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GOTZMANN SQUAREFREE IDEALS

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Abstract. We classify the squarefree ideals which are Gotzmann in a polynomial ring.

1. Introduction

Let $S = k[x_1, \dots, x_n]$ and R be either $S/(x_1^2, \dots, x_n^2)$ or the exterior algebra on the same variables.

The question of what numerical functions can arise as the Hilbert functions of homogeneous ideals in these rings was answered by Macaulay [Ma27] (for S) and Kruskal [Kr63] and Katona [Ka68] (for R). Macaulay's theorem and the Kruskal–Katona theorem have seen widespread application in both commutative algebra and combinatorics. These theorems provide lower bounds on the Hilbert function growth of homogeneous ideals, and describe special ideals, called *lexicographic* or *lex* ideals, which attain these bounds. Lex ideals are defined combinatorially, and are very well understood; for example, their minimal free resolutions are known explicitly.

Lex ideals are important tools in many contexts. In geometry, Hartshorne's proof that the Hilbert scheme is connected [Ha66] uses lex ideals in an essential way. More combinatorially, lex ideals and Macaulay's and Kruskal and Katona's theorems have arisen in Sperner theory, network reliability, and other graph problems; see [En97] or [BL05].

In many of these settings, the only relevant property of the lex ideals is the slow growth of their Hilbert functions, so it is worthwhile to consider other ideals whose Hilbert functions achieve the bounds of Macaulay's (or Kruskal and Katona's) theorem. Such ideals are called *Gotzmann*. Gotzmann ideals share many nice properties with lex ideals; for example, they have componentwise linear resolutions and maximal graded Betti numbers among ideals with the same Hilbert function [HH99].

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However, only a few classes of Gotzmann ideals are known. Murai and Hibi show in [MH08] that all Gotzmann ideals of S with at most n generators have a very specific form. The problem of understanding monomial Gotzmann ideals has proven slightly more tractable. Bonanzinga classifies the principal Borel ideals generated in degree at most four which are Gotzmann [Bo03]. Mermin enumerates in [Me06] the lexlike ideals, ideals which are generated by initial segments of "lexlike" sequences and which share many properties with lex ideals, including minimal Hilbert function growth. Murai studies Hilbert functions for which the only Gotzmann monomial ideals are lex, and classifies the Gotzmann monomial ideals of $k[a, b, c]$, in [Mu07], and Olteanu, Olteanu, and Sorrenti [OOS08] classify the Gotzmann ideals which are generated by (not necessarily initial) segments in the lex order. Finally, in [Ho09], Hoefel shows that a graph has a Gotzmann edge ideal if and only if it is star-shaped. In this paper, we generalize Hoefel's result by classifying all the Gotzmann ideals of S which are generated by squarefree monomials. An immediate consequence of our classification is that all such ideals have at most n generators, so they have the form prescribed by Murai and Hibi.

In Section 2, we define notation which will be used throughout the paper, and recall background information about Gotzmann ideals, squarefree ideals, and the relationship between R and S. In Section 3, we classify the squarefree Gotzmann ideals of S. Finally, in Section 4, we begin to study the monomial Gotzmann ideals of R. Our main result is that a Gotzmann ideal has Gotzmann Alexander dual if and only if all of its degreewise components are lex in some order.

2. Background and notation

Let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring and $R = S/(x_1^2, \ldots, x_n^2)$ be the associated "squarefree ring". Let $\mathbf{m} = (x_1, \ldots, x_n)$ be the homogeneous maximal ideal.

In this paper, all ideals are homogeneous. A *monomial ideal* is an ideal with a generating set consisting of monomials. Every monomial ideal I has a canonical minimal set of monomial generators which we denote $\mathrm{gens}(I)$. When we refer to the generators of a monomial ideal—for example, to count them—we mean those in this canonical generating set. If all the generators of I are squarefree monomials, then we say that I is *squarefree*.

For an ideal I, we write I_d for the vector space of degree d forms in I. When I is a monomial ideal, a basis for I_d is given by the monomials of degree d that are in I.

An important vector space is S_d (or R_d), the space of all forms of degree d. We call subspaces of S_d or R_d *monomial vector spaces* when they have a monomial basis. The (unique) monomial basis of a monomial vector space

V_d will be denoted $\mathrm{gens}(V_d)$. Usually, we indicate the degree of a monomial vector space with a subscript in this way.

The ideal (V_d) generated by a monomial vector space $V_d \subseteq S_d$ is a monomial ideal with with $\mathrm{gens}((V_d)) = \mathrm{gens}(V_d)$. We will often need to consider the monomial vector space

$$\mathbf{m}_1 V_d = \mathrm{span}_{\Bbbk}\{x_i m \mid m \in \mathrm{gens}(V_d)\}.$$

While we treat this as a product of monomial vector spaces, it has a natural interpretation in terms of ideals. If $I = (V_d)$ is the ideal generated by V_d, then $\mathbf{m}_1 V_d = I_{d+1}$.

We will write $|V_d|$ or sometimes $|V|_d$ to denote the vector space dimension of V_d.

Definition 2.1. The *Hilbert function* of an ideal I is the function $\mathrm{HF}_I : \mathbb{N} \to \mathbb{N}$ which gives the dimension of each component of I:

$$\mathrm{HF}_I(d) = |I_d|.$$

When I is a monomial ideal $|I_d| = |\mathrm{gens}(I_d)|$ is simply the number of degree d monomials in I. If I is squarefree, we write $I^{\mathrm{sf}} = IR$ for the corresponding ideal of R. Thus $|I^{\mathrm{sf}}_d|$ is the number of squarefree monomials in I of degree d.

The *Hilbert series* and *squarefree Hilbert series* of I are $\mathrm{HS}_I(t) = \sum_{d=0}^{\infty} |I_d| t^d$ and $\mathrm{HS}^{\mathrm{sf}}_I(t) = \sum_{d=0}^{\infty} |I^{\mathrm{sf}}_d| t^d$. If I is squarefree, these are related by the formula

$$\mathrm{HS}_I(t) = \mathrm{HS}^{\mathrm{sf}}_I\left(\frac{t}{1-t}\right).$$

which follows from [Pe11, Proposition 51.3].

Definition 2.2. We say that an ideal I of S (or, with the obvious changes, of R) is *Gotzmann* if for all degrees d and all ideals J satisfying $|I_d| = |J_d|$ we have $|\mathbf{m}_1 I_d| \le |\mathbf{m}_1 J_d|$.

The Gotzmann property may be viewed degreewise as a property of vector spaces. We say that a vector space $V_d \subseteq S_d$ (or R_d) is *Gotzmann* if for all subspaces $W_d \subseteq S_d$ (resp. R_d) with $|V_d| = |W_d|$ we have $|\mathbf{m}_1 V_d| \le |\mathbf{m}_1 W_d|$. Thus an ideal I is Gotzmann if and only if each component I_d is Gotzmann.

Definition 2.3. A vector space $L_d \subseteq S_d$ (or R_d) is a *lex segment* if it is spanned by an initial segment of the degree d monomials in lexicographic order. An ideal is *lex* if each of its components are lex segments. We say that a squarefree ideal L is *squarefree lex* if L^{sf} is lex in R.

Lex ideals are important because they are very well understood combinatorially, and because of the following theorem of Macaulay [Ma27] (in S) and Kruskal and Katona [Kr63], [Ka68] (in R):

Theorem 2.4 (Macaulay, Kruskal, Katona). *Lex ideals are Gotzmann.*

COROLLARY 2.5. *For every homogeneous ideal I there is a unique lex ideal L with* $\mathrm{HF}_L = \mathrm{HF}_I$.

Macaulay's theorem allows the following alternative characterization of Gotzmann ideals:

PROPOSITION 2.6. *A degree d monomial vector space V_d is Gotzmann if and only if $|\mathbf{m}_1 V_d| = |\mathbf{m}_1 L_d|$ where L_d is the degree d lex segment of dimension $|V_d|$.*

COROLLARY 2.7. *Let L be the lex ideal with the same Hilbert function as I. Then I is Gotzmann if and only if L and I have the same number of generators in each degree.*

Proof. The number of degree d generators of I and L is given by $|I_d| - |\mathbf{m}_1 I_{d-1}|$ and $|L_d| - |\mathbf{m}_1 L_{d-1}|$, respectively. Since I and L have the same Hilbert function, they have the same number of generators in degree d if and only if I_{d-1} is Gotzmann. $\square$

Gotzmann's persistance theorem [Go78] states that if V_d is a Gotzmann vector space then $\mathbf{m}_1 V_d$ is also Gotzmann. Thus, to check if an ideal is Gotzmann, one needs to check only that its components are Gotzmann in the degrees of its minimal generators.

THEOREM 2.8 (Gotzmann's persistence theorem, vector space version). *Suppose that I_d is a Gotzmann vector space. Then $\mathbf{m}_1 I_d$ is Gotzmann.*

THEOREM 2.9 (Gotzmann's persistence theorem, ideal version). *Suppose that every generator of I has degree at most d, and let L be the lex ideal with the same Hilbert function as I. If L has no generators of degree $d+1$, then all generators of L have degree at most d. In particular, if I and L have the same number of generators in every degree less than or equal to $d+1$, then I is Gotzmann.*

The persistence theorem holds in both S and R [AHH97], [FG86].

3. Gotzmann squarefree ideals of the polynomial ring

We will classify the squarefree ideals of S which are Gotzmann. To do this, we compare squarefree ideals with appropriate squarefree lex ideals and exploit the interaction between S and R.

In [Ho09], Hoefel proved that a squarefree quadric ideal is Gotzmann if and only if it is the edge ideal of a *star-shaped* graph. We generalize this result as follows:

DEFINITION 3.1. Let H be a pure d-dimensional hypergraph. We say that H is *star-shaped* if there exists a $(d-1)$-simplex which is contained in every edge of H. More generally, we say that a d-dimensional simplicial complex Δ

is a *supernova* if there exists a chain of faces $\varnothing \subset F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ such that every i-dimensional facet of Δ contains the $(i-1)$-dimensional face F_{i-1}.

We show in Theorem 3.9 that a squarefree ideal is Gotzmann if and only if it is the edge ideal of a supernova. In particular, a purely d-generated squarefree ideal is Gotzmann if and only if it is the edge ideal of a $(d-1)$-dimensional supernova.

A consequence of Theorem 3.9 is that all Gotzmann squarefree ideals of $S = \Bbbk[x_1, \dots, x_n]$ have at most n generators. The Gotzmann ideals of S with at most n generators are classified by Murai and Hibi [MH08]; it is clear from their classification that any Gotzmann squarefree ideal with at most n generators must have the form prescribed by Theorem 3.9. Thus, if this bound on the number of generators could be easily proved, Theorem 3.9 would be a corollary of [MH08, Theorem 1.1]. We have been unable to find a simple proof of this bound. Regardless, the smaller scope of our investigation allows a simpler proof than that given in [MH08].

Definition 3.2. The *squarefree lexification* of a squarefree ideal $I \subseteq S$ is the squarefree lex ideal L in S with the same Hilbert function as I.

The existence of squarefree lexifications follows from the following construction: Let $J^{\mathrm{sf}} \subseteq R$ be the lex ideal having the same Hilbert function as I^{sf}. Then let L be the ideal of S with $L^{\mathrm{sf}} = J^{\mathrm{sf}}$ (that is, L is generated by the monomials of J^{sf}). Then L is squarefree lex and has the same Hilbert function as I because

$$\mathrm{HS}_I(t) = \mathrm{HS}_I^{\mathrm{sf}}\left(\frac{t}{1-t}\right) = \mathrm{HS}_J^{\mathrm{sf}}\left(\frac{t}{1-t}\right) = \mathrm{HS}_L(t).$$

The following proposition is a consequence of Kruskal and Katona's theorem:

Proposition 3.3 (Aramova, Avramov and Herzog [AAH00]). *If $I \subseteq S$ is a Gotzmann squarefree ideal then I^{sf} is Gotzmann in R.*

Lemma 3.4. *If $I \subseteq S$ is a Gotzmann squarefree ideal then its squarefree lexification L is Gotzmann.*

Proof. By Proposition 3.3, I^{sf} is Gotzmann in R. Thus, applying Corollary 2.7, I^{sf} and L^{sf} have the same number of minimal generators in every degree. Now I and I^{sf} have the same generating set, as do L and L^{sf}, so I and L have the same number of generators in every degree. Applying Corollary 2.7 again, L must be Gotzmann in S. $\square$

The next general lemma is the squarefree version of a common inductive technique for studying Hilbert functions in terms of lex ideals.

Lemma 3.5. *Let $I \subseteq S$ be a squarefree ideal and let L be its squarefree lexification. Then $L \subseteq (x_1)$ if and only if $I \subseteq (x_i)$ for some variable x_i.*

Proof. If $I \subseteq (x_i)$ then $I^{\text{sf}} \subseteq (x_i)^{\text{sf}}$ and hence

$$\left|L_d^{\text{sf}}\right| = \left|I_d^{\text{sf}}\right| \leq \left|(x_i)_d^{\text{sf}}\right| = \left|(x_1)_d^{\text{sf}}\right|.$$

As $(x_1)_d^{\text{sf}}$ is a lex segment in R, we have $L_d^{\text{sf}} \subseteq (x_1)_d^{\text{sf}}$ and hence every generator of L is divisible by x_1.

Conversely, assume that $L \subseteq (x_1)$. We have $|L_{n-1}^{\text{sf}}| \leq n-1$, so there is at least one squarefree monomial m of degree $n-1$ which is not in I. Write $m = \frac{x_1 \cdots x_n}{x_i}$. We claim that $I \subseteq (x_i)$. Indeed, every squarefree monomial outside of (x_i) divides m, so no such monomial can appear in I^{sf}. Thus every generator of I is in (x_i) and, in particular, $I \subseteq (x_i)$. $\square$

LEMMA 3.6. *If $I \subseteq S$ is a squarefree Gotzmann ideal, then either $I \subseteq (x_i)$ for some variable x_i or $(x_i) \subseteq I$ for some variable x_i.*

Proof. Suppose to the contrary that I is Gotzmann but, for all i, $I \nsubseteq (x_i)$ and $(x_i) \nsubseteq I$. We will show that L, the squarefree lexification of I, is not Gotzmann, contradicting Lemma 3.4.

It follows from Lemma 3.5 that $L \nsubseteq (x_1)$. Therefore we may choose a generator m of L which is not divisible by x_1. Let d be the degree of m.

Since I contains no variable, L cannot contain x_1. Choose a squarefree monomial $m' \in (x_1) \smallsetminus L$ of maximal degree d'. As L is squarefree lex and m is not divisible by x_1, L contains all squarefree monomials that are divisible by x_1 and have degree d or larger. Thus, $d' < d$.

Let T be the ideal generated by $\text{gens}(L) \cup \{x_1^{d-d'}m'\} \setminus \{m\}$. Note that $|T_d| = |L_d|$.

Let $A = \text{gens}(\mathbf{m}_1 L_d)$ and $B = \text{gens}(\mathbf{m}_1 T_d)$ be the sets of degree $d+1$ monomials lying above L_d and T_d, respectively. If L were Gotzmann, it would follow that $|A| \leq |B|$. We will show that instead $|B| < |A|$.

We claim that $B \smallsetminus A = \{x_1^{d-d'}m'x_i : x_i \text{ divides } m'\}$. Indeed, let $\mu \in B \smallsetminus A$ be a monomial. Then μ is divisible by $x_1^{d-d'}m'$, so it has the form $x_1^{d-d'}m'x_i$ for some i. If x_i divides m', then the support of $x_1^{d-d'}m'x_i$ is m' and hence μ is not in A. On the other hand, if x_i does not divide m', then $m'x_i$ is a squarefree monomial of degree $d'+1$ which is divisible by x_1. By the choice of m', we have $m'x_i \in L$ and hence $x_1^{d-d'}m'x_i \in A$, proving the claim. In particular, $|B \smallsetminus A| = d'$.

Similarly, monomials in $A \smallsetminus B$ must have the form $x_i m$ for some i. If x_i divides m, then $x_i m$ has support m and hence is not in B. Thus,

$$A \smallsetminus B \supseteq \{x_i m : x_i \text{ divides } m\}$$

which has cardinality at least d.

As $|B \smallsetminus A| = d' < d \leq |A \smallsetminus B|$, it follows that $|\mathbf{m}_1 T_d| = |B| < |A| = |\mathbf{m}_1 L_d|$, and so L is not Gotzmann. $\square$

LEMMA 3.7. *Let $I \subseteq S$ be a Gotzmann squarefree monomial ideal with $I \subseteq (x_i)$. Then $\frac{1}{x_i} I$ is Gotzmann in S.*

Proof. Let L be the (non-squarefree) lexification of I. It is clear that $L \subseteq (x_1) : (x_1)$ is the lexification of (x_i), which contains I.

Now multiplication by x_i is a degree one module isomorphism from $\frac{1}{x_i} I$ to I, and similarly for L. Applying Corollary 2.7 twice, we have that $\frac{1}{x_i} I$ is Gotzmann with the same Hilbert function as $\frac{1}{x_1} L$. □

LEMMA 3.8. *Let $I \subseteq S$ be a Gotzmann squarefree monomial ideal with $(x_i) \subseteq I$. The image of I in the quotient ring $S/(x_i)$ is a Gotzmann squarefree monomial ideal.*

Proof. By renaming the variables if necessary, we may assume that $(x_1) \subseteq I$. Let $\bar{I}$ be the image of I in $S/(x_1)$ (or, equivalently, the squarefree monomial ideal of $\Bbbk[x_2, \ldots, x_n]$ generated by every generator of I other than x_1).

Let L be the (non-squarefree) lexification of I in S. We have $(x_1) \subseteq L$. Let $\bar{L}$ be the image of L in $S/(x_1)$. Then $\bar{L}$ is the lexification of $\bar{I}$. Observe that $\operatorname{gens}(\bar{I}) = \operatorname{gens}(I) \smallsetminus \{x_1\}$ and similarly for L. Thus, applying Corollary 2.7 twice, $\bar{I}$ is Gotzmann. □

Lemma 3.6 allows us to characterize the squarefree ideals which are Gotzmann.

THEOREM 3.9. *Suppose $I \subseteq S$ is a squarefree ideal. Then I is Gotzmann if and only if*

$$\begin{aligned} I = {} & m_1(x_{i_{1,1}}, \ldots, x_{i_{1,r_1}}) + m_1 m_2(x_{i_{2,1}}, \ldots, x_{i_{2,r_2}}) \\ & + \cdots + m_1 \cdots m_s(x_{i_{s,1}}, \ldots, x_{i_{s,r_s}}) \end{aligned}$$

for some squarefree monomials $m_1, \ldots, m_s$ and variables $x_{i,j}$ all having pairwise disjoint support.

Proof. Suppose that I has the given form. Then I is a canonical critical ideal (see [MH08]), and so is Gotzmann by [MH08, Theorem 1.1]. Alternatively, I is a lexlike ideal (see [Me06]) and hence Gotzmann.

Now suppose that I is Gotzmann. By Lemma 3.6, either $(x_j) \subseteq I$ or $I \subseteq (x_j)$ for some j.

If $I \subseteq (x_j)$, then $\frac{1}{x_j} I$ is Gotzmann in S and its generators are supported on $\{x_1, \ldots, \hat{x}_j, \ldots, x_n\}$. Inducting on the number of variables, $\frac{1}{x_j} I$ may be written as

$$m_1(x_{i_{1,1}}, \ldots, x_{i_{1,r_1}}) + m_1 m_2(x_{i_{2,1}}, \ldots, x_{i_{2,r_2}}) + \cdots + m_1 \cdots m_s(x_{i_{s,1}}, \ldots, x_{i_{s,r_s}})$$

where x_j does not appear in this expression. Thus, I can be expressed in the desired form by replacing m_1 with $x_j m_1$.

Alternatively, suppose that $(x_j) \subseteq I$, so, without loss of generality, $I = (x_j) + J$, where J is Gotzmann in the ring $\Bbbk[x_1, \ldots, \hat{x}_j, \ldots, x_n]$. By induction on the number of variables, J may be written in the desired form and so $I=(x_j)+J$ has the desired form as well (with $m_1=1$). $\square$

Using Theorem 3.9, it is possible, if difficult, to count the Gotzmann squarefree ideals of S. We begin by counting these ideals up to symmetry. (This is the same as counting the "universally squarefree lex" ideals: the squarefree lex ideals which are still squarefree lex in $S[y]$.)

PROPOSITION 3.10. *If $n\geq 2$, the following are all equal to 2^{n-2}:*

(i) *The number of ordered partitions of n into an even number of summands.*
(ii) *The number of ordered partitions of n into an odd number of summands.*
(iii) *The number of Gotzmann squarefree ideals, up to a reordering of the variables, which contain no linear forms and are not generated by monomials in fewer than all of the variables.*
(iv) *The number of Gotzmann squarefree ideals, up to a reordering of the variables, which do contain linear forms and are not generated by monomials in fewer than all of the variables.*
(v) *The number of Gotzmann squarefree ideals, up to a reordering of the variables, which contain no linear forms and are generated by monomials in fewer than all of the variables.*
(vi) *The number of Gotzmann squarefree ideals, up to a reordering of the variables, which do contain linear forms and are generated by monomials in fewer than all of the variables.*

In particular, there are 2^n nonunit Gotzmann squarefree ideals up to symmetry.

Proof. For (i) and (ii), recall from [St97, p. 14] that the number of ordered partitions of n with k parts is $\binom{n-1}{k-1}$. It follows that (i) and (ii) are both 2^{n-2}.

For (iii) through (vi), we describe a bijection to the ordered partitions. Given a partition, we partition the variables, in order, into sets of the given sizes. Using the notation of Theorem 3.9, we will alternate these sets between the supports of the monomials m_i and the sets $\{x_{i_{j,1}},\dots,x_{i_{j,r_j}}\}$. We begin with the monomial if we are counting without linear forms, and with the set if we are counting with linear forms (because $m_1=1$ in these cases). If we are counting ideals contained in a subalgebra, we do not use the last summand. Note that the parity of the partition is fixed in each case.

For example, consider the ordered partition $11=3+4+3+1$. This partitions the variables $\{x_1,\dots,x_{11}\}$ into four sets (up to reordering): $\{x_1,x_2,x_3\}$, $\{x_4,x_5,x_6,x_7\}$, $\{x_8,x_9,x_{10}\}$, and $\{x_{11}\}$. This induces two Gotzmann ideals:

$$I_{(\mathrm{iii})}=x_1x_2x_3(x_4,x_5,x_6,x_7)+x_1x_2x_3x_8x_9x_{10}(x_{11}),$$

which contains no linear forms and uses every variable in its generators, and

$$I_{(\mathrm{vi})}=1(x_1,x_2,x_3)+x_4x_5x_6x_7(x_8,x_9,x_{10}),$$

which contains linear forms and is generated by monomials involving a proper subset of the variables (x_{11} is not used). Any other even partition of 11 will similarly create ideals which are counted toward (iii) and (vi), while odd partitions will create ideals that are counted toward (iv) and (v).

Combining cases (iii) through (vi) shows that there are $4(2^{n-2}) = 2^n$ non-unit Gotzmann squarefree monomial ideals in the polynomial ring in $n \geq 2$ variables. This formula also holds when $n = 0$ and $n = 1$. $\square$

We use the same idea of partitioning the variables and alternating between monomials m_i and sets $\{x_{i_{j,1}}, \ldots, x_{i_{j,r_j}}\}$ to count the Gotzmann squarefree ideals of S without symmetry. The difficulty is that it is easy to overcount those ideals with $r_s = 1$.

Let G_n be the set of all Gotzmann squarefree ideals of $\Bbbk[x_1, \ldots, x_n]$ and G the disjoint union of all G_n. We define a weight function $\omega : \mathsf{G} \to \mathbb{N}$ by $\omega(I) = n$ if $I \in \mathsf{G}_n$.

We will show that the exponential generating function (e.g.f.) of G is

$$g(t) = \sum_{I \in \mathsf{G}} \frac{t^{\omega(I)}}{\omega(I)!} = e^t \left(\frac{2(1-t)}{2 - e^t} + t \right).$$

The coefficients this e.g.f. count the number of Gotzmann squarefree ideals in polynomial rings in n variables for each value of n.

We begin with notation for ordered set partitions. In Proposition 3.13, we relate them to the set $\mathsf{H} \subset \mathsf{G}$ of all Gotzmann squarefree ideals with full support (i.e., $I \in \mathsf{H}$ means I uses all n variables where $n = \omega(I)$).

NOTATION 3.11 (Ordered set partitions). An ordered set partition of $[n] = \{1, \ldots, n\}$ is a sequence $\sigma = (\sigma_1, \ldots, \sigma_k)$ of sets σ_i which partition $[n]$. Each σ_i is called a *block* of σ.

Let P_n be the set of ordered set partitions of $[n]$ and P be the union of all P_n. On the set P we define a weight function $\nu : \mathsf{P} \to \mathbb{N}$ by $\nu(\sigma) = n$ where $\sigma \in \mathsf{P}_n$.

We will use the e.g.f. of P which counts the number of ordered set partitions of $[n]$:

$$f(t) = \sum_{\sigma \in \mathsf{P}} \frac{t^{\nu(\sigma)}}{\nu(\sigma)!} = \frac{1}{2 - e^t}.$$

This e.g.f. is entry A670 in the on-line encyclopedia of integer sequences [Sl03].

LEMMA 3.12. *Let P' be the set of ordered set partitions that have last blocks of size greater than one. The e.g.f. of P' with weight ν is $(1-t)/(2-e^t)$.*

Proof. Let $\sigma \in \mathsf{P} \smallsetminus \mathsf{P}'$ be an ordered set partition with $\nu(\sigma) = n + 1$. Then the last block of σ is the singleton $\{i\}$ for some $i = 1, \ldots, n + 1$. Removing the last block from this partition gives a bijection between ordered set partitions ending in $\{i\}$ and ordered set partitions of a set of size n. Thus, the exponential generating function of $\mathsf{P} \setminus \mathsf{P}'$ is $tf(t) = \frac{t}{2-e^t}$ and hence the e.g.f. of P' is $f(t) - tf(t) = \frac{1-t}{2-e^t}$. □

The next proposition describes the relationship between ordered set partitions and the set H of Gotzmann squarefree ideals with full support.

PROPOSITION 3.13. *The e.g.f. of* H *with weight* ω *is*

$$h(t) = \sum_{I \in \mathsf{H}} \frac{t^{\omega(I)}}{\omega(I)!} = \frac{2(1-t)}{2-e^t} + t.$$

Proof. Every ideal in H is of the form

$$m_1(x_{i_{1,1}}, \ldots, x_{i_{1,r_1}}) + m_1 m_2(x_{i_{2,1}}, \ldots, x_{i_{2,r_2}}) + \cdots + m_1 \cdots m_s(x_{i_{s,1}}, \ldots, x_{i_{s,r_s}})$$

for some m_j and $x_{i_{j,k}}$ all distinct. Let $\mu_d(I)$ be the number of generators of I of degree d, and let $\operatorname{top}(I)$ be the largest degree of a generator of I. (Because I has a linear resolution, $\operatorname{top}(I)$ is equal to the Castelnuovo–Mumford regularity of I.) We partition H into five subsets $\mathsf{H} = \bigcup_{i=0}^{4} \mathsf{H}_i$ where

$\mathsf{H}_0 = \big\{(x_1)\big\}$,
$\mathsf{H}_1 = \big\{I \in H \mid I \text{ contains a linear form and } \mu_{\operatorname{top}(I)}(I) = 1 \text{ and } \operatorname{top}(I) \neq 1\big\}$,
$\mathsf{H}_2 = \big\{I \in H \mid I \text{ contains a linear form and } \mu_{\operatorname{top}(I)}(I) > 1\big\}$,
$\mathsf{H}_3 = \big\{I \in H \mid I \text{ does not contains a linear form and } \mu_{\operatorname{top}(I)}(I) = 1\big\}$ and
$\mathsf{H}_4 = \big\{I \in H \mid I \text{ does not contains a linear form and } \mu_{\operatorname{top}(I)}(I) > 1\big\}$.

Recall that we use P' to denote the set of ordered set partitions whose last block is not a singleton. There is a weight preserving bijection between P' and $\mathsf{H}_1 \cup \mathsf{H}_2$ that maps $(\sigma_1, \ldots, \sigma_k)$ to

$$(\sigma_1) + \Big(\prod \sigma_2\Big)(\sigma_3) + \cdots + \left(\prod_{i=1}^{(k-1)/2} \prod \sigma_{2i}\right)(\sigma_k) \quad \text{for } k \text{ odd and}$$
$$(\sigma_1) + \Big(\prod \sigma_2\Big)(\sigma_3) + \cdots + \left(\prod_{i=1}^{k/2} \prod \sigma_{2i}\right) \quad \text{for } k \text{ even.}$$

Similarly, there is a weight preserving bijection between P' and $\mathsf{H}_3 \cup \mathsf{H}_4$ that sends $(\sigma_1, \ldots, \sigma_k)$ to

$$\Big(\prod \sigma_1\Big)(\sigma_2) + \Big(\prod \sigma_1\Big)\Big(\prod \sigma_3\Big)(\sigma_4) + \cdots + \left(\prod_{i=1}^{k/2} \prod \sigma_{2i-1}\right)(\sigma_k) \quad \text{for } k \text{ even and}$$
$$\Big(\prod \sigma_1\Big)(\sigma_2) + \Big(\prod \sigma_1\Big)\Big(\prod \sigma_3\Big)(\sigma_4) + \cdots + \left(\prod_{i=1}^{(k+1)/2} \prod \sigma_{2i-1}\right) \quad \text{for } k \text{ odd.}$$

The desired formula follows from Lemma 3.12. □

COROLLARY 3.14. *The exponential generating function for* G, *the set of all Gotzmann squarefree monomial ideals, is*

$$g(t) = \sum_{I \in \mathsf{G}} \frac{t^{\omega(I)}}{\omega(I)!} = e^t \left(\frac{2(1-t)}{2-e^t} + t \right)$$

Proof. For each Gotzmann squarefree monomial ideal with full support in a polynomial ring over k variables there are $\binom{n}{k}$ Gotzmann squarefree monomial ideals in a polynomial ring over n variables with support of size k. Thus, we apply the inverse binomial transform to the previous proposition (i.e., multiply the e.g.f. by e^t). □

From this generating function, one can extract the number of Gotzmann squarefree ideals in $\Bbbk[x_1, \dots, x_n]$. For $0 \le n \le 5$, these numbers are 2, 3, 6, 19, 96, and 669.

4. Gotzmann ideals of the squarefree ring

The problem of classifying all Gotzmann monomial ideals of the squarefree ring R turns out to be much more difficult. We might hope to prove some squarefree analog of Lemma 3.6; then, arguing as in the previous section, we would be able to prove that Gotzmann ideals of R are lex segments (if generated in one degree) or initial segments in a lexlike tower (see [Me06]) in general. Unfortunately such an approach is doomed to fail, as the following examples show.

EXAMPLE 4.1. The ideal $I = (ab, ac, bd, cd)$ is Gotzmann in R but is not lex.

The ideal I above is (up to symmetry) the only monomial Gotzmann ideal of $\Bbbk[a,b,c,d]/(a^2,b^2,c^2,d^2)$ which is not lex in some order. Thus we might hope that it is the only such ideal, or at least is the first instance of a one-parameter family of exceptions. This hope is dashed as well as soon as we add a fifth variable.

EXAMPLE 4.2. The ideal $I = (abc, abd, abe, acd, ace, bcd, bce)$ is Gotzmann in R but is not lex.

Since the Alexander duals of lex ideals are lex, we might hope that the Alexander duals of Gotzmann ideals are Gotzmann. However, the duals of the two examples above are not Gotzmann. We will see in Theorem 4.17 that a Gotzmann ideal has Gotzmann dual if and only if it is in some sense morally lex.

Throughout the section, all ideals will be monomial ideals of R. Since we no longer work with the polynomial ring, we can dispense with the notation I^{sf} to indicate that an ideal lives in R, and will simply write I, J, etc. Many of our arguments are technical, so for ease of notation we work mostly with monomial vector spaces rather than ideals. Recall that a vector space $V \subset R_d$ is Gotzmann if $|\mathbf{m}_1 V|$ is minimal given $|V|$ and d, and that an ideal I is Gotzmann if and only if $|I_d|$ is Gotzmann for all d.

4.1. Decomposing Gotzmann ideals of R. In this section we show every Gotzmann monomial vector space $V \subseteq R_d$ can be decomposed as the direct sum of two monomial vector spaces which are Gotzmann in a squarefree ring with one fewer generator. This decomposition relates to the operation of compression (see [MP06] or [Me08]). We begin by recalling the necessary notation.

Given a (fixed) variable x_i, let $\mathbf{n} = (x_1, \ldots, \hat{x}_i, \ldots, x_n)$ be the maximal ideal in $Q = R/(x_i)$ which is a squarefree ring on $n-1$ variables.

Definition 4.3 (x_i-decomposition). Let $V \subseteq R_d$ be a monomial vector space and fix a variable x_i. The monomial basis of V can be partitioned as $A \cup B$ where A contains the monomials divisible by x_i and B contains those not divisible by x_i.

Let V_0 be the monomial vector space spanned by B and let V_1 be the monomial vector space spanned by $\{m \mid x_i m \in A\}$. We write V as the direct sum

$$V = V_0 \oplus x_i V_1$$

which we call the *x_i-decomposition* of V.

We view the monomial vector spaces V_0 and V_1 as subspaces of Q_d and Q_{d-1} respectively.

Definition 4.4 (x_i-compression). Let $V = V_0 \oplus x_i V_1$ be the x_i-decomposition of the monomial vector space V. Let L_0 and L_1 be the squarefree lex-segments in Q with the same degrees and dimensions as V_0 and V_1. The *x_i-compression* of V is the monomial vector space

$$T = L_0 \oplus x_i L_1.$$

We recall the following important fact about compression from [MP06]:

Proposition 4.5 ([MP06]). *If T is the x_i-compression of the monomial vector space $V \subseteq R_d$, then*

$$|\mathbf{m}_1 T| \leq |\mathbf{m}_1 V|.$$

Lemma 4.6. *If $V = V_0 \oplus x_i V_1$ then the x_i-decomposition of $\mathbf{m}_1 V$ is*

$$\mathbf{m}_1 V = \mathbf{n}_1 V_0 \oplus x_i (V_0 + \mathbf{n}_1 V_1).$$

Proof. Since $x_i^2 = 0$, we have $\mathbf{m}_1(x_i V_1) = \mathbf{n}_1(x_i V_1)$. Thus,

$$\begin{aligned} \mathbf{m}_1 V &= \mathbf{m}_1(V_0 + x_i V_1) \\ &= \mathbf{n}_1 V_0 + x_i V_0 + x_i \mathbf{n}_1 V_1 \\ &= \mathbf{n}_1 V_0 \oplus x_i (V_0 + \mathbf{n}_1 V_1). \end{aligned}$$

This sum is direct since the second summand is contained in (x_i) while the first summand is not. $\square$

LEMMA 4.7. *Let $V \subseteq R_d$ be a Gotzmann monomial vector space and let L be its x_i-compression. If $V = V_0 \oplus x_i V_1$ and $L = L_0 \oplus x_i L_1$ are the x_i-decompositions of V and L, then*

$$|\mathbf{n}_1 V_0| + |V_0 + \mathbf{n}_1 V_1| = |\mathbf{n}_1 L_0| + |L_0 + \mathbf{n}_1 L_1|.$$

Proof. As V is Gotzmann $|\mathbf{m}_1 V| \leq |\mathbf{m}_1 L|$ and so $|\mathbf{m}_1 V| = |\mathbf{m}_1 L|$ by Proposition 4.5. The desired equality now follows from the previous lemma. $\square$

PROPOSITION 4.8. *Let $V \subseteq R_d$ be a Gotzmann monomial vector space and let $V = V_0 \oplus x_i V_1$ be its x_i-decomposition. Then V_0 is Gotzmann in Q.*

Proof. Let L be the x_i-compression of V and let $L = L_0 \oplus x_i L_1$ be its x_i-decomposition.

Since L_1 and $\mathbf{n}_1 L_0$ are lex segments of the same degree, it follows that one is contained in the other. If $\mathbf{n}_1 L_1 \subseteq L_0$, then

$$|L_0 + \mathbf{n}_1 L_1| = |L_0| = |V_0| \leq |V_0 + \mathbf{n}_1 V_1|.$$

Similarly, if $L_0 \subseteq \mathbf{n}_1 L_1$ then

$$|L_0 + \mathbf{n}_1 L_1| = |\mathbf{n}_1 L_1| \leq |\mathbf{n}_1 V_1| \leq |V_0 + \mathbf{n}_1 V_1|.$$

In both cases $|L_0 + \mathbf{n}_1 L_1| \leq |V_0 + \mathbf{n}_1 V_1|$. From Lemma 4.7, we see that $|\mathbf{n}_1 V_0| \leq |\mathbf{n}_1 L_0|$ and hence V_0 is Gotzmann by Proposition 2.6. $\square$

LEMMA 4.9. *Let V be Gotzmann in R with x_i-decomposition $V = V_0 \oplus x_i V_1$ and let $L = L_0 \oplus x_i L_1$ be its x_i-compression. Then either V_1 is Gotzmann in Q or $\mathbf{n}_1 L_1 \subsetneq L_0$.*

Proof. We know from the previous proposition that V_0 is Gotzmann in Q and hence $|\mathbf{n}_1 V_0| = |\mathbf{n}_1 L_0|$. Thus, Lemma 4.7 gives

$$|V_0 + \mathbf{n}_1 V_1| = |L_0 + \mathbf{n}_1 L_1|.$$

If $\mathbf{n}_1 L_1 \not\subset L_0$ then $L_0 \subseteq \mathbf{n}_1 L_1$ as they are both lex segments. Thus,

$$|\mathbf{n}_1 V_1| \leq |V_0 + \mathbf{n}_1 V_1| = |L_0 + \mathbf{n}_1 L_1| = |\mathbf{n}_1 L_1|$$

which proves that V_1 is Gotzmann. $\square$

If $\mathbf{n}_1 L_1 \subset L_0$, then V_1 need not be Gotzmann. For example,

$$V = \operatorname{span}_{\Bbbk}\{abc, abd, acd, bcd, bce, bde, cde\}$$

is Gotzmann in $R = \Bbbk[a,b,c,d,e]/(a^2,\ldots,e^2)$, but $V_1 = \operatorname{span}_{\Bbbk}\{bc, bd, cd\}$ from the a-decomposition of V is not Gotzmann in $Q = R/(a)$.

However, we will see that it is always possible to choose x_i such that V_1 is Gotzmann.

LEMMA 4.10. *Let V be Gotzmann with x_i-decomposition $V = V_0 \oplus x_i V_1$ and compression $L = L_0 \oplus x_i L_1$. If $\mathbf{n}_1 L_1 \subseteq L_0$, then V satisfies the property*:

> *Let $m \in V$ be a monomial such that x_i divides m, and let x_j be any variable not dividing m. Then $\frac{x_j}{x_i} m \in V$.*

Proof. Applying Lemma 4.7, we have $|\mathbf{n}_1 V_1 + V_0| = |\mathbf{n}_1 L_1 + L_0| = |L_0| = |V_0|$, i.e., $\mathbf{n}_1 V_1 \subseteq V_0$. The desired property follows. □

THEOREM 4.11. *Suppose $V \subset R_d$ is a Gotzmann monomial vector space. Then x_i may be chosen so that both summands V_1 and V_0 of the x_i-decomposition of V are Gotzmann in Q and $V_0 \subseteq \mathbf{n}_1 V_1$.*

Proof. Suppose that x_i cannot be chosen so that the summands L_1 and L_0 of the x_i-compression satisfy $L_0 \subseteq \mathbf{n}_1 L_1$. Then Lemma 4.10 applies for all x_i, so V satisfies the property:

> Let $m \in V$ be a monomial, and suppose that x_i divides m and x_j does not. Then $\frac{x_j}{x_i} m \in V$ as well.

The only subspaces of R_d satisfying this property are (0) and R_d. In either case, we have $L_0 \subseteq \mathbf{n}_1 L_1$ for any x_i.

Thus, x_i may be chosen such that $L_0 \subseteq \mathbf{n}_1 L_1$. Then, by Proposition 4.8 and Lemma 4.9, V_1 and V_0 are Gotzmann in Q. Applying Lemma 4.7, we have $|V_0 + \mathbf{n}_1 V_1| = |L_0 + \mathbf{n}_1 L_1| = |\mathbf{n}_1 L_1| = |\mathbf{n}_1 V_1|$, i.e., $V_0 \subseteq \mathbf{n}_1 V_1$. □

In fact, the obvious choice of variable works:

LEMMA 4.12. *Suppose $V \subset R_d$ is a Gotzmann monomial vector space, and let x_i be such that $|V \cap (x_i)|$ is maximal. Let $V = V_0 \oplus x_i V_1$ be the x_i-decomposition of V. Then V_0 and V_1 are both Gotzmann in Q and $V_0 \subseteq \mathbf{n}_1 V_1$.*

Proof. Let L_0 and L_1 be the lexifications in Q of V_0 and V_1, respectively.

By Theorem 4.11, there exists a variable x_j such that we may decompose $V = W_0 \oplus x_j W_1$ with both W_0 and W_1 Gotzmann in Q and $W_0 \subseteq \mathbf{n}_1 W_1$.

We have

$$|L_0| \leq |W_0| \leq |\mathbf{n}_1 W_1| \leq |\mathbf{n}_1 L_1|,$$

the first inequality by construction, the second by Theorem 4.11, and the third because $|W_1| \leq |L_1|$ and both are Gotzmann. By Proposition 4.8 and Lemma 4.9, V_0 and V_1 are Gotzmann. Again applying Lemma 4.7, we have $|V_0 + \mathbf{n}_1 V_1| = |L_0 + \mathbf{n}_1 L_1| = |\mathbf{n}_1 L_1| = |\mathbf{n}_1 V_1|$, i.e., $V_0 \subseteq \mathbf{n}_1 V_1$. □

Unfortunately the converse to Theorem 4.11 does not hold in general. For example, let $V = \operatorname{span}_{\Bbbk}\{ab, ac, bc\}$ in $R = \Bbbk[a,b,c,d]/(a^2, \ldots, d^2)$. Then V is not Gotzmann in R but, decomposing with respect to a, $V_0 = \operatorname{span}_{\Bbbk}\{bc\}$ and $V_1 = \operatorname{span}_{\Bbbk}\{b, c\}$ are both Gotzmann in $Q = \Bbbk[b,c,d]/(b^2, c^2, d^2)$.

However, we can prove the following partial converse.

THEOREM 4.13. *Let V_0 and V_1 be Gotzmann monomial vector spaces in Q with $V_0 = \mathbf{n}_1 V_1$. Then $V = V_0 \oplus x_i V_1$ is Gotzmann in R.*

Proof. By permuting the variables, we may assume that $i = n$ so that $Q = R/(x_n)$ and $V = V_0 \oplus x_n V_1$ is the x_n-decomposition of V. It will be important that x_n is the lex-last variable.

Let $L = L_0 \oplus x_n L_1$ be the x_n-compression of V. By the definition of compression, $|L_1| = |V_1|$ and $|L_0| = |V_0| = |\mathbf{n}_1 V_1|$. Since V_1 and L_1 are Gotzmann, $|\mathbf{n_1} L_1| = |\mathbf{n_1} V_1| = |L_0|$. Thus, $L_0 = \mathbf{n_1} L_1$. Therefore,

$$\begin{aligned} |\mathbf{m}_1 V| &= |\mathbf{n}_1 V_0| + |V_0 + \mathbf{n}_1 V_1| \\ &= |\mathbf{n}_1 V_0| + |V_0| \\ &= |\mathbf{n}_1 L_0| + |L_0| \\ &= |\mathbf{n}_1 L_0| + |L_0 + \mathbf{n}_1 L_1| \\ &= |\mathbf{m}_1 L|. \end{aligned}$$

Thus, it suffices to show that L is lex.

Indeed, suppose that $u \in L$ and v is a monomial of the same degree which precedes u in the lex order. If both or neither of u, v are divisible by x_n, then clearly $v \in L$. Now suppose that u is divisible by x_n but v is not. Then we may write $u = u' x_n$. By construction, v precedes u' in the (ungraded) lex order. Let $v' = \frac{v}{x_j}$, where x_j is the lex-last variable dividing v. Then v' precedes u' in the lex order as well, so $u' \in L_1$ implies $v' \in L_1$ and in particular $v \in \mathbf{n}_1 L_1 = L_0$. A similar argument shows that $v \in L$ if v is divisible by x_n but u is not. □

EXAMPLE 4.14. Consider the Gotzmann vector space

$$V_1 = \operatorname{span}_{\Bbbk}\{ab, bc, cd, ad\}$$

in $Q = \Bbbk[a, b, c, d]/(a^2, \ldots, d^2)$. Let $V_0 = \mathbf{n}_1 V_1$:

$$V_0 = \operatorname{span}_{\Bbbk}\{abc, abd, acd, bcd\}.$$

In $R = \Bbbk[a, b, c, d, e]/(a^2, \ldots, e^2)$, the monomial vector space $V = V_0 + eV_1$ is Gotzmann but is not lex with respect to any order of the variables.

4.2. Alexander duality. Recall that for a monomial vector space $V \subseteq R_d$, the *Alexander dual* of V is the subspace $V^\vee \subset R_{n-d}$ spanned by the monomials $\{\frac{\mathbf{x}}{m} : m \notin V\}$ where $\mathbf{x}$ is the product of all the variables. For a monomial ideal $I \subset R$, the Alexander dual is $I^\vee = \oplus (I_d)^\vee$. This duality corresponds to topological Alexander duality under the Stanley–Reisner correspondence, and turns out to have many nice algebraic properties. For example, duality turns generators into associated primes, and the duals of lex or Borel ideals are always lex or Borel, respectively. Thus, we would like to understand ideals whose duals are Gotzmann.

DEFINITION 4.15. We say that a monomial vector space V is *co-Gotzmann* if $V^\vee$ is Gotzmann.

THEOREM 4.16. *Let V be co-Gotzmann in R. Then x_i may be chosen so that both summands V_0 and V_1 of the x_i-decomposition are co-Gotzmann in Q, and $(V_0 : \mathbf{n}_1) \subseteq V_1$.*

Proof. Let $W = V^\vee$. Then Theorem 4.11 applies to W, so we may choose x_i such that W_0 and W_1 are Gotzmann in Q and $W_0 \subseteq \mathbf{n}_1 W_1$.

We compute $V_0 = (W_1)^\vee$ and $V_1 = (W_0)^\vee$. In particular, V_0 and V_1 are co-Gotzmann. Finally, suppose that $m \in (V_0 : \mathbf{n}_1)$. We will show that $m \in V_1$. By construction, $mx_j \in V_0$ for all $x_j \neq x_i$ and not dividing m, so $\frac{\mathbf{x}}{mx_j} \notin W_1$ for any such x_j. Hence, $\frac{\mathbf{x}}{m} \notin \mathbf{n}_1 W_1$. Since $W_0 \subseteq \mathbf{n}_1 W_1$, we have $\frac{\mathbf{x}}{m} \notin W_0$. Thus, $m \in V_1$, as desired. □

Thus, any recursive enumeration of co-Gotzmann ideals should look similar to any recursive enumeration of Gotzmann ideals. However, they will not be identical. In fact, ideals which are simultaneously Gotzmann and co-Gotzmann are quite rare, as the next theorem shows.

THEOREM 4.17. *Suppose that $V \subset R_d$ is both Gotzmann and co-Gotzmann. Then V is lex in some order.*

Proof. Suppose not. Then there exists a counterexample $V \subset R_d$ where $R = \mathbb{k}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ with n minimal. Let x_i be such that $|V \cap (x_i)|$ is maximal. Then $|V^\vee \cap (x_i)|$ is maximal as well, and Lemma 4.12 applies to both V and $V^\vee$. Thus V_0 and V_1 are both Gotzmann and co-Gotzmann, so, by the minimality of n, both are lex in Q. Since V is not lex, we have $V_0 \neq 0$ and $V_1 \neq Q_{d-1}$.

Since $V_0 \neq 0$, we claim that $\mathbf{m}_{n-d-1} V = R_{n-1}$. Indeed, suppose that some monomial m of degree $n-1$ were not contained in $\mathbf{m}_{n-d-1} V$. Write $m = \frac{x_1 \cdots x_n}{x_j}$ for some j; it follows that $V \subset (x_j)$. But then we would have $|V \cap (x_j)| = |V|$, while $|V \cap (x_i)| = |V| - |V_0|$, contradicting the choice of i.

Hence, the lexification of V (in any order where x_i comes first) must contain at least one monomial not divisible by x_i. Similarly, the lexification of $V^\vee$ must contain at least one monomial not divisible by x_i. Thus, if L and $L^\vee$ are the lexifications of V and $V^\vee$, respectively, we have

$$
\begin{aligned}
|L| + \left|L^\vee\right| &\geq |Q_{d-1}| + 1 + |Q_{n-d-1}| + 1 \\
&\gneq |Q_{d-1}| + |Q_d| \\
&= |R_d|.
\end{aligned}
$$

On the other hand, $|L| + |L^\vee| = |V| + |V^\vee| = |R_d|$. Thus, such a minimal counterexample cannot exist. □

Note that Theorem 4.17 is not a theorem about ideals. If an ideal I is both Gotzmann and co-Gotzmann, then Theorem 4.17 guarantees that every degreewise component I_d is lex in some order, but does not guarantee a consistent order. For example, the ideal $I = (bc, abd, abe, acd, ace, ade) \subset$

$\Bbbk[a,b,c,d,e]/(a^2,b^2,c^2,d^2,e^2)$ is Gotzmann and co-Gotzmann, but is not lex in any order. The component I_p is lex with respect to the order $a > b > c > d > e$ for $p \neq 2$, and with respect to the order $b > c > a > d > e$ for $p < 3$, but no lex order works in both degrees two and three.

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