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PARALLEL CALIBRATIONS AND MINIMAL SUBMANIFOLDS

COLLEEN ROBLES

Abstract. Given a parallel calibration $\varphi \in \Omega^p(M)$ on a Riemannian manifold M, I prove that the φ-critical submanifolds with nonzero critical value are minimal submanifolds. I also show that the φ-critical submanifolds are precisely the integral manifolds of a $\mathscr{C}^\infty(M)$-linear subspace $\mathscr{P} \subset \Omega^p(M)$. In particular, the calibrated submanifolds are necessarily integral submanifolds of the system. (Examples of parallel calibrations include the special Lagrangian calibration on Calabi–Yau manifolds, (co)associative calibrations on G_2-manifolds, and the Cayley calibration on Spin(7)-manifolds.)

1. Introduction

1.1. Calibrated geometry. Let's begin by setting notation and reviewing (briefly) calibrated geometry. See [11] for a through introduction.

Let V be a real, n-dimensional vector space equipped with an inner product. Throughout $\{e_1, \ldots, e_n\} \subset V$ will denote a set of orthonormal vectors. Let

$$\mathrm{Gr}_o(p, V) := \{e_1 \wedge \cdots \wedge e_p\} \subset \textstyle\bigwedge^p V$$

denote the unit decomposable (or simple) p-vectors. Notice that $\mathrm{Gr}_o(p, V)$ is a double cover of the Grassmannian $\mathrm{Gr}(p, V)$ of p-planes in V. Given $\xi \in \mathrm{Gr}_o(p, V)$, let $[\xi] \in \mathrm{Gr}(p, V)$ denote the corresponding p-plane. I will abuse terminology by referring to elements of both $\mathrm{Gr}_o(p, V)$ and $\mathrm{Gr}(p, V)$ as p-planes. (Properly, elements of $\mathrm{Gr}_o(p, V)$ are oriented p-planes.)

Let M be an n-dimensional Riemannian manifold. Let $\mathrm{Gr}(p, TM)$ denote the Grassmann bundle of tangent p-planes on M, and $\mathrm{Gr}_o(p, TM)$ the double

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cover of $\mathrm{Gr}(p, TM)$ of decomposable unit p-vectors. Let $\Omega^p(M)$ denote the space of smooth p-forms on M.

Note that, given a p-form $\varphi \in \Omega^p(M)$ and $\xi = e_1 \wedge \cdots \wedge e_p \in \mathrm{Gr}_o(p, TM)$, $\varphi(\xi) := \varphi(e_1, \ldots, e_p)$ is well defined. If φ is closed and $\varphi \leq 1$ on $\mathrm{Gr}_o(p, TM)$, then φ is a *calibration*. The condition that $\varphi \leq 1$ on $\mathrm{Gr}_o(p, TM)$ is often expressed as $\varphi_{|\xi} \leq \mathrm{vol}_{|\xi}$. Assume φ is a calibration. Let

$$\mathrm{Gr}(\varphi) := \left\{ \xi \in \mathrm{Gr}_o(p, TM) \mid \varphi(\xi) = 1 \right\}$$

denote the set of *(oriented) calibrated planes*, and $\mathrm{Gr}(\varphi)_x$ the fibre over $x \in M$. An oriented p-dimensional submanifold $N \subset M$ is *calibrated* if $T_xN \in \mathrm{Gr}(\varphi)_x$, for all $x \in N$. That is, $\varphi_{|N} = \mathrm{vol}_N$. Compact calibrated submanifolds have the property that they are globally volume minimizing in their homology classes [11]. The first step in the identification or construction of calibrated submanifolds is the determination of $\mathrm{Gr}(\varphi)$. However, this is often a difficult problem, even in the case that $\phi \in \bigwedge^p V$ is a constant coefficient calibration on a vector space. See, for example, [3], [5], [12], [16].

Notice that elements of $\mathrm{Gr}(\varphi)_x$ are critical points of $\varphi_x : \mathrm{Gr}_o(p, T_xM) \to \mathbb{R}$. However, it is not the case that every critical point is an element of $\mathrm{Gr}(\varphi)_x$. (See Section 3.7 below.) Let $C(\varphi)_x \subset \mathrm{Gr}_o(p, T_xM)$ denote the set of critical points of φ_x, and $C(\varphi) \subset \mathrm{Gr}_o(p, TM)$ the associated sub-bundle. An oriented p-dimensional submanifold $N \subset M$ is *φ-critical* if $T_xN \subset C(\varphi)_x$, for all $x \in N$. While the calibrated submanifolds are prized as volume minimizers in their homology classes, the φ-critical submanifolds are also interesting. Unal showed that if the corresponding critical value is a local maximum, then the φ-critical submanifold is minimal [19, Theorem 2.1.2]. See also the work on Hong Van Le on the stability of minimal surfaces [14]. I will prove (Theorem 1.2): if φ is parallel, then the φ-critical submanifolds with nonzero critical value are minimal. I will also show that the φ-critical submanifolds are characterized by an exterior differential system $\mathscr{P}$ (Theorem 1.1).

1.2. Contents. We begin in Section 2.1 with the simple case of a constant coefficient calibration $\phi \in \bigwedge^p V^*$. In Proposition 2.2, I identify the critical points $C(\phi) \supset \mathrm{Gr}(\phi)$ as the annihilator of a linear subspace $\Phi \subset \bigwedge^p V^*$. In the case that ϕ is invariant under a Lie subgroup $H \subset \mathrm{O}(V)$, Φ is a H-submodule of $\bigwedge^p V^*$ (Lemma 3.1). (Of course, every ϕ is invariant under the trivial group $\{\mathrm{Id}\} \subset \mathrm{O}(V)$.) Several examples are discussed in Section 3, and a vector-product variation of Proposition 2.2 is given in Proposition 3.4.

In Section 4, Proposition 2.2 is generalized to a parallel calibrations on a connected, n-dimensional, Riemannian manifold M^n. Given an n-dimensional H-manifold M, a H-invariant $\phi \in \bigwedge^p V^*$ naturally defines a parallel p-form φ on M. Conversely, every parallel p-form φ on a Riemannian manifold arises in this fashion. (See Section 4.3 for a description of the construction.) As a parallel form, φ is a priori closed and thus a calibration on M. Similarly, Φ

defines a sub-bundle $\Phi_M \subset \bigwedge^p T^*M$. Let $\mathscr{P} \subset \Omega^p(M)$ denote smooth sections of Φ_M. A p-dimensional submanifold $N^p \subset M$ is an *integral submanifold of* $\mathscr{P}$ if $\mathscr{P}_{|N} = \{0\}$.

THEOREM 1.1. *Assume that M^n is a connected Riemannian manifold, and φ a parallel calibration. A submanifold N^p is φ-critical if and only if N is an integral manifold of $\mathscr{P}$. In particular, every calibrated submanifold of M is an integral manifold of $\mathscr{P}$.*

Proposition 3.4 (the vector-product variant) easily generalizes to give an alternative formulation of the φ-critical submanifolds as those submanifolds N with the property that T_xN is closed under an alternating $(p-1)$-fold vector product $\rho : \bigwedge^{p-1}TM \to TM$.

If $N \subset M$ is φ-critical, then $\varphi_{|N} = \varphi_o \,\mathrm{vol}_N$, where φ_o is a constant. Refer to this constant as the *critical value of φ on N*.

THEOREM 1.2. *Assume that M is a Riemannian manifold, $\varphi \in \Omega^p(M)$ a parallel calibration, and $N \subset M$ a φ-critical submanifold. If the critical value of φ on N is nonzero, then N is a minimal submanifold of M.*

Theorems 1.1 and 1.2 are proven in Sections 4.3 and 4.4, respectively.

Finally in Section 5 it is shown that the ideal $\mathscr{I} \subset \Omega(M)$ algebraically generated by $\mathscr{P}$ is differentially closed and that, in general, the system fails to be involutive.

Notation. Fix index ranges

$$i, j \in \{1, \ldots, n\}, \qquad a, b \in \{1, \ldots, p\}, \qquad s, t \in \{p+1, \ldots, n\}.$$

The summation convention holds: when an index appears as both a subscript and superscript in an expression, it is summed over.

2. The infinitesimal picture

2.1. The basics. Let $\phi \in \bigwedge^p V^*$ and $\xi = e_1 \wedge \cdots \wedge e_p \in \mathrm{Gr}_o(p, V)$. Then $\phi(\xi) = \phi(e_1, \ldots, e_p)$ is a well-defined function on $\mathrm{Gr}_o(p, V)$. Fix a nonzero $\phi \in \bigwedge^p V^*$, with the property that $\max_{\mathrm{Gr}_o(p,V)} \phi = 1$. The set of (oriented) *calibrated p-planes* is

$$\mathrm{Gr}(\phi) := \big\{\xi \in \mathrm{Gr}_o(p, V) \mid \phi(\xi) = 1\big\}.$$

Let $C(\phi) \subset \mathrm{Gr}_o(p, V)$ denote the critical points of ϕ. Then

$$\mathrm{Gr}(\phi) \subset C(\phi).$$

Let $\mathcal{F}_V$ denote the set of orthonormal bases (or frames) of V. Given $e = (e_1, \ldots, e_n) \in \mathcal{F}_V$, let $e^* = (e^1, \ldots, e^n)$ denote the dual coframe. Then

$$\phi = \phi_{i_1 \cdots i_p} e^{i_1} \wedge \cdots \wedge e^{i_p},$$

uniquely determines functions $\phi_{i_1\cdots i_p}$, skew-symmetric in the indices, on $\mathcal{F}_V$. Note that $|\phi_{i_1\cdots i_p}|\le 1$, and $\xi=e_{i_1}\wedge\cdots\wedge e_{i_p}\in\mathrm{Gr}(\phi)$ if and only if equality holds.

Next we compute $\mathrm{d}\phi_{|\xi}$. Let $\mathrm{O}(V)$ denote the Lie group of linear transformations $V\to V$ preserving the inner product, and let $\mathfrak{o}(V)$ denote its Lie algebra. Let θ denote the $\mathfrak{o}(V)$-valued Maurer–Cartan form on $\mathcal{F}_V$: at $e\in\mathcal{F}_V$, $\theta_e=\theta^j_k e_j\otimes e^k$, where the coefficient 1-forms $\theta^j_k=-\theta^k_j$ are defined by $\mathrm{d}e_j=\theta^k_j e_k$. Then $\{\theta^i_j\mid i<j\}$ is a basis for the 1-forms on $\mathcal{F}_V$.

If $\xi=e_{i_1}\wedge\cdots\wedge e_{i_p}$ is viewed as a map $\mathcal{F}_V\to\mathrm{Gr}_o(p,V)$, then

$$\mathrm{d}\xi=\sum_{1\le a\le p} e_{i_1}\wedge\cdots\wedge e_{i_{a-1}}\wedge\theta^k_{i_a}e_k\wedge e_{i_{a+1}}\wedge\cdots\wedge e_{i_p}.$$

Thus

$$\begin{aligned}\mathrm{d}\phi_\xi&=\mathrm{d}\phi(e_{i_1},\ldots,e_{i_p})\\&=\sum_{1\le a\le p}\phi\big(e_{i_1},\ldots,e_{i_{a-1}},\theta^k_{i_a}e_k,e_{i_{a+1}},\ldots,e_{i_p}\big)\\&=\sum_{1\le a\le p}\theta^k_{i_a}\phi(e_{i_1},\ldots,e_{i_{a-1}},e_k,e_{i_{a+1}},\ldots,e_{i_p})\\&=\sum_{1\le a\le p}\phi_{i_1\cdots i_{a-1}ki_{a+1}\cdots i_p}\theta^k_{i_a}.\end{aligned}$$

The skew-symmetry of ϕ and θ imply that $\phi_{i_1\cdots i_{a-1}ki_{a+1}\cdots i_p}\theta^k_{i_a}$ vanishes if $k\in\{i_1,\ldots,i_p\}$. The $\{\theta^k_{i_a}\mid 1\le a\le p, k\notin\{i_1,\ldots,i_p\}\}$ are linearly independent on $\mathcal{F}_V$, and may be naturally identified with linearly independent 1-forms on $\mathrm{Gr}_{\mathrm{o}}(p,V)$ at ξ. Consequently, $\mathrm{d}\phi_\xi=0$, and

$$\begin{aligned}&\xi=e_{i_1}\wedge\cdots\wedge e_{i_p}\text{ is a critical point}\\&\qquad\text{if and only if }\phi_{i_1\cdots i_{a-1}ki_{a+1}\cdots i_p}\theta^k_{i_a}=0.\end{aligned}\tag{2.1}$$

An equivalent, index-free formulation of this observation is given by the lemma below.

Lemma 2.1. *A p-plane ξ is a critical point of ϕ if and only if $(v\lrcorner\phi)_{|\xi}=0$ for all $v\in\xi^\perp$.*

Remark. The lemma was first observed by Harvey and Lawson (cf. Remark on page 78 of [11]), and is often referred to as the First Cousin Principle.

The lemma allows us to characterize the critical points $\xi\in\mathrm{Gr}_o(p,V)$ of ϕ as the p-planes on which a linear subspace $\Phi\subset\bigwedge^pV^*$ vanishes. Forget, for a moment, that θ is a 1-form on $\mathcal{F}_V$ and regard it simply as an element of $\mathfrak{o}(V)$. Let $\theta.\phi$ denote the action of θ on ϕ. The action yields a map $\mathsf{P}:\mathfrak{o}(V)\to\bigwedge^pV^*$ sending $\theta\mapsto\theta.\phi$. Define

$$\Phi:=\mathsf{P}\big(\mathfrak{o}(V)\big)\subset{\textstyle\bigwedge}^pV^*.$$

Notice that the $e^{i_1} \wedge \cdots \wedge e^{i_p}$-coefficient of $\theta.\phi$ is $\phi_{i_1 \cdots i_{a-1} k i_{a+1} \cdots i_p} \theta^k_{i_a}$. From this observation, (2.1), and the fact that the Maurer–Cartan form $\theta_e : T_e \mathcal{F}_V \to \mathfrak{o}(V)$ is a linear isomorphism, we deduce the following.

PROPOSITION 2.2. *The set of ϕ-critical planes is $C(\phi) = \mathrm{Gr}_o(p, V) \cap \mathrm{Ann}(\Phi)$.*

REMARK. The map P is the restriction of the map $\lambda_\phi : \mathrm{End}(V) \to \bigwedge^p V^*$ in [10] to $\mathfrak{o}(V)$. Corollary 2.6 of [10] is precisely the observation that elements of Φ vanish on $\mathrm{Gr}(\phi) \subset C(\phi)$. Indeed, Proposition 2.2 above follows from Proposition A.4 of that paper. This is seen by observing that if $A \in \mathfrak{o}(V) \subset \mathrm{End}(V)$, then $\mathrm{tr}_\xi A = 0$. Then their (A.2) reads $\lambda_\phi(A)(\xi) = \phi(D_{\tilde{A}} \xi)$. It now suffices to note that their $\{\lambda_\phi(A) \mid A \in \mathfrak{o}(V)\}$ is our Φ, and that $\{D_{\tilde{A}} \xi \mid A \in \mathfrak{o}(V)\} = T_\xi \, \mathrm{Gr}_o(p, V)$.

REMARK. Each $\phi \in \bigwedge^p V^*$ naturally determines an alternating $(p-1)$-fold vector product ρ on V. An equivalent formulation of Proposition 2.2 is given by Proposition 3.4 which asserts that $\xi \in C(\phi)$ and only if $[\xi] \in \mathrm{Gr}(p, V)$ is ρ-closed.

3. Examples and the product characterization

3.1. Invariant forms. Let G denote the stabilizer of ϕ in $\mathrm{O}(V)$. Many of the calibrations that we are interested in have a nontrivial stabilizer; but, of course, all statements hold for trivial G. Observe that Φ is a $\mathfrak{g}$-module. This is seen as follows. Let $\mathfrak{g}$ denote the Lie algebra of G. As a $\mathfrak{g}$-module $\mathfrak{o}(V)$ admits a decomposition of the form $\mathfrak{o}(V) = \mathfrak{g} \oplus \mathfrak{g}^\perp$. By definition, the kernel of P is $\mathfrak{g}$. In particular, $\Phi = \mathsf{P}(\mathfrak{g}^\perp)$. It is straightforward to check that P is G-equivariant, and we have the following lemma.

LEMMA 3.1. *The subspace $\Phi = \mathsf{P}(\mathfrak{g}^\perp) \subset \bigwedge^p V^*$ is isomorphic to $\mathfrak{g}^\perp$ as a G-module.*

Below I identify Φ for some well-known examples. The calibrations ϕ and characterizations of $\mathrm{Gr}(\phi)$ in Sections 3.2–3.5 were introduced in [11].

3.2. Associative calibration. Consider the standard action of the exceptional $G = G_2$ on the imaginary octonions $V = \mathrm{Im}\,\mathbb{O} = \mathbb{R}^7$. As a G_2-module the third exterior power decomposes as $\bigwedge^3 V^* = \mathbb{R} \oplus V^3_{1,0} \oplus V^3_{2,0}$. (Cf. [6, Lemma 3.2] or [1, p. 542].) Here $V^3_{1,0} = V$ as G_2-modules. The trivial subrepresentation $\mathbb{R} \subset \bigwedge^3 V^*$ is spanned by an invariant 3-form ϕ, the associative calibration. It is known that $\xi \in \mathrm{Gr}(\phi)$ if and only if the forms $V^3_{1,0} = \{*(\phi \wedge \alpha) \mid \alpha \in V^*\}$ vanish on ξ [11, Corollary 1.7]. Here $*(\phi \wedge \alpha)$ denotes the Hodge star operation on the 4-form $\phi \wedge \alpha$. As $\Phi = V^3_{1,0}$, we have $C(\phi) = \mathrm{Gr}(\phi)$.

3.3. Coassociative calibration. Again we consider the standard action of G_2 on $V = \mathrm{Im}\,\mathbb{O} = V_{1,0}$. The Hodge star commutes with the G_2 action. So the fourth exterior power decomposes as $\bigwedge^4 V^* = V^4_{0,0} \oplus V^4_{1,0} \oplus V^4_{2,0}$, with $V^4_{a,b} = *V^3_{a,b}$. The trivial subrepresentation is spanned by the invariant coassociative calibration $*\phi$. A 4-plane ξ is calibrated by $*\phi$ if and only if $\phi_{|\xi} \equiv 0$ [11, Corollary 1.19]. Equivalently, the 4-forms of $V^4_{1,0} = \{\phi \wedge \alpha \mid \alpha \in V^*\}$ vanish on ξ. As $\Phi = V^4_{1,0}$, we again have $C(\phi) = \mathrm{Gr}(\phi)$.

3.4. Cayley calibration. Consider the standard action of $G = B_3 = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$ on the octonions $V = \mathbb{O} = \mathbb{R}^8$. The fourth exterior power decomposes as $\bigwedge^4 V^* = V^4_{0,0,0} \oplus V^4_{1,0,0} \oplus V^4_{2,0,0} \oplus V^4_{0,0,2}$. (Cf. [1, p. 548] or [7, Lemma 3.3].) The trivial subrepresentation $V^4_{0,0,0}$ is spanned by the invariant, self-dual Cayley 4-form $\phi = *\phi$. It is known that $\xi \in \mathrm{Gr}(\phi)$ if and only if the forms $V^4_{1,0,0} = \{\alpha.\phi \mid \alpha \in V^2_{1,0,0}\}$ vanish on ξ [11, Proposition 1.25]; here $V^2_{1,0,0} = \{\alpha \in \bigwedge^2 V^* \mid *(\alpha \wedge \phi) = 3\alpha\} \simeq \mathfrak{g}^\perp$. As $\Phi = V^4_{1,0,0}$, we have $C(\phi) = \mathrm{Gr}(\phi)$.

3.5. Special Lagrangian calibration. Regard $V := \mathbb{C}^m$ as a real vector space. Given the standard coordinates $z = x + \mathrm{i}y$,

$$V^* = \mathrm{span}_{\mathbb{R}}\left\{ \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z}), -\frac{\mathrm{i}}{2}(\mathrm{d}z - \mathrm{d}\bar{z}) \right\}.$$

Set

$$\begin{aligned}\sigma &= -\frac{\mathrm{i}}{2}\big(\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 + \cdots + \mathrm{d}z^m \wedge \mathrm{d}\bar{z}^m\big),\\ \Upsilon &= \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^m.\end{aligned}$$

The special Lagrangian calibration is $\mathrm{Re}\,\Upsilon$. An m-dimensional submanifold $i : M \to V$ is calibrated if and only if $i^*\sigma = 0 = i^*\,\mathrm{Im}\,\Upsilon$. (Recall that $i^*\sigma = 0$ characterizes the m-dimensional Lagrangian submanifolds.)

The special Lagrangian example is distinct from those above in that

$$\mathfrak{su}(m)^\perp = \mathbb{R} \oplus W \subset \textstyle\bigwedge^2 V$$

is reducible as an $\mathfrak{su}(m)$-module. The trivial subrepresentation is spanned by σ.

The $\mathfrak{su}(m)$ module Φ decomposes as $\Phi_0 \oplus \Phi_W$, where $\Phi_0 = \mathrm{span}_{\mathbb{R}}\{\mathrm{Im}\,\Upsilon\}$ and $\Phi_W = W.(\mathrm{Re}\,\Upsilon)$. The elements of the sub-module Φ_W may be described as follows. Let $J \subset \{1, \ldots, m\}$ be a multi-index of length $|J| = \ell$, and $\mathrm{d}z^J := dz^{j_1} \wedge \cdots \wedge \mathrm{d}z^{j_\ell}$. The reader may confirm that $\Phi_W = \mathrm{span}_{\mathbb{R}}\{\mathrm{Re}\,\mathrm{d}z^J \wedge \sigma$, $\mathrm{Im}\,\mathrm{d}z^J \wedge \sigma : |J| = m-2\}$.

In the remark of [11, p. 90] Harvey and Lawson showed that an m-plane ζ is Lagrangian if and only if the forms $\Psi := \{\mathrm{d}z^J \wedge \sigma^p : 2p + |J| = m, p > 0\} \supset \Phi_W$ vanish on ζ. So $\pm\xi \in \mathrm{Gr}(\mathrm{Re}\,\Upsilon)$ if and only if $\mathrm{Im}\,\Upsilon_{|\xi} = 0 = \Psi_{|\xi}$, while $\xi \in C(\mathrm{Re}\,\Upsilon)$ if and only if $\mathrm{Im}\,\Upsilon_{|\xi} = 0 = \Phi_{W|\xi}$. So it seems a priori that a critical ξ need not be calibrated. Nonetheless, Zhou [20, Theorem 3.1] has shown that $\pm\,\mathrm{Gr}(\mathrm{Re}\,\Upsilon) = C(\mathrm{Re}\,\Upsilon)$.

3.6. Squared spinors. In [4], Dadok and Harvey construct calibrations $\phi \in \bigwedge^{4p} V^*$ on vector spaces of dimension $n = 8m$ by squaring spinors. Let me assume the notation of that paper: in particular, $\mathbb{P} = \mathbb{S}^+ \oplus \mathbb{S}^-$ is the decomposition of the space of pinors into positive and negative spinors, ε an inner product on $\mathbb{P}$, and $\mathrm{Cl}(V) \simeq \mathrm{End}_{\mathbb{R}}(\mathbb{P})$ the Clifford algebra of V. Given $x, y, z \in \mathbb{P}$, $x \circ y \in \mathrm{End}_{\mathbb{R}}(\mathbb{P})$ is the linear map $z \mapsto \varepsilon(y, z)x$.

Given a unit $x \in \mathbb{S}^+$, $\underline{\phi} = 16^m x \circ x \in \mathrm{End}_{\mathbb{R}}(\mathbb{S}^+) \subset \mathrm{End}_{\mathbb{R}}(\mathbb{P})$ may be viewed as an element of $\bigwedge V^* \simeq \mathrm{Cl}(V)$. Let $\phi_k \in \bigwedge^k V^*$ be the degree k component of $\underline{\phi}$. Each ϕ_k is a calibration, and ϕ_k vanishes unless $k = 4p$. (Also, $\phi_0 = 1$ and $\phi_n = \mathrm{vol}_V$.) The Cayley calibration of Section 3.4 is an example of such a calibration; see [4, Proposition 3.2].

Given such a calibration $\phi = \phi_{4p}$, Dadok and Harvey construct $4p$-forms $\Psi_1, \ldots, \Psi_N$, $N = \frac{1}{2}(16)^m - 1$, that characterize $\mathrm{Gr}(\phi)$; that is, $\xi \in \mathrm{Gr}(\phi)$ if and only if $\Psi_j(\xi) = 0$ [4, Theorem 1.1].

Lemma 3.2. *The span of the Ψ_j is our Φ. In particular, $C(\phi) = \mathrm{Gr}(\phi)$.*

Proof. Continuing to borrow the notation of [4], the proof may be sketched as follows. Complete $x = x_0$ to an orthogonal basis $\{x_0, x_1, \ldots, x_N\}$ of $\mathbb{S}^+$. Then Ψ_j is the degree $4p$ component of $16^m x_j \circ x_0 \in \mathrm{End}_{\mathbb{R}}(\mathbb{S}^+) \subset \bigwedge V^*$. Our Φ is spanned by γ_j, the degree $4p$ component of $16^m(x_j \circ x_0 + x_0 \circ x_j)$. Let $\langle x \circ y, \xi\rangle$ denote the extension of the inner product on V to $\mathrm{End}_{\mathbb{R}}(\mathbb{P}) \simeq \mathrm{Cl}(V) \simeq \bigwedge V^*$. (See [4].) Given $\xi \in \mathrm{Gr}_o(4p, V)$,

$$\Psi_j(\xi) = 16^m \langle x_j \circ x_0, \xi\rangle,$$
$$\gamma_j(\xi) = 16^m \langle x_j \circ x_0 + x_0 \circ x_j, \xi\rangle.$$

To see that $\Phi = \mathrm{span}\{\Psi_1, \ldots, \Psi_N\}$ it suffices to note that

$$16^m \langle x_0 \circ x_j, \xi\rangle = \varepsilon(x_0, \xi x_j) = \varepsilon(x_j, \xi x_0) = 16^m \langle x_j \circ x_0, \xi\rangle,$$

when $\xi \in \bigwedge^{4p} V^*$. Hence $\gamma_j = 2\Psi_j$. □

Remark. Zhou showed that $C(\phi) = \mathrm{Gr}(\phi)$ for many well-known calibrations [20]. As the following example illustrates, this need not be the case.

3.7. Cartan 3-form on $\mathfrak{g}$. Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$. Set $V = \mathfrak{g}$ and consider the adjoint action. Every simple Lie algebra admits an (nonzero) invariant 3-form, the Cartan form ϕ, defined as follows. Given $u, v \in \mathfrak{g}$, let $[u, v] \in \mathfrak{g}$ and $\langle u, v\rangle \in \mathbb{R}$ denote the Lie bracket and invariant inner product, respectively. Then $\phi(u, v, w) = c\langle u, [v, w]\rangle$, with $\frac{1}{c}$ the length of a highest root δ. It is immediate from Lemma 2.1 that ξ is a critical point if and only if ξ is a subalgebra of $\mathfrak{g}$.

PROPOSITION 3.3. *A 3-plane ξ is ϕ-critical if and only if it is a subalgebra of $\mathfrak{g}$.*

REMARK. The proposition generalizes to arbitrary ϕ. See Proposition 3.4.

The $\mathfrak{su}(2)'s$ in $G(3,\mathfrak{g})$ corresponding to a highest root all lie in the same $\mathrm{Ad}(G)$-orbit and Tasaki [17] showed that this orbit is $\mathrm{Gr}(\phi)$. (Thi [18] had observed that the corresponding SU(2) are volume minimizing in their homology class in the case that $G=\mathrm{SU}(n)$.) If the rank of $\mathfrak{g}$ is greater than 1, then $\mathfrak{g}$ contains 3-dimensional subalgebras that are not associated to a highest root. Thus, $\mathrm{Gr}(\phi) \subsetneqq C(\phi)$. More generally, Hông Vân Lê [15] has introduced the notion of a manifold admitting a Cartan 3-form, and investigated the algebraic types of these structures.

REMARK. The quaternionic calibration on $\mathbb{H}^n$ also satisfies $\mathrm{Gr}(\phi) \subsetneqq C(\phi)$; see [19] for details.

3.8. Product version of Proposition 2.2. Proposition 3.3 asserts that a 3-plane ξ is ϕ-critical, ϕ the Cartan 3-form, if and only if ξ is closed under the Lie bracket. This is merely a rephrasing of Proposition 2.2, and an analogous statement holds for any calibration.

Given a p-form $\phi \in \bigwedge^p V^*$, define a $(p-1)$-fold alternating vector product ρ on V by

$$\phi(u, v_2, \ldots, v_p) =: \big\langle u, \rho(v_2, \ldots, v_p)\big\rangle. \tag{3.1}$$

EXAMPLE. In the case that $V=\mathfrak{g}$ and ϕ is the Cartan 3-form, ρ is a multiple of the Lie bracket.

The following proposition is a reformulation of Lemma 2.1.

PROPOSITION 3.4. *Let $\phi \in \bigwedge^p V^*$, and let ρ denote the associated $(p-1)$-fold alternating product defined in (3.1). Then a p-plane $\xi \in \mathrm{Gr}_o(p,V)$ is ϕ-critical if and only if ξ is ρ-closed.*

EXAMPLE. When $V=\mathbb{O}$ and ϕ is the Cayley calibration, then ρ is a multiple of the triple cross product. See [11, Section IV.1.C] where it is shown that a 4-plane is Cayley if and only if it is closed under the triple cross product.

Note that

$$\rho(v_2, \ldots, v_p) \quad \text{is orthogonal to } v_2, \ldots, v_p. \tag{3.2}$$

In particular, ρ may be viewed as a generalization of Gray's vector cross product, satisfying [8, (2.1)] but not necessarily [8, (2.2)].

Assume that $\xi = e_1 \wedge \cdots \wedge e_p \in C(\phi)$. Then (3.2) and Proposition 3.4 imply $\rho(e_2, \ldots, e_p) = \phi(\xi) e_1$. This yields the following.

COROLLARY 3.5. *Let $\xi \in \mathrm{Gr}_o(p,V)$. The product ρ vanishes on $[\xi] \in \mathrm{Gr}(p, V)$ if and only if $\xi \in C(\phi)$ and $\phi(\xi)=0$.*

4. Parallel calibrations

4.1. Orthonormal coframes on M. Let V be an n-dimensional Euclidean vector space. Let M be an n-dimensional connected Riemannian manifold, and let $\pi : \mathcal{F} \to M$ denote the *bundle of orthogonal coframes*. Given $x \in M$, the elements of the fibre $\pi^{-1}(x)$ are the linear isometries $u : T_xM \to V$. Given $g \in \mathrm{O}(V)$, the right-action $u \cdot g := g^{-1} \circ u$ makes $\mathcal{F}$ a principle right $\mathrm{O}(V)$-bundle.

The canonical V-valued 1-form ω on $\mathcal{F}$ is defined by

$$\omega_u(v) := u(\pi_* v),$$

$v \in T_u\mathcal{F}$. Let ϑ denote the unique torsion-free, $\mathfrak{o}(V)$-valued connection 1-form on $\mathcal{F}$ (the Levi–Civita connection form). Fix an orthonormal basis $\{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ of V. Then we may define 1-forms ω^i on $\mathcal{F}$ by

$$\omega_u =: \omega^i_u \mathsf{v}_i.$$

Let $\mathsf{v}^1, \ldots, \mathsf{v}^n$ denote the dual basis of V^*, and define ϑ^i_j by $\vartheta = \vartheta^i_j \mathsf{v}_i \otimes \mathsf{v}^j$. Then

$$\vartheta^i_j + \vartheta^j_i = 0 \quad \text{and} \quad \mathrm{d}\omega^i = -\vartheta^i_j \wedge \omega^j.$$

Given $u \in \mathcal{F}$, let $\{e_1, \ldots, e_n\}$, $e_i = e_i(u) := u^{-1}(\mathsf{v}_i)$, denote the corresponding orthonormal basis of T_xM.

4.2. H-manifolds. Suppose $H \subset \mathrm{O}(V)$ is a Lie subgroup. If the bundle of orthogonal coframes over $\mathcal{F} \to M$ admits a sub-bundle $\mathcal{E} \to M$ with fibre group H, then we say M carries a *H-structure*. The H-structure is *torsion-free* if $\mathcal{E}$ is preserved under parallel transport by the Levi–Civita connection in $\mathcal{F}$. In this case, we say M is a *H-manifold*.

When pulled-back to $\mathcal{E}$, the forms ω^i remain linearly independent, but ϑ takes values in the Lie algebra $\mathfrak{h} \subset \mathfrak{o}(V)$ of H.

4.3. The construction of φ and Φ_M. I now prove Theorem 1.1. Assume that M is a H-manifold. Let $\pi_* : T_u\mathcal{E} \to T_xM$ denote the differential of $\pi : \mathcal{E} \to M$. Any $\phi \in \bigwedge^p V^*$ induces a p-form φ on $\mathcal{E}$ by $\varphi_u(v_1, \ldots, v_p) = \phi(\omega_u(v_1), \ldots, \omega_u(v_p))$. Assume ϕ is H-invariant. Then φ descends to a well-defined p-form on M. Since $\mathcal{E} \subset \mathcal{F}$ is preserved under parallel transport, φ is parallel and therefore closed. Conversely, every parallel p-form φ arises in such a fashion: fix $x_o \in M$, and take $V = T_{x_o}M$ and $\phi = \varphi_{x_o}$.

Assume that $\max_{\mathrm{Gr}_o(p,V)} \phi = 1$. Then φ is a calibration on M.

Since H is a subgroup of the stabilizer G of ϕ, Lemma 3.1 implies $\Phi \subset \bigwedge^p V^*$ is a H-module. It follows that Φ defines a sub-bundle $\Phi_M \subset \bigwedge^p T^*M$. Explicitly, given $u \in \mathcal{E}_x$, $\Phi_{M,x} := (u^{-1})^*(\Phi) \subset \bigwedge^p T^*_xM$. The fact that Φ is an H-module implies that the definition of $\Phi_{M,x}$ is independent of our choice of $u \in \mathcal{E}_x$.

Let $\mathscr{P} \subset \Omega^p(M)$ denote space of smooth sections of Φ_M. Theorem 1.1 now follows from Proposition 2.2.

REMARK. Note that Proposition 3.4 also extends to parallel calibrations in a straightforward manner.

4.4. Proof of Theorem 1.2. Recall the notation of Section 4.1; in particular the framing $e = e(u)$ associated to $u \in \mathcal{F}$. Given a p-form $\psi \in \Omega^p(M)$, define functions $\psi_{i_1\cdots i_p} : \mathcal{F} \to \mathbb{R}$ by $\psi_{i_1\cdots i_p}(u) := \psi(e_{i_1}, \ldots, e_{i_p})$. The fact that φ is parallel implies

$$\mathrm{d}\varphi_{i_1\cdots i_p} = (\vartheta.\varphi)_{i_1\cdots i_p}, \tag{4.1}$$

where $\vartheta.\varphi$ denotes the $\mathfrak{o}(n)$-action of ϑ on φ.

The following notation will be convenient. Let $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ and $\{a_1, \ldots, a_m\} \subset \{1, \ldots, p\}$. If the $\{a_1, \ldots, a_m\}$ are pairwise distinct, then let $\psi^{a_1\cdots a_m}_{i_1\cdots i_m}$ denote the function obtained from $\psi_{12\cdots p}$ by replacing the indices a_ℓ with i_ℓ, $1 \le \ell \le m$. Otherwise, $\psi^{a_1\cdots a_m}_{i_1\cdots i_m} = 0$. For example, $\psi^2_s = \psi_{1s3\cdots p}$ and $\psi^{13}_{st} = \psi_{s2t4\cdots p}$. Note that $\psi^{a_1\cdots a_m}_{i_1\cdots i_m}$ is skew-symmetric in both the upper indices and the lower indices; for example, $\psi^{abc}_{rst} = -\psi^{bac}_{rst} = -\psi^{abc}_{tsr}$.

Define

$$\mathcal{C} := \big\{ u \in \mathcal{F} \mid e_1 \wedge \cdots \wedge e_p \in C(\varphi_x), x = \pi(u), e = e(u) \big\}.$$

It is a consequence of Lemma 2.1 that

$$\mathcal{C} = \big\{ u \in \mathcal{F} \mid \varphi^a_s(u) = 0 \ \forall 1 \le a \le p < s \le n \big\}.$$

Given a p-dimensional submanifold $N \subset M$, a *local adapted framing of M on N* is a section $\sigma : U \to \mathcal{F}$, defined on an open subset $U \subset N$ with the property that $\mathrm{span}\{e_1(x), \ldots, e_p(x)\} = T_xN \subset T_xM$, $e_a(x) := e_a \circ \sigma(x)$, for all $x \in U$. When pulled-back to $\sigma(U)$,

$$\omega^s = 0 \quad \forall p < s \le n \quad \text{and} \quad \omega^1 \wedge \cdots \wedge \omega^p \ne 0. \tag{4.2}$$

Conversely every p-dimensional integral submanifold $\tilde{U} \subset \mathcal{F}$ of (4.2) is locally the image $\sigma(U)$ of an adapted framing over a p-dimensional submanifold $U \subset M$.

Given N, let $\mathcal{F}_N \subset \mathcal{F}$ denote the bundle of adapted frames of M over N. As noted above $\omega^s{}_{|\mathcal{F}_N} = 0$. Differentiating this equation and an application of Cartan's lemma yields

$$\theta^s_a = h^s_{ab}\omega^a$$

for functions $h^s_{ab} = h^s_{ba} : \mathcal{F}_N \to \mathbb{R}$. The h^s_{ab} are the coefficients of the *second fundamental form of $N \subset M$*.

Observe that N is φ-critical if and only if $\mathcal{F}_N \subset \mathcal{C}$. Assume that N is φ-critical. Then $\varphi^a_s = 0$ on $\mathcal{F}_N$. Differentiating this equation yields $0 = \mathrm{d}\varphi^a_s = (\vartheta.\varphi)^a_s = \varphi_o\vartheta^a_s + \varphi^{ab}_{st}\vartheta^t_b$, where

$$\varphi_o := \varphi_{12\cdots p} = \varphi(e_1, \ldots, e_p)$$

is the (constant) critical value of φ on N. Equivalently, $\varphi_o h^s_{ac} = \varphi^{ab}_{st} h^t_{bc}$. Recalling that φ^{ab}_{st} is skew-symmetric and h^s_{ab} is symmetric in the indices a, b

yields $\sum_a \varphi_o h^s_{aa} = \varphi^{ab}_{st} h^t_{ab} = 0$. If $\varphi_o \neq 0$, then $\sum_a h^s_{aa} = 0$ and N is a minimal submanifold of M. This establishes Theorem 1.2.

Remark. Note that a φ-critical submanifold with $\varphi_o = 0$ need not be minimal. As an example, consider $M = \mathbb{R}^n$ with the standard Euclidean metric and coordinates $x = (x^1, \ldots, x^n)$, $n \geq 4$. The form $\varphi = \mathrm{d}x^1 \wedge \mathrm{d}x^2$ is a parallel calibration on M. Any 2-dimensional $N \subset \{x^1 = x^2 = 0\}$ is φ-critical with critical value $\varphi_o = 0$, but in general will not be a minimal submanifold of $\mathbb{R}^n$.

5. The system $\mathscr{P}$

5.1. The ideal $\mathscr{I} = \langle \mathscr{P} \rangle$. Let $\mathscr{I} \subset \Omega(M)$ be the ideal (algebraically) generated by $\mathscr{P}$.

Lemma. *The ideal $\mathscr{I}$ is differentially closed. That is,* $\mathrm{d}\mathscr{I} \subset \mathscr{I}$.

Proof. Let ϑ be the $\mathfrak{h}$-valued, torsion-free connection on M. Let $\{u^1, \ldots, u^n\}$ be a local H-coframe. Note that the coefficients $\varphi_{i_1 i_2 \cdots i_p}$ of φ with respect to the coframe are constant. The space Φ_M is spanned by forms of the form $\{\gamma = \theta.\varphi \mid \theta \in \mathfrak{g}^\perp \subset \mathfrak{h}^\perp\}$. In particular, the coefficients of these spanning γ are also constant. Consequently the covariant derivative is $\nabla\gamma = \vartheta.\gamma$. Since ϑ is $\mathfrak{h}$-valued and Φ is $\mathfrak{h}$-invariant, $\nabla\gamma$ may be viewed as a 1-form taking values in Φ_M. As the exterior derivative $\mathrm{d}\gamma$ is the skew-symmetrization of the covariant derivative $\nabla\gamma$, it follows that $\mathrm{d}\gamma \in \mathscr{I}$. □

5.2. Involutivity. This section assumes that reader is familiar with exterior differential systems. Excellent references are [2] and [13].

In general, the exterior differential system defined by $\mathscr{I}$ will fail to be involutive. In fact, involutivity always fails when $p > \frac{1}{2}n$. This is seen as follows. Let $\mathscr{I}^k = \mathscr{I} \cap \Omega^k(M)$. Note that $\mathscr{I}^a = \{0\}$, for $a < p$. Let $\mathscr{V}_k(\mathscr{I}) \subset \mathrm{Gr}(k, TM)$ denote the k-dimensional integral elements E of $\mathscr{I}$. Then,

$$\mathscr{V}_a(\mathscr{I}) = \mathrm{Gr}(a, TM), \quad \forall a < p, \quad \text{and} \quad \mathscr{V}_p(\mathscr{I}) = \big\{[\xi] \mid \xi \in C(\varphi)\big\}.$$

Let $\mathscr{V}_k(\mathscr{I})_x \subset \mathrm{Gr}(k, T_xM)$ denote the fibre over $x \in M$. Given an integral element $E \in \mathscr{V}_k(\mathscr{I})_x$ spanned by $\{e_1, \ldots, e_k\} \subset T_xM$, the *polar space* of E is

$$H(E) := \big\{v \in T_xM \mid \psi(e_1, \ldots, e_k, v) = 0, \forall \psi \in \mathscr{I}^{k+1}\big\} \supset E.$$

Suppose that $E_p = [\xi] \in \mathscr{V}_p(\mathscr{I})_x$. Let $\{e_1, \ldots, e_p\}$ be an orthonormal basis of E and set $E_a = \mathrm{span}\{e_1, \ldots, e_a\}$, $1 \leq a \leq p$. Since $\mathscr{I}^a = \{0\}$, $a < p$, we have $H(E_a) = T_xM$ and $c_a := \mathrm{codim}\, H(E_a) = 0$ for $1 \leq a \leq p-2$.

Note that $0 \neq v \in H(E_{p-1}) \backslash E_{p-1}$ if and only if $\{v, e_1, \ldots, e_{p-1}\}$ spans a φ-critical plane. Proposition 3.4 implies that the span of $\{v, e_1, \ldots, e_{p-1}\}$ is closed under the product ρ. Suppose that $\varphi_o = \varphi(\xi) = \varphi(e_1, \ldots, e_p) \neq 0$. Then (3.2) implies $\rho(e_1, \ldots, e_{p-1}) = \phi(E) e_p \neq 0$, and this forces $H(E_{p-1}) = E$. So

$c_{p-1} := \operatorname{codim} H(E_{p-1}) = n - p$. Cartan's test (cf. [13, Theorem 7.4.1] or [2, Theorem III.1.11]) implies that

$$\operatorname{codim}_E \mathscr{V}_p(\mathscr{I}) \geq n - p. \tag{5.1}$$

Note that the Hodge dual $*\varphi \in \Omega^{n-p}$ is also a parallel calibration on M; the associated ideal is $*\mathscr{I}$, the Hodge dual of $\mathscr{I}$. In particular $\mathscr{V}_{n-p}(*\mathscr{I}) = \{E^\perp \mid E \in \mathscr{V}_p(\mathscr{I})\}$, so that $\operatorname{codim}_{E^\perp} \mathscr{V}_{n-p}(*\mathscr{I}) = \operatorname{codim}_E \mathscr{V}_p(\mathscr{I})$. It follows that equality fails in (5.1) when $p > \frac{1}{2}n$: the system $\mathscr{I}$ is not involutive.

Remark. For example, $\mathscr{I}$ fails to be involutive in the case that M is a G_2-manifold and φ is the coassociative calibration of Section 3.3. Here, $n = 7$ and $p = 4$, so that $n - p = 3$, while $\operatorname{codim}_E \mathscr{V}_4(\mathscr{I}) = 4$. It fact, $\mathscr{P} = \{\alpha \wedge (*\varphi) \mid \alpha \in \Omega^1(M)\}$, where $*\varphi \in \Omega^3(M)$ is the associative calibration. As is well-known, coassociative submanifolds are integral manifolds of $\{*\varphi = 0\}$, and this system is involutive.

Remark. If the critical value $\varphi_o = \varphi(\xi)$ equals zero, then Corollary 3.5 implies that the ρ vanishes on E. In this case, $H(E_{p-1}) = \{v \in T_xM \mid \rho(v, a_1, \ldots, a_{p-2}) = 0 \forall \{a_1, \ldots, a_{p-2}\} \subset \{1, \ldots, p\}\}$.

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Colleen Robles, Department of Mathematics, Mail-stop 3368, Texas A&M University, College Station, TX 77843-3368, USA

E-mail address: robles@math.tamu.edu