{\frac{1}{q}}\|_{\Phi_0, P}^{p'} \|u^{\frac{1}{q'}}\|_{\Phi_0, P}^{p'} \sigma(P) \\ & \lesssim \left(|P|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, P} \left(\int_P \sigma \, dx \right)^{\frac{1}{p'}} \right)^{p'} \cdot \phi \left(\frac{\|u^{\frac{1}{q}}\|_{q, P}}{\|u^{\frac{1}{q}}\|_{\Phi, P}} \right)^{p'} \cdot \|u^{\frac{1}{q'}}\|_{\Phi_0, P} |P|^{\frac{p'}{q'}}. \end{aligned}$$

To estimate this term, we need to further divide the sum over $P \in \mathcal{C}^a$. Recall that if $P \in \mathcal{C}^a$, then

$$2^{a-1} < |P|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\int_P u \, dx \right)^{\frac{1}{q}} \left(\int_P \sigma \, dx \right)^{\frac{1}{p'}} \leq 2^a.$$

Moreover, if $P \in \mathcal{C}^a$ then since $\|u^{\frac{1}{q}}\|_{q, P} \leq \|u^{\frac{1}{q}}\|_{\Phi, P}$, there exists an integer c , $a \leq c \leq \log_2 K$, such that

$$(35) \quad 2^{c-1} < |P|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, P} \left(\int_P \sigma \, dx \right)^{\frac{1}{p'}} \leq 2^c.$$

For each such a and c , let \mathcal{C}_c^a be the collection of all cubes $P \in \mathcal{C}^a$ such that (35) holds. Then we can estimate as follows:

$$\begin{aligned} & \left(\sum_{P \in \mathcal{C}^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(P) \right)^{\frac{1}{p'}} \\ & = \left(\sum_{\substack{c \in \Omega(K) \\ c \geq a}} \sum_{P \in \mathcal{C}_c^a} \left(|P|^{\frac{\alpha}{n}} \int_P u \, dx \right)^{p'} \cdot \sigma(P) \right)^{\frac{1}{p'}} \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{\substack{c \in \Omega(K) \\ c \geq a}} \left(\sum_{P \in \mathcal{C}_c^a} \left(|P|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\Phi, P} \left(\int_P \sigma \, dx \right)^{\frac{1}{p'}} \right)^{p'} \right. \\ &\quad \cdot \phi \left(\frac{\|u^{\frac{1}{q}}\|_{q, P}}{\|u^{\frac{1}{q}}\|_{\Phi, P}} \right)^{p'} \times \|u^{\frac{1}{q'}}\|_{\bar{\Phi}_0, P} |P|^{\frac{p'}{q'}} \Big)^{\frac{1}{p'}} \\ &\lesssim \sum_{\substack{c \in \Omega(K) \\ c \geq a}} 2^c \left(\sum_{P \in \mathcal{C}_c^a} \phi \left(\frac{\|u^{\frac{1}{q}}\|_{q, P}}{\|u^{\frac{1}{q}}\|_{\Phi, P}} \right)^{p'} \cdot \|u^{\frac{1}{q'}}\|_{\bar{\Phi}_0, P}^p |P|^{\frac{p'}{q'}} \right)^{\frac{1}{p'}}. \end{aligned}$$

If $P \in \mathcal{C}_c^a$, then

$$\frac{\|u^{\frac{1}{q}}\|_{q, P}}{\|u^{\frac{1}{q}}\|_{\Phi, P}} \simeq 2^{a-c},$$

and so

$$\phi \left(\frac{\|u^{\frac{1}{q}}\|_{q, P}}{\|u^{\frac{1}{q}}\|_{\Phi, P}} \right) \simeq \frac{1}{(1 + c - a)^\kappa}.$$

Given this the rest of proof proceeds exactly as before: we have that

$$\left(\sum_{P \in \mathcal{C}_c^a} \|u^{\frac{1}{q'}}\|_{\bar{\Phi}_0, P}^{p'} |P|^{\frac{p'}{q'}} \right)^{\frac{1}{p'}} \lesssim \left(\int_{\mathbb{R}^n} M_{\bar{\Phi}_0} (u^{\frac{1}{q'}} \chi_R)^{q'} \, dx \right)^{1/q'} \lesssim u(R)^{1/q'},$$

since

$$\bar{\Phi}_0(t) \simeq \frac{t^{q'}}{\log(e + t) \log \log(e^e + t)^{1 + \frac{\delta}{2(q-1)}}}$$

belongs to $B_{q'}$. Moreover, the double sum

$$\sum_{a \in \Omega(K)} \sum_{\substack{c \in \Omega(K) \\ c \geq a}} \frac{2^c}{(1 + c - a)^\kappa} = \sum_{c=-\infty}^{\log_2 K} 2^c \sum_{a=-\infty}^c \frac{1}{(1 + c - a)^\kappa} \lesssim K$$

converges if we assume that δ is large enough that $\kappa > 1$. □

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