

TWO INEQUALITIES ON THE AREAL MAHLER MEASURE

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ABSTRACT. Recent work of Pritsker defines and studies an areal version of the Mahler measure. We further explore this function with a particular focus on the case where its value is small, as this is most relevant to Lehmer’s conjecture. In this situation, we provide improvements to two inequalities established in Pritsker’s original paper.

1. Introduction

For a polynomial $P(z)$ with complex coefficients, the (*logarithmic*) Mahler measure of P is given by

$$\log M(P) = \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{it})| dt.$$

In view of the well-known identity

$$(1.1) \quad \log M(P) = \lim_{p \rightarrow 0^+} \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^p dt \right)^{1/p},$$

the Mahler measure is often called the H^0 Hardy space norm. If $P(z) = a_N \prod_{n=1}^N (z - z_n)$, then we may apply Jensen’s formula to find that

$$(1.2) \quad \log M(P) = \log |a_N| + \sum_{|z_n| > 1} \log |z_n|,$$

where the sum is taken over all roots of P that lie outside the closed unit disk.

If we consider only polynomials $P \in \mathbb{Z}[z]$, then it follows from (1.2) that $M(P) \geq 1$. Kronecker’s theorem implies that $M(P) = 1$ if and only if P is a

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product of cyclotomic polynomials and $\pm z$. As part of a famous 1933 article constructing large prime numbers, Lehmer [8] asked whether there exists a constant $c > 1$ such that $M(P) \geq c$ in all other cases. In his investigations, he noted that

$$\ell(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

has Mahler measure equal to $1.17\dots$, although he found no polynomial of measure smaller than this. In view of this observation, we are led to the following conjecture.

CONJECTURE 1.1 (Lehmer’s conjecture). There exists $c > 1$ such that $M(P) \geq c$ whenever $P \in \mathbb{Z}[z]$ is not a product of cyclotomic polynomials and $\pm z$.

Although Lehmer’s conjecture remains open, numerous special cases of it have been resolved (see, for instance, [1], [2], [12], [14]). The best known universal lower bound is

$$\log M(P) \gg \left(\frac{\log \log \deg P}{\log \deg P} \right)^3,$$

which arises from the work of Dobrowolski [3].

Inspired by Lehmer’s original paper, various authors have constructed analogues and generalizations of the Mahler measure. Dubickas and Smyth [4], [5] defined and examined the *metric Mahler measure*, which is a function $m_1 : \overline{\mathbb{Q}}^\times / \text{Tor}(\overline{\mathbb{Q}}^\times) \rightarrow [0, \infty)$ that is closely related to the classical Mahler measure. They note that $(\alpha, \beta) \mapsto m_1(\alpha\beta^{-1})$ defines a metric on $\overline{\mathbb{Q}}^\times / \text{Tor}(\overline{\mathbb{Q}}^\times)$ which induces the discrete topology if and only if Lehmer’s conjecture holds. An ultrametric version of m_1 was explored by Fili and the second author [6], while both functions were later generalized by the second author [10], [11].

In a different direction, Sinclair [13] defined the notion of a *multiplicative distance function*. This is a map $\phi : \mathbb{C}[z] \rightarrow [1, \infty)$ of which the non-logarithmic Mahler measure M is an example. Other examples of such functions are discussed in [13].

Returning now to the situation where $P(z)$ is an arbitrary polynomial over \mathbb{C} , Pritsker [9] recently defined the *areal Mahler measure* of $P \in \mathbb{C}[z]$ by

$$(1.3) \quad \log \|P\|_0 = \frac{1}{\pi} \iint_D \log |P(z)| \, dA,$$

where D is the closed unit disk. It is noted that $\|P\|_0$ satisfies an analogue of (1.1)

$$\log \|P\|_0 = \lim_{p \rightarrow 0^+} \left(\frac{1}{\pi} \iint_D |P(z)|^p \, dA \right)^{1/p}.$$

Although we will focus on (1.3) almost exclusively in this article, we note the work of Huang [7] which studies a Fock space version of the Mahler measure analogous to $\|P\|_0$.

In Theorem 1.1 of [9], Pritsker establishes the important identity

$$(1.4) \quad \log \|P\|_0 = \log M(P) + \frac{1}{2} \sum_{|z_n| < 1} (|z_n|^2 - 1),$$

which we should view as the analogue of (1.2) for the areal Mahler measure. Indeed, by adding and subtracting $\sum_{|z_n| < 1} |z_n|$ to the right-hand side of (1.2), we quickly obtain that

$$(1.5) \quad \log M(P) = \log |a_0| - \sum_{|z_n| < 1} \log |z_n|,$$

where a_0 is the constant term of P . Combining this with (1.4), we find that

$$(1.6) \quad \log \|P\|_0 = \log |a_0| + \sum_{|z_n| < 1} \left(\frac{|z_n|^2 - 1}{2} - \log |z_n| \right).$$

We interpret this identity as the areal Mahler measure version of (1.5).

In view of (1.4), we obtain the inequalities (see Corollary 1.2 in [9])

$$(1.7) \quad -\frac{\deg P}{2} + \log M(P) \leq \log \|P\|_0 \leq \log M(P).$$

Equality occurs in the first inequality precisely when P has no zeros except at the origin, and we obtain equality in the second inequality whenever P has no zeros inside the open unit disk. If we wish to study Lehmer’s conjecture, then we will most often consider polynomials of small Mahler measure. In this case, the left-hand side of (1.7) is negative, and therefore, weaker than even the most trivial bound that $\|P\|_0 \geq |a_0|$.

Using only elementary techniques, we anticipate that further improvements to (1.7) are possible when $M(P)$ is small and $P \in \mathbb{Z}[z]$. Indeed, examining the individual terms of the sums in (1.5) and (1.6), we find the power series expansions

$$-\log x = \sum_{k=1}^{\infty} \frac{(1-x)^k}{k} \quad \text{and} \quad \frac{x^2 - 1}{2} - \log x = (1-x)^2 + \sum_{k=3}^{\infty} \frac{(1-x)^k}{k}.$$

Of course, this implies that

$$\log^2 x = (1-x)^2 + \sum_{k=3}^{\infty} \left(\sum_{\ell=1}^{k-1} \frac{1}{\ell(k-\ell)} \right) (1-x)^k,$$

so that the expansions for $\log^2 x$ and $(x^2 - 1)/2 - \log x$ have the same main terms. In view of these observations, we find it reasonable to guess that $\log^2 M(P)$ behaves similarly to $\log \|P\|_0$ when these values are close to zero.

Nevertheless, these remarks are only heuristic and must be made precise. In this article, we provide upper and lower bounds on $\log^2 M(P)$ in terms of $\log \|P\|_0$. In both cases, our results constitute improvements to (1.7) when $P \in \mathbb{Z}[z]$ and $\|P\|_0$ is sufficiently small. Our first result is the lower bound.

THEOREM 1.2. *If $P \in \mathbb{C}[z]$ is such that $|P(0)| = 1$, then $\log \|P\|_0 \leq \log^2 M(P)$.*

If P has integer coefficients, $P(0) \neq 0$, and $\|P\|_0 < 2$, then the hypotheses of Theorem 1.2 are automatically satisfied. Indeed, it follows from (1.6) that $0 < |P(0)| < 2$ for all $P \in \mathbb{C}[z]$. Therefore, if $P \in \mathbb{Z}[z]$, then $|P(0)| = 1$. Furthermore, if $M(P) \leq e$ then $\log^2 M(P) \leq \log M(P)$, so that Theorem 1.2 does indeed provide an improvement to the upper bound in (1.7).

Pritsker further notes that the analogue of Lehmer’s conjecture is false for $\|P\|_0$, providing the example $P_n(z) = nz^n - 1$. Here, we have that

$$\log M(P_n) = \log n \quad \text{and} \quad \log \|P_n\|_0 = O\left(\frac{\log^2 n}{n}\right).$$

Now let $Z(P)$ denote the number of roots of P that lie inside the open unit disk. In this example, and in another provided by Pritsker, $Z(P_n)$ tends to ∞ as $n \rightarrow \infty$.

For the remainder of this article, we will always assume that $Z(P) \geq 1$. Otherwise Pritsker’s work [9] establishes that $M(P) = \|P\|_0 = |P(0)|$ and there is little new information to discover. Our next result gives an upper bound on $\log^2 M(P)/Z(P)$ in terms of $\|P\|_0$.

THEOREM 1.3. *If $P \in \mathbb{C}[z]$ is such that $|P(0)| = 1$ and $\|P\|_0 < e$ then*

$$(1.8) \quad \frac{\log^2 M(P)}{Z(P)} \leq \log \|P\|_0 \left(1 + \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{k+1} \frac{1}{\ell(k+2-\ell)} - \frac{1}{k+2} \right) \log^{k/2} \|P\|_0 \right).$$

We have trivially that $0 \leq \sum_{\ell=1}^{k-1} 1/(\ell(k-\ell)) \leq 1$ so that the interior sum on the right hand side of (1.8) converges. Moreover, if we are willing to sacrifice a small amount of sharpness, we obtain a more concrete result.

COROLLARY 1.4. *If $P \in \mathbb{C}[z]$ is such that $|P(0)| = 1$ and $\|P\|_0 < e$ then*

$$(1.9) \quad \frac{\log^2 M(P)}{Z(P)} \leq \log \|P\|_0 \left(1 + \frac{2}{3} \cdot \frac{\sqrt{\log \|P\|_0}}{1 - \sqrt{\log \|P\|_0}} \right).$$

The right-hand term in (1.9) should be regarded as an error term as $\|P\|_0 \rightarrow 1$, showing that $\log^2 M(P)/Z(P)$ is asymptotically bounded above by $\log \|P\|_0$. Pritsker’s aforementioned example establishes that the main term in this estimate is best possible. Indeed, taking $P_n(z) = nz^n - 1$, we see that $M(P) = Z(P) = n$, implying that the left-hand side of (1.8) equals $(\log^2 n)/n$. We also observe that

$$\log \|P_n\|_0 = \log n + \frac{n(n^{-2/n} - 1)}{2} = \log n + \frac{n}{2} \left(\exp\left(\frac{-2 \log n}{n}\right) - 1 \right).$$

Since

$$\exp\left(\frac{-2 \log n}{n}\right) - 1 = \sum_{k=1}^{\infty} \frac{(-2)^k \log^k n}{k! n^k} = \frac{-2 \log n}{n} + \frac{2 \log^2 n}{n^2} + O\left(\frac{\log^3 n}{n^3}\right),$$

we obtain that

$$\log \|P_n\|_0 = \frac{\log^2 n}{n} + O\left(\frac{\log^3 n}{n^2}\right).$$

In other words, we have shown that

$$(1.10) \quad \frac{\log^2 M(P_n)}{Z(P_n)} = \log \|P_n\|_0 \left(1 + O\left(\frac{\log n}{n}\right)\right).$$

Therefore, the main term in (1.9) provides the best possible upper bound. We must acknowledge, however, that improvements to the error term still seem likely. In this example, Theorem 1.3 gives

$$\frac{\log^2 M(P_n)}{Z(P_n)} \leq \log \|P_n\|_0 \left(1 + O\left(\frac{\log n}{\sqrt{n}}\right)\right),$$

the right-hand side of which has a larger error term than that of (1.10).

We also note that all roots of $P_n = nz^n - 1$ that lie inside the unit circle have the same absolute value. For a general polynomial $P \in \mathbb{C}[z]$ satisfying this property, we will say that P is *balanced*. In this case, we can obtain a lower bound on $\log^2 M(P)$ analogous to Theorem 1.3.

THEOREM 1.5. *If P is a balanced polynomial over \mathbb{C} such that $|P(0)| = 1$ and $\|P\|_0 < e$ then*

$$\frac{\log^2 M(P)}{Z(P)} \geq \log \|P\|_0.$$

Applying Corollary 1.4 and Theorem 1.5 to any sequence of balanced polynomials $P_n(z)$ with $\|P_n\|_0 \rightarrow 1$, we find that

$$\frac{\log^2 M(P_n)}{Z(P_n)} = \log \|P_n\|_0 (1 + O(\sqrt{\log \|P_n\|_0})).$$

In the case of $P_n(z) = nz^n - 1$, this formula becomes

$$\frac{\log^2 M(P_n)}{Z(P_n)} = \log \|P_n\|_0 \left(1 + O\left(\frac{\log n}{\sqrt{n}}\right)\right),$$

yielding an asymptotic formula having the same main term as (1.10). Although we cannot obtain the best error term in this specific example, the advantage of Theorems 1.3 and 1.5 is that they apply far more generally.

2. Proofs

For this section, we will always write $f(z) = (z^2 - 1)/2 - \log z$ and note that

$$(2.1) \quad \log \|P\|_0 = \sum_{|z_n| < 1} f(|z_n|),$$

provided that $|P(0)| = 1$ and z_n are the roots of P . By applying the power series expansion for the logarithm, we find that

$$(2.2) \quad f(z) = (1 - z)^2 + \sum_{k=3}^{\infty} \frac{(1 - z)^k}{k}$$

holds for all $|z - 1| < 1$. Moreover, it can be easily verified that

$$(2.3) \quad \log^2 z = f(z) + \sum_{k=3}^{\infty} \left(\sum_{\ell=1}^{k-1} \frac{1}{\ell(k - \ell)} - \frac{1}{k} \right) (1 - z)^k$$

in the same disk. This information alone is enough to prove Theorem 1.2.

Proof of Theorem 1.2. We assume that

$$P(z) = \sum_{n=0}^N a_n z^n = a_N \prod_{n=1}^N (z - z_n),$$

where $|a_0| = 1$ and observe immediately from (1.5) that

$$\log^2 M(P) = \left(\sum_{|z_n| < 1} -\log |z_n| \right)^2 \geq \sum_{|z_n| < 1} \log^2 |z_n|$$

because $-\log |z_n| > 0$ for $|z_n| < 1$. If $0 < x < 1$, then it is clear from (2.3) that $\log^2 x \geq f(x)$, which gives

$$\log^2 M(P) \geq \sum_{|z_n| < 1} f(|z_n|) = \log \|P\|_0. \quad \square$$

In our proof of Theorem 1.3, it is important to have a lower bound on the roots of P in terms of the areal Mahler measure. If $\|P\|_0$ is small, then Theorem 2.3(a) of [9] shows the roots of P are close to the unit circle. Our next lemma gives a quantitative result in this direction.

LEMMA 2.1. *Suppose that $P(z) = a_N \prod_{n=1}^N (z - z_n) \in \mathbb{C}[z]$ where $|P(0)| = 1$ and $\|P\|_0 < e$. Then $|z_n| \geq 1 - \sqrt{\log \|P\|_0}$ for all n .*

Proof. Let $f(x) = (x^2 - 1)/2 - \log x$. Since $f(x) > 0$ is decreasing on $(0, 1)$, we find from (2.1) that

$$(2.4) \quad \log \|P\|_0 = \sum_{|z_n| < 1} f(|z_n|) \geq \max_{|z_n| < 1} \{f(|z_n|)\} = f\left(\min_{|z_n| < 1} |z_n|\right).$$

By (2.2), we find that $f(x) \geq (1 - x)^2$ for all $x \in (0, 1)$. Combining this with the above inequality, we obtain that

$$\min_{|z_n| < 1} |z_n| \geq 1 - \sqrt{\log \|P\|_0}$$

completing the proof. □

Assume that $P_N \in \mathbb{C}[z]$ is any sequence of polynomials of degree N , each having roots $z_{N,1}, z_{N,2}, \dots, z_{N,N}$. If $\lim_{N \rightarrow \infty} \log \|P_N\|_0 = 0$, then Pritsker showed in Theorem 2.3(a) of [9] that

$$\liminf_{N \rightarrow \infty} \min_{1 \leq n \leq N} |z_{N,n}| \geq 1.$$

The special case involving sequences P_N with $|P_N(0)| = 1$ is a consequence of Lemma 2.1. Now we present our proof of Theorem 1.3.

Proof of Theorem 1.3. Applying Cauchy’s inequality, we obtain immediately that

$$(2.5) \quad \log^2 M(P) \leq Z(P) \cdot \sum_{|z_n| < 1} \log^2 |z_n|.$$

By (2.3), we see that

$$\begin{aligned} (2.6) \quad \sum_{|z_n| < 1} \log^2 |z_n| &= \sum_{|z_n| < 1} \left(f(|z_n|) + \sum_{k=3}^{\infty} \left(\sum_{\ell=1}^{k-1} \frac{1}{\ell(k-\ell)} - \frac{1}{k} \right) (1 - |z_n|)^k \right) \\ &= \log \|P\|_0 + \sum_{|z_n| < 1} \sum_{k=3}^{\infty} \left(\sum_{\ell=1}^{k-1} \frac{1}{\ell(k-\ell)} - \frac{1}{k} \right) (1 - |z_n|)^k \\ &= \log \|P\|_0 + \sum_{|z_n| < 1} (1 - |z_n|)^2 \\ &\quad \times \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{k+1} \frac{1}{\ell(k+2-\ell)} - \frac{1}{k+2} \right) (1 - |z_n|)^k. \end{aligned}$$

Now let $g : (0, 1) \rightarrow \mathbb{R}$ be given by

$$g(x) = \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{k+1} \frac{1}{\ell(k+2-\ell)} - \frac{1}{k+2} \right) (1 - x)^k.$$

Since $g'(x) < 0$ for all $x \in (0, 1)$, we know that g is decreasing on this interval, and Lemma 2.1 implies that $g(|z_n|) \leq g(1 - \sqrt{\log \|P\|_0})$. Using this observation in (2.6), we see that

$$\begin{aligned} & \sum_{|z_n| < 1} \log^2 |z_n| \\ & \leq \log \|P\|_0 + g(1 - \sqrt{\log \|P\|_0}) \sum_{|z_n| < 1} (1 - |z_n|)^2 \\ & \leq \log \|P\|_0 + g(1 - \sqrt{\log \|P\|_0}) \sum_{|z_n| < 1} \left((1 - |z_n|)^2 + \sum_{k=3}^{\infty} \frac{(1 - |z_n|)^k}{k} \right) \\ & = \log \|P\|_0 + g(1 - \sqrt{\log \|P\|_0}) \log \|P\|_0 \end{aligned}$$

and the result follows from (2.5). □

Although the proof of Theorem 1.3 uses several estimates, we believe the weakest of these are the use of Cauchy’s Inequality and Lemma 2.1. Our application of Cauchy’s Inequality in (2.5) admits equality whenever the values $|z_n|$ are all equal (i.e., whenever P is balanced). On the other hand, inequality (2.4) in the proof of Lemma 2.1 is sharpest when one root of P is somewhat far from the unit circle while the others are close. To summarize, these estimates are sharpest in opposing cases, so that their simultaneous use leads to a weaker estimate that should be expected.

Nevertheless, Lemma 2.1 is used only to estimate the error term in Theorem 1.3. In Pritsker’s sequence $P_n(z) = nz^n - 1$, all roots of P_n have the same absolute value, meaning that we have equality in (2.5). Indeed, we found that

$$\frac{\log^2 M(P_n)}{Z(P_n)} = \log \|P_n\|_0 \left(1 + O\left(\frac{\log n}{n}\right) \right),$$

while Theorem 1.3 gives

$$\frac{\log^2 M(P_n)}{Z(P_n)} \leq \log \|P_n\|_0 \left(1 + O\left(\frac{\log n}{\sqrt{n}}\right) \right).$$

As expected, these estimates have the same main term, while differing in their error terms.

Proof of Corollary 1.4. Let $\phi(x) = \sum_{\ell=1}^{x-1} \frac{1}{\ell(x-\ell)} - \frac{1}{x}$ so we must show that $\phi(x) \leq 2/3$ for all integers $x \geq 3$. We verify easily that $\phi(3) = \phi(4) = 2/3$ so it is enough to show that $\phi(x) \leq 2/3$ for all $x \geq 5$. Set

$$\psi(x) = \frac{2}{x-1} + \frac{x-3}{2(x-2)} - \frac{1}{x}$$

and observe that $\phi(x) \leq \psi(x)$ for all $x \geq 3$. We have that $\psi(5) = 19/30 < 2/3$, and it is a standard calculus problem to show that $\psi(x)$ is decreasing on $[5, \infty)$. We now obtain that $\phi(x) \leq 2/3$ for all integers $x \geq 3$.

Therefore, Theorem 1.3 yields

$$\frac{\log^2 M(P)}{Z(P)} \leq \log \|P\|_0 \left(1 + \frac{2}{3} \sum_{k=1}^{\infty} \log^{k/2} \|P\|_0 \right)$$

and the result follows by summing the geometric series. \square

Proof of Theorem 1.5. Since all roots of P have the same absolute value, we obtain equality in (2.5) giving

$$\log^2 M(P) = Z(P) \cdot \sum_{|z_n| < 1} \log^2 |z_n|.$$

Therefore, we may apply (2.6) to obtain that

$$\begin{aligned} \frac{\log^2 M(P)}{Z(P)} &= \log \|P\|_0 \\ &+ \sum_{|z_n| < 1} (1 - |z_n|)^2 \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{k+1} \frac{1}{\ell(k+2-\ell)} - \frac{1}{k+2} \right) (1 - |z_n|)^k. \end{aligned}$$

It is straightforward to verify that

$$\sum_{\ell=1}^{k+1} \frac{1}{\ell(k+2-\ell)} - \frac{1}{k+2} \geq 0$$

for all $k \geq 1$, so the result follows. \square

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