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NEWTONIAN LORENTZ METRIC SPACES

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Abstract. This paper studies Newtonian Sobolev–Lorentz spaces. We prove that these spaces are Banach. We also study the global p,q-capacity and the p,q-modulus of families of rectifiable curves. Under some additional assumptions (that is, X carries a doubling measure and a weak Poincaré inequality), we show that when $1 \le q < p$ the Lipschitz functions are dense in those spaces; moreover, in the same setting we show that the p,q-capacity is Choquet provided that $q > 1$. We also provide a counterexample to the density result in the Euclidean setting when $1 < p \le n$ and $q = \infty$.

1. Introduction

In this paper, (X, d) is a complete metric space endowed with a nontrivial Borel regular measure μ. We assume that μ is finite and nonzero on nonempty bounded open sets. In particular, this implies that the measure μ is σ-finite. Further restrictions on the space X and on the measure μ will be imposed later.

The Sobolev–Lorentz relative p,q-capacity was studied in the Euclidean setting by Costea [6] and Costea and Maz'ya [8]. The Sobolev p-capacity was studied by Maz'ya [24] and Heinonen, Kilpeläinen and Martio [16] in $\mathbf{R}^n$ and by Costea [7] and Kinnunen and Martio [21] and [22] in metric spaces. The relative Sobolev p-capacity in metric spaces was introduced by J. Björn in [2] when studying the boundary continuity properties of quasiminimizers.

After recalling the definition of p,q-Lorentz spaces, we study some useful properties of the p,q-modulus of families of curves needed to give the notion of p,q-weak upper gradients. Then, following the approach of Shanmugalingam

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in [27] and [28], we generalize the notion of Newtonian Sobolev spaces to the Lorentz setting. There are several other definitions of Sobolev-type spaces in the metric setting when $p=q$; see Hajłasz [12], Heinonen and Koskela [17], Cheeger [4], and Franchi, Hajłasz and Koskela [11]. It has been shown that under reasonable hypotheses, the majority of these definitions yields the same space; see Franchi, Hajłasz and Koskela [11] and Shanmugalingam [27].

We prove that these spaces are Banach. In order to do this, we develop a theory of the Sobolev p,q-capacity. Some of the ideas used here when proving the properties of the p,q-capacity follow Kinnunen and Martio [21] and [22] and Costea [7]. We also use this theory to prove that, in the case $1\le q<p$, Lipschitz functions are dense in the Newtonian Sobolev–Lorentz space if the space X carries a doubling measure μ and a weak $(1,L^{p,q})$-Poincaré inequality. Newtonian Banach-valued Sobolev–Lorentz spaces were studied by Podbrdsky in [26].

We prove that under certain restrictions (when $1<q\le p$ and the space (X,d) carries a doubling measure μ and a certain weak Poincaré inequality) this capacity is a Choquet set function.

We recall the standard notation and definitions to be used throughout this paper. We denote by $B(x,r)=\{y\in X : d(x,y)<r\}$ the open ball with center $x\in X$ and radius $r>0$, while $\overline{B}(x,r)=\{y\in X : d(x,y)\le r\}$ is the closed ball with center $x\in X$ and radius $r>0$. For a positive number λ, $\lambda B(a,r)=B(a,\lambda r)$ and $\lambda\overline{B}(a,r)=\overline{B}(a,\lambda r)$.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. $C(a,b,\ldots)$ is a constant that depends only on the parameters $a,b,\ldots$. For $E\subset X$, the boundary, the closure, and the complement of E with respect to X will be denoted by ∂E, $\overline{E}$, and $X\setminus E$, respectively; $\operatorname{diam} E$ is the diameter of E with respect to the metric d.

2. Lorentz spaces

Let $f : X\to[-\infty,\infty]$ be a μ-measurable function. We define $\mu_{[f]}$, the *distribution function* of f as follows (see Bennett and Sharpley [1, Definition II.1.1]):

$$\mu_{[f]}(t)=\mu\big(\big\{x\in X : \big|f(x)\big|>t\big\}\big),\quad t\ge 0.$$

We define f^*, the *nonincreasing rearrangement* of f by

$$f^*(t)=\inf\big\{v : \mu_{[f]}(v)\le t\big\},\quad t\ge 0.$$

(See Bennett and Sharpley [1, Definition II.1.5].) We note that f and f^* have the same distribution function. For every positive α, we have

$$\big(|f|^\alpha\big)^*=\big(|f|^*\big)^\alpha$$

and if $|g| \le |f|$ μ-almost everywhere on X, then $g^* \le f^*$. (See [1, Proposition II.1.7].) We also define f^{**}, the *maximal function* of f^* by

$$f^{**}(t) = m_{f^*}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \quad t > 0.$$

(See [1, Definition II.3.1].)

Throughout the paper, we denote by p' the Hölder conjugate of $p \in [1,\infty]$.

The *Lorentz space* $L^{p,q}(X,\mu)$, $1 < p < \infty$, $1 \le q \le \infty$, is defined as follows:

$$L^{p,q}(X,\mu) = \big\{f : X \to [-\infty,\infty] : f \text{ is } \mu\text{-measurable}, \ \|f\|_{L^{p,q}(X,\mu)} < \infty\big\},$$

where

$$\|f\|_{L^{p,q}(X,\mu)} = \|f\|_{p,q} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty, \\ \sup_{t>0} t\mu_{[f]}(t)^{1/p} = \sup_{s>0} s^{1/p} f^*(s), & q = \infty. \end{cases}$$

(See Bennett and Sharpley [1, Definition IV.4.1] and Stein and Weiss [29, p. 191].)

If $1 \le q \le p$, then $\|\cdot\|_{L^{p,q}(X,\mu)}$ represents a norm, but for $p < q \le \infty$ it represents a quasinorm, equivalent to the norm $\|\cdot\|_{L^{(p,q)}(X,\mu)}$, where

$$\|f\|_{L^{(p,q)}(X,\mu)} = \|f\|_{(p,q)} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty, \\ \sup_{t>0} t^{1/p} f^{**}(t), & q = \infty. \end{cases}$$

(See [1, Definition IV.4.4].) Namely, from [1, Lemma IV.4.5] we have that

$$\|f\|_{L^{p,q}(X,\mu)} \le \|f\|_{L^{(p,q)}(X,\mu)} \le p'\|f\|_{L^{p,q}(X,\mu)}$$

for every $q \in [1,\infty]$ and every μ-measurable function $f : X \to [-\infty,\infty]$.

It is known that $(L^{p,q}(X,\mu), \|\cdot\|_{L^{p,q}(X,\mu)})$ is a Banach space for $1 \le q \le p$, while $(L^{p,q}(X,\mu), \|\cdot\|_{L^{(p,q)}(X,\mu)})$ is a Banach space for $1 < p < \infty$, $1 \le q \le \infty$. In addition, if the measure μ is nonatomic, the aforementioned Banach spaces are reflexive when $1 < q < \infty$. (See Hunt [18, pp. 259–262] and Bennett and Sharpley [1, Theorem IV.4.7 and Corollaries I.4.3 and IV.4.8].) (A measure μ is called *nonatomic* if for every measurable set A of positive measure there exists a measurable set $B \subset A$ such that $0 < \mu(B) < \mu(A)$.)

Definition 2.1 (See [1, Definition I.3.1]). Let $1 < p < \infty$ and $1 \le q \le \infty$. Let $Y = L^{p,q}(X,\mu)$. A function f in Y is said to have absolutely continuous norm in Y if and only if $\|f\chi_{E_k}\|_Y \to 0$ for every sequence E_k of μ-measurable sets satisfying $E_k \to \emptyset$ μ-almost everywhere.

Let Y_a be the subspace of Y consisting of functions of absolutely continuous norm and let Y_b be the closure in Y of the set of simple functions. It is known that $Y_a = Y_b$ whenever $1 \le q \le \infty$. (See Bennett and Sharpley [1, Theorem I.3.13].) Moreover, since (X,μ) is a σ-finite measure space, we have $Y_b = Y$ whenever $1 \le q < \infty$. (See Hunt [18, pp. 258–259].)

We recall (see Costea [6]) that in the Euclidean setting (that is, when $\mu = m_n$ is the n-dimensional Lebesgue measure and d is the Euclidean distance on $\mathbb{R}^n$) we have $Y_a \neq Y$ for $Y = L^{p,\infty}(X, m_n)$ whenever X is an open subset of $\mathbb{R}^n$. Let $X = B(0,2) \setminus \{0\}$. As in Costea [6] we define $u : X \to \mathbb{R}$,

$$u(x) = \begin{cases} |x|^{-\frac{n}{p}} & \text{if } 0 < |x| < 1, \\ 0 & \text{if } 1 \leq |x| \leq 2. \end{cases} \tag{1}$$

It is easy to see that $u \in L^{p,\infty}(X, m_n)$ and moreover,

$$\|u\chi_{B(0,\alpha)}\|_{L^{p,\infty}(X,m_n)} = \|u\|_{L^{p,\infty}(X,m_n)} = m_n\big(B(0,1)\big)^{1/p}$$

for every $\alpha > 0$. This shows that u does not have absolutely continuous weak L^p-norm and therefore $L^{p,\infty}(X, m_n)$ does not have absolutely continuous norm.

Remark 2.2. It is also known (see [1, Proposition IV.4.2]) that for every $p \in (1, \infty)$ and $1 \leq r < s \leq \infty$ there exists a constant $C(p, r, s)$ such that

$$\|f\|_{L^{p,s}(X,\mu)} \leq C(p, r, s)\|f\|_{L^{p,r}(X,\mu)} \tag{2}$$

for all measurable functions $f \in L^{p,r}(X, \mu)$. In particular, the embedding $L^{p,r}(X, \mu) \hookrightarrow L^{p,s}(X, \mu)$ holds.

Remark 2.3. By using the results contained in Bennett and Sharpley [1, Proposition II.1.7 and Definition IV.4.1] it is easy to see that for every $p \in (1, \infty)$, $q \in [1, \infty]$ and $0 < \alpha \leq \min(p, q)$, we have

$$\|f\|^{\alpha}_{L^{p,q}(X,\mu)} = \|f^{\alpha}\|_{L^{\frac{p}{\alpha},\frac{q}{\alpha}}(X,\mu)}$$

for every nonnegative function $f \in L^{p,q}(X, \mu)$.

2.1. The subadditivity and superadditivity of the Lorentz quasinorms. We recall the known results and present new results concerning the superadditivity and the subadditivity of the Lorentz p, q-quasinorm. For the convenience of the reader, we will provide proofs for the new results and for some of the known results.

The superadditivity of the Lorentz p, q-norm in the case $1 \leq q \leq p$ was stated in Chung, Hunt and Kurtz [5, Lemma 2.5].

Proposition 2.4 (See [5, Lemma 2.5]). *Let (X, μ) be a measure space. Suppose that $1 \leq q \leq p$. Let $\{E_i\}_{i\geq 1}$ be a collection of pairwise disjoint μ-measurable subsets of X with $E_0 = \bigcup_{i\geq 1} E_i$ and let $f \in L^{p,q}(X, \mu)$. Then*

$$\sum_{i\geq 1} \|\chi_{E_i} f\|^p_{L^{p,q}(X,\mu)} \leq \|\chi_{E_0} f\|^p_{L^{p,q}(X,\mu)}.$$

A similar result concerning the superadditivity was obtained in Costea and Maz'ya [8, Proposition 2.4] for the case $1 < p < q < \infty$ when $X = \Omega$ was an open set in $\mathbb{R}^n$ and μ was an arbitrary measure. That result is valid for a general measure space (X, μ).

Proposition 2.5. *Let* (X,μ) *be a measure space. Suppose that* $1<p<q<\infty$. *Let* $\{E_i\}_{i\geq 1}$ *be a collection of pairwise disjoint* μ*-measurable subsets of* X *with* $E_0=\bigcup_{i\geq 1} E_i$ *and let* $f\in L^{p,q}(X,\mu)$. *Then*

$$\sum_{i\geq 1}\|\chi_{E_i} f\|^q_{L^{p,q}(X,\mu)}\leq \|\chi_{E_0} f\|^q_{L^{p,q}(X,\mu)}.$$

Proof. We mimic the proof of Proposition 2.4 from Costea and Maz'ya [8]. We replace Ω with X. □

We have a similar result for the subadditivity of the Lorentz p,q-quasinorm. When $1<p<q\leq\infty$ we obtain a result that generalizes Theorem 2.5 from Costea [6].

Proposition 2.6. *Let* (X,μ) *be a measure space. Suppose that* $1<p<q\leq\infty$. *Suppose* $f_i, i=1,2,\ldots,$ *is a sequence of functions in* $L^{p,q}(X,\mu)$ *and let* $f_0=\sup_{i\geq 1}|f_i|$. *Then*

$$\|f_0\|^p_{L^{p,q}(X,\mu)}\leq \sum_{i=1}^{\infty}\|f_i\|^p_{L^{p,q}(X,\mu)}.$$

Proof. Without loss of generality we can assume that all the functions $f_i, i=1,2,\ldots$ are nonnegative. We have to consider two cases, depending on whether $p<q<\infty$ or $q=\infty$.

Let $\mu_{[f_i]}$ be the distribution function of f_i for $i=0,1,2,\ldots$. It is easy to see that

$$\mu_{[f_0]}(s)\leq \sum_{i=1}^{\infty}\mu_{[f_i]}(s)\quad \text{for every } s\geq 0. \tag{3}$$

Suppose that $p<q<\infty$. We have (see Kauhanen, Koskela and Malý [20, Proposition 2.1])

$$\|f_i\|^p_{L^{p,q}(X,\mu)}=\left(p\int_0^\infty s^{q-1}\mu_{[f_i]}(s)^{\frac{q}{p}}\,ds\right)^{\frac{p}{q}} \tag{4}$$

for $i=0,1,2,\ldots$. From this and (3), we obtain

$$\begin{aligned}\|f_0\|^p_{L^{p,q}(\Omega,\mu)}&=\left(p\int_0^\infty s^{q-1}\mu_{[f_0]}(s)^{\frac{q}{p}}\,ds\right)^{\frac{p}{q}}\\&\leq \sum_{i\geq 1}\left(p\int_0^\infty s^{q-1}\mu_{[f_i]}(s)^{\frac{q}{p}}\,ds\right)^{\frac{p}{q}}=\sum_{i\geq 1}\|f_i\|^p_{L^{p,q}(\Omega,\mu)}.\end{aligned}$$

Now, suppose that $q=\infty$. From (3), we obtain

$$s^p\mu_{[f_0]}(s)\leq \sum_{i\geq 1}\big(s^p\mu_{[f_i]}(s)\big)\quad \text{for every } s>0,$$

which implies

$$s^p \mu_{[f_0]}(s) \le \sum_{i\ge 1} \|f_i\|^p_{L^{p,\infty}(X,\mu)} \quad \text{for every } s>0. \tag{5}$$

By taking the supremum over all $s>0$ in (5), we get the desired conclusion. This finishes the proof. ☐

We recall a few results concerning Lorentz spaces.

THEOREM 2.7 (See [6, Theorem 2.6]). *Suppose* $1<p<q\le\infty$ *and* $\varepsilon\in(0,1)$. *Let* $f_1, f_2 \in L^{p,q}(X,\mu)$. *We denote* $f_3 = f_1+f_2$. *Then* $f_3\in L^{p,q}(X,\mu)$ *and*

$$\|f_3\|^p_{L^{p,q}(X,\mu)} \le (1-\varepsilon)^{-p}\|f_1\|^p_{L^{p,q}(X,\mu)} + \varepsilon^{-p}\|f_2\|^p_{L^{p,q}(X,\mu)}.$$

Proof. The proof of Theorem 2.6 from Costea [6] carries verbatim. We replace Ω with X. ☐

Theorem 2.7 has an useful corollary.

COROLLARY 2.8 (See [6, Corollary 2.7]). *Suppose* $1<p<\infty$ *and* $1\le q\le\infty$. *Let* f_k *be a sequence of functions in* $L^{p,q}(X,\mu)$ *converging to* f *with respect to the* p,q*-quasinorm and pointwise* μ*-almost everywhere in* X. *Then*

$$\lim_{k\to\infty} \|f_k\|_{L^{p,q}(X,\mu)} = \|f\|_{L^{p,q}(X,\mu)}.$$

Proof. The proof of Corollary 2.7 from Costea [6] carries verbatim. We replace Ω with X. ☐

3. p,q-modulus of the path family

In this section, we establish some results about the p,q-modulus of families of curves. Here (X,d,μ) is a metric measure space. We say that a curve γ in X is rectifiable if it has finite length. Whenever γ is rectifiable, we use the arc length parametrization $\gamma:[0,\ell(\gamma)]\to X$, where $\ell(\gamma)$ is the length of the curve γ.

Let Γ_{rect} denote the family of all nonconstant rectifiable curves in X. It may well be that $\Gamma_{\text{rect}}=\emptyset$, but we will be interested in metric spaces for which Γ_{rect} is sufficiently large.

DEFINITION 3.1. For $\Gamma\subset\Gamma_{\text{rect}}$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho: X\to[0,\infty]$ such that

$$\int_\gamma \rho \ge 1 \quad \text{for every } \gamma\in\Gamma.$$

Now for each $1<p<\infty$ and $1\le q\le\infty$ we define

$$\mathrm{Mod}_{p,q}(\Gamma) = \inf_{\rho\in F(\Gamma)} \|\rho\|^p_{L^{p,q}(X,\mu)}.$$

The number $\mathrm{Mod}_{p,q}(\Gamma)$ is called the *p,q-modulus* of the family Γ.

3.1. Basic properties of the p,q-modulus. Usually, a modulus is a monotone and subadditive set function. The following theorem will show, among other things, that this is true in the case of the p,q-modulus.

THEOREM 3.2. *Suppose* $1<p<\infty$ *and* $1\le q\le\infty$. *The set function* $\Gamma\to \mathrm{Mod}_{p,q}(\Gamma)$, $\Gamma\subset\Gamma_{\mathrm{rect}}$, *enjoys the following properties*:

(i) $\mathrm{Mod}_{p,q}(\emptyset)=0$.
(ii) *If* $\Gamma_1\subset\Gamma_2$, *then* $\mathrm{Mod}_{p,q}(\Gamma_1)\le\mathrm{Mod}_{p,q}(\Gamma_2)$.
(iii) *Suppose* $1\le q\le p$. *Then*

$$\mathrm{Mod}_{p,q}\left(\bigcup_{i=1}^{\infty}\Gamma_i\right)^{q/p}\le\sum_{i=1}^{\infty}\mathrm{Mod}_{p,q}(\Gamma_i)^{q/p}.$$

(iv) *Suppose* $p<q\le\infty$. *Then*

$$\mathrm{Mod}_{p,q}\left(\bigcup_{i=1}^{\infty}\Gamma_i\right)\le\sum_{i=1}^{\infty}\mathrm{Mod}_{p,q}(\Gamma_i).$$

Proof. (i) $\mathrm{Mod}_{p,q}(\emptyset)=0$ because $\rho\equiv 0\in F(\emptyset)$.

(ii) If $\Gamma_1\subset\Gamma_2$, then $F(\Gamma_2)\subset F(\Gamma_1)$ and hence $\mathrm{Mod}_{p,q}(\Gamma_1)\le\mathrm{Mod}_{p,q}(\Gamma_2)$.

(iii) Suppose that $1\le q\le p$. The case $p=q$ corresponds to the p-modulus and the claim certainly holds in that case. (See, for instance, Hajłasz [13, Theorem 5.2 (3)].) So we can look at the case $1\le q<p$.

We can assume without loss of generality that

$$\sum_{i=1}^{\infty}\mathrm{Mod}_{p,q}(\Gamma_i)^{q/p}<\infty.$$

Let $\varepsilon>0$ be fixed. Take $\rho_i\in F(\Gamma_i)$ such that

$$\|\rho_i\|^q_{L^{p,q}(X,\mu)}<\mathrm{Mod}_{p,q}(\Gamma_i)^{q/p}+\varepsilon 2^{-i}.$$

Let $\rho:=(\sum_{i=1}^{\infty}\rho_i^q)^{1/q}$. We notice via Bennett-Sharpley [1, Proposition II.1.7 and Definition IV.4.1] and Remark 2.3 applied with $\alpha=q$ that

$$\rho_i^q\in L^{\frac{p}{q},1}(X,\mu)\quad\text{and}\quad \|\rho_i^q\|_{L^{\frac{p}{q},1}(X,\mu)}=\|\rho_i\|^q_{L^{p,q}(X,\mu)}. \tag{6}$$

for every $i=1,2,\dots$. By using (6) and Remark 2.3 together with the definition of ρ and the fact that $\|\cdot\|_{L^{\frac{p}{q},1}(X,\mu)}$ is a norm when $1\le q\le p$, it follows that $\rho\in F(\Gamma)$ and

$$\mathrm{Mod}_{p,q}(\Gamma_i)^{q/p}\le\|\rho\|^q_{L^{p,q}(X,\mu)}\le\sum_{i=1}^{\infty}\|\rho_i\|^q_{L^{p,q}(X,\mu)}<\sum_{i=1}^{\infty}\mathrm{Mod}_{p,q}(\Gamma_i)^{q/p}+2\varepsilon.$$

Letting $\varepsilon\to 0$, we complete the proof when $1\le q\le p$.

(iv) Suppose now that $p < q \le \infty$. We can assume without loss of generality that

$$\sum_{i=1}^{\infty} \operatorname{Mod}_{p,q}(\Gamma_i) < \infty.$$

Let $\varepsilon > 0$ be fixed. Take $\rho_i \in F(\Gamma_i)$ such that

$$\|\rho_i\|^p_{L^{p,q}(X,\mu)} < \operatorname{Mod}_{p,q}(\Gamma_i) + \varepsilon 2^{-i}.$$

Let $\rho := \sup_{i \ge 1} \rho_i$. Then $\rho \in F(\Gamma)$. Moreover, from Proposition 2.6 it follows that $\rho \in L^{p,q}(X,\mu)$ and

$$\operatorname{Mod}_{p,q}(\Gamma) \le \|\rho\|^p_{L^{p,q}(X,\mu)} \le \sum_{i=1}^{\infty} \|\rho_i\|^p_{L^{p,q}(X,\mu)} < \sum_{i=1}^{\infty} \operatorname{Mod}_{p,q}(\Gamma_i) + 2\varepsilon.$$

Letting $\varepsilon \to 0$, we complete the proof when $p < q \le \infty$. □

So we proved that the modulus is a monotone function. Also, the shorter the curves, the larger the modulus. More precisely, we have the following lemma.

LEMMA 3.3. *Let* $\Gamma_1, \Gamma_2 \subset \Gamma_{\text{rect}}$. *If each curve* $\gamma \in \Gamma_1$ *contains a subcurve that belongs to* Γ_2, *then* $\operatorname{Mod}_{p,q}(\Gamma_1) \le \operatorname{Mod}_{p,q}(\Gamma_2)$.

Proof. $F(\Gamma_2) \le F(\Gamma_1)$. □

The following theorem provides an useful characterization of path families that have p,q-modulus zero.

THEOREM 3.4. *Let* $\Gamma \subset \Gamma_{\text{rect}}$. *Then* $\operatorname{Mod}_{p,q}(\Gamma) = 0$ *if and only if there exists a Borel measurable function* $0 \le \rho \in L^{p,q}(X,\mu)$ *such that* $\int_\gamma \rho = \infty$ *for every* $\gamma \in \Gamma$.

Proof. Sufficiency. We notice that $\rho/n \in F(\Gamma)$ for every n and hence

$$\operatorname{Mod}_{p,q}(\Gamma) \le \lim_{n\to\infty} \|\rho/n\|^p_{L^{p,q}(X,\mu)} = 0.$$

Necessity. There exists $\rho_i \in F(\Gamma)$ such that $\|\rho_i\|_{L^{(p,q)}(X,\mu)} < 2^{-i}$ and $\int_\gamma \rho_i \ge 1$ for every $\gamma \in \Gamma$. Then $\rho := \sum_{i=1}^{\infty} \rho_i$ has the required properties. □

COROLLARY 3.5. *Suppose* $1 < p < \infty$ *and* $1 \le q \le \infty$ *are given. If* $0 \le g \in L^{p,q}(X,\mu)$ *is Borel measurable, then* $\int_\gamma g < \infty$ *for* p,q*-almost every* $\gamma \in \Gamma_{\text{rect}}$.

The following theorem is also important.

THEOREM 3.6. *Let* $u_k : X \to \overline{\mathbb{R}} = [-\infty, \infty]$ *be a sequence of Borel functions which converge to a Borel function* $u : X \to \overline{\mathbb{R}}$ *in* $L^{p,q}(X,\mu)$. *Then there is a subsequence* $(u_{k_j})_j$ *such that*

$$\int_\gamma |u_{k_j} - u| \to 0 \quad \textit{as } j \to \infty,$$

for p,q*-almost every curve* $\gamma \in \Gamma_{\text{rect}}$.

Proof. We follow Hajłasz [13]. We take a subsequence $(u_{k_j})_j$ such that

$$\|u_{k_j} - u\|_{L^{p,q}(X,\mu)} < 2^{-2j}. \tag{7}$$

Set $g_j = |u_{k_j} - u|$, and let $\Gamma \subset \Gamma_{\mathrm{rect}}$ be the family of curves such that

$$\limsup_{j\to\infty} \int_\gamma g_j > 0.$$

We want to show that $\mathrm{Mod}_{p,q}(\Gamma) = 0$. Denote by Γ_j the family of curves in Γ_{rect} for which $\int_\gamma g_j > 2^{-j}$. Then $2^j g_j \in F(\Gamma_j)$ and hence $\mathrm{Mod}_{p,q}(\Gamma_j) < 2^{-pj}$ as a consequence of (7). We notice that

$$\Gamma \subset \bigcap_{i=1}^\infty \bigcup_{j=i}^\infty \Gamma_j.$$

Thus,

$$\mathrm{Mod}_{p,q}(\Gamma)^{1/p} \le \sum_{j=i}^\infty \mathrm{Mod}_{p,q}(\Gamma_j)^{1/p} \le \sum_{j=i}^\infty 2^{-j} = 2^{1-i}$$

for every integer $i \ge 1$, which implies $\mathrm{Mod}_{p,q}(\Gamma) = 0$. □

3.2. Upper gradient.

DEFINITION 3.7. Let $u : X \to [-\infty, \infty]$ be a Borel function. We say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if for *every* rectifiable curve γ parametrized by arc length parametrization we have

$$\big|u\big(\gamma(0)\big) - u\big(\gamma\big(\ell(\gamma)\big)\big)\big| \le \int_\gamma g \tag{8}$$

whenever both $u(\gamma(0))$ and $u(\gamma(\ell(\gamma)))$ are finite and $\int_\gamma g = \infty$ otherwise. We say that g is a p,q-weak upper gradient of u if (8) holds on p,q-almost every curve $\gamma \in \Gamma_{\mathrm{rect}}$.

The weak upper gradients were introduced in the case $p = q$ by Heinonen and Koskela in [17]. See also Heinonen [15] and Shanmugalingam [27] and [28].

If g is an upper gradient of u and $\widetilde{g} = g$, μ-almost everywhere, is another nonnegative Borel function, then it might happen that $\widetilde{g}$ is not an upper gradient of u. However, we have the following result.

LEMMA 3.8. *If g is a p,q-weak upper gradient of u and $\widetilde{g}$ is another nonnegative Borel function such that $\widetilde{g} = g$ μ-almost everywhere, then $\widetilde{g}$ is a p,q-weak upper gradient of u.*

Proof. Let $\Gamma_1 \subset \Gamma_{\mathrm{rect}}$ be the family of all nonconstant rectifiable curves $\gamma : [0, \ell(\gamma)] \to X$ for which $\int_\gamma |g - \widetilde{g}| > 0$. The constant sequence $g_n = |g - \widetilde{g}|$ converges to 0 in $L^{p,q}(X, \mu)$, so from Theorem 3.6 it follows that $\mathrm{Mod}_{p,q}(\Gamma_1) =$

0 and $\int_\gamma |g-\widetilde{g}|=0$ for every nonconstant rectifiable curve $\gamma\colon[0,\ell(\gamma)]\to X$ that is not in Γ_1.

Let $\Gamma_2\subset\Gamma_{\mathrm{rect}}$ be the family of all nonconstant rectifiable curves $\gamma: [0,\ell(\gamma)]\to X$ for which the inequality

$$\left|u(\gamma(0))-u(\gamma(\ell(\gamma)))\right|\le\int_\gamma g$$

is *not* satisfied. Then $\mathrm{Mod}_{p,q}(\Gamma_2)=0$. Thus $\mathrm{Mod}_{p,q}(\Gamma_1\cup\Gamma_2)=0$. For every $\gamma\in\Gamma_{\mathrm{rect}}$ not in $\Gamma_1\cup\Gamma_2$ we have

$$\left|u(\gamma(0))-u(\gamma(\ell(\gamma)))\right|\le\int_\gamma g=\int_\gamma\widetilde{g}.$$

This finishes the proof. □

The next result shows that p,q-weak upper gradients can be nicely approximated by upper gradients. The case $p=q$ was proved by Koskela and MacManus [23].

LEMMA 3.9. *If g is a p,q-weak upper gradient of u which is finite μ-almost everywhere, then for every $\varepsilon>0$ there exists an upper gradient g_ε of u such that*

$$g_\varepsilon\ge g\quad\text{everywhere on } X \text{ and } \|g_\varepsilon-g\|_{L^{p,q}(X,\mu)}\le\varepsilon.$$

Proof. Let $\Gamma\subset\Gamma_{\mathrm{rect}}$ be the family of all nonconstant rectifiable curves $\gamma:[0,\ell(\gamma)]\to X$ for which the inequality

$$\left|u(\gamma(0))-u(\gamma(\ell(\gamma)))\right|\le\int_\gamma g$$

is *not* satisfied. Then $\mathrm{Mod}_{p,q}(\Gamma)=0$ and hence, from Theorem 3.4 it follows that there exists $0\le\rho\in L^{p,q}(X,\mu)$ such that $\int_\gamma\rho=\infty$ for every $\gamma\in\Gamma$. Take $g_\varepsilon=g+\varepsilon\rho/\|\rho\|_{L^{p,q}(X,\mu)}$. Then g_ε is a nonnegative Borel function and

$$\left|u(\gamma(0))-u(\gamma(\ell(\gamma)))\right|\le\int_\gamma g_\varepsilon$$

for *every* curve $\gamma\in\Gamma_{\mathrm{rect}}$. This finishes the proof. □

If A is a subset of X let Γ_A be the family of all curves in Γ_{rect} that intersect A and let Γ_A^+ be the family of all curves in Γ_{rect} such that the Hausdorff one-dimensional measure $\mathcal{H}_1(|\gamma|\cap A)$ is positive. Here and throughout the paper $|\gamma|$ is the image of the curve γ.

The following lemma will be useful later in this paper.

LEMMA 3.10. *Let $u_i:X\to\mathbb{R}$, $i\ge1$, be a sequence of Borel functions such that $g\in L^{p,q}(X)$ is a p,q-weak upper gradient for every $u_i, i\ge1$. We define $u(x)=\lim_{i\to\infty}u_i(x)$ and $E=\{x\in X:|u(x)|=\infty\}$. Suppose that $\mu(E)=0$ and that u is well-defined outside E. Then g is a p,q-weak upper gradient for u.*

Proof. For every $i \geq 1$, we define $\Gamma_{1,i}$ to be the set of all curves $\gamma \in \Gamma_{\mathrm{rect}}$ for which

$$\big|u_i\big(\gamma(0)\big) - u_i\big(\gamma\big(\ell(\gamma)\big)\big)\big| \leq \int_\gamma g$$

is *not* satisfied. Then $\mathrm{Mod}_{p,q}(\Gamma_{1,i}) = 0$ and hence $\mathrm{Mod}_{p,q}(\Gamma_{1,0}) = 0$, where $\Gamma_{1,0} = \bigcup_{i=1}^{\infty} \Gamma_{1,i}$. Let $\Gamma_1 \subset \Gamma_{\mathrm{rect}}$ be the collection of all curves having a subcurve in $\Gamma_{1,0}$. Then $F(\Gamma_{1,0}) \subset F(\Gamma_1)$ and hence $\mathrm{Mod}_{p,q}(\Gamma_1) \leq \mathrm{Mod}_{p,q}(\Gamma_{1,0}) = 0$.

Let Γ_0 be the collection of all paths $\gamma \in \Gamma_{\mathrm{rect}}$ such that $\int_\gamma g = \infty$. Then we have via Theorem 3.4 that $\mathrm{Mod}_{p,q}(\Gamma_0) = 0$ since $g \in L^{p,q}(X,\mu)$.

Since $\mu(E) = 0$, it follows that $\mathrm{Mod}_{p,q}(\Gamma_E^+) = 0$. Indeed, $\infty \cdot \chi_E \in F(\Gamma_E^+)$ and $\|\infty \cdot \chi_E\|_{L^{p,q}(X,\mu)} = 0$. Therefore, $\mathrm{Mod}_{p,q}(\Gamma_0 \cup \Gamma_E^+ \cup \Gamma_1) = 0$.

For any path γ in the family $\Gamma_{\mathrm{rect}} \setminus (\Gamma_0 \cup \Gamma_E^+ \cup \Gamma_1)$, by the fact that the path is not in Γ_E^+, there exists a point y in $|\gamma|$ such that y is not in E, that is $y \in |\gamma|$ and $|u(y)| < \infty$. For any point $x \in |\gamma|$, we have (since γ has no subcurves in $\Gamma_{1,0}$)

$$\big|u_i(x)\big| - \big|u_i(y)\big| \leq \big|u_i(x) - u_i(y)\big| \leq \int_\gamma g < \infty.$$

Therefore,

$$\big|u_i(x)\big| \leq \big|u_i(y)\big| + \int_\gamma g.$$

Taking limits on both sides and using the facts that $|u(y)| < \infty$ and that γ is not in $\Gamma_0 \cup \Gamma_1$, we see that

$$\lim_{i\to\infty} \big|u_i(x)\big| \leq \lim_{i\to\infty} \big|u_i(y)\big| + \int_\gamma g = \big|u(y)\big| + \int_\gamma g < \infty$$

and therefore x is not in E. Thus $\Gamma_E \subset \Gamma_0 \cup \Gamma_E^+ \cup \Gamma_1$ and $\mathrm{Mod}_{p,q}(\Gamma_E) = 0$.

Next, let γ be a path in $\Gamma_{\mathrm{rect}} \setminus (\Gamma_0 \cup \Gamma_E^+ \cup \Gamma_1)$. The above argument showed that $|\gamma|$ does not intersect E. If we denote by x and y the endpoints of γ, we have

$$\big|u(x) - u(y)\big| = \Big|\lim_{i\to\infty} u_i(x) - \lim_{i\to\infty} u_i(y)\Big| = \lim_{i\to\infty} \big|u_i(x) - u_i(y)\big| \leq \int_\gamma g.$$

Therefore, g is a p,q-weak upper gradient for u as well. □

The following proposition shows how the upper gradients behave under a change of variable.

PROPOSITION 3.11. *Let $F : \mathbb{R} \to \mathbb{R}$ be C^1 and let $u : X \to \mathbb{R}$ be a Borel function. If $g \in L^{p,q}(X,\mu)$ is a p,q-weak upper gradient for u, then $|F'(u)|g$ is a p,q-weak upper gradient for $F \circ u$.*

Proof. Let Γ_0 to be the set of all curves $\gamma \in \Gamma_{\mathrm{rect}}$ for which

$$\big|u\big(\gamma(0)\big) - u\big(\gamma\big(\ell(\gamma)\big)\big)\big| \le \int_\gamma g$$

is *not* satisfied. Then $\mathrm{Mod}_{p,q}(\Gamma_0) = 0$. Let $\Gamma_1 \subset \Gamma_{\mathrm{rect}}$ be the collection of all curves having a subcurve in Γ_0. Then $F(\Gamma_0) \subset F(\Gamma_1)$ and hence $\mathrm{Mod}_{p,q}(\Gamma_1) \le \mathrm{Mod}_{p,q}(\Gamma_0) = 0$.

Let Γ_2 be the set of curves $\gamma \in \Gamma_{\mathrm{rect}}$ for which $\int_\gamma g = \infty$. Then we have via Theorem 3.4 that $\mathrm{Mod}_{p,q}(\Gamma_2) = 0$ since $g \in L^{p,q}(X, \mu)$. Thus, $\mathrm{Mod}_{p,q}(\Gamma_1 \cup \Gamma_2) = 0$.

The claim will follow immediately after we show that

$$\big|(F \circ u)\big(\gamma(0)\big) - (F \circ u)\big(\gamma\big(\ell(\gamma)\big)\big)\big| \le \int_0^{\ell(\gamma)} \big(\big|F'\big(u\big(\gamma(s)\big)\big)\big| + \varepsilon\big) g\big(\gamma(s)\big)\, ds \tag{9}$$

for all curves $\gamma \in \Gamma_{\mathrm{rect}} \setminus (\Gamma_1 \cup \Gamma_2)$ and for every $\varepsilon > 0$.

So fix $\varepsilon > 0$ and choose a curve $\gamma \in \Gamma_{\mathrm{rect}} \setminus (\Gamma_1 \cup \Gamma_2)$. Let $\ell = \ell(\gamma)$. We notice immediately that $u \circ \gamma$ is uniformly continuous on $[0, \ell]$ and F' is uniformly continuous on the compact interval $I := (u \circ \gamma)([0, \ell])$. Let $\delta, \delta_1 > 0$ be chosen such that

$$\big|\big(F' \circ u \circ \gamma\big)(t) - \big(F' \circ u \circ \gamma\big)(s)\big| + |(u \circ \gamma)(t) - (u \circ \gamma)(s)| < \delta_1$$

for all $t, s \in [0, \ell]$ with $|t - s| < \delta$ and such that

$$\big|F'(u) - F'(v)\big| < \varepsilon \quad \text{for all } u, v \in I \text{ with } |u - v| < \delta_1.$$

Fix an integer $n > 1/\delta$ and put $\ell_i = (i\ell)/n, i = 0, \dots, n-1$. For every $i = 0, \dots, n-1$ we have

$$\begin{aligned}\big|(F \circ u \circ \gamma)(\ell_{i+1}) - (F \circ u \circ \gamma)(\ell_i)\big| &= \big|F'(t_{i,i+1})\big|\big|(u \circ \gamma)(\ell_{i+1}) - (u \circ \gamma)(\ell_i)\big| \\ &\le \big|F'(t_{i,i+1})\big| \int_{\ell_i}^{\ell_{i+1}} g\big(\gamma(s)\big)\, ds,\end{aligned}$$

where $t_{i,i+1} \in I_{i,i+1} := (u \circ \gamma)((\ell_i, \ell_{i+1}))$. From the choice of δ, it follows that

$$\big|(F \circ u \circ \gamma)(\ell_{i+1}) - (F \circ u \circ \gamma)(\ell_i)\big| \le \int_{\ell_i}^{\ell_{i+1}} \big(\big|F'\big(u\big(\gamma(s)\big)\big)\big| + \varepsilon\big) g\big(\gamma(s)\big)\, ds,$$

for every $i = 0, \dots, n-1$. If we sum over i, we obtain easily (9). This finishes the proof. □

As a direct consequence of Proposition 3.11, we have the following corollaries.

COROLLARY 3.12. *Let $r \in (1, \infty)$ be fixed. Suppose $u : X \to \mathbb{R}$ is a bounded nonnegative Borel function. If $g \in L^{p,q}(X, \mu)$ is a p,q-weak upper gradient of u, then $ru^{r-1}g$ is a p,q-weak upper gradient for u^r.*

Proof. Let $M>0$ be such that $0\le u(x)<M$ for all $x\in X$. We apply Proposition 3.11 to any C^1 function $F:\mathbb{R}\to\mathbb{R}$ satisfying $F(t)=t^r, 0\le t\le M$. $\square$

COROLLARY 3.13. *Let $r\in(0,1)$ be fixed. Suppose that $u:X\to\mathbb{R}$ is a nonnegative Borel function that has a p,q-weak upper gradient $g\in L^{p,q}(X,\mu)$. Then $r(u+\varepsilon)^{r-1}g$ is a p,q-weak upper gradient for $(u+\varepsilon)^r$ for all $\varepsilon>0$.*

Proof. Fix $\varepsilon>0$. We apply Proposition 3.11 to any C^1 function $F:\mathbb{R}\to\mathbb{R}$ satisfying $F(t)=t^r, \varepsilon\le t<\infty$. $\square$

COROLLARY 3.14. *Suppose $1\le q\le p<\infty$. Let u_1,u_2 be two nonnegative bounded real-valued Borel functions defined on X. Suppose $g_i\in L^{p,q}(X,\mu), i=1,2$ are p,q-weak upper gradients for $u_i, i=1,2$. Then $L^{p,q}(X,\mu)\ni g:=(g_1^q+g_2^q)^{1/q}$ is a p,q-weak upper gradient for $u:=(u_1^q+u_2^q)^{1/q}$.*

Proof. The claim is obvious when $q=1$, so we assume without loss of generality that $1<q\le p$. We prove first that $g\in L^{p,q}(X,\mu)$. Indeed, via Remark 2.3 it is enough to show that $g^q\in L^{\frac{p}{q},1}(X,\mu)$. But $g^q=g_1^q+g_2^q$ and $g_i^q\in L^{\frac{p}{q},1}(X,\mu)$ since $g_i\in L^{p,q}(X,\mu)$. (See Remark 2.3.) This, the fact that $\|\cdot\|_{L^{\frac{p}{q},1}(X,\mu)}$ is a norm whenever $1<q\le p$, and another appeal to Remark 2.3 yield $g\in L^{p,q}(X,\mu)$ with

$$\begin{aligned}\|g\|^q_{L^{p,q}(X,\mu)} &= \|g^q\|_{L^{\frac{p}{q},1}(X,\mu)} \le \|g_1^q\|_{L^{\frac{p}{q},1}(X,\mu)}+\|g_2^q\|_{L^{\frac{p}{q},1}(X,\mu)}\\ &= \|g_1\|^q_{L^{p,q}(X,\mu)}+\|g_2\|^q_{L^{p,q}(X,\mu)}.\end{aligned}$$

For $i=1,2$ let $\Gamma_{i,1}$ be the family of nonconstant rectifiable curves γ in Γ_{rect} for which

$$\big|u_i\big(\gamma(0)\big)-u_i\big(\gamma\big(\ell(\gamma)\big)\big)\big|\le\int_\gamma g_i$$

is *not* satisfied. Then $\mathrm{Mod}_{p,q}(\Gamma_{i,1})=0$ since g_i is a p,q-weak upper gradient for $u_i, i=1,2$. Let $\Gamma_{0,1}$ be the family of nonconstant rectifiable curves γ in Γ_{rect} having a subcurve in $\Gamma_{1,1}\cup\Gamma_{2,1}$. Then $F(\Gamma_{1,1}\cup\Gamma_{2,1})\subset F(\Gamma_{0,1})$ and hence $\mathrm{Mod}_{p,q}(\Gamma_{0,1})\le\mathrm{Mod}_{p,q}(\Gamma_{1,1}\cup\Gamma_{2,1})=0$.

Let $\Gamma_{i,2}$ be the family of nonconstant rectifiable curves γ in Γ_{rect} for which $\int_\gamma g_i=\infty$. Then for $i=1,2$ we have $\mathrm{Mod}_{p,q}(\Gamma_{i,2})=0$ via Theorem 3.4 because by hypothesis $g_i\in L^{p,q}(X,\mu), i=1,2$. Let $\Gamma_0=\Gamma_{0,1}\cup\Gamma_{1,2}\cup\Gamma_{2,2}$. Then $\mathrm{Mod}_{p,q}(\Gamma_0)=0$.

Fix $\varepsilon>0$. By applying Corollary 3.12 with $r=q$, $u=u_i$ and $g=g_i$, $i=1,2$, we see that $L^{p,q}(X,\mu)\ni q(u_i+\varepsilon)^{q-1}g_i$ is a p,q-weak upper gradient of $(u_i+\varepsilon)^q$ for $i=1,2$. Thus, via Hölder's inequality it follows that G_ε is a p,q-weak upper gradient for U_ε, where

$$\begin{aligned}&G_\varepsilon:=q\big((u_1+\varepsilon)^q+(u_2+\varepsilon)^q\big)^{(q-1)/q}\big(g_1^q+g_2^q\big)^{1/q}\quad\text{and}\\ &U_\varepsilon:=(u_1+\varepsilon)^q+(u_2+\varepsilon)^q.\end{aligned}$$

We notice that $G_\varepsilon \in L^{p,q}(X,\mu)$. Indeed, $G_\varepsilon = qU_\varepsilon^{(q-1)/q} g$, with U_ε nonnegative a bounded and $g \in L^{p,q}(X,\mu)$, so $G_\varepsilon \in L^{p,q}(X,\mu)$.

Now we apply Corollary 3.13 with $r = 1/q$, $u = U_\varepsilon$ and $g = G_\varepsilon$ to obtain that $u_\varepsilon := U_\varepsilon^{1/q}$ has $1/qU_\varepsilon^{(1-q)/q} G_\varepsilon = g$ as a p,q-weak upper gradient that belongs to $L^{p,q}(X,\mu)$. In fact, by looking at the proof of Proposition 3.11, we see that

$$\big|u_\varepsilon\big(\gamma(0)\big) - u_\varepsilon\big(\gamma\big(\ell(\gamma)\big)\big)\big| \le \int_\gamma g$$

for every curve $\gamma \in \Gamma_{\mathrm{rect}}$ that is not in Γ_0. Letting $\varepsilon \to 0$, we obtain the desired conclusion. This finishes the proof of the corollary. □

LEMMA 3.15. *If $u_i, i = 1,2$ are nonnegative real-valued Borel functions in $L^{p,q}(X,\mu)$ with corresponding p,q-weak upper gradients $g_i \in L^{p,q}(X,\mu)$, then $g := \max(g_1,g_2) \in L^{p,q}(X,\mu)$ is a p,q-weak upper gradient for $u := \max(u_1, u_2) \in L^{p,q}(X,\mu)$.*

Proof. It is easy to see that $u, g \in L^{p,q}(X,\mu)$. For $i = 1,2$ let $\Gamma_{0,i} \subset \Gamma_{\mathrm{rect}}$ be the family of nonconstant rectifiable curves γ for which $\int_\gamma g_i = \infty$. Then we have via Theorem 3.4 that $\mathrm{Mod}_{p,q}(\Gamma_{0,i}) = 0$ because $g_i \in L^{p,q}(X,\mu)$. Thus $\mathrm{Mod}_{p,q}(\Gamma_0) = 0$, where $\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}$.

For $i = 1,2$ let $\Gamma_{1,i} \subset \Gamma_{\mathrm{rect}}$ be the family of curves $\gamma \in \Gamma_{\mathrm{rect}} \setminus \Gamma_0$ for which

$$\big|u_i\big(\gamma(0)\big) - u_i\big(\gamma\big(\ell(\gamma)\big)\big)\big| \le \int_\gamma g_i$$

is *not* satisfied. Then $\mathrm{Mod}_{p,q}(\Gamma_{1,i}) = 0$ since g_i is a p,q-weak upper gradient for u_i, $i = 1,2$. Thus, $\mathrm{Mod}_{p,q}(\Gamma_1) = 0$, where $\Gamma_1 = \Gamma_{1,1} \cup \Gamma_{1,2}$.

It is easy to see that

$$\big|u(x) - u(y)\big| \le \max\big(\big|u_1(x) - u_1(y)\big|, \big|u_2(x) - u_2(y)\big|\big). \tag{10}$$

On every curve $\gamma \in \Gamma_{\mathrm{rect}} \setminus (\Gamma_0 \cup \Gamma_1)$ we have

$$\big|u_i\big(\gamma(0)\big) - u_i\big(\gamma\big(\ell(\gamma)\big)\big)\big| \le \int_\gamma g_i \le \int_\gamma g.$$

This and (10) show that

$$\big|u\big(\gamma(0)\big) - u\big(\gamma\big(\ell(\gamma)\big)\big)\big| \le \int_\gamma g$$

on every curve $\gamma \in \Gamma_{\mathrm{rect}} \setminus (\Gamma_0 \cup \Gamma_1)$. This finishes the proof. □

LEMMA 3.16. *Suppose $g \in L^{p,q}(X,\mu)$ is a p,q-weak upper gradient for a nonnegative Borel function $u \in L^{p,q}(X,\mu)$. Let $\lambda \ge 0$ be fixed. Then $u_\lambda := \min(u,\lambda) \in L^{p,q}(X,\mu)$ and g is a p,q-weak upper gradient for u_λ.*

Proof. Obviously $0 \le u_\lambda \le u$ on X, so it follows via Bennett and Sharpley [1, Proposition I.1.7] and Kauhanen, Koskela and Malý [20, Proposition 2.1] that $u_\lambda \in L^{p,q}(X,\mu)$ with $\|u_\lambda\|_{L^{p,q}(X,\mu)} \le \|u\|_{L^{p,q}(X,\mu)}$. The second claim follows immediately since $|u_\lambda(x) - u_\lambda(y)| \le |u(x) - u(y)|$ for every $x, y \in X$. $\square$

4. Newtonian $L^{p,q}$ spaces

We denote by $\widetilde{N}^{1,L^{p,q}}(X,\mu)$ the space of all Borel functions $u \in L^{p,q}(X,\mu)$ that have a p,q-weak upper gradient $g \in L^{p,q}(X,\mu)$. We note that the space $\widetilde{N}^{1,L^{p,q}}(X,\mu)$ is a vector space, since if $\alpha, \beta \in \mathbb{R}$ and $u_1, u_2 \in \widetilde{N}^{1,L^{p,q}}(X,\mu)$ with respective p,q-weak upper gradients $g_1, g_2 \in L^{p,q}(X,\mu)$, then $|\alpha|g_1 + |\beta|g_2$ is a p,q-weak upper gradient of $\alpha u_1 + \beta u_2$.

Definition 4.1. If u is a function in $\widetilde{N}^{1,L^{p,q}}(X,\mu)$, let

$$\|u\|_{\widetilde{N}^{1,L^{p,q}}} := \begin{cases} (\|u\|^q_{L^{p,q}(X,\mu)} + \inf_g \|g\|^q_{L^{p,q}(X,\mu)})^{1/q}, & 1 \le q \le p, \\ (\|u\|^p_{L^{p,q}(X,\mu)} + \inf_g \|g\|^p_{L^{p,q}(X,\mu)})^{1/p}, & p < q \le \infty, \end{cases}$$

where the infimum is taken over all p,q-integrable p,q-weak upper gradients of u.

Similarly, let

$$\|u\|_{\widetilde{N}^{1,L^{(p,q)}}} := \begin{cases} (\|u\|^q_{L^{(p,q)}(X,\mu)} + \inf_g \|g\|^q_{L^{(p,q)}(X,\mu)})^{1/q}, & 1 \le q \le p, \\ (\|u\|^p_{L^{(p,q)}(X,\mu)} + \inf_g \|g\|^p_{L^{(p,q)}(X,\mu)})^{1/p}, & p < q \le \infty, \end{cases}$$

where the infimum is taken over all p,q-integrable p,q-weak upper gradients of u.

If u, v are functions in $\widetilde{N}^{1,L^{p,q}}(X,\mu)$, let $u \sim v$ if $\|u - v\|_{\widetilde{N}^{1,L^{p,q}}} = 0$. It is easy to see that $\sim$ is an equivalence relation that partitions $\widetilde{N}^{1,L^{p,q}}(X,\mu)$ into equivalence classes. We define the space $N^{1,L^{p,q}}(X,\mu)$ as the quotient $\widetilde{N}^{1,L^{p,q}}(X,\mu)/\sim$ and

$$\|u\|_{N^{1,L^{p,q}}} = \|u\|_{\widetilde{N}^{1,L^{p,q}}} \quad \text{and} \quad \|u\|_{N^{1,L^{(p,q)}}} = \|u\|_{\widetilde{N}^{1,L^{(p,q)}}}$$

Remark 4.2. Via Lemma 3.9 and Corollary 2.8, it is easy to see that the infima in Definition 4.1 could as well be taken over all p,q-integrable upper gradients of u. We also notice (see the discussion before Definition 2.1) that $\|\cdot\|_{N^{1,L^{(p,q)}}}$ is a norm whenever $1 < p < \infty$ and $1 \le q \le \infty$, while $\|\cdot\|_{N^{1,L^{p,q}}}$ is a norm when $1 \le q \le p < \infty$. Moreover (see the discussion before Definition 2.1),

$$\|u\|_{N^{1,L^{p,q}}} \le \|u\|_{N^{1,L^{(p,q)}}} \le p' \|u\|_{N^{1,L^{p,q}}}$$

for every $1 < p < \infty$, $1 \le q \le \infty$ and $u \in N^{1,L^{p,q}}(X,\mu)$.

Definition 4.3. Let $u : X \to [-\infty, \infty]$ be a given function. We say that (i) u is absolutely continuous along a rectifiable curve γ if $u \circ \gamma$ is absolutely continuous on $[0, \ell(\gamma)]$.

(ii) u is absolutely continuous on p,q-almost every curve (has $ACC_{p,q}$ property) if for p,q-almost every $\gamma \in \Gamma_{\rm rect}$, $u \circ \gamma$ is absolutely continuous.

PROPOSITION 4.4. *If u is a function in $\widetilde{N}^{1,L^{p,q}}(X,\mu)$, then u is $ACC_{p,q}$.*

Proof. We follow Shanmugalingam [27]. By the definition of $\widetilde{N}^{1,L^{p,q}}(X,\mu)$, u has a p,q-weak upper gradient $g \in L^{p,q}(X,\mu)$. Let Γ_0 be the collection of all curves in $\Gamma_{\rm rect}$ for which

$$|u(\gamma(0)) - u(\gamma(\ell(\gamma)))| \le \int_\gamma g$$

is *not* satisfied. Then by the definition of p,q-weak upper gradients, we have that $\mathrm{Mod}_{p,q}(\Gamma_0) = 0$. Let Γ_1 be the collection of all curves in $\Gamma_{\rm rect}$ that have a subcurve in Γ_0. Then $\mathrm{Mod}_{p,q}(\Gamma_1) \le \mathrm{Mod}_{p,q}(\Gamma_0) = 0$.

Let Γ_2 be the collection of all curves in $\Gamma_{\rm rect}$ such that $\int_\gamma g = \infty$. Then $\mathrm{Mod}_{p,q}(\Gamma_2) = 0$ because $g \in L^{p,q}(X,\mu)$. Hence, $\mathrm{Mod}_{p,q}(\Gamma_1 \cup \Gamma_2) = 0$. If γ is a curve in $\Gamma_{\rm rect} \setminus (\Gamma_1 \cup \Gamma_2)$, then γ has no subcurves in Γ_0, and hence

$$|u(\gamma(\beta)) - u(\gamma(\alpha))| \le \int_\alpha^\beta g(\gamma(t))\,dt, \quad \text{provided } [\alpha,\beta] \subset [0,\ell(\gamma)].$$

This implies the absolute continuity of $u \circ \gamma$ as a consequence of the absolute continuity of the integral. Therefore, u is absolutely continuous on every curve γ in $\Gamma_{\rm rect} \setminus (\Gamma_1 \cup \Gamma_2)$. ☐

LEMMA 4.5. *Suppose $u \in \widetilde{N}^{1,L^{p,q}}(X,\mu)$ is such that $\|u\|_{L^{p,q}(X,\mu)} = 0$. Then the family*

$$\Gamma = \{\gamma \in \Gamma_{\rm rect} : u(x) \neq 0 \textit{ for some } x \in |\gamma|\}$$

has zero p,q-modulus.

Proof. We follow Shanmugalingam [27]. Since $\|u\|_{L^{p,q}(X,\mu)} = 0$, the set $E = \{x \in X : u(x) \neq 0\}$ has measure zero. With the notation introduced earlier, we have

$$\Gamma = \Gamma_E = \Gamma_E^+ \cup (\Gamma_E \setminus \Gamma_E^+).$$

We can disregard the family Γ_E^+, since

$$\mathrm{Mod}_{p,q}(\Gamma_E^+) \le \|\infty \cdot \chi_E\|^p_{L^{p,q}(X,\mu)} = 0,$$

where χ_E is the characteristic function of the set E. The curves γ in $\Gamma_E \setminus \Gamma_E^+$ intersect E only on a set of linear measure zero, and hence with respect to the linear measure almost everywhere on γ the function u is equal to zero. Since γ also intersects E, it follows that u is *not* absolutely continuous on γ. By Proposition 4.4, we have $\mathrm{Mod}_{p,q}(\Gamma_E \setminus \Gamma_E^+) = 0$, yielding $\mathrm{Mod}_{p,q}(\Gamma) = 0$. This finishes the proof. ☐

LEMMA 4.6. *Let F be a closed subset of X. Suppose that $u : X \to [-\infty,\infty]$ is a Borel $ACC_{p,q}$ function that is constant μ-almost everywhere on F. If $g \in L^{p,q}(X,\mu)$ is a p,q-weak upper gradient of u, then $g\chi_{X\setminus F}$ is a p,q-weak upper gradient of u.*

Proof. We can assume without loss of generality that $u = 0$ μ-almost everywhere on F. Let $E = \{x \in F : u(x) \neq 0\}$. Then by assumption $\mu(E) = 0$. Hence, $\mathrm{Mod}_{p,q}(\Gamma_E^+) = 0$ because $\infty \cdot \chi_E \in F(\Gamma_E^+)$.

Let $\Gamma_0 \subset \Gamma_{\mathrm{rect}}$ be the family of curves on which u is not absolutely continuous or on which

$$\big|u\big(\gamma(0)\big) - u\big(\gamma\big(\ell(\gamma)\big)\big)\big| \le \int_\gamma g$$

is *not* satisfied. Then $\mathrm{Mod}_{p,q}(\Gamma_0) = 0$. Let $\Gamma_1 \subset \Gamma_{\mathrm{rect}}$ be the family of curves that have a subcurve in Γ_0. Then $F(\Gamma_0) \subset F(\Gamma_1)$ and thus $\mathrm{Mod}_{p,q}(\Gamma_1) \le \mathrm{Mod}_{p,q}(\Gamma_0) = 0$.

Let $\Gamma_2 \subset \Gamma_{\mathrm{rect}}$ be the family of curves on which $\int_\gamma g = \infty$. Then via Theorem 3.4 we have $\mathrm{Mod}_{p,q}(\Gamma_2) = 0$ because $g \in L^{p,q}(X,\mu)$.

Let $\gamma : [0,\ell(\gamma)] \to X$ be a curve in $\Gamma_{\mathrm{rect}} \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_E^+)$ connecting x and y. We show that

$$\big|u(x) - u(y)\big| \le \int_\gamma g\chi_{X\setminus F}$$

for every such curve γ.

The cases $|\gamma| \subset F \setminus E$ and $|\gamma| \subset (X \setminus F) \cup E$ are trivial. So is the case when both x and y are in $F \setminus E$. Let $K := (u \circ \gamma)^{-1}(\{0\})$. Then K is a compact subset of $[0,\ell(\gamma)]$ because $u \circ \gamma$ is continuous on $[0,\ell(\gamma)]$. Hence, K contains its lower bound c and its upper bound d. Let $x_1 = \gamma(c)$ and $y_1 = \gamma(d)$.

Suppose that both x and y are in $(X \setminus F) \cup E$. Then we see that $[c,d] \subset (0,\ell(\gamma))$ and $\gamma([0,c) \cup (d,\ell(\gamma)]) \subset (X \setminus F) \cup E$.

Moreover, since γ is not in Γ_1 and $u(x_1) = u(y_1)$, we have

$$\begin{aligned}\big|u(x) - u(y)\big| &\le \big|u(x) - u(x_1)\big| + \big|u(y_1) - u(y)\big| \\ &\le \int_{\gamma([0,c])} g + \int_{\gamma([d,\ell(\gamma)])} g \le \int_\gamma g\chi_{X\setminus F}\end{aligned}$$

because the subcurves $\gamma|_{[0,c]}$ and $\gamma|_{[d.\ell(\gamma)]}$ intersect E on a set of Hausdorff 1-measure zero.

Suppose now by symmetry that $x \in (X \setminus F) \cup E$ and $y \in F \setminus E$. This means in terms of our notation that $c > 0$ and $d = \ell(\gamma)$. We notice that $\gamma([0,c)) \subset (X \setminus F) \cup E$ and $u(x_1) = u(y)$ and thus

$$\big|u(x) - u(y)\big| = \big|u(x) - u(x_1)\big| \le \int_{\gamma([0,c])} g \le \int_\gamma g\chi_{X\setminus F}$$

because the subcurve $\gamma|_{[0,c]}$ intersects E on a set of Hausdorff 1-measure zero.

This finishes the proof of the lemma. $\square$

Lemma 4.7. *Assume that* $u \in N^{1,L^{p,q}}(X,\mu)$, *and that* $g,h \in L^{p,q}(X,\mu)$ *are* p,q*-weak upper gradients of* u. *If* $F \subset X$ *is a closed set, then*

$$\rho = g\chi_F + h\chi_{X\setminus F}$$

is a p,q*-weak upper gradient of* u *as well.*

Proof. We follow Hajłasz [13]. Let $\Gamma_1 \subset \Gamma_{\mathrm{rect}}$ be the family of curves on which $\int_\gamma (g+h) = \infty$. Then via Theorem 3.4 it follows that $\mathrm{Mod}_{p,q}(\Gamma_1) = 0$ because $g+h \in L^{p,q}(X,\mu)$.

Let $\Gamma_2 \subset \Gamma_{\mathrm{rect}}$ be the family of curves on which u is *not* absolutely continuous. Then via Proposition 4.4 we see that $\mathrm{Mod}_{p,q}(\Gamma_2) = 0$.

Let $\Gamma_3' \subset \Gamma_{\mathrm{rect}}$ be the family of curves on which

$$\left|u(\gamma(0)) - u\big(\gamma(\ell(\gamma))\big)\right| \le \min\left(\int_\gamma g, \int_\gamma h\right)$$

is *not* satisfied. Let $\Gamma_3 \subset \Gamma_{\mathrm{rect}}$ be the family of curves which contain subcurves belonging to Γ_3'. Since $F(\Gamma_3') \subset F(\Gamma_3)$, we have $\mathrm{Mod}_{p,q}(\Gamma_3) \le \mathrm{Mod}_{p,q}(\Gamma_3') = 0$. Now it remains to show that

$$\left|u(\gamma(0)) - u\big(\gamma(\ell(\gamma))\big)\right| \le \int_\gamma \rho$$

for all $\gamma \in \Gamma_{\mathrm{rect}} \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$. If $|\gamma| \subset F$ or $|\gamma| \subset X \setminus F$, then the inequality is obvious. Thus, we can assume that the image $|\gamma|$ has a nonempty intersection both with F and with $X \setminus F$.

The set $\gamma^{-1}(X \setminus F)$ is open and hence it consists of a countable (or finite) number of open and disjoint intervals. Assume without loss of generality that there are countably many such intervals. Denote these intervals by $((t_i,s_i))_{i=1}^\infty$. Let $\gamma_i = \gamma|_{[t_i,s_i]}$. We have

$$\begin{aligned}\left|u(\gamma(0)) - u\big(\gamma(\ell(\gamma))\big)\right| &\le \left|u(\gamma(0)) - u(\gamma(t_1))\right| + \left|u(\gamma(t_1)) - u(\gamma(s_1))\right| \\ &\quad + \left|u(\gamma(s_1)) - u\big(\gamma(\ell(\gamma))\big)\right| \\ &\le \int_{\gamma\setminus\gamma_1} g + \int_{\gamma_1} h,\end{aligned}$$

where $\gamma \setminus \gamma_1$ denotes the two curves obtained from γ by removing the interior part γ_1, that is the curves $\gamma|_{[0,t_1]}$ and $\gamma|_{[s_1,b]}$. Similarly, we can remove a larger number of subcurves of γ. This yields

$$\left|u(\gamma(0)) - u\big(\gamma(\ell(\gamma))\big)\right| \le \int_{\gamma\setminus\bigcup_{i=1}^n \gamma_i} g + \int_{\bigcup_{i=1}^n \gamma_i} h$$

for each positive integer n. By applying Lebesgue dominated convergence theorem to the curve integral on γ, we obtain

$$\left|u(\gamma(0)) - u\big(\gamma(\ell(\gamma))\big)\right| \le \int_\gamma g\chi_F + \int_\gamma h\chi_{X\setminus F} = \int_\gamma \rho. \qquad \square$$

Next we show that when $1 < p < \infty$ and $1 \le q < \infty$, every function $u \in N^{1,L^{p,q}}(X,\mu)$ has a 'smallest' p,q-weak upper gradient. For the case $p = q$, see Kallunki and Shanmugalingam [19] and Shanmugalingam [28].

THEOREM 4.8. *Suppose that $1 < p < \infty$ and $1 \le q < \infty$. For every $u \in N^{1,L^{p,q}}(X,\mu)$, there exists the least p,q-weak upper gradient $g_u \in L^{p,q}(X,\mu)$ of u. It is smallest in the sense that if $g \in L^{p,q}(X,\mu)$ is another p,q-weak upper gradient of u, then $g \ge g_u$ μ-almost everywhere.*

Proof. We follow Hajłasz [13]. Let $m = \inf_g \|g\|_{L^{p,q}(X,\mu)}$, where the infimum is taken over the set of all p,q-weak upper gradients of u. It suffices to show that there exists a p,q-weak upper gradient g_u of u such that $\|g_u\|_{L^{p,q}(X,\mu)} = m$. Indeed, if we suppose that $g \in L^{p,q}(X,\mu)$ is another p,q-weak upper gradient of u such that the set $\{g < g_u\}$ has positive measure, then by the inner regularity of the measure μ there exists a closed set $F \subset \{g < g_u\}$ such that $\mu(F) > 0$. Via Lemma 4.7 it follows that the function $\rho := g\chi_F + g_u\chi_{X\setminus F}$ is a p,q-weak upper gradient. Via Kauhanen, Koskela and Malý [20, Proposition 2.1] that would give $\|\rho\|_{L^{p,q}(X,\mu)} < \|g_u\|_{L^{p,q}(X,\mu)} = m$, in contradiction with the minimality of $\|g_u\|_{L^{p,q}(X,\mu)}$.

Thus, it remains to prove the existence of a p,q-weak upper gradient g_u such that $\|g_u\|_{L^{p,q}(X,\mu)} = m$. Let $(g_i)_{i=1}^\infty$ be a sequence of p,q-weak upper gradients of u such that $\|g_i\|_{L^{p,q}(X,\mu)} < m + 2^{-i}$. We will show that it is possible to modify the sequence (g_i) in such a way that we will obtain a new sequence of p,q-weak upper gradients (ρ_i) of u satisfying

$$\|\rho_i\|_{L^{p,q}(X,\mu)} < m + 2^{1-i}, \quad \rho_1 \ge \rho_2 \ge \rho_3 \ge \cdots \ \mu\text{-almost everywhere.}$$

The sequence $(\rho_i)_{i=1}^\infty$ will be defined by induction. We set $\rho_1 = g_1$. Suppose the p,q-weak upper gradients $\rho_1, \rho_2, \ldots, \rho_i$ have already been chosen. We will now define ρ_{i+1}. Since $\rho_i \in L^{p,q}(X,\mu)$, the measure μ is inner regular and the (p,q)-norm has the absolute continuity property whenever $1 < p < \infty$ and $1 \le q < \infty$ (see the discussion after Definition 2.1), there exists a closed set $F \subset \{g_{i+1} < \rho_i\}$ such that

$$\left\|\rho_i \chi_{\{g_{i+1}<\rho_i\}\setminus F}\right\|_{L^{p,q}(X,\mu)} < 2^{-i-1}.$$

Now, we set $\rho_{i+1} = g_{i+1}\chi_F + \rho_i\chi_{X\setminus F}$. Then

$$\rho_{i+1} \le \rho_i \quad \text{and} \quad \rho_{i+1} \le g_{i+1}\chi_{F\cup\{g_{i+1}\ge\rho_i\}} + \rho_i\chi_{\{g_{i+1}<\rho_i\}\setminus F}.$$

We show that $m \le \|\rho_{i+1}\|_{L^{p,q}(X,\mu)} < m + 2^{-i}$. Suppose first that $1 \le q \le p$. Since $\|\cdot\|_{L^{p,q}(X,\mu)}$ is a norm, we see that

$$\begin{aligned}\|\rho_{i+1}\|_{L^{p,q}(X,\mu)} &\le \|g_{i+1}\chi_{F\cup\{g_{i+1}\ge\rho_i\}}\|_{L^{p,q}(X,\mu)} + \|\rho_i\chi_{\{g_{i+1}<\rho_i\}\setminus F}\|_{L^{p,q}(X,\mu)} \\ &< m + 2^{-i-1} + 2^{-i-1} = m + 2^{-i}.\end{aligned}$$

Suppose now that $p<q<\infty$. Then we have via Proposition 2.6

$$\begin{aligned}\|\rho_{i+1}\|^p_{L^{p,q}(X,\mu)} &\le \|g_{i+1}\chi_{F\cup\{g_{i+1}\ge\rho_i\}}\|^p_{L^{p,q}(X,\mu)} + \|\rho_i\chi_{\{g_{i+1}<\rho_i\}\setminus F}\|^p_{L^{p,q}(X,\mu)}\\ &< \bigl(m+2^{-i-1}\bigr)^p + 2^{-p(i+1)} < \bigl(m+2^{-i}\bigr)^p.\end{aligned}$$

The sequence of p,q-weak upper gradients $(\rho_i)_{i=1}^{\infty}$ converges pointwise to a function ρ. The absolute continuity of the (p,q)-norm (see Bennett and Sharpley [1, Proposition I.3.6] and the discussion after Definition 2.1) yields

$$\lim_{i\to\infty}\|\rho_i-\rho\|_{L^{p,q}(X,\mu)}=0.$$

Obviously $\|\rho\|_{L^{p,q}(X,\mu)}=m$. The proof will be finished as soon as we show that ρ is a p,q-weak upper gradient for u.

By taking a subsequence if necessary, we can assume that $\|\rho_i-\rho\|_{L^{p,q}(X,\mu)}\le 2^{-2i}$ for every $i\ge 1$.

Let $\Gamma_1\subset\Gamma_{\text{rect}}$ be the family of curves on which $\int_\gamma(\rho+\rho_i)=\infty$ for some $i\ge 1$. Then via Theorem 3.4 and the subadditivity of $\operatorname{Mod}_{p,q}(\cdot)^{1/p}$ we see that $\operatorname{Mod}_{p,q}(\Gamma_1)=0$ since $\rho+\rho_i\in L^{p,q}(X,\mu)$ for every $i\ge 1$.

For any integer $i\ge 1$ let $\Gamma_{2,i}\subset\Gamma_{\text{rect}}$ be the family of curves for which

$$\bigl|u\bigl(\gamma(0)\bigr)-u\bigl(\gamma\bigl(\ell(\gamma)\bigr)\bigr)\bigr|\le\int_\gamma\rho_i$$

is *not* satisfied. Then $\operatorname{Mod}_{p,q}(\Gamma_{2,i})=0$ because ρ_i is a p,q-weak upper gradient for u. Let $\Gamma_2=\bigcup_{i=1}^{\infty}\Gamma_{2,i}$.

Let $\Gamma_3\subset\Gamma_{\text{rect}}$ be the family of curves for which $\limsup_{i\to\infty}\int_\gamma|\rho_i-\rho|>0$. Then it follows via Theorem 3.6 that $\operatorname{Mod}_{p,q}(\Gamma_3)=0$.

Let γ be a curve in $\Gamma_{\text{rect}}\setminus(\Gamma_1\cup\Gamma_2\cup\Gamma_3)$. On any such curve we have (since γ is not in $\Gamma_{2,i}$)

$$\bigl|u\bigl(\gamma(0)\bigr)-u\bigl(\gamma\bigl(\ell(\gamma)\bigr)\bigr)\bigr|\le\int_\gamma\rho_i\quad\text{for every } i\ge 1.$$

By letting $i\to\infty$, we obtain (since γ is not in $\Gamma_1\cup\Gamma_3$)

$$\bigl|u\bigl(\gamma(0)\bigr)-u\bigl(\gamma\bigl(\ell(\gamma)\bigr)\bigr)\bigr|\le\lim_{i\to\infty}\int_\gamma\rho_i=\int_\gamma\rho<\infty.$$

This finishes the proof of the theorem. □

5. Sobolev p,q-capacity

In this section, we establish a general theory of the Sobolev–Lorentz p,q-capacity in metric measure spaces. If (X,d,μ) is a metric measure space, then the Sobolev p,q-capacity of a set $E\subset X$ is

$$\operatorname{Cap}_{p,q}(E)=\inf\bigl\{\|u\|^p_{N^{1,L^{p,q}}}:u\in\mathcal{A}(E)\bigr\},$$

where

$$\mathcal{A}(E)=\bigl\{u\in N^{1,L^{p,q}}(X,\mu):u\ge 1\text{ on }E\bigr\}.$$

We call $\mathcal{A}(E)$ the set of *admissible functions* for E. If $\mathcal{A}(E)=\emptyset$, then $\operatorname{Cap}_{p,q}(E)=\infty$.

Remark 5.1. It is easy to see that we can consider only admissible functions u for which $0\le u\le 1$. Indeed, for $u\in\mathcal{A}(E)$, let $v:=\min(u_+,1)$, where $u_+=\max(u,0)$. We notice that $|v(x)-v(y)|\le|u(x)-u(y)|$ for every x,y in X, which implies that every p,q-weak upper gradient for u is also a p,q-weak upper gradient for v. This implies that $v\in\mathcal{A}(E)$ and $\|v\|_{N^{1,L^{p,q}}}\le\|u\|_{N^{1,L^{p,q}}}$.

5.1. Basic properties of the Sobolev p,q-capacity. A capacity is a monotone, subadditive set function. The following theorem expresses, among other things, that this is true for the Sobolev p,q-capacity.

Theorem 5.2. *Suppose that $1<p<\infty$ and $1\le q\le\infty$. Suppose also that (X,d,μ) is a complete metric measure space. The set function $E\mapsto\operatorname{Cap}_{p,q}(E)$, $E\subset X$, enjoys the following properties*:

(i) *If $E_1\subset E_2$, then $\operatorname{Cap}_{p,q}(E_1)\le\operatorname{Cap}_{p,q}(E_2)$.*

(ii) *Suppose that μ is nonatomic. Suppose that $1<q\le p$. If $E_1\subset E_2\subset\cdots\subset E=\bigcup_{i=1}^\infty E_i\subset X$, then*

$$\operatorname{Cap}_{p,q}(E)=\lim_{i\to\infty}\operatorname{Cap}_{p,q}(E_i).$$

(iii) *Suppose that $p<q\le\infty$. If $E=\bigcup_{i=1}^\infty E_i\subset X$, then*

$$\operatorname{Cap}_{p,q}(E)\le\sum_{i=1}^\infty\operatorname{Cap}_{p,q}(E_i).$$

(iv) *Suppose that $1\le q\le p$. If $E=\bigcup_{i=1}^\infty E_i\subset X$, then*

$$\operatorname{Cap}_{p,q}(E)^{q/p}\le\sum_{i=1}^\infty\operatorname{Cap}_{p,q}(E_i)^{q/p}.$$

Proof. Property (i) is an immediate consequence of the definition.

(ii) Monotonicity yields

$$L:=\lim_{i\to\infty}\operatorname{Cap}_{p,q}(E_i)\le\operatorname{Cap}_{p,q}(E).$$

To prove the opposite inequality, we may assume without loss of generality that $L<\infty$. The reflexivity of $L^{p,q}(X,\mu)$ (guaranteed by the nonatomicity of μ whenever $1<q\le p<\infty$) will be used here in order to prove the opposite inequality.

Let $\varepsilon>0$ be fixed. For every $i=1,2,\dots$ we choose $u_i\in\mathcal{A}(E_i)$, $0\le u_i\le 1$ and a corresponding upper gradient g_i such that

$$\|u_i\|^q_{N^{1,L^{p,q}}}<\operatorname{Cap}_{p,q}(E_i)^{q/p}+\varepsilon\le L^{q/p}+\varepsilon. \tag{11}$$

We notice that u_i is a bounded sequence in $N^{1,L^{p,q}}(X,\mu)$. Hence there exists a subsequence, which we denote again by u_i and functions $u,g\in L^{p,q}(X,\mu)$

such that $u_i \to u$ weakly in $L^{p,q}(X,\mu)$ and $g_i \to g$ weakly in $L^{p,q}(X,\mu)$. It is easy to see that

$$u \geq 0 \quad \mu\text{-almost everywhere} \quad \text{and} \quad g \geq 0 \quad \mu\text{-almost everywhere.}$$

Indeed, since u_i converges weakly to u in $L^{p,q}(X,\mu)$ which is the dual of $L^{p',q'}(X,\mu)$ (see Hunt [18, p. 262]), we have

$$\lim_{i\to\infty} \int_X u_i(x)\varphi(x)\,d\mu(x) = \int_X u(x)\varphi(x)\,d\mu(x)$$

for all $\varphi \in L^{p',q'}(X,\mu)$. For nonnegative functions $\varphi \in L^{p',q'}(X,\mu)$, this yields

$$0 \leq \lim_{i\to\infty} \int_X u_i(x)\varphi(x)\,d\mu(x) = \int_X u(x)\varphi(x)\,d\mu(x),$$

which easily implies $u \geq 0$ μ-almost everywhere on X. Similarly, we have $g \geq 0$ μ-almost everywhere on X.

From the weak-$*$ lower semicontinuity of the p,q-norm (see Bennett and Sharpley [1, Proposition II.4.2, Definition IV.4.1 and Theorem IV.4.3] and Hunt [18, p. 262]), it follows that

$$\begin{aligned} &\|u\|_{L^{p,q}(X,\mu)} \leq \liminf_{i\to\infty} \|u_i\|_{L^{p,q}(X,\mu)} \quad \text{and} \\ &\|g\|_{L^{p,q}(X,\mu)} \leq \liminf_{i\to\infty} \|g_i\|_{L^{p,q}(X,\mu)}. \end{aligned} \tag{12}$$

Using Mazur's lemma simultaneously for u_i and g_i, we obtain sequences v_i with correspondent upper gradients $\widetilde{g}_i$ such that $v_i \in \mathcal{A}(E_i)$, $v_i \to u$ in $L^{p,q}(X,\mu)$ and μ-almost everywhere and $\widetilde{g}_i \to g$ in $L^{p,q}(X,\mu)$ and μ-almost everywhere. These sequences can be found in the following way. Let i_0 be fixed. Since every subsequence of (u_i, g_i) converges to (u,g) weakly in the reflexive space $L^{p,q}(X,\mu) \times L^{p,q}(X,\mu)$, we may use the Mazur lemma (see Yosida [30, p. 120]) for the subsequence $(u_i, g_i), i \geq i_0$.

We obtain finite convex combinations v_{i_0} and $\widetilde{g}_{i_0}$ of the functions u_i and g_i, $i \geq i_0$ as close as we want in $L^{p,q}(X,\mu)$ to u and g, respectively. For every $i = i_0, i_0+1, \ldots$, we see that $u_i = 1$ in $E_i \supset E_{i_0}$. The intersection of finitely many supersets of E_{i_0} contains E_{i_0}. Therefore, v_{i_0} equals 1 on E_{i_0}. It is easy to see that $\widetilde{g}_{i_0}$ is an upper gradient for v_{i_0}. Passing to subsequences if necessary, we may assume that v_i converges to u pointwise μ-almost everywhere, that $\widetilde{g}_i$ converges to g pointwise μ-almost everywhere and that for every $i = 1, 2, \ldots$ we have

$$\|v_{i+1} - v_i\|_{L^{p,q}(X,\mu)} + \|\widetilde{g}_{i+1} - \widetilde{g}_i\|_{L^{p,q}(X,\mu)} \leq 2^{-i}. \tag{13}$$

Since v_i converges to u in $L^{p,q}(X,\mu)$ and pointwise μ-almost everywhere on X while $\widetilde{g}_i$ converges to g in $L^{p,q}(X,\mu)$ and pointwise μ-almost everywhere on X it follows via Corollary 2.8 that

$$\begin{aligned} &\lim_{i\to\infty} \|v_i\|_{L^{p,q}(X,\mu)} = \|u\|_{L^{p,q}(X,\mu)} \quad \text{and} \\ &\lim_{i\to\infty} \|\widetilde{g}_i\|_{L^{p,q}(X,\mu)} = \|g\|_{L^{p,q}(X,\mu)}. \end{aligned} \tag{14}$$

This, (11) and (12) yield

$$\|u\|^q_{L^{p,q}(X,\mu)} + \|g\|^q_{L^{p,q}(X,\mu)} = \lim_{i\to\infty} \|v_i\|^q_{N^{1,L^{p,q}}} \le L^{q/p} + \varepsilon. \tag{15}$$

For $j = 1, 2, \dots$ we set

$$w_j = \sup_{i\ge j} v_i \quad \text{and} \quad \widehat{g}_j = \sup_{i\ge j} \widetilde{g}_i.$$

It is easy to see that $w_j = 1$ on E. We claim that $\widehat{g}_j$ is a p,q-weak upper gradient for w_j. Indeed, for every $k > j$, let

$$w_{j,k} = \sup_{k\ge i\ge j} v_i.$$

Via Lemma 3.15 and finite induction, it follows easily that $\widehat{g}_j$ is a p,q-weak upper gradient for every $w_{j,k}$ whenever $k > j$. It is easy to see that $w_j = \lim_{k\to\infty} w_{j,k}$ pointwise in X. This and Lemma 3.10 imply that $\widehat{g}_j$ is indeed a p,q-weak upper gradient for w_j.

Moreover,

$$w_j \le v_j + \sum_{i=j}^{\infty} |v_{i+1} - v_i| \quad \text{and} \quad \widehat{g}_j \le \widetilde{g}_j + \sum_{i=j}^{k-1} |\widetilde{g}_{i+1} - \widetilde{g}_i|. \tag{16}$$

Thus,

$$\begin{aligned}\|w_j\|_{L^{p,q}(X,\mu)} &\le \|v_j\|_{L^{p,q}(X,\mu)} + \sum_{i=j}^{\infty} \|v_{i+1} - v_i\|_{L^{p,q}(X,\mu)} \\ &\le \|v_j\|_{L^{p,q}(X,\mu)} + 2^{-j+1}\end{aligned}$$

and

$$\begin{aligned}\|\widehat{g}_j\|_{L^{p,q}(X,\mu)} &\le \|\widetilde{g}_j\|_{L^{p,q}(X,\mu)} + \sum_{i=j}^{\infty} \|\widetilde{g}_{i+1} - \widetilde{g}_i\|_{L^{p,q}(X,\mu)} \\ &\le \|\widetilde{g}_j\|_{L^{p,q}(X,\mu)} + 2^{-j+1},\end{aligned}$$

which implies that $w_j, \widehat{g}_j \in L^{p,q}(X,\mu)$. Thus, $w_j \in \mathcal{A}(E)$ with p,q-weak upper gradient $\widehat{g}_j$. We notice that $0 \le g = \inf_{j\ge1} \widehat{g}_j$ μ-almost everywhere on X and $0 \le u = \inf_{j\ge1} w_j$ μ-almost everywhere on X. Since w_1 and $\widehat{g}_1$ are in $L^{p,q}(X,\mu)$, the absolute continuity of the p,q-norm (see Bennett and Sharpley [1, Proposition I.3.6] and the discussion after Definition 2.1) yields

$$\lim_{j\to\infty} \|w_j - u\|_{L^{p,q}(X,\mu)} = 0 \quad \text{and} \quad \lim_{j\to\infty} \|\widehat{g}_j - g\|_{L^{p,q}(X,\mu)} = 0. \tag{17}$$

By using (15), (17), and Corollary 2.8, we see that

$$\mathrm{Cap}_{p,q}(E)^{q/p} \le \lim_{j\to\infty} \|w_j\|^q_{N^{1,L^{p,q}}} = \|u\|^q_{L^{p,q}(X,\mu)} + \|g\|^q_{L^{p,q}(X,\mu)} \le L^{q/p} + \varepsilon.$$

By letting $\varepsilon \to 0$, we get the converse inequality so (ii) is proved.

(iii) We can assume without loss of generality that

$$\sum_{i=1}^{\infty} \mathrm{Cap}_{p,q}(E_i)^{q/p} < \infty.$$

For $i = 1, 2, \ldots$ let $u_i \in \mathcal{A}(E_i)$ with upper gradient g_i such that

$$0 \le u_i \le 1 \quad \text{and} \quad \|u_i\|^q_{N^{1,L^{p,q}}} < \mathrm{Cap}_{p,q}(E_i)^{q/p} + \varepsilon 2^{-i}.$$

Let $u := (\sum_{i=1}^{\infty} u_i^q)^{1/q}$ and $g := (\sum_{i=1}^{\infty} g_i^q)^{1/q}$. We notice that $u \ge 1$ on E. By repeating the argument from the proof of Theorem 3.2 (iii), we see that $u, g \in L^{p,q}(X, \mu)$ and

$$\begin{aligned}\|u\|^q_{L^{p,q}(X,\mu)} + \|g\|^q_{L^{p,q}(X,\mu)} &\le \sum_{i=1}^{\infty} \big(\|u_i\|^q_{L^{p,q}(X,\mu)} + \|g_i\|^q_{L^{p,q}(X,\mu)}\big) \\ &\le 2\varepsilon + \sum_{i=1}^{\infty} \mathrm{Cap}_{p,q}(E_i)^{q/p}.\end{aligned}$$

We are done with the case $1 \le q \le p$ as soon as we show that $u \in \mathcal{A}(E)$ and that g is a p,q-weak upper gradient for u. It follows easily via Corollary 3.14 and finite induction that g is a p,q-weak upper gradient for $\widetilde{u}_n := (\sum_{1 \le i \le n} u_i^q)^{1/q}$ for every $n \ge 1$. Since $u(x) = \lim_{i\to\infty} \widetilde{u}_i(x) < \infty$ on $X \setminus F$, where $F = \{x \in X : u(x) = \infty\}$ it follows from Lemma 3.10 combined with the fact that $u \in L^{p,q}(X, \mu)$ that g is in fact a p,q-weak upper gradient for u. This finishes the proof for the case $1 \le q \le p$.

(iv) We can assume without loss of generality that

$$\sum_{i=1}^{\infty} \mathrm{Cap}_{p,q}(E_i) < \infty.$$

For $i = 1, 2, \ldots$ let $u_i \in \mathcal{A}(E_i)$ with upper gradients g_i such that

$$0 \le u_i \le 1 \quad \text{and} \quad \|u_i\|^p_{N^{1,L^{p,q}}} < \mathrm{Cap}_{p,q}(E_i) + \varepsilon 2^{-i}.$$

Let $u := \sup_{i\ge 1} u_i$ and $g := \sup_{i \ge 1} g_i$. We notice that $u = 1$ on E. Moreover, via Proposition 2.6 it follows that $u, g \in L^{p,q}(X, \mu)$ with

$$\begin{aligned}\|u\|^p_{L^{p,q}(X,\mu)} + \|g\|^p_{L^{p,q}(X,\mu)} &\le \sum_{i=1}^{\infty} \big(\|u_i\|^p_{L^{p,q}(X,\mu)} + \|g_i\|^p_{L^{p,q}(X,\mu)}\big) \\ &\le 2\varepsilon + \sum_{i=1}^{\infty} \mathrm{Cap}_{p,q}(E_i).\end{aligned}$$

We are done with the case $p < q \le \infty$ as soon as we show that $u \in \mathcal{A}(E)$ and that g is a p,q-weak upper gradient for u. Via Lemma 3.15 and finite induction, it follows that g is a p,q-weak upper gradient for $\widetilde{u}_n := \max_{1\le i \le n} u_i$ for every $n \ge 1$. Since $u(x) = \lim_{i\to\infty} \widetilde{u}_i(x)$ pointwise on X, it follows via

Lemma 3.10 that g is in fact a p,q-weak upper gradient for u. This finishes the proof for the case $p<q\leq\infty$. $\square$

REMARK 5.3. We make a few remarks.

(i) Suppose μ is nonatomic and $1<q<\infty$. By mimicking the proof of Theorem 5.2 (ii) and working with the (p,q)-norm and the (p,q)-capacity, we can also show that

$$\lim_{i\to\infty}\operatorname{Cap}_{(p,q)}(E_i)=\operatorname{Cap}_{(p,q)}(E)$$

whenever $E_1\subset E_2\subset\cdots\subset E=\bigcup_{i=1}^{\infty}E_i\subset X$.

(ii) Moreover, if $\operatorname{Cap}_{p,q}$ is an outer capacity then it follows immediately that

$$\lim_{i\to\infty}\operatorname{Cap}_{p,q}(K_i)=\operatorname{Cap}_{p,q}(K)$$

whenever $(K_i)_{i=1}^{\infty}$ is a decreasing sequence of compact sets whose intersection set is K. We say that $\operatorname{Cap}_{p,q}$ is an outer capacity if for every $E\subset X$ we have

$$\operatorname{Cap}_{p,q}(E)=\inf\big\{\operatorname{Cap}_{p,q}(U):E\subset U\subset X,U\text{ open}\big\}.$$

(iii) Any outer capacity satisfying properties (i) and (ii) of Theorem 5.2 is called a Choquet capacity. (See Appendix II in Doob [9].)

We recall that if $A\subset X$, then Γ_A is the family of curves in Γ_{rect} that intersect A and Γ_A^+ is the family of all curves in Γ_{rect} such that the Hausdorff one-dimensional measure $\mathcal{H}_1(|\gamma|\cap A)$ is positive. The following lemma will be useful later in this paper.

LEMMA 5.4. *If* $F\subset X$ *is such that* $\operatorname{Cap}_{p,q}(F)=0$, *then* $\operatorname{Mod}_{p,q}(\Gamma_F)=0$.

Proof. We follow Shanmugalingam [27]. We can assume without loss of generality that $q\neq p$. Since $\operatorname{Cap}_{p,q}(F)=0$, for each positive integer i there exists a function $v_i\in\mathcal{A}(F)$ such that $0\leq v_i\leq 1$ and such that $\|v_i\|_{N^{1,L^{(p,q)}}}\leq 2^{-i}$. Let $u_n:=\sum_{i=1}^{n}v_i$. Then $u_n\in N^{1,L^{(p,q)}}(X,\mu)$ for each n, $u_n(x)$ is increasing for each $x\in X$, and for every $m>n$ we have

$$\|u_n-u_m\|_{N^{1,L^{(p,q)}}}\leq\sum_{i=m+1}^{n}\|v_i\|_{N^{1,L^{(p,q)}}}\leq 2^{-m}\to 0,\quad\text{as } m\to\infty.$$

Therefore, the sequence $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $N^{1,L^{(p,q)}}(X,\mu)$.

Since $\{u_n\}_{n=1}^{\infty}$ Cauchy in $N^{1,L^{(p,q)}}(X,\mu)$, it follows that it is Cauchy in $L^{p,q}(X,\mu)$. Hence by passing to a subsequence if necessary, there is a function $\widetilde{u}$ in $L^{p,q}(X,\mu)$ to which the subsequence converges both pointwise μ-almost everywhere and in the $L^{(p,q)}$ norm. By choosing a further subsequence, again denoted by $\{u_i\}_{i=1}^{\infty}$ for simplicity, we can assume that

$$\|u_i-\widetilde{u}\|_{L^{(p,q)}(X,\mu)}+\|g_{i,i+1}\|_{L^{(p,q)}(X,\mu)}\leq 2^{-2i},$$

where $g_{i,j}$ is an upper gradient of $u_i - u_j$ for $i<j$. If g_1 is an upper gradient of u_1, then $u_2 = u_1 + (u_2 - u_1)$ has an upper gradient $g_2 = g_1 + g_{12}$. In general,

$$u_i = u_1 + \sum_{k=1}^{i-1}(u_{k+1} - u_k)$$

has an upper gradient

$$g_i = g_1 + \sum_{k=1}^{i-1} g_{k,k+1}$$

for every $i \geq 2$. For $j<i$ we have

$$\begin{aligned}\|g_i - g_j\|_{L^{(p,q)}(X,\mu)} &\leq \sum_{k=j}^{i-1} \|g_{k,k+1}\|_{L^{(p,q)}(X,\mu)} \leq \sum_{k=j}^{i-1} 2^{-2k} \\ &\leq 2^{1-2j} \to 0 \quad \text{as } j \to \infty.\end{aligned}$$

Therefore, $\{g_i\}_{i=1}^{\infty}$ is also a Cauchy sequence in $L^{(p,q)}(X,\mu)$, and hence converges in the $L^{(p,q)}$ norm to a nonnegative Borel function g. Moreover, we have

$$\|g_j - g\|_{L^{(p,q)}(X,\mu)} \leq 2^{1-2j}$$

for every $j \geq 1$.

We define u by $u(x) = \lim_{i\to\infty} u_i(x)$ wherever the definition makes sense. Since $u_i \to \widetilde{u}$ μ-almost everywhere, it follows that $u = \widetilde{u}$ μ-almost everywhere and thus $u \in L^{p,q}(X,\mu)$. Let

$$E = \Big\{x \in X : \lim_{i\to\infty} u_i(x) = \infty\Big\}.$$

The function u is well defined outside of E. In order for the function u to be in the space $N^{1,L^{p,q}}(X,\mu)$, the function u has to be defined on almost all paths by Proposition 4.4. To this end, it is shown that the p,q-modulus of the family Γ_E is zero. Let Γ_1 be the collection of all paths from Γ_{rect} such that $\int_\gamma g = \infty$. Then we have via Theorem 3.4 that $\mathrm{Mod}_{p,q}(\Gamma_1) = 0$ since $g \in L^{p,q}(X,\mu)$.

Let Γ_2 be the family of all curves from Γ_{rect} such that $\limsup_{j\to\infty} \int_\gamma |g_j - g| > 0$. Since $\|g_j - g\|_{L^{p,q}}(X,\mu) \leq 2^{1-2j}$ for all $j \geq 1$, it follows via Theorem 3.6 that $\mathrm{Mod}_{p,q}(\Gamma_2) = 0$.

Since $u \in L^{p,q}(X,\mu)$ and $E = \{x \in X : u(x) = \infty\}$, it follows that $\mu(E) = 0$ and thus $\mathrm{Mod}_{\Gamma_E^+} = 0$. Therefore, $\mathrm{Mod}_{p,q}(\Gamma_1 \cup \Gamma_2 \cup \Gamma_E^+) = 0$. For any path, γ in the family $\Gamma_{\text{rect}} \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_E^+)$, by the fact that γ is not in Γ_E^+, there exists a point y in $|\gamma| \setminus E$. For any point x in $|\gamma|$, since g_i is an upper gradient of u_i, it follows that

$$u_i(x) - u_i(y) \leq \big|u_i(x) - u_i(y)\big| \leq \int_\gamma g_i.$$

Therefore,

$$u_i(x) \le u_i(y) + \int_\gamma g_i.$$

Taking limits on both sides and using the fact that γ is not in $\Gamma_1 \cup \Gamma_2$, it follows that

$$\lim_{i\to\infty} u_i(x) \le \lim_{i\to\infty} u_i(y) + \int_\gamma g = u(y) + \int_\gamma g < \infty,$$

and therefore x is not in E. Thus $\Gamma_E \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_E^+$ and $\mathrm{Mod}_{p,q}(\Gamma_E) = 0$. Therefore, g is a p,q-weak upper gradient of u, and hence $u \in N^{1,L^{p,q}}(X,\mu)$. For each x not in E, we can write $u(x) = \lim_{i\to\infty} u_i(x) < \infty$. If $F \setminus E$ is nonempty, then

$$u|_{F\setminus E} \ge u_n|_{F\setminus E} = \sum_{i=1}^n v_i|_{F\setminus E} = n$$

for arbitrarily large n, yielding that $u|_{F\setminus E} = \infty$. But this impossible, since x is not in the set E. Therefore $F \subset E$, and hence $\Gamma_F \subset \Gamma_E$. This finishes the proof of the lemma. $\square$

Next, we prove that $(N^{1,L^{p,q}}(X,\mu), \|\cdot\|_{N^{1,L^{(p,q)}}})$ is a Banach space.

THEOREM 5.5. *Suppose* $1 < p < \infty$ *and* $1 \le q \le \infty$. *Then* $(N^{1,L^{p,q}}(X,\mu), \|\cdot\|_{N^{1,L^{(p,q)}}})$ *is a Banach space.*

Proof. We follow Shanmugalingam [27]. We can assume without loss of generality that $q \ne p$. Let $\{u_i\}_{i=1}^\infty$ be a Cauchy sequence in $N^{1,L^{p,q}}(X,\mu)$. To show that this sequence is convergent in $N^{1,L^{p,q}}(X,\mu)$, it suffices to show that some subsequence is convergent in $N^{1,L^{p,q}}(X,\mu)$. Passing to a further subsequence if necessary, it can be assumed that

$$\|u_{i+1} - u_i\|_{L^{(p,q)}(X,\mu)} + \|g_{i,i+1}\|_{L^{(p,q)}(X,\mu)} \le 2^{-2i},$$

where $g_{i,j}$ is an upper gradient of $u_i - u_j$ for $i < j$. Let

$$E_j = \big\{x \in X : |u_{j+1}(x) - u_j(x)| \ge 2^{-j}\big\}.$$

Then $2^j|u_{j+1} - u_j| \in \mathcal{A}(E_j)$ and hence

$$\mathrm{Cap}_{p,q}(E_j)^{1/p} \le 2^j \|u_{j+1} - u_j\|_{N^{1,L^{p,q}}} \le 2^{-j}.$$

Let $F_j = \bigcup_{k=j}^\infty E_k$. Then

$$\mathrm{Cap}_{p,q}(E_j)^{1/p} \le \sum_{k=j}^\infty \mathrm{Cap}_{p,q}(E_k)^{1/p} \le 2^{1-j}.$$

Let $F = \bigcap_{j=1}^\infty F_j$. We notice that $\mathrm{Cap}_{p,q}(F) = 0$. If x is a point in $X \setminus F$, there exists $j \ge 1$ such that x is not in $F_j = \bigcup_{k=j}^\infty E_k$. Hence for all $k \ge j$, x is not in E_k. Thus, $|u_{k+1}(x) - u_k(x)| \leq 2^{-k}$ for all $k \geq j$. Therefore, whenever $l \geq k \geq j$ we have that

$$\big|u_k(x) - u_l(x)\big| \leq 2^{1-k}.$$

Thus, the sequence $\{u_k(x)\}_{k=1}^{\infty}$ is Cauchy for every $x \in X \setminus F$. For every $x \in X \setminus F$, let $u(x) = \lim_{i\to\infty} u_i(x)$. For $k < m$,

$$u_m = u_k + \sum_{n=k}^{m-1} (u_{n+1} - u_n).$$

Therefore for each x in $X \setminus F$,

$$u(x) = u_k(x) + \sum_{n=k}^{\infty} \big(u_{n+1}(x) - u_n(x)\big). \tag{18}$$

Noting by Lemma 5.4 that $\operatorname{Mod}_{p,q}(\Gamma_F) = 0$ and that for each path γ in $\Gamma_{\text{rect}} \setminus \Gamma_F$ equation (18) holds pointwise on $|\gamma|$, we conclude that $\sum_{n=k}^{\infty} g_{n,n+1}$ is a p,q-weak upper gradient of $u - u_k$. Therefore,

$$\begin{aligned}
\|u - u_k\|_{N^{1,L^{(p,q)}}} &\leq \|u - u_k\|_{L^{(p,q)}(X,\mu)} + \sum_{n=k}^{\infty} \|g_{n,n+1}\|_{L^{(p,q)}(X,\mu)} \\
&\leq \|u - u_k\|_{L^{(p,q)}(X,\mu)} + \sum_{n=k}^{\infty} 2^{-2n} \\
&\leq \|u - u_k\|_{L^{(p,q)}(X,\mu)} + 2^{1-2k} \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Therefore, the subsequence converges in the norm of $N^{1,L^{p,q}}(X,\mu)$ to u. This completes the proof of the theorem. □

6. Density of Lipschitz functions in $N^{1,L^{p,q}}(X,\mu)$

6.1. Poincaré inequality. Now we define the weak $(1, L^{p,q})$-*Poincaré inequality.* Podbrdsky in [26] introduced a stronger Poincaré inequality in the case of Banach-valued Newtonian Lorentz spaces.

Definition 6.1. The space (X, d, μ) is said to support a *weak* $(1, L^{p,q})$-*Poincaré inequality* if there exist constants $C > 0$ and $\sigma \geq 1$ such that for all balls B with radius r, all μ-measurable functions u on X and all upper gradients g of u we have

$$\frac{1}{\mu(B)} \int_B |u - u_B|\, d\mu \leq Cr \frac{\|g\chi_{\sigma B}\|_{L^{p,q}(X,\mu)}}{\mu(\sigma B)^{1/p}}. \tag{19}$$

Here

$$u_B = \frac{1}{\mu(B)} \int_B u(x)\, d\mu(x)$$

whenever u is a locally μ-integrable function on X.

In the above definition, we can equivalently assume via Lemma 3.9 and Corollary 2.8 that g is a p,q-weak upper gradient of u. When $p=q$, we have the weak $(1,p)$-Poincaré inequality. For more about the Poincaré inequality in the case $p=q$, see Hajłasz and Koskela [14] and [17].

A measure μ is said to be *doubling* if there exists a constant $C \geq 1$ such that

$$\mu(2B) \leq C\mu(B)$$

for every ball $B=B(x,r)$ in X. A metric measure space (X,d,μ) is called *doubling* if the measure μ is doubling. Under the assumption that the measure μ is doubling, it is known that (X,d,μ) is proper (that is, closed bounded subsets of X are compact) if and only if it is complete.

Now we prove that if $1 \leq q \leq p$, the measure μ is doubling, and the space (X,d,μ) carries a weak $(1,L^{p,q})$-Poincaré inequality, the Lipschitz functions are dense in $N^{1,L^{p,q}}(X,\mu)$.

In order to prove that we need a few definitions and lemmas.

Definition 6.2. Suppose (X,d) is a metric space that carries a doubling measure μ. For $1<p<\infty$ and $1 \leq q \leq \infty$, we define the noncentered maximal function operator by

$$M_{p,q}u(x) = \sup_{B \ni x} \frac{\|u\chi_B\|_{L^{p,q}(X,\mu)}}{\mu(B)^{1/p}},$$

where $u \in L^{p,q}(X,\mu)$.

Lemma 6.3. *Suppose (X,d) is a metric space that carries a doubling measure μ. If $1 \leq q \leq p$, then $M_{p,q}$ maps $L^{p,q}(X,\mu)$ to $L^{p,\infty}(X,\mu)$ boundedly and moreover,*

$$\lim_{\lambda \to \infty} \lambda^p \mu\big(\{x \in X : M_{p,q}u(x) > \lambda\}\big) = 0.$$

Proof. We can assume without loss of generality that $1 \leq q < p$. For every $R>0$ let $M_{p,q}^R$ be the restricted maximal function operator defined on $L^{p,q}(X,\mu)$ by

$$M_{p,q}^R u(x) = \sup_{B \ni x, \operatorname{diam}(B) \leq R} \frac{\|u\chi_B\|_{L^{p,q}(X,\mu)}}{\mu(B)^{1/p}}.$$

Denote $G_\lambda = \{x \in X : M_{p,q}u(x) > \lambda\}$ and $G_\lambda^R = \{x \in X : M_{p,q}^R u(x) > \lambda\}$. It is easy to see that $G_\lambda^{R_1} \subset G_\lambda^{R_2}$ if $0<R_1<R_2<\infty$ and $G_\lambda^R \to G_\lambda$ as $R \to \infty$.

Fix $R>0$. For every $x \in G_\lambda^R$, $\lambda>0$, there exists a ball $B(y_x,r_x)$ with diameter at most R such that $x \in B(y_x,r_x)$ and such that

$$\|u\chi_{B(y_x,r_x)}\|_{L^{p,q}(X,\mu)}^p > \lambda^p \mu\big(B(y_x,r_x)\big).$$

We notice that $B(y_x,r_x) \subset G_\lambda^R$. The set G_λ^R is covered by such balls and then by Heinonen [15, Theorem 1.2] it follows that there exists a countable disjoint subcollection $\{B(x_i,r_i)\}_{i=1}^\infty$ such that the collection $\{B(x_i,5r_i)\}_{i=1}^\infty$ covers G_λ^R. Hence,

$$\begin{aligned}\mu\big(G_\lambda^R\big) &\le \sum_{i=1}^\infty \mu\big(B(x_i,5r_i)\big) \le C\left(\sum_{i=1}^\infty \mu\big(B(x_i,r_i)\big)\right)\\ &\le \frac{C}{\lambda^p}\left(\sum_{i=1}^\infty \|u\chi_{B(x_i,r_i)}\|^p_{L^{p,q}(X,\mu)}\right) \le \frac{C}{\lambda^p}\|u\chi_{G_\lambda^R}\|^p_{L^{p,q}(X,\mu)}.\end{aligned}$$

The last inequality in the sequence was obtained by applying Proposition 2.4. (See also Chung, Hunt and Kurtz [5, Lemma 2.5].)

Thus,

$$\mu\big(G_\lambda^R\big) \le \frac{C}{\lambda^p}\|u\chi_{G_\lambda^R}\|^p_{L^{p,q}(X,\mu)} \le \frac{C}{\lambda^p}\|u\chi_{G_\lambda}\|^p_{L^{p,q}(X,\mu)}$$

for every $R>0$. Since $G_\lambda = \bigcup_{R>0} G_\lambda^R$, we obtain (by taking the limit as $R\to\infty$)

$$\mu(G_\lambda) \le \frac{C}{\lambda^p}\|u\chi_{G_\lambda}\|^p_{L^{p,q}(X,\mu)}.$$

The absolute continuity of the p,q-norm (see the discussion after Definition 2.1), the p,q-integrability of u and the fact that $G_\lambda\to\emptyset$ μ-almost everywhere as $\lambda\to\infty$ yield the desired conclusion. □

QUESTION 6.4. Is Lemma 6.3 true when $p<q<\infty$?

The following proposition is necessary in the sequel.

PROPOSITION 6.5. *Suppose $1<p<\infty$ and $1\le q<\infty$. If u is a nonnegative function in $N^{1,L^{p,q}}(X,\mu)$, then the sequence of functions $u_k=\min(u,k)$, $k\in\mathbb{N}$, converges in the norm of $N^{1,L^{p,q}}(X,\mu)$ to u as $k\to\infty$.*

Proof. We notice (see Lemma 3.16) that $u_k\in L^{p,q}(X,\mu)$. That lemma also yields easily $u_k\in N^{1,L^{p,q}}(X,\mu)$ and moreover $\|u_k\|_{N^{1,L^{p,q}}}\le\|u\|_{N^{1,L^{p,q}}}$ for all $k\ge1$.

Let $E_k=\{x\in X: u(x)>k\}$. Since μ is a Borel regular measure, there exists an open set O_k such that $E_k\subset O_k$ and $\mu(O_k)\le\mu(E_k)+2^{-k}$. In fact the sequence $(O_k)_{k=1}^\infty$ can be chosen such that $O_{k+1}\subset O_k$ for all $k\ge1$. Since $\mu(E_k)\le\frac{C(p,q)}{k^p}\|u\|^p_{L^{p,q}(X,\mu)}$, it follows that

$$\mu(O_k)\le\mu(E_k)+2^{-k}\le\frac{C(p,q)}{k^p}\|u\|^p_{L^{p,q}(X,\mu)}+2^{-k}.$$

Thus, $\lim_{k\to\infty}\mu(O_k)=0$. We notice that $u=u_k$ on $X\setminus O_k$. Thus, $2g\chi_{O_k}$ is a p,q-weak upper gradient of $u-u_k$ whenever g is an upper gradient for u and u_k. See Lemma 4.6. The fact that $O_k\to\emptyset$ μ-almost everywhere and the absolute continuity of the (p,q)-norm yield

$$\begin{aligned}&\limsup_{k\to\infty}\|u-u_k\|_{N^{1,L^{(p,q)}}}\\ &\quad\le 2\limsup_{k\to\infty}\big(\|u\chi_{O_k}\|_{L^{(p,q)}(X,\mu)}+\|g\chi_{O_k}\|_{L^{(p,q)}(X,\mu)}\big)=0.\end{aligned}$$

□

COUNTEREXAMPLE 6.6. For $q=\infty$, Proposition 6.5 is not true. Indeed, let $n\geq 2$ be an integer and let $1<p\leq n$ be fixed. Let $X=B(0,1)\setminus\{0\}\subset\mathbb{R}^n$, endowed with the Euclidean metric and the Lebesgue measure.

Suppose first that $1<p<n$. Let u_p and g_p be defined on X by

$$u_p(x)=|x|^{1-\frac{n}{p}}-1 \quad\text{and}\quad g_p(x)=\left(\frac{n}{p}-1\right)|x|^{-\frac{n}{p}}.$$

It is easy to see that $u_p, g_p\in L^{p,\infty}(X,m_n)$. Moreover (see, for instance, Hajłasz [13, Proposition 6.4]), g_p is the minimal upper gradient for u_p. Thus $u_p\in N^{1,L^{p,\infty}}(X,m_n)$. For every integer $k\geq 1$, we define $u_{p,k}$ and $g_{p,k}$ on X by

$$u_{p,k}(x)=\begin{cases} k & \text{if } 0<|x|\leq (k+1)^{\frac{p}{p-n}},\\ |x|^{1-\frac{n}{p}}-1 & \text{if } (k+1)^{\frac{p}{p-n}}<|x|<1\end{cases}$$

and

$$g_{p,k}(x)=\begin{cases} (\frac{n}{p}-1)|x|^{-\frac{n}{p}} & \text{if } 0<|x|<(k+1)^{\frac{p}{p-n}},\\ 0 & \text{if } (k+1)^{\frac{p}{p-n}}\leq |x|<1.\end{cases}$$

We notice that $u_{p,k}\in N^{1,L^{p,\infty}}(X,m_n)$ for all $k\geq 1$. Moreover, via [13, Proposition 6.4] and Lemma 4.6, we see that $g_{p,k}$ is the minimal upper gradient for $u_p-u_{p,k}$ for every $k\geq 1$. Since $g_{p,k}\searrow 0$ on X as $k\to\infty$ and $\|g_{p,k}\|_{L^{p,\infty}(X,m_n)}=\|g_p\|_{L^{p,\infty}(X,m_n)}=C(n,p)>0$ for all $k\geq 1$, it follows that $u_{p,k}$ does *not* converge to u_p in $N^{1,L^{p,\infty}}(X,m_n)$ as $k\to\infty$.

Suppose now that $p=n$. Let u_n and g_n be defined on X by

$$u_n(x)=\ln\frac{1}{|x|} \quad\text{and}\quad g_n(x)=\frac{1}{|x|}.$$

It is easy to see that $u_n, g_n\in L^{p,\infty}(X,m_n)$. Moreover (see, for instance, Hajłasz [13, Proposition 6.4]), g_n is the minimal upper gradient for u_n. Thus, $u_n\in N^{1,L^{n,\infty}}(X,m_n)$. For every integer $k\geq 1$ we define $u_{n,k}$ and $g_{n,k}$ on X by

$$u_{n,k}(x)=\begin{cases} k & \text{if } 0<|x|\leq e^{-k},\\ \ln\frac{1}{|x|} & \text{if } e^{-k}<|x|<1\end{cases}$$

and

$$g_{n,k}(x)=\begin{cases} \frac{1}{|x|} & \text{if } 0<|x|<e^{-k},\\ 0 & \text{if } e^{-k}\leq |x|<1.\end{cases}$$

We notice that $u_{n,k}\in N^{1,L^{n,\infty}}(X,m_n)$ for all $k\geq 1$. Moreover, via [13, Proposition 6.4] and Lemma 4.6 we see that $g_{n,k}$ is the minimal upper gradient for $u_n-u_{n,k}$ for every $k\geq 1$. Since $g_{n,k}\searrow 0$ on X as $k\to\infty$ and $\|g_{n,k}\|_{L^{p,\infty}(X,m_n)}=\|g_n\|_{L^{n,\infty}(X,m_n)}=C(n)>0$ for all $k\geq 1$, it follows that $u_{n,k}$ does *not* converge to u_n in $N^{1,L^{n,\infty}}(X,m_n)$ as $k\to\infty$.

The following lemma will be used in the paper.

Lemma 6.7. *Let $f_1 \in N^{1,L^{p,q}}(X,\mu)$ be a bounded Borel function with p,q-weak upper gradient $g_1 \in L^{p,q}(X,\mu)$ and let f_2 be a bounded Borel function with p,q-weak upper gradient $g_2 \in L^{p,q}(X,\mu)$. Then $f_3 := f_1 f_2 \in N^{1,L^{p,q}}(X,\mu)$ and $g_3 := |f_1| g_2 + |f_2| g_1$ is a p,q-weak upper gradient of f_3.*

Proof. It is easy to see that f_3 and g_3 are in $L^{p,q}(X,\mu)$. Let $\Gamma_0 \subset \Gamma_{\mathrm{rect}}$ be the family of curves on which $\int_\gamma (g_1+g_2) = \infty$. Then it follows via Theorem 3.4 that $\mathrm{Mod}_{p,q}(\Gamma_0) = 0$ because $g_1 + g_2 \in L^{p,q}(X,\mu)$.

Let $\Gamma_{1,i} \subset \Gamma_{\mathrm{rect}}, i = 1,2$ be the family of curves for which

$$\left|f_i\big(\gamma(0)\big) - f_i\big(\gamma\big(\ell(\gamma)\big)\big)\right| \le \int_\gamma g_i$$

is *not* satisfied. Then $\mathrm{Mod}_{\Gamma_{1,i}} = 0, i = 1,2$. Let $\Gamma_1 \subset \Gamma_{\mathrm{rect}}$ be the family of curves that have a subcurve in $\Gamma_{1,1} \cup \Gamma_{1,2}$. Then $F(\Gamma_{1,1} \cup \Gamma_{1,2}) \subset F(\Gamma_1)$ and thus $\mathrm{Mod}_{p,q}(\Gamma_1) \le \mathrm{Mod}_{p,q}(\Gamma_{1,1} \cup \Gamma_{1,2}) = 0$. We notice immediately that $\mathrm{Mod}_{p,q}(\Gamma_0 \cup \Gamma_1) = 0$.

Fix $\varepsilon > 0$. By using the argument from Lemma 1.7 in Cheeger [4], we see that

$$\begin{aligned}&\left|f_3\big(\gamma(0)\big) - f_3\big(\gamma\big(\ell(\gamma)\big)\big)\right| \\ &\quad \le \int_0^{\ell(\gamma)} \Big(\big|f_1\big(\gamma(s)\big)\big| + \varepsilon\Big) g_2\big(\gamma(s)\big) + \Big(\big|f_2\big(\gamma(s)\big)\big| + \varepsilon\Big) g_1\big(\gamma(s)\big)\, ds\end{aligned}$$

for every curve γ in $\Gamma_{\mathrm{rect}} \setminus (\Gamma_0 \cup \Gamma_1)$. Letting $\varepsilon \to 0$ we obtain the desired claim. $\square$

Fix $x_0 \in X$. For each integer $j > 1$ we consider the function

$$\eta_j(x) = \begin{cases} 1 & \text{if } d(x_0,x) \le j-1, \\ j - d(x_0,x) & \text{if } j-1 < d(x_0,x) \le j, \\ 0 & \text{if } d(x_0,x) > j. \end{cases}$$

Lemma 6.8. *Suppose $1 \le q < \infty$. Let u be a bounded function in the space $N^{1,L^{p,q}}(X,\mu)$. Then the function $v_j = u\eta_j$ is also in $N^{1,L^{p,q}}(X,\mu)$ where η_j is defined as above. Furthermore, the sequence v_j converges to u in $N^{1,L^{p,q}}(X,\mu)$.*

Proof. If X is bounded, the claims of the lemma are trivial. Thus, we can assume without loss of generality that X is unbounded. Moreover, we can also assume without loss of generality that $u \ge 0$. Let $g \in L^{p,q}(X,\mu)$ be an upper gradient for u. It is easy to see by invoking Lemma 4.6 that $h_j := \chi_{B(x_0,j) \setminus \overline{B}(x_0,j-1)}$ is a p,q-weak upper gradient for η_j and for $1 - \eta_j$. By using Lemma 6.7, we see that $v_j \in N^{1,L^{p,q}}(X,\mu)$ and that $g_j := uh_j + g\eta_j$ is a p,q-weak upper gradient for v_j. By using Lemma 6.7, we notice that $\widetilde{h}_j := uh_j + g(1-\eta_j)$ is a p,q-weak upper gradient for $u - v_j$. We have in fact

$$0 \le u - v_j \le u\chi_{X \setminus B(x_0,j-1)} \quad \text{and} \quad \widetilde{h}_j \le (u+g)\chi_{X \setminus B(x_0,j-1)} \tag{20}$$

for every $j > 1$. The absolute continuity of the (p,q)-norm when $1 \le q < \infty$ (see the discussion after Definition 2.1) together with the p,q-integrability of u, g and (20) yield the desired claim. $\square$

Now we prove the density of the Lipschitz functions in $N^{1,L^{p,q}}(X,\mu)$ when $1 \le q < p$. The case $q = p$ was proved by Shanmugalingam. (See [27] and [28].)

THEOREM 6.9. *Let $1 \le q \le p < \infty$. Suppose that (X,d,μ) is a doubling metric measure space that carries a weak $(1,L^{p,q})$-Poincaré inequality. Then the Lipschitz functions are dense in $N^{1,L^{p,q}}(X,\mu)$.*

Proof. We can consider only the case $1 \le q < p$ because the case $q = p$ was proved by Shanmugalingam in [27] and [28]. We can assume without loss of generality that u is nonnegative. Moreover, via Lemmas 6.5 and 6.7 we can assume without loss of generality that u is bounded and has compact support in X. Choose $M > 0$ such that $0 \le u \le M$ on X. Let $g \in L^{p,q}(X,\mu)$ be a p,q-weak upper gradient for u. Let $\sigma \ge 1$ be the constant from the weak $(1,L^{p,q})$-Poincaré inequality.

Let $G_\lambda := \{x \in X : M_{p,q}g(x) > \lambda\}$. If x is a point in the closed set $X \setminus G_\lambda$, then for all $r > 0$ one has that

$$\begin{aligned}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |u - u_{B(x,r)}|\,d\mu &\le Cr\frac{\|g\chi_{B(x,\sigma r)}\|_{L^{p,q}(X,\mu)}}{\mu(B(x,\sigma r))^{1/p}}\\ &\le CrM_{p,q}g(x) \le C\lambda r.\end{aligned}$$

Hence, for $s \in [r/2, r]$ one has that

$$\begin{aligned}|u_{B(x,s)} - u_{B(x,r)}| &\le \frac{1}{\mu(B(x,s))}\int_{B(x,s)} |u - u_{B(x,r)}|\,d\mu\\ &\le \frac{\mu(B(x,r))}{\mu(B(x,s))}\cdot\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |u - u_{B(x,r)}|\,d\mu \le C\lambda r\end{aligned}$$

whenever x is in $X \setminus G_\lambda$. For a fixed $s \in (0, r/2)$ there exists an integer $k \ge 1$ such that $2^{-k}r \le 2s < 2^{-k+1}r$. Then

$$\begin{aligned}|u_{B(x,s)} - u_{B(x,r)}| &\le |u_{B(x,s)} - u_{B(x,2^{-k}r)}| + \sum_{i=0}^{k-1} |u_{B(x,2^{-i-1}r)} - u_{B(x,2^{-i}r)}|\\ &\le C\lambda\left(\sum_{i=0}^{k} 2^{-i}r\right) \le C\lambda r.\end{aligned}$$

For any sequence $r_i \searrow 0$ we notice that $(u_{B(x,r_i)})_{i=1}^{\infty}$ is a Cauchy sequence for every point x in $X \setminus G_\lambda$. Thus, on $X \setminus G_\lambda$ we can define the function

$$u_\lambda(x) := \lim_{r\to 0} u_{B(x,r)}.$$

We notice that $u_\lambda(x) = u(x)$ for every Lebesgue point x in $X \setminus G_\lambda$.

For every x, y in $X \setminus G_\lambda$ we consider the chain of balls $\{B_i\}_{i=-\infty}^{\infty}$, where

$$B_i = B\big(x, 2^{1+i} d(x,y)\big), \quad i \le 0 \quad \text{and} \quad B_i = B\big(y, 2^{1-i} d(x,y)\big), \quad i > 0.$$

For every two such points x and y, we have that they are Lebesgue points for u_λ by construction and hence

$$\big|u_\lambda(x) - u_\lambda(y)\big| \le \sum_{i=-\infty}^{\infty} |u_{B_i} - u_{B_{i+1}}| \le C\lambda d(x,y),$$

where C depends only on the data on X. Thus, u_λ is $C\lambda$-Lipschitz on $X \setminus G_\lambda$. By construction it follows that $0 \le u_\lambda \le M$ on $X \setminus G_\lambda$. Extend u_λ as a $C\lambda$-Lipschitz function on X (see McShane [25]) and denote the extension by v_λ. Then $v_\lambda \ge 0$ on X since $u_\lambda \ge 0$ on $X \setminus G_\lambda$. Let $w_\lambda := \min(v_\lambda, M)$. We notice that w_λ is a nonnegative $C\lambda$-Lipschitz function on X since v_λ is. Moreover, $w_\lambda = v_\lambda = u_\lambda$ on $X \setminus G_\lambda$ whenever $\lambda > M$.

Since $u = w_\lambda$ μ-almost everywhere on $X \setminus G_\lambda$ whenever $\lambda > M$, we have

$$\begin{aligned}\|u - w_\lambda\|_{L^{p,q}(X,\mu)} &= \big\|(u - w_\lambda)\chi_{G_\lambda}\big\|_{L^{p,q}(X,\mu)} \\ &\le \|u\chi_{G_\lambda}\|_{L^{p,q}(X,\mu)} + C(p,q)\lambda\mu(G_\lambda)^{1/p}\end{aligned}$$

whenever $\lambda > M$. The absolute continuity of the p,q-norm when $1 \le q \le p$ together with Lemma 6.3 imply that

$$\lim_{\lambda\to\infty} \|u - w_\lambda\|_{L^{p,q}(X,\mu)} = 0.$$

Since $u - w_\lambda = 0$ μ-almost everywhere on the closed set G_λ, it follows via Lemma 4.6 that $(C\lambda + g)\chi_{G_\lambda}$ is a p,q-weak upper gradient for $u - w_\lambda$. By using the absolute continuity of the p,q-norm when $1 \le q \le p$ together with Lemma 6.3, we see that

$$\lim_{\lambda\to\infty} \big\|(C\lambda + g)\chi_{G_\lambda}\big\|_{L^{p,q}(X,\mu)} = 0.$$

This finishes the proof of the theorem. □

Theorem 6.9 yields the following result.

Proposition 6.10. *Let $1 \le q \le p < \infty$. Suppose that (X, d, μ) satisfies the hypotheses of Theorem 6.9. Then $\mathrm{Cap}_{p,q}$ is an outer capacity.*

In order to prove Proposition 6.10, we need to state and prove the following proposition, thus generalizing Proposition 1.4 from Björn, Björn and Shanmugalingam [3].

Proposition 6.11 (See [3, Proposition 1.4]). *Let $1 < p < \infty$ and $1 \le q < \infty$. Suppose that (X, d, μ) is a proper metric measure space. Let $E \subset X$ be such that $\mathrm{Cap}_{p,q}(E) = 0$. Then for every $\varepsilon > 0$ there exists an open set $U \supset E$ with $\mathrm{Cap}_{p,q}(U) < \varepsilon$.*

Proof. We adjust the proof of Proposition 1.4 in Björn, Björn and Shanmugalingam [3] to the Lorentz setting with some modifications. It is enough to consider the case when $q \neq p$. Due to the countable subadditivity of $\mathrm{Cap}_{p,q}(\cdot)^{1/p}$ we can assume without loss of generality that E is bounded. Moreover, we can also assume that E is Borel. Since $\mathrm{Cap}_{p,q}(E) = 0$, we have $\chi_E \in N^{1,L^{p,q}}(X,\mu)$ and $\|\chi_E\|_{N^{1,L^{p,q}}} = 0$. Let $\varepsilon \in (0,1)$ be arbitrary. Via Lemma 3.9 and Corollary 2.8, there exists $g \in L^{p,q}(X,\mu)$ such that g is an upper gradient for χ_E and $\|g\|_{L^{p,q}(X,\mu)} < \varepsilon$. By adapting the proof of the Vitali–Carathéodory theorem to the Lorentz setting (see Folland [10, Proposition 7.14]) we can find a lower semicontinuous function $\rho \in L^{p,q}(X,\mu)$ such that $\rho \geq g$ and $\|\rho - g\|_{L^{p,q}(X,\mu)} < \varepsilon$. Since $\mathrm{Cap}_{p,q}(E) = 0$, it follows immediately that $\mu(E) = 0$. By using the outer regularity of the measure μ and the absolute continuity of the (p,q)-norm, there exists a bounded open set $V \supset E$ such that

$$\|\chi_V\|_{L^{p,q}(X,\mu)} + \big\|(\rho+1)\chi_V\big\|_{L^{p,q}(X,\mu)} < \frac{\varepsilon}{2}.$$

Let

$$u(x) = \min\left\{1, \inf_\gamma \int_\gamma (\rho+1)\right\},$$

where the infimum is taken over all the rectifiable (including constant) curves connecting x to the closed set $X \setminus V$. We notice immediately that $0 \leq u \leq 1$ on X and $u = 0$ on $X \setminus V$. By Björn, Björn and Shanmugalingam [3, Lemma 3.3] it follows that u is lower semicontinuous on X and thus the set $U = \{x \in X : u(x) > \frac{1}{2}\}$ is open. We notice that for $x \in E$ and every curve connecting x to some $y \in X \setminus V$, we have

$$\int_\gamma (\rho+1) \geq \int_\gamma \rho \geq \chi_E(x) - \chi_E(y) = 1.$$

Thus, $u = 1$ on E and $E \subset U \subset V$. From [3, Lemmas 3.1 and 3.2] it follows that $(\rho+1)\chi_V$ is an upper gradient of u. Since $0 \leq u \leq \chi_V$ and u is lower semicontinuous, it follows that $u \in N^{1,L^{p,q}}(X,\mu)$. Moreover, $2u \in \mathcal{A}(U)$ and thus

$$\begin{aligned}\mathrm{Cap}_{p,q}(U)^{1/p} &\leq 2\|u\|_{N^{1,L^{p,q}}} \leq 2\big(\|u\|_{L^{p,q}(X,\mu)} + \big\|(\rho+1)\chi_V\big\|_{L^{p,q}(X,\mu)}\big)\\ &\leq 2\big(\|\chi_V\|_{L^{p,q}(X,\mu)} + \big\|(\rho+1)\chi_V\big\|_{L^{p,q}(X,\mu)}\big) < \varepsilon.\end{aligned}$$

This finishes the proof of Proposition 6.11. □

Now we prove Proposition 6.10.

Proof. We start the proof of Proposition 6.10 by showing that every function u in $N^{1,L^{p,q}}(X,\mu)$ is continuous outside open sets of arbitrarily small p,q-capacity. (Such a function is called p,q-quasicontinuous.) Indeed, let u

be a function in $N^{1,L^{p,q}}(X,\mu)$. From Theorem 6.9 there exists a sequence $\{u_j\}_{j=1}^\infty$ of Lipschitz functions on X such that

$$\|u_j - u\|_{N^{1,L^{p,q}}} < 2^{-2j} \quad \text{for every integer } j \ge 1.$$

For every integer $j \ge 1$ let

$$E_j = \big\{x \in X : \big|u_{j+1}(x) - u_j(x)\big| > 2^{-j}\big\}.$$

Then all the sets E_j are open because the all functions u_j are Lipschitz. By letting $F = \bigcap_{j=1}^\infty \bigcup_{k=j}^\infty E_k$ and applying the argument from Theorem 5.5 to the sequence $\{u_k\}_{k=1}^\infty$ which is Cauchy in $N^{1,L^{p,q}}(X,\mu)$, we see that $\mathrm{Cap}_{p,q}(F) = 0$ and the sequence $\{u_k\}$ converges in $N^{1,L^{p,q}}(X,\mu)$ to a function $\widetilde{u}$ whose restriction on $X \setminus F$ is continuous. Thus, $\|u - \widetilde{u}\|_{N^{1,L^{p,q}}} = 0$ and hence if we let $E = \{x \in X : u(x) \neq \widetilde{u}(x)\}$, we have $\mathrm{Cap}_{p,q}(E) = 0$. Therefore $\mathrm{Cap}_{p,q}(E \cup F) = 0$ and hence, via Proposition 6.11 we have that $u = \widetilde{u}$ outside open supersets of $E \cup F$ of arbitrarily small p,q-capacity. This shows that u is quasicontinuous.

Now we fix $E \subset X$ and we show that

$$\mathrm{Cap}_{p,q}(E) = \inf\big\{\mathrm{Cap}_{p,q}(U), E \subset U \subset X, U \text{ open}\big\}.$$

For a fixed $\varepsilon \in (0,1)$ we choose $u \in \mathcal{A}(E)$ such that $0 \le u \le 1$ on X and such that

$$\|u\|_{N^{1,L^{p,q}}} < \mathrm{Cap}_{p,q}(E)^{1/p} + \varepsilon.$$

We have that u is p,q-quasicontinuous and hence there is an open set V such that $\mathrm{Cap}_{p,q}(V)^{1/p} < \varepsilon$ and such that $u|_{X \setminus V}$ is continuous. Thus, there exists an open set U such that $U \setminus V = \{x \in X : u(x) > 1 - \varepsilon\} \setminus V \supset E \setminus V$. We see that $U \cup V = (U \setminus V) \cup V$ is an open set containing $E \cup V = (E \setminus V) \cup V$. Therefore,

$$\begin{aligned}
\mathrm{Cap}_{p,q}(E)^{1/p} &\le \mathrm{Cap}_{p,q}(U \cup V)^{1/p} \le \mathrm{Cap}_{p,q}(U \setminus V)^{1/p} + \mathrm{Cap}_{p,q}(V)^{1/p} \\
&\le \frac{1}{1-\varepsilon}\|u\|_{N^{1,L^{p,q}}} + \mathrm{Cap}_{p,q}(V)^{1/p} \\
&\le \frac{1}{1-\varepsilon}\big(\mathrm{Cap}_{p,q}(E)^{1/p} + \varepsilon\big) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ finishes the proof of Proposition 6.10. □

Theorems 5.2 and 6.9 together with Proposition 6.10 and Remark 5.3 yield immediately the following capacitability result. (See also Appendix II in Doob [9].)

THEOREM 6.12. *Let $1 < q \le p < \infty$. Suppose that (X,d,μ) satisfies the hypotheses of Theorem 6.9. The set function $E \mapsto \mathrm{Cap}_{p,q}(E)$ is a Choquet capacity. In particular, all Borel subsets (in fact, all analytic subsets) E of X are capacitable, that is*

$$\mathrm{Cap}_{p,q}(E) = \sup\big\{\mathrm{Cap}_{p,q}(K) : K \subset E, K \textit{ compact}\big\}$$

whenever $E \subset X$ is Borel (or analytic).

REMARK 6.13. Counterexample 6.6 can be used to construct a counterexample to the density result for $N^{1,L^{p,\infty}}$ in the Euclidean setting for $1 < p \le n$ and $q = \infty$.

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