

INVERSE SEMIGROUP EXPANSIONS AND THEIR ACTIONS ON C^* -ALGEBRAS

ALCIDES BUSS AND RUY EXEL

ABSTRACT. In this work, we give a presentation of the prefix expansion $\mathbf{Pr}(G)$ of an inverse semigroup G as recently introduced by Lawson, Margolis and Steinberg which is similar to the universal inverse semigroup defined by the second named author in case G is a group. The inverse semigroup $\mathbf{Pr}(G)$ classifies the partial actions of G on spaces. We extend this result and prove that Fell bundles over G correspond bijectively to saturated Fell bundles over $\mathbf{Pr}(G)$. In particular, this shows that twisted partial actions of G (on C^* -algebras) correspond to twisted (global) actions of $\mathbf{Pr}(G)$. Furthermore, we show that this correspondence preserves C^* -algebra crossed products.

1. Introduction

Modifying the construction of the Birget–Rhodes prefix expansion of a semigroup [1], Lawson, Margolis and Steinberg [8] introduced a generalized prefix expansion of an inverse semigroup G as follows: let $\mathbf{Pr}(G)$ be the collection of all pairs (A, t) , where $t \in G$, and A is a finite subset of G satisfying

$$tt^*, t \in A \quad \text{and} \quad ss^* = tt^* \quad \text{for all } s \in A.$$

With the operation

$$(1.1) \quad (A, t)(B, s) = (ts^*t^*A \cup tB, ts),$$

$\mathbf{Pr}(G)$ turns out to be an inverse semigroup, and in the main application presented in [8], it is proved that $\mathbf{Pr}(G)$ possesses a universal property with re-

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spect to partial actions of G : partial actions (resp. representations/homomorphisms) of G correspond bijectively to global actions (resp. representations/homomorphisms) of $\mathbf{Pr}(G)$.

The above explicit construction of $\mathbf{Pr}(G)$ is suggested via a McAlister–O’Carroll triple [7] constructed from G in a somewhat involving way.

The first goal of this paper is to obtain an alternate description of $\mathbf{Pr}(G)$ in terms of generators and relations which emphasizes its role as governing partial representations (or homomorphisms) of G . Our method for doing so is independent of [8] and is very similar to the one adopted in [4], where an inverse semigroup $\mathcal{S}(G)$ was constructed from a group G , possessing a universal property related to partial representations and which was later shown by Kellendonk and Lawson [6] to be precisely the Birget–Rhodes expansion of G . Needless to say, in case G is a group, the prefix expansion $\mathbf{Pr}(G)$ coincides with the inverse semigroup $\mathcal{S}(G)$ constructed in [4].

The interest in approaching $\mathbf{Pr}(G)$ from a different route is, in part, to show that its structure, including the product operation (1.1) above, is a direct consequence of the concept of partial representations. In addition, we answer some natural questions about the expansion $\mathbf{Pr}(G)$ which did not take place in [8], as for instance what happens with $\mathbf{Pr}(\mathbf{Pr}(G))$ as well as the order of $\mathbf{Pr}(G)$ if G is finite.

As already observed in [8], the assignment $G \mapsto \mathbf{Pr}(G)$ is a functor on the category Inv of inverse semigroups (with homomorphisms as morphisms). It may also be viewed as a functor from the category PInv of inverse semigroups with partial homomorphisms as morphisms to Inv . When viewed in this way, we show that it is the left adjoint of the forgetful functor $\text{Inv} \rightarrow \text{PInv}$. This is related to the result in [11] where Szendrei shows that $G \mapsto \mathbf{Pr}(G)$, when considered as a functor from the category of groups to F -inverse semigroups, is left adjoint of to the functor assigning to an inverse semigroup its maximal homomorphic group image.

One of the main goals in this paper is to study the relation between Fell bundles and crossed products by G and $\mathbf{Pr}(G)$. We prove that Fell bundles over G correspond bijectively to *saturated* Fell bundles over $\mathbf{Pr}(G)$ in a somewhat canonical way. In fact, this correspondence extends to a functor and gives an equivalence between the categories of Fell bundles over G and saturated Fell bundles over $\mathbf{Pr}(G)$. Moreover, this equivalence preserves the associated (full and reduced) cross-sectional C^* -algebras. In particular, the functor $G \mapsto \mathbf{Pr}(G)$ preserves classical partial crossed products, that is, $A \rtimes G \cong A \rtimes \mathbf{Pr}(G)$ for any partial action of G on a C^* -algebra A and the corresponding (global) action of $\mathbf{Pr}(G)$ on A . A similar result still holds for twisted partial actions. In fact, all these results are special cases of the more general theorem we obtain for Fell bundles and their corresponding C^* -algebras.

2. The expansion

Throughout this section, we will let G be an inverse semigroup.

DEFINITION 2.1. We shall let $\mathbf{Pr}(G)$ denote the universal semigroup defined via generators and relations as follows: to each element s in G we take a generator $[s]$ (from any fixed set having as many elements as G), and for every $s, t \in G$ we consider the relations

- (i) $[s][t][t^*] = [st][t^*]$,
- (ii) $[s^*][s][t] = [s^*][st]$,
- (iii) $[s][s^*][s] = [s]$.

The following result is inspired by [4, Proposition 2.4].

PROPOSITION 2.2. For every t in G , let $\varepsilon_t = [t][t^*]$. Then, for each t and s in G ,

- (i) ε_t is idempotent,
- (ii) $[t]\varepsilon_s = \varepsilon_{ts}[t]$,
- (iii) ε_t and ε_s commute.

Proof. Notice that (i) follows immediately from Definition 2.1(iii). With respect to (iii), we have

$$(2.1) \quad \begin{aligned} \varepsilon_s \varepsilon_t &= [s][s^*][t][t^*] = [s][s^*t][t^*] \\ &= [s][s^*t][t^*s][t^*] = [ss^*t][t^*s][s^*tt^*]. \end{aligned}$$

Interchanging the roles of s and t , the above computation gives

$$\begin{aligned} \varepsilon_t \varepsilon_s &= [tt^*s][s^*t][t^*ss^*] = [tt^*s][s^*t][t^*s][s^*t][t^*ss^*] \\ &= [tt^*ss^*t][t^*s][s^*tt^*ss^*] \\ &= [ss^*tt^*t][t^*s][s^*ss^*tt^*] = [ss^*t][t^*s][s^*tt^*] \stackrel{(2.1)}{=} \varepsilon_s \varepsilon_t. \end{aligned}$$

Addressing (ii) observe that

$$\begin{aligned} [t]\varepsilon_s &= [t][s][s^*] = [t][t^*][t][s][s^*] \stackrel{(iii)}{=} [t][s][s^*][t^*][t] \\ &= [ts][s^*][t^*][t] = [ts][s^*t^*][t] = \varepsilon_{ts}[t]. \end{aligned} \quad \square$$

In the next result, we write $E(G)$ for the idempotent semilattice of G .

PROPOSITION 2.3. Given $e \in E(G)$ and $s \in G$ one has that:

- (i) $\varepsilon_e = [e]$, and hence $[e]$ is idempotent,
- (ii) $[e][s] = [es]$, and $[s][e] = [se]$,
- (iii) $\varepsilon_e \varepsilon_s = \varepsilon_{es}$.

Proof. (i) This follows from

$$[e] = [e][e^*][e] = [e][e^*e] = [e][e^*] = \varepsilon_e.$$

In order to prove (ii), we have by (i) that

$$\begin{aligned} [e][s] &= [e][e][s] = [e][es] = [e][es][(es)^*][es] = [ees][(es)^*][es] \\ &= [es][(es)^*][es] = [es]. \end{aligned}$$

The second part of (ii) follows similarly. As for (iii), we have

$$\varepsilon_e \varepsilon_s = \varepsilon_e \varepsilon_s \varepsilon_e \stackrel{(i)}{=} [e][s][s^*][e] \stackrel{(ii)}{=} [es][s^*e] = \varepsilon_{es}. \quad \square$$

LEMMA 2.4. *Given $s_1, \dots, s_n, t \in G$, let $x = \varepsilon_{s_1} \cdots \varepsilon_{s_n}[t]$. Then*

$$x = \varepsilon_{s'_1} \varepsilon_{s'_2} \cdots \varepsilon_{s'_n}[t'],$$

where $t' = pt$, $s'_i = ps_i$, for all $i = 1, \dots, n$, and $p = s_1 s_1^* s_2 s_2^* \cdots s_n s_n^* t t^*$.

Proof. Letting $f = tt^*$ and $e_i = s_i s_i^*$, for all $i = 1, \dots, n$, observe that, for each i , one has

$$\varepsilon_{s_i} = \varepsilon_{e_i s_i} \stackrel{\text{Proposition 2.3(iii)}}{=} \varepsilon_{e_i} \varepsilon_{s_i},$$

while

$$[t] = [t][t^*][t] = [tt^*][t] = [f][t] \stackrel{\text{Proposition 2.3(i)}}{=} \varepsilon_f[t].$$

So

$$\begin{aligned} x &= \varepsilon_{s_1} \varepsilon_{s_2} \cdots \varepsilon_{s_n}[t] = \varepsilon_{e_1} \varepsilon_{s_1} \varepsilon_{e_2} \varepsilon_{s_2} \cdots \varepsilon_{e_n} \varepsilon_{s_n} \varepsilon_f[t] \\ &= \varepsilon_{e_1} \varepsilon_{e_2} \cdots \varepsilon_{e_n} \varepsilon_f \varepsilon_{s_1} \varepsilon_{s_2} \cdots \varepsilon_{s_n}[t] \stackrel{\text{Proposition 2.3(iii)}}{=} \varepsilon_p \varepsilon_{s_1} \varepsilon_{s_2} \cdots \varepsilon_{s_n}[t] \\ &= \varepsilon_p \varepsilon_{s_1} \varepsilon_p \varepsilon_{s_2} \cdots \varepsilon_p \varepsilon_{s_n} \varepsilon_p[t] = \varepsilon_{ps_1} \varepsilon_{ps_2} \cdots \varepsilon_{ps_n}[pt]. \end{aligned} \quad \square$$

PROPOSITION 2.5. *Every element $x \in \mathbf{Pr}(G)$ may be written as a product*

$$(2.2) \quad x = \varepsilon_{s_1} \cdots \varepsilon_{s_n}[t],$$

where $s_1, \dots, s_n, t \in G$. One may moreover assume that

- (a) $s_1 s_1^* = \cdots = s_n s_n^* = tt^*$, and
- (b) t and tt^* belong to the set $\{s_1, \dots, s_n\}$.

Proof. Given $x \in \mathbf{Pr}(G)$ write $x = [r_1][r_2] \cdots [r_m]$, where the $r_i \in G$. We shall prove the first sentence of the statement by induction on m . The case $m = 1$ follows from

$$x = [r_1] = [r_1][r_1^*][r_1] = \varepsilon_{r_1}[r_1].$$

Assuming that $m \geq 2$, use the induction hypothesis to write

$$[r_1][r_2] \cdots [r_{m-1}] = \varepsilon_{s_1} \cdots \varepsilon_{s_n}[t],$$

with $s_1, \dots, s_n, t \in G$. Next observe that

$$(2.3) \quad [t][r_m] = [t][t^*][t][r_m] = [t][t^*][tr_m] = \varepsilon_t[tr_m].$$

Therefore

$$x = [r_1][r_2] \cdots [r_m] = \varepsilon_{s_1} \cdots \varepsilon_{s_n}[t][r_m] \stackrel{(2.3)}{=} \varepsilon_{s_1} \cdots \varepsilon_{s_n} \varepsilon_t[tr_m],$$

proving the first sentence in the statement.

In order to prove (a) write $x = \varepsilon_{s_1} \varepsilon_{s_2} \cdots \varepsilon_{s_n} [t]$, using what we have already proved, and let t', s'_i , and p , be as in Lemma 2.4. Then the condition in (a) is verified since

$$s'_i s_i'^* = p s_i s_i^* p = p = p t t^* p = t' t'^*.$$

To conclude, we need to check (b) but this follows from

$$[t] = [t][t^*][t] = \varepsilon_t [t] = \varepsilon_t [t t^* t] \stackrel{\text{Proposition 2.3(ii)}}{=} \varepsilon_t [t t^*] [t] = \varepsilon_t \varepsilon_{t t^*} [t]. \quad \square$$

DEFINITION 2.6. If $x \in \mathbf{Pr}(G)$ is written as in (2.2) in such a way that Proposition 2.5(a), (b) are satisfied, we will say that x is in *normal form*.

Suppose that an element $x \in \mathbf{Pr}(G)$ is written as $x = \varepsilon_{s_1} \varepsilon_{s_2} \cdots \varepsilon_{s_n} [t]$, as in (2.2), but not necessarily in normal form. Letting $A = \{s_1, s_2, \dots, s_n\}$ we will use the notation

$$(2.4) \quad \varepsilon_A := \varepsilon_{s_1} \varepsilon_{s_2} \cdots \varepsilon_{s_n},$$

which is unambiguously defined since the ε_{s_i} commute with each other by Proposition 2.2(iii). Employing this notation, we may therefore write

$$(2.5) \quad x = \varepsilon_A [t].$$

Assuming now that x is in normal form, notice that Proposition 2.5(a), (b) may be rephrased as

$$(2.6) \quad t, t t^* \in A \quad \text{and} \quad s s^* = t t^* \quad \text{for all } s \in A.$$

DEFINITION 2.7. Given $e \in E(G)$ we will say that a subset $A \subseteq G$ is an *e*-set if

- (i) $e \in A$, and
- (ii) $s s^* = e$, for all $s \in A$.

Observe that condition (2.6) may be rephrased by saying that A is a finite $t t^*$ -set containing t . We may therefore summarize our findings as follows:

PROPOSITION 2.8. *Every element $x \in \mathbf{Pr}(G)$ may be written as*

$$x = \varepsilon_A [t],$$

where $t \in G$, and A is a finite $t t^*$ -set containing t , in which case we say that x is in normal form.

Let us now give an expression for multiplying elements of $\mathbf{Pr}(G)$.

PROPOSITION 2.9. *Let $A, B \subseteq G$ be finite subsets, let $t, s \in G$, and let $x = \varepsilon_A [t]$ and $y = \varepsilon_B [s]$. Assuming that $t \in A$, one has that*

$$xy = \varepsilon_{A \cup (tB)} [ts] = \varepsilon_{(t s s^* t^* A) \cup (tB)} [ts].$$

Assuming that both x and y are in normal form (in which case the above requirement that $t \in A$ is automatically satisfied) one moreover has that the second expression for xy above is in normal form.

Proof. Using Proposition 2.2(ii), it is easy to see that $[t]\varepsilon_B = \varepsilon_{(tB)}[t]$, so

$$xy = \varepsilon_A[t]\varepsilon_B[s] = \varepsilon_A\varepsilon_{(tB)}[t][s] = \varepsilon_{A\cup(tB)}[t][s] = \dots$$

Letting $B = A \cup (tB)$, the above equals

$$\dots = \varepsilon_B[t][t^*][t][s] = \varepsilon_B[t][t^*][ts] = \varepsilon_B\varepsilon_t[ts] = \varepsilon_{B\cup\{t\}}[ts] = \varepsilon_B[ts],$$

proving the first expression for xy above. Next, observe that

$$[ts] = [ts][s^*t^*][ts] = [tss^*t^*][ts] = \varepsilon_q[ts],$$

where $q = tss^*t^*$. Therefore,

$$\begin{aligned} xy &= \varepsilon_{A\cup(tB)}\varepsilon_q[ts] = \varepsilon_q\varepsilon_A\varepsilon_{(tB)}[ts] \\ &\stackrel{\text{Proposition 2.3(iii)}}{=} \varepsilon_{(qA)}\varepsilon_{(tB)}[ts] = \varepsilon_{(qA)\cup(tB)}[ts], \end{aligned}$$

proving the second expression for xy above. Assuming that $x_i = \varepsilon_{A_i}[t_i]$ is in normal form for $i = 1, 2$, we have for all $s \in A$ that $ss^* = tt^*$, so

$$qs(qs)^* = qss^* = tss^*t^*tt^* = tss^*t^*,$$

while for $s \in B$, one has that $ss^* = ss^*$, so

$$ts(ts)^* = tss^*t^* = tss^*t^*.$$

Moreover, since $s \in B$,

$$ts \in tB \subseteq (qA) \cup (tB),$$

and, because $tt^* \in A$,

$$tss^*t^* = tss^*t^*tt^* \in qA \subseteq (qA) \cup (tB),$$

hence proving the last assertion. □

Proposition 2.5 may be used to prove that $\mathbf{Pr}(G)$ is a regular semigroup.

PROPOSITION 2.10. *For every $x \in \mathbf{Pr}(G)$ there exists $\bar{x} \in \mathbf{Pr}(G)$ such that $x\bar{x}x = x$ and $\bar{x}x\bar{x} = \bar{x}$.*

Proof. Given x in G , use Proposition 2.5 to write $x = \varepsilon_{s_1} \dots \varepsilon_{s_n}[t]$, with $s_1, \dots, s_n, t \in G$, and set $\bar{x} = [t^*]\varepsilon_{s_1} \dots \varepsilon_{s_n}$. Then, using Proposition 2.2(iii), we get

$$\begin{aligned} x\bar{x}x &= \varepsilon_{s_1} \dots \varepsilon_{s_n}[t][t^*]\varepsilon_{s_1} \dots \varepsilon_{s_n}\varepsilon_{s_1} \dots \varepsilon_{s_n}[t] \\ &= \varepsilon_{s_1} \dots \varepsilon_{s_n}[t][t^*][t] = \varepsilon_{s_1} \dots \varepsilon_{s_n}[t] = x. \end{aligned}$$

The proof that $\bar{x}x\bar{x} = \bar{x}$ is just as easy. □

Before we proceed, we need to introduce the following important concept:

DEFINITION 2.11. Let H be a semigroup. A *partial homomorphism* of G in H is a map

$$\pi : G \rightarrow H$$

such that for all $s, t \in G$, one has that

- (i) $\pi(s)\pi(t)\pi(t^*) = \pi(st)\pi(t^*)$,
- (ii) $\pi(s^*)\pi(s)\pi(t) = \pi(s^*)\pi(st)$,
- (iii) $\pi(s)\pi(s^*)\pi(s) = \pi(s)$.

Notice that the definition of partial homomorphisms does not require H to be an *inverse* semigroup. However, should H happen to be an inverse semigroup, then Definition 2.11(iii) applied to both s and s^* , together with the uniqueness of inverses, immediately implies that

$$(2.7) \quad \pi(s^*) = \pi(s)^* \quad \text{for all } s \in G.$$

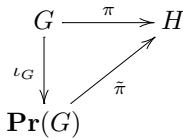
Hence, if H is an inverse semigroup, the axioms (i)–(iii) in Definition 2.11 are equivalent to (i)–(ii) plus (2.7). If H is the semigroup $\mathcal{L}(\mathcal{H})$ of all bounded linear maps on a Hilbert space \mathcal{H} with composition of operators as the semigroup multiplication, partial homomorphisms $G \rightarrow H$ are also called *partial representations* of G on \mathcal{H} .

It is evident that the map

$$(2.8) \quad \iota_G : G \rightarrow \mathbf{Pr}(G), \quad s \in G \mapsto [s] \in \mathbf{Pr}(G)$$

is a partial homomorphism of G in $\mathbf{Pr}(G)$, and we will call it the *canonical partial homomorphism*. The universal property of $\mathbf{Pr}(G)$ tells us that every partial homomorphism of G factors through the canonical one. More precisely:

PROPOSITION 2.12. *For every semigroup H and every partial homomorphism $\pi : G \rightarrow H$, there exists a unique semigroup homomorphism $\tilde{\pi} : \mathbf{Pr}(G) \rightarrow H$, such that the diagram*



commutes, where the vertical arrow is the canonical partial homomorphism.

Proof. Obvious from the definition of $\mathbf{Pr}(G)$. □

This result has a very useful consequence, as follows.

COROLLARY 2.13. *Let $\pi : G \rightarrow H$ be a partial homomorphism of G in a semigroup H . Then, given any $e \in E(G)$ one has that*

- (i) $\pi(e)$ is idempotent,
- (ii) if $s \in G$, then $\pi(e)\pi(s) = \pi(es)$, and $\pi(s)\pi(e) = \pi(se)$.

Proof. Letting $\tilde{\pi}$ be as in Proposition 2.12 we have

$$\pi(e)\pi(s) = \tilde{\pi}([e])\tilde{\pi}([s]) = \tilde{\pi}([e][s]) \stackrel{\text{Proposition 2.3(ii)}}{=} \tilde{\pi}([es]) = \pi(es),$$

proving the first part of (ii), while the second part follows similarly. As for (i), it follows from (ii) by taking $s = e$. □

If we take π to be the identity map on G , then it is evidently a partial homomorphism of G in itself, so Proposition 2.12 provides a semigroup homomorphism

$$(2.9) \quad \partial : \mathbf{Pr}(G) \rightarrow G,$$

called the *degree map*, such that $\partial([s]) = s$, for all s in G .

PROPOSITION 2.14. *Let $x \in \mathbf{Pr}(G)$ be written in normal form as $x = \varepsilon_A[t]$. Then $\partial(x) = t$. Moreover, x is idempotent if and only if $\partial(x) = t$ is idempotent, in which case we have $x = \varepsilon_A$. Hence ∂ is an essentially injective homomorphism in the sense that pre-images of idempotents are idempotents.¹ In particular, the idempotents of $\mathbf{Pr}(G)$ have the form ε_A for some finite subset $A \subseteq G$ which can be chosen to be an e -set for some $e \in E(G)$.*

Proof. We have

$$\partial(x) = \partial(\varepsilon_{s_1} \cdots \varepsilon_{s_n}[t]) = \partial([s_1][s_1^*] \cdots [s_n][s_n^*][t]) = s_1 s_1^* \cdots s_n s_n^* t = t t^* t = t.$$

In particular, if x is idempotent then so is t . In this case, $t = \varepsilon_t$ (by Proposition 2.3(i)) so that $x = \varepsilon_A \varepsilon_t = \varepsilon_{A \cup \{t\}} = \varepsilon_A$ because A contains t . \square

THEOREM 2.15. $\mathbf{Pr}(G)$ is an inverse semigroup.

Proof. We have already seen in Proposition 2.10 that $\mathbf{Pr}(G)$ is regular. It therefore suffices to prove that the idempotent elements of $\mathbf{Pr}(G)$ commute (see [7, Chapter 1, Theorem 3]). By Proposition 2.14, every idempotent of $\mathbf{Pr}(G)$ has the form ε_A for some $A \subseteq G$. The fact that idempotents commute then follows from Proposition 2.2(iii). \square

Recall that an inverse semigroup G is E -unitary if for all $s \in G$ and $e \in E(G)$,

$$es \in E(G) \quad \Rightarrow \quad s \in E(G).$$

PROPOSITION 2.16. *The inverse semigroup $\mathbf{Pr}(G)$ is E -unitary if and only if G is E -unitary.*

Proof. Suppose that G is E -unitary, and let $x \in \mathbf{Pr}(G)$ and $y \in E(\mathbf{Pr}(G))$ with $yx \in E(\mathbf{Pr}(G))$. Letting $e := \partial(y) \in E(G)$ and $s := \partial(x) \in G$ we then get $es = \partial(yx) \in E(G)$. Since G is E -unitary, we have $\partial(x) = s \in E(G)$, so that $x \in E(\mathbf{Pr}(G))$ because ∂ is essentially injective.

Conversely, assume that $\mathbf{Pr}(G)$ is E -unitary and let $s \in G$ and $e \in E(G)$ with $es \in E(G)$. Then $[e][s] = [es] \in E(\mathbf{Pr}(G))$ (by Proposition 2.3(i)) so that $[s] \in E(\mathbf{Pr}(G))$ because $\mathbf{Pr}(G)$ is E -unitary. Therefore, $s = \partial([s]) \in E(G)$. \square

¹ Essentially injective homomorphisms are also called *idempotent pure* homomorphisms in [8].

Before ending this section, let us remark at this point that our inverse semigroup $\mathbf{Pr}(G)$ is canonically isomorphic to the inverse semigroup

$$H = \{(A, t) : A \subseteq G \text{ finite, } t, tt^* \in A \text{ and } ss^* = tt^* \text{ for all } s \in A\}$$

introduced in [8] (see our [Introduction](#)). In fact, this obviously follows from the same universal property (of classifying partial homomorphisms) that both have. An explicit isomorphism $\mathbf{Pr}(G) \cong H$ is given by taking an element $x = \epsilon_A[t] \in \mathbf{Pr}(G)$ in normal form and sending it to the pair $(A, t) \in H$. This is the same map one gets by first considering the map $\pi : G \rightarrow H, t \mapsto (\{tt^*, t\}, t)$ and checking that π is a partial homomorphism. The corresponding homomorphism $\tilde{\pi} : \mathbf{Pr}(G) \rightarrow H$ obtained from the universal property of $\mathbf{Pr}(G)$ is then equal to the map $\epsilon_A[t] \mapsto (A, t)$ just mentioned. In particular, this shows that this map is well-defined and, even more important, that normal forms are unique: if $x = \epsilon_A[t] = \epsilon_B[s]$ are two ways of writing x in normal form, then $A = B$ and $s = t$. Later, in Section 5, we are going to prove uniqueness of normal forms without using the inverse semigroup H above: we construct a canonical partial action of G that “separates” normal forms.

3. Partial homomorphisms and partial actions

In Definition 2.11, we defined partial homomorphisms of inverse semigroups. Now we will find equivalent conditions for checking that a map into an inverse semigroup is a partial homomorphism. As before, we fix an inverse semigroup G .

PROPOSITION 3.1. *Let H be an inverse semigroup and let $\pi : G \rightarrow H$ be a map. Then π is a partial homomorphism if and only if, for all s and t in G , one has that*

- (i) $\pi(s^*) = \pi(s)^*$,
- (ii) $\pi(s)\pi(t) \leq \pi(st)$, and
- (iii) $\pi(s) \leq \pi(t)$ whenever $s \leq t$.

Proof. Assume that π is a partial homomorphism. Then (i) holds by (2.7). Next pick $s, t \in G$, and let $f = \pi(t^*)\pi(t)$. We then have that f is idempotent by (i), and

$$\pi(st)f = \pi(st)\pi(t^*)\pi(t) = \pi(s)\pi(t)\pi(t^*)\pi(t) = \pi(s)\pi(t),$$

proving (ii). Assuming that $s \leq t$, write $s = te$, for some idempotent e . Then $\pi(e)$ is idempotent by Corollary 2.13(i) and

$$\pi(t)\pi(e) \stackrel{\text{Corollary 2.13(ii)}}{=} \pi(te) = \pi(s),$$

proving (iii).

Conversely, suppose that (i)–(iii) hold. Given $s, t \in G$, we have

$$\begin{aligned} \pi(s)\pi(t)\pi(t^*) &\leq \pi(st)\pi(t^*) = \pi(st)\pi(t^*)\pi(t)\pi(t^*) \\ &\leq \pi(stt^*)\pi(t)\pi(t^*) \leq \pi(s)\pi(t)\pi(t^*), \end{aligned}$$

where in the last step we have used (iii) and the fact that $stt^* \leq s$. This gives Definition 2.11(i), and 2.11(ii) follows similarly. The last axiom of Definition 2.11 follows immediately from (i). □

REMARK 3.2. Maps $\pi : G \rightarrow H$ satisfying conditions (i)–(iii) of Proposition 3.1 are sometimes called *dual pre-homomorphisms* by some authors (as for instance, in [8]).

Let us specialize to partial homomorphisms in symmetric inverse semigroups, but first we would like to introduce some terminology.

DEFINITION 3.3. Let G be an inverse semigroup and let X be a set. By a *partial action* of G on X we shall mean a partial homomorphism

$$\pi : G \rightarrow \mathcal{I}(X),$$

where $\mathcal{I}(X)$ denotes the symmetric inverse semigroup on X .

Given a partial action π , as above, we shall write π_s for $\pi(s)$.

PROPOSITION 3.4. *Let X be a set and let $\pi : G \rightarrow \mathcal{I}(X)$ be a map. For each $s \in G$, let X_s be the range of π_s . Then π is a partial action if and only if, for all $s, t \in G$, the following holds:*

- (i) $\pi_s^{-1} = \pi_{s^*}$ (in particular this implies that the domain of π_s coincides with the range of π_{s^*} , namely X_{s^*}),
- (ii) $\pi_s(X_{s^*} \cap X_t) = X_s \cap X_{st}$,
- (iii) for every $x \in X_{t^*} \cap X_{t^*s^*}$, one has that $\pi_s(\pi_t(x)) = \pi_{st}(x)$.

Proof. Initially observe that, under (i) and (ii), the composition $\pi_s(\pi_t(x))$ appearing in (iii) is meaningful, since

$$\pi_t(x) \in \pi_t(X_{t^*} \cap X_{t^*s^*}) = \pi_t(\pi_{t^*}(X_t \cap X_{s^*})) = X_t \cap X_{s^*} \subseteq X_{s^*}.$$

Assume first that (i)–(iii) hold. In order to prove that π is a partial action we will use Proposition 3.1. Since Proposition 3.1(i) is granted, let us attack Proposition 3.1(ii). For this let $s, t \in G$, and notice that the domain of $\pi_s\pi_t$ coincides with

$$\pi_{t^*}(X_t \cap X_{s^*}) = X_{t^*} \cap X_{t^*s^*},$$

by (ii). Evidently this is contained in $X_{t^*s^*}$, also known as the domain of π_{st} . By (iii) we see that $\pi_s\pi_t$ coincides with π_{st} on the domain of the former, which means that $\pi_s\pi_t \leq \pi_{st}$, proving Proposition 3.1(ii).

Let us now study the behavior of π_e , for an idempotent $e \in E(G)$. By (iii) with $s = t = e$, we deduce that for every $x \in X_e$, one has that $\pi_e(\pi_e(x)) =$

$\pi_e(x)$, but since $\pi_e(x)$ is injective it must be that $\pi_e(x) = x$. In other words, π_e is the identity map on X_e .

Now let $s, t \in G$ with $s \leq t$, so that $s = ts^*s$. As seen above π_{s^*s} is the identity on X_{s^*s} , so

$$\begin{aligned} X_{s^*s} \cap X_{t^*} &= \pi_{s^*s}(X_{s^*s} \cap X_{t^*}) \stackrel{(i)}{=} X_{s^*s} \cap X_{s^*st^*} = X_{s^*s} \cap X_{s^*} \\ &\stackrel{(ii)}{=} \pi_{s^*}(X_s \cap X_s) = \pi_{s^*}(X_s) = X_{s^*}. \end{aligned}$$

This implies that $X_{s^*} \subseteq X_{t^*}$ and, for every $x \in X_{s^*}$, we have $x \in X_{s^*s} \cap X_{s^*st^*}$, hence by (iii)

$$\pi_s(x) = \pi_{ts^*s}(x) = \pi_t(\pi_{s^*s}(x)) = \pi_t(x),$$

proving that $\pi_s \leq \pi_t$.

Now assume that π is a partial action, so Proposition 3.1(i)–(iii) hold. Then (i) is granted so let us prove (ii). Given $s, t \in G$, observe that the domain of $\pi_s\pi_t$ is given by $\pi_{t^*}(X_t \cap X_{s^*})$, which is therefore contained in the domain of π_{st} , namely $X_{t^*s^*}$, by Proposition 3.1(ii), that is,

$$\pi_{t^*}(X_t \cap X_{s^*}) \subseteq X_{t^*s^*}.$$

Since the set on the left-hand side above is obviously also contained in X_{t^*} , we deduce that

$$(3.1) \quad \pi_{t^*}(X_t \cap X_{s^*}) \subseteq X_{t^*} \cap X_{t^*s^*}.$$

We claim that (3.1) is actually an equality of sets. In order to prove it, notice that a suitable change of variables in (3.1) yields

$$(3.2) \quad \pi_t(X_{t^*} \cap X_{t^*s^*}) \subseteq X_t \cap X_{tt^*s^*}.$$

Evidently $stt^* \leq s$, so the domain of π_{stt^*} is contained in the domain of π_s by Proposition 3.1(iii) or, in other words, $X_{tt^*s^*} \subseteq X_{s^*}$. Plugging this into (3.2) one obtains

$$(3.3) \quad \pi_t(X_{t^*} \cap X_{t^*s^*}) \subseteq X_t \cap X_{s^*}.$$

Applying the inverse of π_t to both sides then gives

$$X_{t^*} \cap X_{t^*s^*} \subseteq \pi_{t^*}(X_t \cap X_{s^*}),$$

which happens to be precisely the converse of the inclusion in (3.1). Therefore

$$\pi_{t^*}(X_t \cap X_{s^*}) = X_{t^*} \cap X_{t^*s^*},$$

and our claim is proved. Suitably changing variables in the above equality immediately yields (ii).

In order to prove (iii) let $x \in X_{t^*} \cap X_{t^*s^*}$. Then by (3.3) one has that $\pi_t(x) \in X_{s^*}$, so x is in the domain of $\pi_s\pi_t$, and hence by Proposition 3.1(ii) one has that

$$\pi_s(\pi_t(x)) = \pi_{st}(x). \quad \square$$

Let us now prove a result very similar to the above, except that it is tailored to require minimal effort for checking a map to be a partial action.

PROPOSITION 3.5. *Let X be a set and let $\pi : G \rightarrow \mathcal{I}(X)$ be a map. For each $s \in G$, let X_s be the range of π_s . Then π is a partial action if and only if, for all $s, t \in G$, the following holds:*

- (i) $\pi_s^{-1} = \pi_{s^*}$,
- (ii) $\pi_s(X_{s^*} \cap X_t) \subseteq X_{st}$,
- (iii) if $s \leq t$, then $X_s \subseteq X_t$,
- (iv) for every $x \in X_{t^*} \cap X_{t^*s^*}$, one has that $\pi_s(\pi_t(x)) = \pi_{st}(x)$.

Proof. The reader should have noticed that the conditions above are very similar to the conditions given in Proposition 3.4, except that the equality in Proposition 3.4(ii) is replaced by the weaker inclusion in (ii) above, while (iii) is new.

The remark made at the beginning of the proof of Proposition 3.4 applies here as well, although with a different argument: under (i)–(iii), let $x \in X_{t^*} \cap X_{t^*s^*}$. Then

$$\pi_t(x) \in \pi_t(X_{t^*} \cap X_{t^*s^*}) \stackrel{(ii)}{\subseteq} X_{tt^*s^*} \stackrel{(iii)}{\subseteq} X_{s^*},$$

hence the composition $\pi_s(\pi_t(x))$ appearing in (iv) is meaningful.

Let us first observe that conditions (i)–(iv) above are necessary for π to be a partial action: clearly (i), (ii), and (iv) immediately follow from Proposition 3.4, while (iii) is a direct consequence of Proposition 3.1(iii).

Conversely, let us prove that π is a partial action under the assumption that the above conditions hold. Using Proposition 3.4, it is clearly enough to verify Proposition 3.4(ii). Since the range of π_s is precisely X_s one immediately deduces from (ii) that

$$(3.4) \quad \pi_s(X_{s^*} \cap X_t) \subseteq X_s \cap X_{st}.$$

Applying this with a suitable change of variables, we get

$$\pi_{s^*}(X_s \cap X_{st}) \subseteq X_{s^*} \cap X_{s^*st} \stackrel{(iii)}{\subseteq} X_{s^*} \cap X_t.$$

Because π_s is the inverse of π_{s^*} the above implies that

$$X_s \cap X_{st} \subseteq \pi_s(X_{s^*} \cap X_t),$$

which combines with (3.4) to give the desired Proposition 3.4(ii). □

4. The expansion as an adjunction of categories

Let Inv be the category of inverse semigroups with usual homomorphisms as morphisms, and let PInv be the category of inverse semigroups with partial homomorphisms as morphisms. Note that composition of partial homomorphisms is again a partial homomorphism (this can be seen from Proposition 3.1) so that PInv is indeed a category. Given two inverse semigroups G and H , we write $\text{Hom}(G, H)$ for the set of all homomorphisms $G \rightarrow H$ (that is, the hom-set in Inv) and $\text{PHom}(G, H)$ for the set of all partial homomorphisms $G \rightarrow H$ (that is, the hom-set in PInv). Observe that $G \mapsto \mathbf{Pr}(G)$ is a

functor $\mathbf{PInv} \rightarrow \mathbf{Inv}$: given a partial homomorphism $\pi : G \rightarrow H$ the induced homomorphism $\mathbf{Pr}(\pi) : \mathbf{Pr}(G) \rightarrow \mathbf{Pr}(H)$ is given by $S(\pi) = \iota_H \circ \tilde{\pi}$, where $\iota_H : H \rightarrow \mathbf{Pr}(H)$ is the canonical inclusion $h \mapsto [h]$ and $\tilde{\pi} : \mathbf{Pr}(G) \rightarrow H$ is the morphism provided by Proposition 2.12. In other words, $\mathbf{Pr}(\pi)[g] = [\pi(g)]$ for all $g \in G$. By Proposition 2.12, there is a bijective correspondence between partial homomorphisms $G \rightarrow H$ and homomorphisms $\mathbf{Pr}(G) \rightarrow H$, that is, we have a canonical bijection $\text{Hom}(G, H) \cong \text{PHom}(\mathbf{Pr}(G), H)$. It is easy to see that this correspondence is natural in both variables G and H . This already says that we have an adjunction between the prefix expansion functor $\mathbf{Pr} : \mathbf{PInv} \rightarrow \mathbf{Inv}$, $G \rightarrow \mathbf{Pr}(G)$ and the forgetful functor $\mathbf{Fg} : \mathbf{Inv} \rightarrow \mathbf{PInv}$.

Now we describe the above adjunction in terms of units and counits. Given an inverse semigroup G , we write $\partial_G : \mathbf{Pr}(G) \rightarrow G$ for the degree map and $\iota_G : G \rightarrow \mathbf{Pr}(G)$ for the canonical inclusion map, that is, $\partial_G([g]) = g$ and $\iota_G(g) = [g]$ for all $g \in G$. Then ∂ may be interpreted as a natural transformation from the functor $\mathbf{Pr} \circ \mathbf{Fg} : \mathbf{Inv} \rightarrow \mathbf{Inv}$ to the identity functor $\text{Id}_{\mathbf{Inv}}$ on \mathbf{Inv} , and ι may be viewed as a natural transformation from the identity functor $\text{Id}_{\mathbf{PInv}}$ to the functor $\mathbf{Fg} \circ \mathbf{Pr} : \mathbf{PInv} \rightarrow \mathbf{PInv}$. Moreover, (ι, ∂) is the pair unit-counit for the adjunction between \mathbf{Pr} and \mathbf{Fg} . To see this, one has to check the equations

$$1_{\mathbf{Pr}} = \partial \mathbf{Pr} \circ \mathbf{Pr} \iota \quad \text{and} \quad 1_{\mathbf{Fg}} = \mathbf{Fg} \partial \circ \iota \mathbf{Fg},$$

write 1 denotes identity natural transformations. More explicitly, this means that the composition

$$\partial_{\mathbf{Pr}(G)} \circ \mathbf{Pr}(\iota_G) : \mathbf{Pr}(G) \rightarrow \mathbf{Pr}(\mathbf{Pr}(G)) \rightarrow \mathbf{Pr}(G)$$

is the identity homomorphism $\text{Id}_{\mathbf{Pr}(G)} : \mathbf{Pr}(G) \rightarrow \mathbf{Pr}(G)$; and the composition

$$\partial_G \circ \iota_G : G \rightarrow \mathbf{Pr}(G) \rightarrow G$$

is the identity (partial) homomorphism $\text{Id}_G : G \rightarrow G$. Both assertions are easily checked.

5. The canonical partial action

We again fix an inverse semigroup G . The goal of this section is to exhibit a somewhat canonical partial action of G which will, among other things, enable us to prove that the normal form of each element in $\mathbf{Pr}(G)$ is unique.

DEFINITION 5.1. A nonempty subset $\xi \subseteq G$ will be called a *filter* if for every $e \in E(G)$ and $s \in G$,

$$(5.1) \quad es \in \xi \iff e \in \xi \quad \text{and} \quad s \in \xi.$$

If G is a group, a filter is just a subset $\xi \subseteq G$ containing 1 (the unit of G). In general, filters may contain several idempotents (for instance G is always a filter). If G has a zero $0 \in G$, then the only filter containing 0 is G . Also, observe that $E(S)$ is a filter if and only if S is E -unitary.

REMARK 5.2. In general, a filter does not satisfy the symmetric property of (5.1):

$$se \in \xi \iff e \in \xi \text{ and } s \in \xi.$$

In fact, it may happen that $s \in \xi$ but $s^*s \notin \xi$ and also $s^* \notin \xi$. As a simple example, let G be the inverse semigroup with 0 and one generator $s \neq 0$ satisfying $s^2 = 0$. This is the inverse semigroup with five elements $G = \{0, e, f, s, t\}$, where $t = s^*$, $ss^* = e$ and $s^*s = f$ (with $s^2 = t^2 = 0$). Observe that $\xi = \{e, s\}$ is a filter in G . We have $s \in \xi$ but $s^* = t \notin \xi$ and $s^*s = f \notin \xi$. The same problem happens for the filter $\eta = \{f, t\}$.

Let us use the opportunity to mention that $\mathbf{Pr}(G) = \{\epsilon_0, \epsilon_e, \epsilon_s, \epsilon_f, \epsilon_t, [s], [t]\}$, an inverse semigroup containing $\{\epsilon_0 = [0], \epsilon_s, \epsilon_t, [s], [t]\}$ as an inverse subsemigroup isomorphic to G and two other new idempotents $\epsilon_e \geq \epsilon_s$ and $\epsilon_f \geq \epsilon_t$.

Here is a useful alternative characterization of filters.

PROPOSITION 5.3. *Let ξ be a nonempty subset of G . Then ξ is a filter if and only if*

- (i) $ss^* \in \xi$ for every $s \in \xi$;
- (ii) $ss^*t \in \xi$ for every $s, t \in \xi$; and
- (iii) whenever $s, t \in G$ are such that $s \in \xi$ and $s \leq t$, we have $t \in \xi$.

Proof. Assuming that ξ is a filter, let $s \in \xi$. With $e = ss^*$, we have $es = s \in \xi$, hence $e \in \xi$, proving (i). If one is also given $t \in \xi$, then $ss^*t = et \in \xi$, proving (ii). Next assume that $t \geq s \in \xi$. Again with $e = ss^*$, we have $et = ss^*t = s \in \xi$, so (5.1) implies that $t \in \xi$.

Conversely, suppose that ξ satisfies (i)–(iii) and assume that $es \in \xi$, where e is idempotent. Then

$$\xi \ni es \leq s$$

and hence $s \in \xi$, by (iii). Moreover, by (i),

$$\xi \ni es(es)^* = ess^* \leq e,$$

and hence $e \in \xi$, again by (iii). On the other hand, assuming that $e, s \in \xi$, with e idempotent, we have by (ii) that

$$es = ee^*s \in \xi. \quad \square$$

DEFINITION 5.4. A nonempty subset $\xi \subseteq G$ is said to be a *filter base* if it satisfies Proposition 5.3(i)–(ii).

Observe that Proposition 5.3(i) follows from Proposition 5.3(ii) by taking $s = t$. So a filter base is just a subset $\emptyset \neq \xi \subseteq G$ satisfying Proposition 5.3(ii).

As a relevant example, notice that every e -set (Definition 2.7) is a filter base.

PROPOSITION 5.5. *Let η be a filter base. Then*

$$\langle \eta \rangle := \{t \in G : t \geq s, \text{ for some } s \in \eta\}$$

is a filter.

Proof. Let $e \in E(G)$ and $s \in G$. Supposing that $es \in \langle \eta \rangle$, there exists $t \in \eta$ such that $es \geq t$. But then

$$\eta \ni t \leq es \leq s$$

from where we see that $s \in \langle \eta \rangle$. Moreover

$$\eta \ni tt^* \leq es(es)^* = ess^* \leq e,$$

so $e \in \langle \eta \rangle$. Conversely, suppose that both e and s lie in $\langle \eta \rangle$. Then there are $e_1, s_1 \in \eta$ such that $e_1 \leq e$, and $s_1 \leq s$, so

$$\eta \ni e_1e_1^*s_1 \leq ee^*s = es,$$

and therefore $es \in \langle \eta \rangle$. □

DEFINITION 5.6. If η is a filter base, we will refer to $\langle \eta \rangle$ as the *filter generated by η* . If ξ is a filter and $\langle \eta \rangle = \xi$, we will say that η is a *filter base for ξ* .

If ξ is a filter, then it is evidently also a filter base and $\langle \xi \rangle = \xi$.

DEFINITION 5.7. We will denote by X the collection of all filters, and for $t \in G$ we will set

$$X_t = \{\xi \in X : t \in \xi\}.$$

We next start preparing for the construction of our partial action of G on X .

PROPOSITION 5.8. *If $t \in G$ and η is a filter base containing t^* , then*

$$t\eta := \{ts : s \in \eta\}$$

is a filter base containing t . If moreover η' is another filter base containing t^ , such that $\langle \eta \rangle = \langle \eta' \rangle$, then $\langle t\eta \rangle = \langle t\eta' \rangle$.*

Proof. Given $u \in t\eta$, write $u = ts$, with $s \in \eta$. Then $ss^*t^* \in \eta$, so

$$uu^* = tss^*t^* \in t\eta.$$

If v is another element of $t\eta$, write $v = tr$, with $r \in \eta$, and notice that $ss^*r \in \eta$, so

$$uu^*v = tss^*t^*tr = tt^*tss^*r = tss^*r \in t\eta.$$

This proves that $t\eta$ is a filter base. Since $t^* \in \eta$, we have $t^*t \in \eta$, so

$$t = tt^*t \in t\eta.$$

Given η' as above, suppose that $u \in \langle t\eta \rangle$. Then there exists $s \in \eta$ such that $u \geq ts$. Since

$$s \in \eta \subseteq \langle \eta \rangle = \langle \eta' \rangle,$$

there exists $s' \in \eta'$ such that $s \geq s'$. Therefore,

$$u \geq ts \geq ts' \in t\eta',$$

proving that $u \in \langle t\eta' \rangle$. This shows that $\langle t\eta \rangle \subseteq \langle t\eta' \rangle$, and the reverse inclusion may be proved in a similar way. \square

We are now ready to introduce the partial action which is the main object of this section.

THEOREM 5.9. *For each $t \in G$, define*

$$\pi_t : X_{t^*} \rightarrow X_t,$$

by $\pi_t(\xi) = \langle t\xi \rangle$, for all $\xi \in X_{t^}$. Then π is a partial action of G on X .*

Proof. We will check the conditions in Proposition 3.5. For this, we first claim that $\pi_{s^*}\pi_s$ is the identity on X_{s^*} , for every s in G . Given $\xi \in X_{s^*}$, we have $s \in s\xi$, and $s\xi$ is a filter base for $\pi_s(\xi)$, so we deduce from Proposition 5.8 that $s^*s\xi$ is a filter base for $\pi_{s^*}(\pi_s(\xi))$. Thus, in order to prove our claim, it suffices to show that $\langle s^*s\xi \rangle = \xi$. As $s^* \in \xi$, one has that $s^*s\xi \subseteq \xi$, and hence $\langle s^*s\xi \rangle \subseteq \xi$. Conversely, given $u \in \xi$, we have

$$u \geq s^*su \in s^*s\xi,$$

so $u \in \langle s^*s\xi \rangle$, proving that $\langle s^*s\xi \rangle = \xi$, and therefore that $\pi_{s^*}(\pi_s(\xi)) = \xi$. Reversing the roles of s and s^* we deduce that $\pi_s\pi_{s^*}$ is the identity on X_s , whence $\pi_{s^*} = \pi_s^{-1}$, proving Proposition 3.5(i).

In order to prove Proposition 3.5(ii), let $t, s \in G$ and let $\xi \in X_{s^*} \cap X_t$. This means that $s^*, t \in \xi$, so evidently

$$st \in s\xi \subseteq \langle s\xi \rangle = \pi_s(\xi),$$

which implies that $\pi_s(\xi) \in X_{st}$.

With respect to Proposition 3.5(iii), assume that $s \leq t$, and pick $\xi \in X_s$. Then

$$t \geq s \in \xi,$$

and hence $t \in \xi$, by Proposition 5.3(iii), proving that $\xi \in X_t$.

Finally, assuming that $\xi \in X_{t^*} \cap X_{t^*s^*}$, we claim that

$$\langle st\xi \rangle = \langle s\langle t\xi \rangle \rangle.$$

Clearly $t\xi \subseteq \langle t\xi \rangle$, so $st\xi \subseteq s\langle t\xi \rangle$, and hence $\langle st\xi \rangle \subseteq \langle s\langle t\xi \rangle \rangle$. Conversely, given $u \in \langle s\langle t\xi \rangle \rangle$, there exists $v \in \langle t\xi \rangle$ such that $u \geq sv$, and there exists $w \in \xi$, such that $v \geq tw$. So

$$u \geq sv \geq stw \in st\xi,$$

implying that $u \in \langle st\xi \rangle$. This proves our claim and hence

$$\pi_s(\pi_t(\xi)) = \pi_s(\langle t\xi \rangle) = \langle s\langle t\xi \rangle \rangle = \langle st\xi \rangle = \pi_{st}(\xi).$$

The reader may have thought of another strategy to prove this last fact, using $t\xi$ as a filter base for $\pi_t(\xi)$ when computing $\pi_s(\pi_t(\xi))$. However, there is no guarantee that $s^* \in t\xi$, so this might not work. \square

We shall now employ the above partial action to deduce the promised fact that the normal form of each element in $\mathbf{Pr}(G)$ is unique. Recall that every e -set is a filter base, so if A is an e -set we may form the filter $\langle A \rangle$ generated by A .

LEMMA 5.10. *Let A be an e -set, where $e \in E(G)$, and let $\xi = \langle A \rangle$. Then*

$$A = \{s \in \xi : ss^* = e\}.$$

Proof. It is evident that A is contained in the set in the right-hand side. Conversely, given $s \in \xi$, such that $ss^* = e$, there exists $t \in A$, such that $s \geq t$. Thus,

$$t = tt^*s = es = ss^*s = s,$$

which implies that $s \in A$. \square

THEOREM 5.11. *Let $x, y \in \mathbf{Pr}(G)$ be written in normal form as $x = \varepsilon_A[s]$ and $y = \varepsilon_B[t]$. If $x = y$, then $s = t$ and $A = B$.*

Proof. By Proposition 2.14, we have

$$s = \partial(x) = \partial(y) = t,$$

so $s = t$. Since $s \in A$, and $s = t \in B$, we deduce that

$$\varepsilon_A = \varepsilon_A\varepsilon_s = \varepsilon_A[s][s^*] = x[s^*] = y[s^*] = \varepsilon_B[s][s^*] = \varepsilon_B\varepsilon_s = \varepsilon_B,$$

meaning that $\varepsilon_A = \varepsilon_B$. Next, consider the semigroup homomorphism

$$\tilde{\pi} : \mathbf{Pr}(G) \rightarrow \mathcal{I}(X),$$

provided by Proposition 2.12 in terms of the partial action π introduced in Theorem 5.9. For every $s \in G$, we have

$$\tilde{\pi}(\varepsilon_s) = \tilde{\pi}([s][s^*]) = \tilde{\pi}([s])\tilde{\pi}([s^*]) = \pi_s\pi_{s^*} = \pi_s\pi_s^{-1} = \text{id}_{X_s}.$$

Therefore,

$$\tilde{\pi}(\varepsilon_A) = \prod_{s \in A} \tilde{\pi}(\varepsilon_s) = \prod_{s \in A} \text{id}_{X_s} = \text{id}_{X_A},$$

where by X_A we of course mean $\bigcap_{s \in A} X_s$. Having seen above that $\varepsilon_A = \varepsilon_B$, we then deduce that $X_A = X_B$.

Since $\varepsilon_A[s]$ and $\varepsilon_B[s]$ are in normal form we have by definition that A and B are e -sets, where $e = ss^*$. Recalling that every e -set is a filter base we have by Proposition 5.5 that $\xi := \langle A \rangle$ is a filter, which evidently belongs to X_A , and hence also to X_B , meaning that $B \subseteq \xi$. Using Lemma 5.10, we then obtain

$$A = \{s \in \xi : ss^* = e\} \supseteq B,$$

and a symmetric argument yields $A \subseteq B$, proving that $A = B$ and completing the proof. \square

One natural question is whether $\mathbf{Pr}(\mathbf{Pr}(G)) \cong \mathbf{Pr}(G)$. The following result shows that this happens if and only if G is a semilattice.

PROPOSITION 5.12. *Let G be an inverse semigroup. Then the following assertions are equivalent:*

- (i) every partial homomorphism of G is a homomorphism;
- (ii) the canonical map $\iota_G : g \mapsto [g]$ from G into $\mathbf{Pr}(G)$ is a homomorphism;
- (iii) $\{[g] : g \in G\}$ is an inverse subsemigroup of $\mathbf{Pr}(G)$ (and hence equals $\mathbf{Pr}(G)$);
- (iv) the canonical map $G \rightarrow \mathbf{Pr}(G)$ is an isomorphism (whose inverse is the degree map $\partial : \mathbf{Pr}(G) \rightarrow G$);
- (v) $G = E(G)$, that is, G is a semilattice.

In particular, every inverse semigroup G which is not a semilattice has a partial homomorphism which is not a representation. Moreover, in this case we have a strictly increasing chain of inclusions:

$$G \hookrightarrow \mathbf{Pr}(G) \hookrightarrow \mathbf{Pr}(\mathbf{Pr}(G)) \hookrightarrow \dots$$

Proof. The implication (i) \Rightarrow (ii) is obvious since the canonical map $\iota_G : G \rightarrow \mathbf{Pr}(G)$ is a partial homomorphism. On the other hand, if the canonical map $g \mapsto [g]$ is a homomorphism, then $\{[g] : g \in G\}$ is an inverse subsemigroup of $\mathbf{Pr}(G)$ which generates it, so that $\mathbf{Pr}(G) = \{[g] : g \in G\}$. In this case, ι_G must be an isomorphism from G to $\mathbf{Pr}(G)$ because $\partial : \mathbf{Pr}(G) \rightarrow G$ is a homomorphism satisfying $\partial([g]) = g$ for all $g \in G$. Hence, we have the implications (ii) \Rightarrow (iii) \Rightarrow (iv). To prove (iv) \Rightarrow (v), suppose that $g \mapsto [g]$ is a homomorphism of G . Then we have $\epsilon_g = [g][g^*] = [gg^*] = \epsilon_{gg^*}$. Observe that the map $g \mapsto \epsilon_g$ is injective. In fact, the normal form of ϵ_g is $\epsilon_g \epsilon_{gg^*} [gg^*]$. Thus, if $\epsilon_g = \epsilon_h$, then $\{g, gg^*\} = \{h, hh^*\}$ (by Theorem 5.11) so that $g = h$. In particular, since $\epsilon_g = \epsilon_{gg^*}$, it follows that $g = gg^*$ for all $g \in G$, that is, $G = E(G)$. Finally, to check (v) \Rightarrow (i), suppose that G is a semilattice and π is a partial homomorphism of G . Then $\pi(e f) = \pi(e)\pi(f)$ for all $e, f \in E(G) = G$ by Corollary 2.13, so that π is a homomorphism. \square

PROPOSITION 5.13. *Let G be a finite inverse semigroup. Given $e \in E(G)$, define $G^e := \{s \in G : ss^* = e\}$ and let $p_e = |G^e|$ be the number of elements of G^e . Then $\mathbf{Pr}(G)$ has exactly*

$$(5.2) \quad |E(\mathbf{Pr}(G))| = \sum_{e \in E(G)} 2^{p_e - 1}$$

idempotent elements and exactly

$$(5.3) \quad |\mathbf{Pr}(G) - E(\mathbf{Pr}(G))| = \sum_{e \in E(G)} 2^{p_e - 2} (p_e - 1)$$

non-idempotent elements. In particular, the total number of elements of $\mathbf{Pr}(G)$ is given by

$$(5.4) \quad |\mathbf{Pr}(G)| = \sum_{e \in E(G)} 2^{p_e - 1} + \sum_{e \in E(G)} 2^{p_e - 2}(p_e - 1) = \sum_{e \in E(G)} 2^{p_e - 2}(p_e + 1).$$

Proof. Given $e \in E(G)$, let $X^e = \{A \subseteq G^e : e \in A\}$. Observe that $|X^e| = 2^{p_e - 1}$. We already know (see Proposition 2.14) that the idempotents of $\mathbf{Pr}(G)$ have the form ϵ_A for some $e \in E(G)$ and some $A \in X^e$. And by Theorem 5.11, for $A, B \subseteq G^e$, we have $\epsilon_A = \epsilon_B$ if and only if $A = B$ (observe that the normal form of ϵ_A is $\epsilon_A[e]$ whenever $A \in X^e$). This implies (5.2). To prove (5.3), observe that (by Theorem 5.11) every non-idempotent element of $\mathbf{Pr}(G)$ can be uniquely written as $\epsilon_A[s]$ for some $A \subseteq X^e$ and some $s \in A$ with $s \neq e$. Notice that we have exactly $2^{p_e - 2}$ sets A in X^e with this property. Since we have $p_e - 1$ elements $s \in G^e$ with $s \neq e$, (5.3) follows as well. \square

6. Twisted partial actions on C^* -algebras

In this section, we are going to enrich sets with further structure and require a partial action of an inverse semigroup G to be compatible with this structure. For instance, if X is a topological space, and $\alpha : G \rightarrow \mathcal{I}(X)$ is a partial action of G on X , then it is natural to require the domain $\mathcal{D}_{s^*} := \text{dom}(\alpha_s)$ and the range $\mathcal{D}_s := \text{ran}(\alpha_s)$ of α_s to be open subsets of X and $\alpha_s : \mathcal{D}_{s^*} \rightarrow \mathcal{D}_s$ to be continuous for all $s \in G$. Observe that this implies that each α_s is a homeomorphism with inverse α_{s^*} . Alternatively, we could say that α_s is a partial homeomorphism of X . So, a partial action of G on a topological space X is just a partial homomorphism of G into the inverse semigroup of all partial homeomorphisms of X . We also require the union $\bigcup \mathcal{D}_s$ to be dense in X —otherwise this can be arranged replacing X by the closure of $\bigcup \mathcal{D}_s$.

Similarly, if B is a C^* -algebra and $\beta : G \rightarrow \mathcal{I}(B)$ is a partial action of G on B , it is natural to require each $\beta_s : \mathcal{D}_{s^*} \rightarrow \mathcal{D}_s$ to be a *partial automorphism* between (closed, two-sided) ideals $\mathcal{D}_{s^*}, \mathcal{D}_s$ of B . Of course, by a partial automorphism of B we mean a $*$ -isomorphism $I \xrightarrow{\sim} J$ between ideals I, J of B . Hence, a partial action of G on a C^* -algebra B is just a partial homomorphism from G into the inverse semigroup of all partial automorphisms of B . In addition, we require that the union $\bigcup \mathcal{D}_s$ spans a dense subspace of B . If $B = \mathcal{C}_0(X)$ is a commutative C^* -algebra, then partial automorphisms of B correspond to partial homeomorphisms of X , so partial actions on B correspond to partial actions on X .

The notion of partial actions of inverse semigroups on C^* -algebras appears in [9]. In the case of groups, it can be seen as a special case of the twisted partial actions defined by the second named author in [3]. In [2], we considered a notion of twisted (global) actions for inverse semigroups improving Sieben’s definition in [10]. We already know that partial actions of G correspond bijectively to actions of $\mathbf{Pr}(G)$. A similar result should also hold with twists,

that is, *twisted partial actions* of G should correspond to twisted (global) actions of $\mathbf{Pr}(G)$. In order to prove this, we first have to find the right definition of twisted partial actions in the realm of inverse semigroups. Using the canonical partial homomorphism of G in $\mathbf{Pr}(G)$ to view G as a subset of $\mathbf{Pr}(G)$, a short way to define this is to say that a twisted partial action of G is the restriction of a twisted action of $\mathbf{Pr}(G)$ (in the sense of [2, Definition 4.1]) to G . However, since this concept should be a simultaneous generalization of [2, Definition 4.1] and [3, Definition 2.1], it is not so difficult to guess what should be the right definition:

Given a C^* -algebra B , we write $\mathcal{M}(B)$ for the multiplier algebra of B and $\mathcal{UM}(B)$ for the group of unitary multipliers of B .

DEFINITION 6.1. Let G be an inverse semigroup and let B be a C^* -algebra. A *twisted partial action* of G on B is a pair (β, ω) , where $\beta = \{\beta_s\}_{s \in G}$ is a family of partial automorphisms $\beta_s : \mathcal{D}_{s^*} \rightarrow \mathcal{D}_s$ of B and $\omega = \{\omega(s, t)\}_{s, t \in G}$ is a family of unitary multipliers $\omega(s, t) \in \mathcal{UM}(\mathcal{D}_s \cap \mathcal{D}_{st})$, such that the linear span of the family of ideals $\{\mathcal{D}_s\}_{s \in G}$ is dense in B and the following conditions are satisfied for all $r, s, t \in G$ and $e, f \in E(G)$:

- (i) $\beta_r(\mathcal{D}_{r^*} \cap \mathcal{D}_s) = \mathcal{D}_r \cap \mathcal{D}_{rs}$;
- (ii) $\beta_r(\beta_s(x)) = \omega(r, s)\beta_{rs}(x)\omega(r, s)^*$ for all $x \in \mathcal{D}_{s^*} \cap \mathcal{D}_{s^*r^*}$;
- (iii) $\beta_r(x\omega(s, t))\omega(r, st) = \beta_r(x)\omega(r, s)\omega(rs, t)$ whenever $x \in \mathcal{D}_{r^*} \cap \mathcal{D}_s \cap \mathcal{D}_{st}$;
- (iv) $\omega(e, f) = 1_{ef}$ and $\omega(r, r^*r) = \omega(rr^*, r) = 1_r$, where 1_r is the unit of $\mathcal{M}(\mathcal{D}_r)$;
- (v) $\omega(s^*, e)\omega(s^*e, s)x = \omega(s^*, s)x$ for all $x \in \mathcal{D}_{s^*e}$.

It is not difficult to see that the above definition is in fact a generalization of both [2, Definition 4.1] and [3, Definition 2.1]. Let us observe that the first axiom above is not included in the definition of twisted (global) actions appearing in [2, Definition 4.1], but it is a consequence of the other axioms. In fact, if (β, ω) is a twisted action, then by [2, Lemma 4.6] we have $\beta_r(\mathcal{D}_{r^*} \cap \mathcal{D}_s) = \mathcal{D}_{rs} = \mathcal{D}_r \cap \mathcal{D}_{rs}$ because (by the same lemma) $\mathcal{D}_{rs} = \mathcal{D}_{rs^*r^*} \subseteq \mathcal{D}_{rr^*} = \mathcal{D}_r$.

REMARK 6.2. Axiom (ii) in the above definition is equivalent to

$$(6.1) \quad \beta_r \circ \beta_s = \text{Ad}_{\omega(r, s)} \circ \beta_{rs} \quad (\text{composition of partial automorphisms}),$$

where $\text{Ad}_{\omega(r, s)} : \mathcal{D}_r \cap \mathcal{D}_{rs} \rightarrow \mathcal{D}_r \cap \mathcal{D}_{rs}$ is the partial automorphism defined by $\text{Ad}_{\omega(r, s)}(z) = \omega(r, s)z\omega(r, s)^*$. In fact, observe that (by the properties (iii) and (iv) to be proved in Proposition 6.3 below)

$$\text{dom}(\beta_r \circ \beta_s) = \beta_s^{-1}(\mathcal{D}_s \cap \mathcal{D}_{r^*}) = \mathcal{D}_{s^*} \cap \mathcal{D}_{s^*r^*} = \mathcal{D}_{s^*r^*r} \cap \mathcal{D}_{s^*r^*}.$$

On the other hand,

$$\text{dom}(\text{Ad}_{\omega(r, s)} \circ \beta_{rs}) = \beta_{rs}^{-1}(\mathcal{D}_{rs} \cap \mathcal{D}_r) = \mathcal{D}_{s^*r^*} \cap \mathcal{D}_{s^*r^*r}.$$

Thus, $\text{dom}(\beta_r \circ \beta_s) = \text{dom}(\text{Ad}_{\omega(r,s)} \circ \beta_{rs})$. In the untwisted case (that is, if $\omega(r,s)$ is the unit multiplier of $\mathcal{D}_r \cap \mathcal{D}_{rs}$ for all r,s), we may interpret $\text{Ad}_{\omega(r,s)} \circ \beta_{rs}$ as the restriction of β_{rs} to the domain of $\beta_r \circ \beta_s$. In this case, (6.1) just says that β_{rs} extends $\beta_r \circ \beta_s$.

Here are some properties of twisted partial actions (compare with [2, Lemma 4.6]).

PROPOSITION 6.3. *Let (β, ω) be a twisted partial action of G on B as above. Then the following holds:*

- (i) $\mathcal{D}_s \subseteq \mathcal{D}_{ss^*}$ for all $s \in G$. Moreover, $\mathcal{D}_s = \mathcal{D}_{ss^*}$ for all $s \in G$ if and only if (β, ω) is a twisted (global) action in the sense of [2, Definition 4.1];
- (ii) $\beta_e : \mathcal{D}_e \rightarrow \mathcal{D}_e$ is the identity map for all $e \in E(G)$;
- (iii) $\mathcal{D}_s \cap \mathcal{D}_t = \mathcal{D}_s \cap \mathcal{D}_{ss^*t} = \mathcal{D}_{tt^*s} \cap \mathcal{D}_{ss^*t} = \mathcal{D}_{es} \cap \mathcal{D}_{et}$ for all $s, t \in G$, where $e = ss^*tt^*$; moreover $\mathcal{D}_e \cap \mathcal{D}_s = \mathcal{D}_{es}$ for all $e \in E$ and $s \in S$;
- (iv) $\beta_s^{-1}(\mathcal{D}_s \cap \mathcal{D}_{r^*}) = \mathcal{D}_{s^*} \cap \mathcal{D}_{s^*r^*}$;
- (v) $s \leq t \Rightarrow \mathcal{D}_s \subseteq \mathcal{D}_t$.

Proof. (i) Observe that $\mathcal{D}_s = \beta_s(\mathcal{D}_{s^*}) = \beta_s(\mathcal{D}_{s^*} \cap \mathcal{D}_{s^*}) = \mathcal{D}_s \cap \mathcal{D}_{ss^*}$, so that $\mathcal{D}_s \subseteq \mathcal{D}_{ss^*}$ for all $s \in G$. If $\mathcal{D}_s = \mathcal{D}_{ss^*}$ for all $s \in G$, we have to check that (β, ω) is a twisted global action, that is, the axioms (i)–(iv) appearing in [2, Definition 4.1] are satisfied. But the axioms are essentially the same by looking at property (iii) we are going to prove below.

(ii) By axiom (iii) in Definition 6.1, we have $\omega(e, e) = 1_e$, so that $\beta_e(\beta_e(x)) = \omega(e, e)\beta_e(x)\omega(e, e) = \beta_e(x)$ for all \mathcal{D}_e by Definition 6.1(ii). Since $\beta_e : \mathcal{D}_e \rightarrow \mathcal{D}_e$ is an isomorphism, this implies that $\beta_e = \text{id}_{\mathcal{D}_e}$.

(iii) First, given $f \in E(G)$, since $\beta_f = \text{id}_{\mathcal{D}_f}$, we have $\mathcal{D}_f \cap \mathcal{D}_t = \beta_f(\mathcal{D}_f \cap \mathcal{D}_t) = \mathcal{D}_f \cap \mathcal{D}_{ft}$ by Definition 6.1(i). Since $\mathcal{D}_s \subseteq \mathcal{D}_{ss^*}$, taking $f = ss^*$, we get $\mathcal{D}_s \cap \mathcal{D}_t = \mathcal{D}_s \cap \mathcal{D}_{ss^*} \cap \mathcal{D}_t = \mathcal{D}_s \cap \mathcal{D}_{ss^*t}$. Now, applying this again, we get $\mathcal{D}_s \cap \mathcal{D}_{ss^*t} = \mathcal{D}_{ss^*t} \cap \mathcal{D}_s = \mathcal{D}_{ss^*t} \cap \mathcal{D}_{ss^*tt^*s} = \mathcal{D}_{ss^*t} \cap \mathcal{D}_{tt^*s} = \mathcal{D}_{es} \cap \mathcal{D}_{et}$.

(iv) By Definition 6.1(i) and (iii) above, we have $\beta_s(\mathcal{D}_{s^*} \cap \mathcal{D}_{s^*r^*}) = \mathcal{D}_s \cap \mathcal{D}_{ss^*r^*} = \mathcal{D}_s \cap \mathcal{D}_{r^*}$, whence (iv) follows.

(v) If $s \leq t$, then $s = ss^*t$, so that $\mathcal{D}_s \cap \mathcal{D}_t \stackrel{\text{(iii)}}{=} \mathcal{D}_s \cap \mathcal{D}_{ss^*t} = \mathcal{D}_s$. Hence, $\mathcal{D}_s \subseteq \mathcal{D}_t$. □

Let (β, ω) be a twisted partial action of an inverse semigroup S on a C^* -algebra B as in Definition 6.1. Given an element $x = \epsilon_{r_1} \cdots \epsilon_{r_n}[r]$ of $\mathbf{Pr}(G)$ in normal form, we define $\tilde{\mathcal{D}}_x := \mathcal{D}_{r_1} \cdots \mathcal{D}_{r_n} \cdot \mathcal{D}_r$ and $\tilde{\beta}_x : \tilde{\mathcal{D}}_{x^*} \rightarrow \tilde{\mathcal{D}}_x$ by $\tilde{\beta}_x := \beta_{r_1} \beta_{r_1}^{-1} \cdots \beta_{r_n} \beta_{r_n}^{-1} \beta_r$ (as composition of partial maps). If $y = \epsilon_{s_1} \cdots \epsilon_{s_m}[s]$ is another element of $\mathbf{Pr}(G)$ in normal form, we also define $\tilde{\omega}(x, y)$ as the restriction of the unitary multiplier $\omega(r, s) \in \mathcal{UM}(\mathcal{D}_r \cdot \mathcal{D}_{rs})$ to $\mathcal{D}_{xy} = \mathcal{D}_{r_1} \cdots \mathcal{D}_{r_n} \mathcal{D}_{rs_1} \cdots \mathcal{D}_{rs_m} \cdot \mathcal{D}_r \cdot \mathcal{D}_{rs} \subseteq \mathcal{D}_r \cdot \mathcal{D}_{rs}$.

The following result is a generalization of Theorems 4.2 and 4.3 in [10].

PROPOSITION 6.4. *The pair $(\tilde{\beta}, \tilde{\omega})$ (together with the domains $\tilde{\mathcal{D}}_x$) above defined is a (global) twisted action of $\mathbf{Pr}(G)$ on B . Moreover, the assignment $(\beta, \omega) \mapsto (\tilde{\beta}, \tilde{\omega})$ is a bijective correspondence between twisted partial actions of G on B and twisted (global) actions of $\mathbf{Pr}(G)$ on B .*

Proof. First, let us observe that if $x = \epsilon_{r_1} \cdots \epsilon_{r_n} [r]$ is an element of $\mathbf{Pr}(G)$ (not necessarily in normal form), then $\tilde{\mathcal{D}}_x := \mathcal{D}_{r_1} \cdots \mathcal{D}_{r_n} \cdot \mathcal{D}_r$. In fact, to check this it is enough to put x in normal form and use Proposition 6.3(iii). This justifies the fact that $\tilde{\mathcal{D}}_{xy} = \mathcal{D}_{r_1} \cdots \mathcal{D}_{r_n} \mathcal{D}_{r_{s_1}} \cdots \mathcal{D}_{r_{s_m}} \cdot \mathcal{D}_r \cdot \mathcal{D}_{r_s}$ for $x = \epsilon_{r_1} \cdots \epsilon_{r_n} [r]$ and $y = \epsilon_{s_1} \cdots \epsilon_{s_m} [s]$ because in this case $xy = \epsilon_{r_1} \cdots \epsilon_{r_n} \epsilon_{r_{s_1}} \cdots \epsilon_{r_{s_m}} \epsilon_r [rs]$ (see Proposition 2.9). Also, since $x^* = [r^*] \epsilon_{r_1} \cdots \epsilon_{r_n} = \epsilon_{r^* r_1} \cdots \epsilon_{r^* r_n} [r^*]$ (see Proposition 2.2(ii)), it follows that $\tilde{\mathcal{D}}_{x^*} = \mathcal{D}_{r^*} \cdot \mathcal{D}_{r^* r_1} \cdots \mathcal{D}_{r^* r_n}$. Now, it is not difficult to see that the domain and range of $\beta_{r_1} \beta_{r_1}^{-1} \cdots \beta_{r_n} \beta_{r_n}^{-1} \beta_r$ are $\mathcal{D}_{r^*} \cdot \mathcal{D}_{r^* r_1} \cdots \mathcal{D}_{r^* r_n}$ and $\mathcal{D}_{r_1} \cdots \mathcal{D}_{r_n} \cdot \mathcal{D}_r$. Therefore, $\tilde{\beta}_x : \tilde{\mathcal{D}}_{x^*} \rightarrow \tilde{\mathcal{D}}_x$ is a $*$ -isomorphism. Also, observe that $\tilde{\omega}(x, y)$ is a unitary multiplier of $\tilde{\mathcal{D}}_{xy}$ since $\omega(r, s)$ is a unitary multiplier of $\mathcal{D}_r \cap \mathcal{D}_{r_s}$ which contains $\tilde{\mathcal{D}}_{xy}$ as an ideal. It remains to check the axioms of twisted action in [2, Definition 4.1] for the pair $(\tilde{\beta}, \tilde{\omega})$. But since the $\tilde{\beta}$ and $\tilde{\omega}$ are just restrictions of β and ω , it is an exercise to check that the axioms in Definition 6.1 also hold for the pair $(\tilde{\beta}, \tilde{\omega})$ and hence it is a twisted partial action of $\mathbf{Pr}(G)$ on B . To prove that $(\tilde{\beta}, \tilde{\omega})$ is a (global) twisted action in the sense of [2, Definition 4.1], by Proposition 6.3(i), it is enough to check that $\tilde{\mathcal{D}}_x = \tilde{\mathcal{D}}_{xx^*}$ for all $x \in \mathbf{Pr}(G)$. But if x is written in normal form as $x = \epsilon_{r_1} \cdots \epsilon_{r_n} [r]$, then $xx^* = \epsilon_{r_1} \cdots \epsilon_{r_n} \epsilon_r$ so that $\tilde{\mathcal{D}}_{xx^*} = \mathcal{D}_{r_1} \cdots \mathcal{D}_{r_n} \cdot \mathcal{D}_r = \tilde{\mathcal{D}}_x$.

Therefore, if (β, ω) is a twisted partial action of G on B , then the pair $(\tilde{\beta}, \tilde{\omega})$ above defined is a (global) twisted action of $\mathbf{Pr}(G)$ on B . Conversely, if $(\tilde{\beta}, \tilde{\omega})$ is a twisted action of $\mathbf{Pr}(G)$ on B , then we can define (β, ω) by setting $\mathcal{D}_s := \tilde{\mathcal{D}}_{[s]}$ (here, of course, $\tilde{\mathcal{D}}_x$ denotes the range of $\tilde{\beta}_x$ for all $x \in \mathbf{Pr}(G)$), $\beta_s := \tilde{\beta}_{[s]}$, and $\omega(s, t) := \tilde{\omega}([s], [t]) \in \mathcal{UM}(\tilde{\mathcal{D}}_{[s][t]})$. Observe that by [2, Lemma 4.6(v)], $\tilde{\mathcal{D}}_{[s][t]} = \tilde{\mathcal{D}}_{[s][s^*][st]} = \tilde{\mathcal{D}}_{[s]} \cdot \tilde{\mathcal{D}}_{[st]} = \mathcal{D}_s \cdot \mathcal{D}_{st}$, so that $\omega(s, t)$ is a unitary multiplier of $\mathcal{D}_s \cdot \mathcal{D}_{st}$. Again, it is easy to check the axioms of twisted partial action of the pair (β, ω) . And it is clear that the assignments $(\beta, \omega) \mapsto (\tilde{\beta}, \tilde{\omega})$ and $(\tilde{\beta}, \tilde{\omega}) \mapsto (\beta, \omega)$ are inverse of each other. \square

7. Fell bundles over G and $\mathbf{Pr}(G)$

Recall that a *Fell bundle* over an inverse semigroup G is a family $\mathcal{A} = \{\mathcal{A}_s\}_{s \in G}$ of Banach spaces \mathcal{A}_s together with multiplication maps $\mathcal{A}_s \times \mathcal{A}_t \rightarrow \mathcal{A}_{st}$, involutions $\mathcal{A}_s \rightarrow \mathcal{A}_{s^*}$ for all $s, t \in G$ and inclusions $\mathcal{A}_s \hookrightarrow \mathcal{A}_t$ whenever $s \leq t$ in G . All this structure is required to be compatible in a suitable way. In addition, the norms on the fibers \mathcal{A}_s are required to satisfy the C^* -condition $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}_s$. Observe that $a^*a \in \mathcal{A}_{s^*s}$ whenever $a \in \mathcal{A}_s$. In particular, the fiber \mathcal{A}_e over an idempotent $e \in E(G)$ is a C^* -algebra. One

of the requirements in the definition of a Fell bundle is that a^*a has to be a positive element of the C^* -algebra \mathcal{A}_{s^*s} for all $a \in \mathcal{A}_s$, that is, $a^*a = b^*b$ for some $b \in \mathcal{A}_{s^*s}$ (this is not automatically satisfied in general). There are two canonical C^* -algebras that can be constructed from a Fell bundle \mathcal{A} : one is the *full cross-sectional* C^* -algebra $C^*(\mathcal{A})$ and the other is the *reduced cross-sectional* C^* -algebra $C_r^*(\mathcal{A})$ of \mathcal{A} . We refer the reader to [5] for the precise definition of Fell bundles and their cross-sectional C^* -algebras.

A famous result by Gelfand–Neumark asserts that every C^* -algebra is isomorphic to a concrete $*$ -algebra of operators, that is, to some closed $*$ -subalgebra $A \subseteq \mathcal{L}(\mathcal{H})$ of bounded operators on some Hilbert space \mathcal{H} (this is the reason why C^* -algebras are sometimes also called *operator algebras*). An extension of this result is also true for Fell bundles. A *concrete* Fell bundle over G is a family $\mathcal{A} = \{\mathcal{A}_s\}$ of closed subspaces $\mathcal{A}_s \subseteq \mathcal{L}(\mathcal{H})$, for some fixed Hilbert space \mathcal{H} , such that

- $\mathcal{A}_s\mathcal{A}_t \subseteq \mathcal{A}_{st}$ for all $s, t \in G$;
- $\mathcal{A}_s^* \subseteq \mathcal{A}_{s^*}$ for all $s \in G$; and
- $\mathcal{A}_s \subseteq \mathcal{A}_t$ whenever $s \leq t$ in G .

Every (abstract) Fell bundle \mathcal{A} is isomorphic to a concrete one. Indeed, this can be proved by applying the Gelfand–Neumark theorem mentioned above to the full (or reduced) cross-sectional C^* -algebra $A = C^*(\mathcal{A})$ and using the canonical representation of \mathcal{A} into A (see [5] for more details). Thus, given a Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in G}$, we may assume that there is a Hilbert space \mathcal{H} such that \mathcal{A} is a concrete Fell bundle in $\mathcal{L}(\mathcal{H})$ and such that $A = C^*(\mathcal{A})$ is the C^* -subalgebra of $\mathcal{L}(\mathcal{H})$ defined as the closed linear span of the fibers $\mathcal{A}_s \subseteq \mathcal{L}(\mathcal{H})$. Observe that the fibers \mathcal{A}_e over idempotents $e \in E = E(G)$ are in this way C^* -subalgebras of $A \subseteq \mathcal{L}(\mathcal{H})$. Let $\mathcal{E} = \{\mathcal{A}_e\}_{e \in \mathcal{E}}$ be the restriction of \mathcal{A} to the semilattice E . By [5, Proposition 4.3], the C^* -algebra $B = C^*(\mathcal{E})$ is isomorphic to the closed linear span of the fibers \mathcal{A}_e with $e \in E$. So, we may further assume that B is a C^* -subalgebra of $A \subseteq \mathcal{L}(\mathcal{H})$. Hence, there is no loss of generality in assuming that a given Fell bundle is concrete and we shall usually do so in what follows.

DEFINITION 7.1. Given a C^* -algebra A , we write $\mathcal{P}_\ell(A)$ for the semigroup of all closed subspaces of A with respect to the canonical multiplication:

$$M \cdot N := \overline{\text{span}} MN \quad \text{for all } M, N \in \mathcal{P}_\ell(A).$$

If \mathcal{H} is a Hilbert space, we write $\mathcal{P}_\ell(\mathcal{H})$ for $\mathcal{P}_\ell(\mathcal{L}(\mathcal{H}))$.

Observe that $\mathcal{P}_\ell(A)$ is not an inverse semigroup in general, even if we restrict attention to subspaces $M \subseteq A$ that are ternary ring of operators in the sense that $M \cdot M^* \cdot M = M$. The problem is that $\mathcal{P}_\ell(A)$ is too big and idempotents do not commute in general. One way to find an inverse subsemigroup inside $\mathcal{P}_\ell(A)$ is to consider a fixed C^* -subalgebra $B \subseteq A$ and consider the

subset $\mathcal{P}_\ell(A, B) \subseteq \mathcal{P}_\ell(A)$ consisting of all closed subspaces $M \subseteq A$ satisfying

$$MM^*M \subseteq M, \quad M^*M \subseteq B, \quad MM^* \subseteq B, \\ MB \subseteq M \quad \text{and} \quad BM \subseteq M.$$

Then it is not difficult to prove that $\mathcal{P}_\ell(A, B)$ is in fact an inverse subsemigroup of $\mathcal{P}_\ell(A)$. This is done in the proof of [2, Proposition 8.6]. The inverse of M in $\mathcal{P}_\ell(A, B)$ is the adjoint $M^* = \{m^* : m \in M\}$. If A is $\mathcal{L}(\mathcal{H})$ and B is a C^* -subalgebra of $\mathcal{L}(\mathcal{H})$, we write $\mathcal{P}_\ell(\mathcal{H}, B)$ for $\mathcal{P}_\ell(\mathcal{L}(\mathcal{H}), B)$.

In particular, if $\mathcal{A} = \{\mathcal{A}_s\}_{s \in G}$ is a concrete Fell bundle in $\mathcal{L}(\mathcal{H})$ for some Hilbert space \mathcal{H} , we may consider the C^* -algebra $B = C^*(\mathcal{E}) \subseteq \mathcal{L}(\mathcal{H})$ defined as the closed linear span of the fibers \mathcal{A}_e with $e \in E(S)$. Then $\mathcal{P}_\ell(\mathcal{H}, B)$ is an inverse semigroup containing all the fibers $\mathcal{A}_s \subseteq \mathcal{L}(\mathcal{H})$ (see proof of Proposition 8.6 in [2]). This assertion includes in particular the fact that each \mathcal{A}_s is a ternary ring of operators. Observe that the fibers generate an inverse subsemigroup of $\mathcal{P}_\ell(\mathcal{H}, B)$, namely the inverse semigroup consisting of all finite products (that is, closed linear spans) of the form $\mathcal{A}_{s_1} \cdot \mathcal{A}_{s_2} \cdots \mathcal{A}_{s_n}$ with $s_i \in G$. If \mathcal{A} is saturated, meaning that $\overline{\text{span}} \mathcal{A}_s \mathcal{A}_t = \mathcal{A}_{st}$ for all $s, t \in G$, then in a sense the Fell bundle \mathcal{A} itself may be “confused” with this inverse semigroup.

THEOREM 7.2. *Let $\mathcal{A} = \{\mathcal{A}_s\}_{s \in G}$ be a concrete Fell bundle in $\mathcal{L}(\mathcal{H})$ over an inverse semigroup G . Then the map $\pi : G \rightarrow \mathcal{P}_\ell(\mathcal{H})$ given by $\pi(s) := \mathcal{A}_s$ is a partial homomorphism. Moreover, π is a homomorphism if and only if \mathcal{A} is saturated. Let $\tilde{\pi} : \mathbf{Pr}(G) \rightarrow \mathcal{P}_\ell(\mathcal{H})$ be the representation associated to π as in Proposition 2.12, and let $\tilde{\mathcal{A}}_x := \tilde{\pi}(x)$ for all $x \in \mathbf{Pr}(G)$. Then $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_x\}_{x \in \mathbf{Pr}(G)}$ is a saturated Fell bundle over $\mathbf{Pr}(G)$. Given $s_1, \dots, s_n, s \in G$, the fiber $\tilde{\mathcal{A}}_x$ over $x = \varepsilon_{s_1} \cdots \varepsilon_{s_n}[s]$ is given by*

$$(7.1) \quad \tilde{\mathcal{A}}_x = \mathcal{A}_{s_1} \cdot \mathcal{A}_{s_1^*} \cdots \mathcal{A}_{s_n} \cdot \mathcal{A}_{s_n^*} \cdot \mathcal{A}_s.$$

The assignment $\mathcal{A} \mapsto \tilde{\mathcal{A}}$ is a bijective correspondence between Fell bundles over G and saturated Fell bundles over $\mathbf{Pr}(G)$. Furthermore, the canonical Fell bundle morphism $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ induces C^ -algebra isomorphisms $C^*(\tilde{\mathcal{A}}) \cong C^*(\mathcal{A})$, $C_r^*(\tilde{\mathcal{A}}) \cong C_r^*(\mathcal{A})$ and $C^*(\tilde{\mathcal{E}}) \cong C^*(\mathcal{E})$, where $\tilde{\mathcal{E}}$ and \mathcal{E} are the restrictions of $\tilde{\mathcal{A}}$ and \mathcal{A} to $E(\mathbf{Pr}(G))$ and $E(G)$, respectively.*

Proof. To prove that π is a partial homomorphism, we have to show that

$$\mathcal{A}_s \cdot \mathcal{A}_t \cdot \mathcal{A}_{t^*} = \mathcal{A}_{st} \cdot \mathcal{A}_{t^*} \quad \text{and} \quad \mathcal{A}_{s^*} \cdot \mathcal{A}_s \cdot \mathcal{A}_t = \mathcal{A}_{s^*} \cdot \mathcal{A}_{st} \quad \text{for all } s, t \in G.$$

Since $\mathcal{A}_s \mathcal{A}_t \subseteq \mathcal{A}_{st}$, we have $\mathcal{A}_s \cdot \mathcal{A}_t \cdot \mathcal{A}_{t^*} \subseteq \mathcal{A}_{st} \cdot \mathcal{A}_{t^*}$. On the other hand, observe that $\mathcal{A}_{se} \subseteq \mathcal{A}_s$ for all $e \in E(G)$ because $se \leq s$ in G . Applying this to $e = tt^*$ and using that $\mathcal{A}_{t^*} = \mathcal{A}_{t^*} \cdot \mathcal{A}_t \cdot \mathcal{A}_{t^*}$, we get

$$\mathcal{A}_{st} \cdot \mathcal{A}_{t^*} = \mathcal{A}_{st} \cdot \mathcal{A}_{t^*} \cdot \mathcal{A}_t \cdot \mathcal{A}_{t^*} \subseteq \mathcal{A}_{stt^*} \cdot \mathcal{A}_t \cdot \mathcal{A}_{t^*} \subseteq \mathcal{A}_s \cdot \mathcal{A}_t \cdot \mathcal{A}_{t^*}.$$

Similarly, $\mathcal{A}_{s^*} \cdot \mathcal{A}_s \cdot \mathcal{A}_t = \mathcal{A}_{s^*} \cdot \mathcal{A}_{st}$. Thus, π is a partial homomorphism.

Of course, π is a homomorphism if and only if $\mathcal{A}_{st} = \pi(st) = \pi(s)\pi(t) = \mathcal{A}_s \cdot \mathcal{A}_t$, that is, \mathcal{A} is saturated. Observe that the image of π (and hence of $\tilde{\pi}$) is contained in $\mathcal{P}_\ell(\mathcal{H}, B)$, where $B = C^*(\mathcal{E})$ (see comments above). Since $\tilde{\pi}$ is a representation, we have $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$ and $\tilde{\pi}(x^*) = \tilde{\pi}(x)^*$, that is, we have $\tilde{\mathcal{A}}_{xy} = \tilde{\mathcal{A}}_x \cdot \tilde{\mathcal{A}}_y$ and $\tilde{\mathcal{A}}_{x^*} = \tilde{\mathcal{A}}_x^*$ for all $x, y \in \mathbf{Pr}(G)$. If $x \leq y$ in $\mathbf{Pr}(G)$, then $\tilde{\pi}(x) \leq \tilde{\pi}(y)$ in $\mathcal{P}_\ell(\mathcal{H}, B)$, that is, $\tilde{\mathcal{A}}_x \subseteq \tilde{\mathcal{A}}_y$ whenever $x \leq y$ (the natural order of $\mathcal{P}_\ell(\mathcal{H}, B)$ is given by inclusion of subspaces). This implies that $\tilde{\mathcal{A}}$ is a (concrete) saturated Fell bundle over $\mathbf{Pr}(G)$. If $x = \varepsilon_{s_1} \cdots \varepsilon_{s_n}[s]$, observe that

$$\tilde{\pi}(x) = \tilde{\pi}(\varepsilon_{s_1}) \cdots \tilde{\pi}(\varepsilon_{s_n})\tilde{\pi}(s) = \pi(s_1)\pi(s_1^*) \cdots \pi(s_n)\pi(s_n^*)\pi(s).$$

This is exactly (7.1).

If $\tilde{\mathcal{A}}$ is a saturated Fell bundle over $\mathbf{Pr}(G)$, then we may “restrict” $\tilde{\mathcal{A}}$ to G (using the canonical inclusion map $\iota_G : G \rightarrow \mathbf{Pr}(G)$) and get a Fell bundle \mathcal{A} over G . The two constructions $\mathcal{A} \mapsto \tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}} \mapsto \mathcal{A}$ are easily seen to be inverse of each other, so that Fell bundles over G correspond bijectively to saturated Fell bundles over $\mathbf{Pr}(G)$.

The final assertion will follow from [2, Theorem 8.4], after we check that the canonical morphism

$$(\iota, \partial) : (\tilde{\mathcal{A}}, \mathbf{Pr}(G)) \rightarrow (\mathcal{A}, G)$$

gives $\tilde{\mathcal{A}}$ as a *refinement* of \mathcal{A} in the sense of [2, Definition 8.1], where $\partial : \mathbf{Pr}(G) \rightarrow G$ is the degree map and $\iota : \tilde{\mathcal{A}} \rightarrow \mathcal{A}$ is given by the inclusions $\tilde{\mathcal{A}}_x \hookrightarrow \mathcal{A}_{\partial(x)}$ (observe that $\tilde{\mathcal{A}}_x \subseteq \mathcal{A}_{\partial(x)}$ by (7.1)). First, we already know from Proposition 2.14 that ∂ is (surjective and) essentially injective. And, by definition, $\iota : \tilde{\mathcal{A}}_x \rightarrow \mathcal{A}_{\partial(x)}$ is the inclusion map, so it is injective. Finally, given $s \in G$, the condition

$$\mathcal{A}_s = \overline{\sum_{\partial(x)=s} \iota(\tilde{\mathcal{A}}_x)}$$

appearing in [2, Definition 8.1] is obviously satisfied since $\iota(\tilde{\mathcal{A}}_{[s]}) = \mathcal{A}_s$. \square

REMARK 7.3. The above result improves [2, Proposition 8.6]. To be more precise, in [2, Proposition 8.6] we proved that every Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in G}$ has a saturated refinement $\mathcal{B} = \{\mathcal{B}_t\}_{t \in T}$, where T is another inverse semigroup which, a priori, depends on the Fell bundle \mathcal{A} (see proof of Proposition 8.6 in [2]). Above we have proved that T can be chosen to be $\mathbf{Pr}(G)$, a more natural choice which depends only on G . So, for instance, if we know that G is countable, then so is $T = \mathbf{Pr}(G)$ (this was not clear in [2, Proposition 8.6]). Also, if the fibers \mathcal{A}_s are separable, then so are the fibers \mathcal{B}_t because each one is a subspace of some \mathcal{A}_s . The refinement obtained in the above theorem also preserves (local) regularity (see [2] for the definition of (local) regularity of Fell bundles): in fact, observe that each fiber $\tilde{\mathcal{A}}_x$ is an ideal of $\mathcal{A}_{\partial(x)}$ (in the sense of [3, Definition 6.1]). So, the result follows from [3, Proposition 6.3].

8. Fell bundles and partial twisted actions

Given a twisted partial action (B, β, ω) of G on a C^* -algebra B as in Definition 6.1, there is a canonical associated Fell bundle \mathcal{A} over G as described in [2] for (global) twisted actions (where we get saturated Fell bundles). The Fell bundle \mathcal{A} is defined as follows:

The fiber \mathcal{A}_s over $s \in G$ is defined by $\mathcal{A}_s := \{(a, s) : b \in \mathcal{D}_s\} = \mathcal{D}_s \delta_s$, where we write $a\delta_s$ for the element (a, s) . The operations on \mathcal{A} are defined by

$$(a\delta_s) \cdot (b\delta_t) := \beta_s(\beta_s^{-1}(a)b)\omega(s, t)\delta_{st}$$

and

$$(a\delta_s)^* := \beta_s^{-1}(a^*)\omega(s^*, s)^* \delta_{s^*}$$

and the inclusion maps $j_{t,s} : \mathcal{A}_s \hookrightarrow \mathcal{A}_t$ for $s \leq t$ in G are defined by

$$j_{t,s}(a\delta_s) := a\omega(t, s^*)^* \delta_t.$$

The above operations are exactly the same as the ones defined in [2]. And the proof that \mathcal{A} is in fact a Fell bundle can also be made in the same way. The only difference here is that \mathcal{A} is not necessarily saturated because of the partiality of our twisted action (β, ω) . In fact, it is easy to see that \mathcal{A} is saturated if and only if (β, ω) is a (global) twisted action (as defined in [2]). In any case, the Fell bundle \mathcal{A} is always regular. This means that the fibers \mathcal{A}_s are regular as imprimitivity Hilbert I_s, J_s -bimodules, where $I_s := \mathcal{A}_s \cdot \mathcal{A}_s^*$ and $J_s = \mathcal{A}_s^* \cdot \mathcal{A}_s$ (closed linear spans), that is (see [2] for the precise definition), there is a family $u = (u_s)_{s \in G}$ of unitary multipliers $u_s : J_s \rightarrow \mathcal{A}_s$ for the imprimitivity bimodules \mathcal{A}_s (recall that a multiplier of an imprimitivity Hilbert I, J -bimodule \mathcal{F} is, by definition, an adjointable operator $v : J \rightarrow \mathcal{F}$). For the Fell bundle \mathcal{A} above defined we may take u_s as the multiplier defined by $u_s(x\delta_{s^*}) := \beta_s(x)\delta_s$ for all $x \in \mathcal{D}_{s^*}$. It is easy to see that $I_s = \mathcal{A}_s^* \mathcal{A}_s = \mathcal{D}_{s^*} \delta_s$ so that u_s is a well-defined map from J_s to \mathcal{A}_s . Moreover, u_s is adjointable with adjoint given by $u_s^*(y\delta_s) = \beta_s^{-1}(y)\delta_{s^*}$ for all $y \in \mathcal{D}_s$. Observe that u_s is in fact a unitary multiplier, that is, $u_s^* u_s = 1_{s^*}$ and $u_s u_s^* = 1_s$, where 1_s is the unit multiplier of $J_s \cong \mathcal{D}_s$ (isomorphism of C^* -algebras). Also, observe that $u_e = 1_e$ for all $e \in E(G)$.

Hence, to a twisted partial action (β, ω) we attach a pair (\mathcal{A}, u) consisting of a (regular) Fell bundle \mathcal{A} over G and a family $u = (u_s)_{s \in G}$ of unitary multipliers u_s for \mathcal{A}_s such that $u_e = 1_e$ for all $e \in E(G)$.

Conversely, to such a pair (\mathcal{A}, u) it is possible to construct a twisted partial action (β, ω) using ideas as in [2, Section 3] as follows: given a regular concrete Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in G}$ in $\mathcal{L}(\mathcal{H})$, a family of unitary multipliers $u = \{u_s\}_{s \in G}$ as above may be viewed concretely as partial isometries $u_s \in \mathcal{L}(\mathcal{H})$ satisfying:

$$(8.1) \quad \begin{aligned} u_s \mathcal{A}_s^* &= \mathcal{A}_s \cdot \mathcal{A}_s^*, & \mathcal{A}_s^* u_s &= \mathcal{A}_s^* \cdot \mathcal{A}_s, \\ u_s u_s^* &= 1_s, & u_s^* u_s &= 1_{s^*}, \quad \text{and} \quad u_e = 1_e \end{aligned}$$

for all $s \in G$ and $e \in E(G)$, where 1_s denotes the unit multiplier of $\mathcal{A}_s \cdot \mathcal{A}_s^*$, that is, the (orthogonal) projection onto the subspace $\mathcal{A}_s \cdot \mathcal{A}_s^* \cdot \mathcal{H} \subseteq \mathcal{H}$. The associated twisted partial action (β, ω) is then given by

$$\beta_s : \mathcal{D}_{s^*} \rightarrow \mathcal{D}_s, \quad \beta_s(x) := u_s x u_s^*, \quad \text{and} \quad \omega(s, t) := u_s u_t u_{st}^*,$$

where $\mathcal{D}_s := \mathcal{A}_s \cdot \mathcal{A}_s^*$ (closed linear span). The same ideas as in [2] may be used to prove that the pair (β, ω) above defined is in fact a twisted partial action and the following result (see [2, Corollary 4.16] for the saturated case) may be obtained.

PROPOSITION 8.1. *The above construction $(\beta, \omega) \mapsto (\mathcal{A}, u)$ is a bijective correspondence between (isomorphism classes of) twisted partial actions (β, ω) of G and regular Fell bundles (\mathcal{A}, u) over G .*

The C^* -algebra B above where G acts through (β, ω) is, via the above correspondence, canonically isomorphic to the cross-sectional C^* -algebra $C^*(\mathcal{E})$ of the restriction \mathcal{E} of \mathcal{A} to the idempotents $E = E(G)$. In fact, observe that by the construction of \mathcal{A} , each fiber $\mathcal{A}_e = \mathcal{D}_e \delta_e \cong \mathcal{D}_e$ over an idempotent $e \in E$ embeds into B . Since the ideals \mathcal{D}_s for $s \in G$ span a dense subspace of B , the same is true for \mathcal{D}_e with $e \in E$ because for every $s \in G$ we have $\mathcal{D}_s \subseteq \mathcal{D}_{ss^*}$ (see Proposition 6.3(i)). Hence, the embeddings $\mathcal{A}_e \hookrightarrow B$ extend to an isomorphism $C^*(\mathcal{E}) \xrightarrow{\sim} B$ by [5, Proposition 4.3].

Moreover, the correspondence $(B, G, \beta, \omega) \mapsto \mathcal{A}$ is also compatible with crossed products in the sense that

$$(8.2) \quad B \rtimes_{\beta, \omega} G \cong C^*(\mathcal{A}) \quad \text{and} \quad B \rtimes_{\beta, \omega}^r G \cong C_r^*(\mathcal{A}).$$

We did not define the above full $B \rtimes_{\beta, \omega} G$ and reduced $B \rtimes_{\beta, \omega}^r G$ crossed products for a twisted partial action (β, ω) of G on B , but they can be defined (and the above isomorphisms can be obtained) in the same way as explained in [2, Section 6] for twisted (global) actions.

Now, if \mathcal{A} is a regular Fell bundle concrete in $\mathcal{L}(\mathcal{H})$, then, as already mentioned in Remark 7.3, the corresponding Fell bundle $\tilde{\mathcal{A}}$ over $\mathbf{Pr}(G)$ as in Theorem 7.2 is also regular (and concrete in $\mathcal{L}(\mathcal{H})$). Moreover, if $u = \{u_s\}_{s \in G}$ is a family of partial isometries on \mathcal{H} satisfying (8.1) for \mathcal{A} , then the family $\tilde{u} = \{\tilde{u}_x\}_{x \in \mathbf{Pr}(G)}$ defined by

$$\tilde{u}_x := u_{s_1} u_{s_1}^* \cdots u_{s_n} u_{s_n}^* u_s \quad \text{whenever } x = \epsilon_{s_1} \cdots \epsilon_{s_n} [s] \text{ is in normal form,}$$

also satisfies (8.1) for $\tilde{\mathcal{A}}$. Moreover, it is not difficult to see that if (β, ω) is the twisted partial action corresponding to (\mathcal{A}, u) , then the twisted (global) action $(\tilde{\beta}, \tilde{\omega})$ associated to $(\tilde{\mathcal{A}}, \tilde{u})$ as above is the same as the one already obtained in Proposition 6.4. In particular, Theorem 7.2 and (8.2) yields the following consequence:

COROLLARY 8.2. *Let (β, ω) be a twisted partial action of G on B and let $(\tilde{\beta}, \tilde{\omega})$ be the corresponding twisted action of $\mathbf{Pr}(G)$ on B . Then*

$$B \rtimes_{\beta, \omega} G \cong B \rtimes_{\tilde{\beta}, \tilde{\omega}} \mathbf{Pr}(G) \quad \text{and} \quad B \rtimes_{\beta, \omega}^r G \cong B \rtimes_{\tilde{\beta}, \tilde{\omega}}^r \mathbf{Pr}(G).$$

Of course, this also covers the case of (untwisted) partial actions of inverse semigroups on C^* -algebras (and, in particular, on locally compact spaces):

COROLLARY 8.3. *Let β be a partial action of G on B and let $\tilde{\beta}$ be the corresponding action of $\mathbf{Pr}(G)$ on B . Then*

$$B \rtimes_{\beta} G \cong B \rtimes_{\tilde{\beta}} \mathbf{Pr}(G) \quad \text{and} \quad B \rtimes_{\beta}^r G \cong B \rtimes_{\tilde{\beta}}^r \mathbf{Pr}(G).$$

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ALCIDES BUSS, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE SANTA CATARINA, 88.040-900 FLORIANÓPOLIS-SC, BRASIL

E-mail address: alcides@mtm.ufsc.br

RUY EXEL, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE SANTA CATARINA, 88.040-900 FLORIANÓPOLIS-SC, BRASIL

E-mail address: exel@mtm.ufsc.br