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ORLICZ–SOBOLEV CAPACITY OF BALLS

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ABSTRACT. Our aim in this note is to estimate the Orlicz–Sobolev capacity of balls.

1. Introduction and statement of results

For $0 < \alpha < n$ and a locally integrable function f on $\mathbf{R}^n$, we define the Riesz potential $I_\alpha f$ of order α by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\, dy.$$

In the present note, we treat functions f satisfying an Orlicz condition:

$$\int_{\mathbf{R}^n} \varphi_p\bigl(|f(y)|\bigr)\, dy < \infty. \tag{1.1}$$

Here, $\varphi_p(r)$ is a positive nondecreasing function on the interval $(0,\infty)$ of the form

$$\varphi_p(r) = r^p \varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive monotone function on $(0,\infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

(φ1)

$$c_1^{-1}\varphi(r) \le \varphi\bigl(r^2\bigr) \le c_1 \varphi(r) \quad \text{whenever } r > 0.$$

We set

$$\varphi_p(0) = 0,$$

because we will see from (φ4) below that

$$\lim_{r\to 0+} \varphi_p(r) = 0;$$

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see [14, p. 205]. For an open set $G \subset \mathbf{R}^n$, we denote by $L^{\varphi_p}(G)$ the family of all locally integrable functions g on G such that

$$\int_G \varphi_p\big(|g(x)|\big)\,dx < \infty,$$

and define

$$\|g\|_{\varphi_p,G} = \inf\left\{\lambda > 0 : \int_G \varphi_p\big(|g(x)|/\lambda\big)\,dx \le 1\right\}.$$

This is a quasi-norm in $L^{\varphi_p}(G)$. For $E \subset G$, the (α,φ_p)-capacity is defined by

$$C_{\alpha,\varphi_p}(E;G) = \inf \|f\|_{\varphi_p,G},$$

where the infimum is taken over all functions f such that $f = 0$ outside G and

$$I_\alpha f(x) \ge 1 \quad \text{for all } x \in E$$

(cf. Adams and Hedberg [1], Meyers [10], Ziemer [17] and the second author [11], [12]).

Our aim in the present note is to give an estimate of (α,φ_p)-capacity of balls. We denote by $B(x,r)$ the open ball centered at x of radius r. For $R > 0$, consider

$$\tilde{\varphi}_p(r) = \int_r^R \big[t^{n-\alpha p}\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\,dt/t.$$

As an extension of Adams and Hurri-Syrjänen [3, Theorem 2.11] and Joensuu [9, Corollary 6.3], we state our theorem in the following.

Theorem A. *Suppose $p > 1$ and*

$$\tilde{\varphi}_p(0) = \infty.$$

For $R > 0$, there exists a constant $A > 0$ such that

$$A^{-1}\tilde{\varphi}_p(r)^{-(p-1)/p} \le C_{\alpha,\varphi_p}\big(B(x,r);B(x,R)\big) \le A\tilde{\varphi}_p(r)^{-(p-1)/p}$$

whenever $0 < r < R/2$.

Recently Joensuu [9, Corollary 6.3] treated the case when φ is nondecreasing. His main idea was to use the rearrangement equivalent norm for $\|f\|_{\varphi_p,G}$ ([5], [7], [8]), as an extension of Adams and Hurri-Syrjänen [3, Theorem 2.11] in the case when $\varphi(t) = (\log(e+t))^\beta$ with $p = n/\alpha > 1$ and $0 \le \beta \le p-1$. Our proof will be done straightforward from the definition of capacity, and several technical assumptions posed in [9] are removed.

Throughout this note, let A denote various constants independent of the variables in question and $A(a,b,\ldots)$ be a constant that depends on $a,b,\ldots$.

Remark 1.1. If $\tilde{\varphi}_p(0) < \infty$, then $C_{\alpha,\varphi_p}(\{0\};B(0,R)) > 0$. In this case, $I_\alpha f$ is continuous when $f \in L^{\varphi_p}(\mathbf{R}^n)$ vanishes outside a compact set; for this fact, we refer the reader to the paper [14], [16].

REMARK 1.2. We here introduce another capacity. For a set $E \subset \mathbf{R}^n$ and an open set $G \subset \mathbf{R}^n$, we define

$$B_{\alpha,\varphi_p}(E;G) = \inf \int_G \varphi_p\big(f(y)\big)\,dy,$$

where the infimum is taken over all nonnegative measurable functions f on $\mathbf{R}^n$ such that $f = 0$ outside G and $I_\alpha f(x) \geq 1$ for all $x \in E$. With the aid of Adams and Hurri-Syrjänen [3], Joensuu [7], [8], [9] and Mizuta [12, Section 8.3, Lemma 3.1], [11], one can find a constant $A > 1$ such that

$$A^{-1}\tilde{\varphi}_p(r)^{-(p-1)} \leq B_{\alpha,\varphi_p}\big(B(x,r);B(x,R)\big) \leq A\tilde{\varphi}_p(r)^{-(p-1)}$$

for $0 < r < R/2$ and $x \in \mathbf{R}^n$. Hence, in view of Theorem A, there is a constant $A > 1$ such that

$$\begin{aligned} A^{-1}B_{\alpha,\varphi_p}\big(B(x,r);B(x,R)\big)^{1/p} &\leq C_{\alpha,\varphi_p}\big(B(x,r);B(x,R)\big) \\ &\leq AB_{\alpha,\varphi_p}\big(B(x,r);B(x,R)\big)^{1/p} \end{aligned}$$

for $0 < r < R/2$ and $x \in \mathbf{R}^n$.

We write $f \sim g$ if there exists a constant A so that $A^{-1}g \leq f \leq Ag$.

EXAMPLE 1.3. For $n = \alpha p$, consider the function

$$\varphi(t) = \big(\log(e+t)\big)^\beta.$$

If $\beta < p-1$, then

$$\tilde{\varphi}_p(r) \sim \big(\log(e+1/r)\big)^{-\beta/(p-1)+1}$$

for $0 < r < 1$. In this case,

$$C_{\alpha,\varphi_p}\big(B(x_0,r);B(x_0,R)\big) \sim \big(\log(e+1/r)\big)^{(\beta-p+1)/p}$$

whenever $0 < r < R/2$ and $x_0 \in \mathbf{R}^n$.

If $\beta = p-1$, then

$$\tilde{\varphi}_p(r) \sim \log\big(e + \big(\log(e+1/r)\big)\big)$$

for $0 < r < 1$. In this case,

$$C_{\alpha,\varphi_p}\big(B(x_0,r);B(x_0,R)\big) \sim \big(\log\big(e + \big(\log(e+1/r)\big)\big)\big)^{-(p-1)/p}$$

whenever $0 < r < R/2$ and $x_0 \in \mathbf{R}^n$.

For further related results, see Aissaoui and A. Benkirane [4], Adams and Hurri-Syrjänen [2], Edmunds and Evans [6] and Mizuta and Shimomura [14], [15], [16].

2. Proof of Theorem A

First, we collect properties which follow from condition (φ1) (see [12], [14, Lemma 2.3], [13, Section 7]).

(φ2) φ satisfies the doubling condition, that is, there exists $c_2 > 1$ such that

$$c_2^{-1}\varphi(r) \le \varphi(2r) \le c_2\varphi(r) \quad \text{whenever } r > 0.$$

(φ3) For each $\gamma > 0$, there exists $c_3 = c_3(\gamma) \ge 1$ such that

$$c_3^{-1}\varphi(r) \le \varphi\big(r^\gamma\big) \le c_3\varphi(r) \quad \text{whenever } r > 0.$$

(φ4) For each $\gamma > 0$, there exists $c_4 = c_4(\gamma) \ge 1$ such that

$$s^\gamma\varphi(s) \le c_4 t^\gamma\varphi(t) \quad \text{whenever } 0 < s < t.$$

(φ5) For each $\gamma > 0$, there exists $c_5 = c_5(\gamma) \ge 1$ such that

$$t^{-\gamma}\varphi(t) \le c_5 s^{-\gamma}\varphi(s) \quad \text{whenever } 0 < s < t.$$

(φ6) If φ and φ_1 are positive monotone functions on $[0,\infty)$ satisfying (φ1), then for each $\gamma > 0$ then there exists a constant $c_6 = c_6(\gamma) \ge 1$ such that

$$c_6{}^{-1}\varphi(r) \le \varphi\big(r^\gamma\varphi_1(r)\big) \le c_6\varphi(r) \quad \text{whenever } r > 0.$$

Remark 2.1. For each $A_1 > 0$ there exists $A_2 > 0$ such that

$$A_1\varphi_p(r) \ge \varphi_p(A_2 r) \quad \text{whenever } r > 0. \tag{2.1}$$

Remark 2.2. If $\alpha p < n$, then we see from (φ2) and (φ5) that

$$\tilde{\varphi}_p(r) \sim \big[r^{n-\alpha p}\varphi\big(r^{-1}\big)\big]^{-1/(p-1)} \tag{2.2}$$

whenever $0 < r < R/2$.

Remark 2.3. If $n = \alpha p$ and $0 < R \le 1$, then $\tilde{\varphi}_p$ is of logarithmic type on $[0, R^2]$, that is, there exists $c > 0$ such that

$$c^{-1}\tilde{\varphi}_p(r) \le \tilde{\varphi}_p\big(r^2\big) \le c\tilde{\varphi}_p(r) \quad \text{whenever } 0 \le r \le R^2.$$

In fact, we see from (φ1) that

$$\begin{aligned}
\tilde{\varphi}_p\big(r^2\big) &= \int_{r^2}^{R} \big[\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\,dt/t \\
&= \int_{r^2}^{R^2} \big[\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\,dt/t + \int_{R^2}^{R} \big[\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\,dt/t \\
&= 2\int_{r}^{R} \big[\varphi\big(t^{-2}\big)\big]^{-1/(p-1)}\,dt/t + \int_{R^2}^{R} \big[\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\,dt/t \\
&\le 2c_1^{1/(p-1)}\int_{r}^{R} \big[\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\,dt/t + \int_{R^2}^{R} \big[\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\,dt/t \\
&\le \big(2c_1^{1/(p-1)} + 1\big)\tilde{\varphi}_p(r)
\end{aligned}$$

whenever $0 < r \le R^2$. Since $\tilde{\varphi}_p(r) \le \tilde{\varphi}_p(r^2)$, we see that $\tilde{\varphi}_p$ is of logarithmic type on $[0, R^2]$.

If $R^2 < r < R$, then one sees that $\tilde{\varphi}_p(r) \sim \varphi(R^{-1})^{-1/(p-1)} \log(R/r)$.

Here let us give an upper estimate of (α, φ_p)-capacity of balls.

LEMMA 2.4. *There exists a constant $A > 0$ such that*

$$C_{\alpha,\varphi_p}\big(B(x_0,r); B(x_0,2r)\big) \le A\big[r^{n-\alpha p}\varphi\big(r^{-1}\big)\big]^{1/p}$$

whenever $r > 0$ and $x_0 \in \mathbf{R}^n$.

Proof. Without loss of generality, we may assume that $x_0 = 0$. For simplicity, set

$$\psi(r) = \big[r^{n-\alpha p}\varphi\big(r^{-1}\big)\big]^{1/p}.$$

For $r > 0$, consider the function

$$f_r(y) = |y|^{-\alpha}$$

for $r < |y| < 2r$ and $f_r = 0$ elsewhere. If $x \in B(0,r)$ and $y \in B(0,2r) \setminus B(0,r)$, then $|x - y| < 3r$, so that

$$I_\alpha f_r(x) \ge (3r)^{\alpha-n} \int_{B(0,2r)\setminus B(0,r)} |y|^{-\alpha}\, dy = A_1$$

with a constant $A_1 = A_1(\alpha, n) > 0$. It follows from the definition of capacity that

$$C_{\alpha,\varphi_p}\big(B(0,r); B(0,2r)\big) \le \|f_r/A_1\|_{\varphi_p, B(0,2r)}.$$

Here, in view of $(\varphi 6)$ with $\varphi_1(r) = \varphi(r^{-1})^{-1/p}$, we see that

$$\begin{aligned}\int_{B(0,2r)} \varphi_p\big(f_r(y)/\psi(r)\big)\, dy &\le A_2 \int_{B(0,2r)\setminus B(0,r)} r^{-\alpha p}\psi(r)^{-p}\varphi\big(r^{-1}\big)\, dy \\ &= A_3\end{aligned}$$

with constants $A_2 = A_2(c_6) > 0$ and $A_3 = A_3(c_6, n) > 0$. Hence, in view of (2.1), we can find $A_4 > 0$ such that

$$\|f_r\|_{\varphi_p, B(0,2r)} \le A_4\psi(r).$$

Now we establish

$$\begin{aligned}C_{\alpha,\varphi_p}\big(B(0,r); B(0,2r)\big) &\le A_1^{-1}\|f_r\|_{\varphi_p, B(0,2r)} \\ &\le A_1^{-1}A_4\psi(r),\end{aligned}$$

which proves the lemma. □

For $0 < R \le 1$, we take $r_0 = r_0(R) > 0$ such that $r < r\tilde{\varphi}_p(r)^{1/n} \le \sqrt{r}$ for $0 < r < r_0$ and

$$\int_{r_0}^{R} \big[\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\, dt/t \ge 2\int_{R^2}^{R} \big[\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\, dt/t. \tag{2.3}$$

By Lemma 2.4 and Remark 2.2, we obtain the following result.

COROLLARY 2.5. *Suppose $\alpha p < n$. Then there exists a constant $A > 0$ independent of R such that*

$$C_{\alpha,\varphi_p}\big(B(x_0,r);B(x_0,R)\big) \le A\tilde{\varphi}_p(r)^{-(p-1)/p}$$

whenever $0 < r < R/2$ and $x_0 \in \mathbf{R}^n$.

Next, we prove the following result.

LEMMA 2.6. *Let $\alpha p = n$ and $0 < R \le 1$. Then there exists a constant $A > 0$ independent of R such that*

$$C_{\alpha,\varphi_p}\big(B(x_0,r);B(x_0,R)\big) \le A\tilde{\varphi}_p(r)^{-(p-1)/p}$$

whenever $0 < r < r_0$ and $x_0 \in \mathbf{R}^n$.

Proof. Suppose $\alpha p = n$, $0 < R \le 1$ and $x_0 = 0$. For $0 < r < r_0$ and $0 < K < 1$, consider the function

$$f_{r,K}(y) = |y|^{-\alpha}\big[\varphi\big(K|y|^{-1}\big)\big]^{-1/(p-1)}$$

for $r < |y| < KR$ and $f_{r,K} = 0$ elsewhere. If $x \in B(0,r)$ and $y \in B(0,R) \setminus B(0,r)$, then $|x-y| < 2|y|$, so that

$$\begin{aligned} I_\alpha f_{r,K}(x) &\ge 2^{\alpha-n}\int_{B(0,KR)\setminus B(0,r)} |y|^{\alpha-n} f_{r,K}(y)\,dy \\ &\ge 2^{\alpha-n}\omega_{n-1}\int_r^{KR}\big[\varphi(K/t)\big]^{-1/(p-1)}\,dt/t \\ &= 2^{\alpha-n}\omega_{n-1}\tilde{\varphi}_p(r/K), \end{aligned}$$

where ω_{n-1} is the surface measure of the boundary of the unit ball in $\mathbf{R}^n$. If $K = \tilde{\varphi}_p(r)^{-1/n}(<1)$, then we see from $(\varphi 1)$ and (2.3) that

$$\begin{aligned} \tilde{\varphi}_p(r/K) &= \int_{r/K}^R \big[\varphi(1/t)\big]^{-1/(p-1)}\,dt/t \\ &\ge \int_{\sqrt{r}}^R \big[\varphi(1/t)\big]^{-1/(p-1)}\,dt/t \\ &\ge 2c_1^{-1/(p-1)}\int_r^{R^2}\big[\varphi(1/t)\big]^{-1/(p-1)}\,dt/t \\ &\ge 2c_1^{-1/(p-1)} \\ &\quad\times\left(\int_r^R \big[\varphi(1/t)\big]^{-1/(p-1)}\,dt/t - 2^{-1}\int_{r_0}^R\big[\varphi(1/t)\big]^{-1/(p-1)}\,dt/t\right) \\ &\ge c_1^{-1/(p-1)}\tilde{\varphi}_p(r). \end{aligned}$$

Thus, it follows that

$$I_\alpha f_{r,K}(x) \ge 2^{\alpha-n}\omega_{n-1}c_1^{-1/(p-1)}\tilde{\varphi}_p(r) = A_1\tilde{\varphi}_p(r)$$

with a constant $A_1 = 2^{\alpha-n}\omega_{n-1}c_1^{-1/(p-1)}$, which implies

$$C_{\alpha,\varphi_p}\big(B(0,r);B(0,R)\big) \le \big\| f_{r,K}/\{A_1\tilde\varphi_p(r)\}\big\|_{\varphi_p,B(0,R)} = \{A_1\tilde\varphi_p(r)\}^{-1}\|f_{r,K}\|_{\varphi_p,B(0,R)}.$$

Here note from (φ6) with $\varphi_1(r) = \varphi(r)^{-1/p}$ that

$$\begin{aligned}
&\int_{B(0,KR)} \varphi_p\big(K^\alpha f_{r,K}(y)\big)\,dy \\
&\le c_6 \int_{B(0,KR)\setminus B(0,r)} \big(K/|y|\big)^{\alpha p}\big[\varphi\big(K|y|^{-1}\big)\big]^{-p/(p-1)}\varphi\big(K|y|^{-1}\big)\,dy \\
&= A_2 K^{\alpha p}\int_r^{KR}\big[\varphi(K/t)\big]^{-1/(p-1)}\,dt/t \le A_2
\end{aligned}$$

with $K = \tilde\varphi_p(r)^{-1/n}$ and $A_2 = c_6\omega_{n-1}$. This implies by (2.1) that there exists a constant $A_3 > 0$ such that

$$\|f_{r,K}\|_{\varphi_p,B(0,R)} \le A_3 K^{-\alpha} = A_3\tilde\varphi_p(r)^{1/p}.$$

Now it follows that

$$\begin{aligned}
C_{\alpha,\varphi_p}\big(B(0,r);B(0,R)\big) &\le A_1^{-1}\tilde\varphi_p(r)^{-1}\|f_{r,K}\|_{\varphi_p,B(0,R)} \\
&\le A_1^{-1}A_3\tilde\varphi_p(r)^{-1+1/p}.
\end{aligned}$$

Thus, the lemma is proved. □

By Corollary 2.5 and Lemma 2.6, we find the following result.

THEOREM 2.7. *Suppose $p > 1$ and $0 < R \le 1$. Then there exist constants $A > 0$ independent of R and $r_0 = r_0(R) > 0$ such that*

$$C_{\alpha,\varphi_p}\big(B(x_0,r);B(x_0,R)\big) \le A\tilde\varphi_p(r)^{-(p-1)/p}$$

whenever $0 < r < r_0$ and $x_0 \in \mathbf{R}^n$.

REMARK 2.8. Suppose $p > 1$. Then for each $R > 0$ one can find a constant $A(R) > 0$ such that

$$C_{\alpha,\varphi_p}\big(B(x_0,r);B(x_0,R)\big) \le A(R)\tilde\varphi_p(r)^{-(p-1)/p}$$

whenever $0 < r < R/2$ and $x_0 \in \mathbf{R}^n$.

In fact, if $0 < R \le 1$ and $0 < r < r_0$, then this is a consequence of Theorem 2.7. If $0 < R \le 1$ and $r_0 \le r < R/2$, then

$$C_{\alpha,\varphi_p}\big(B(x_0,r);B(x_0,R)\big) \le C_{\alpha,\varphi_p}\big(B(x_0,R/2);B(x_0,R)\big)$$

and hence one can take $A(R) > 0$ such that

$$C_{\alpha,\varphi_p}\big(B(x_0,R/2);B(x_0,R)\big) \le A(R)\tilde\varphi_p(r_0)^{-(p-1)/p}.$$

The case $R \ge 1$ is similarly treated.

Next, we give a lower estimate of (α,φ_p)-capacity of balls.

THEOREM 2.9. *For $R>0$, there exists a constant $A=A(R)>0$ such that*

$$\tilde{\varphi}_p(r)^{-(p-1)/p} \le A C_{\alpha,\varphi_p}\big(B(x_0,r);B(x_0,R)\big)$$

whenever $0<r<R/2<\infty$ and $x_0 \in \mathbf{R}^n$.

Proof. As above, we assume that $x_0=0$. For $0<r<R/2$, take a nonnegative measurable function f on $B(0,R)$ such that

$$I_\alpha f(x) \ge 1 \quad \text{for } x \in B(0,r).$$

Then we have by Fubini's theorem

$$\begin{aligned}\int_{B(0,r)} dx &\le \int_{B(0,r)} I_\alpha f(x)\,dx \\ &\le \int_{B(0,R)} \left(\int_{B(0,r)} |x-y|^{\alpha-n}\,dx\right) f(y)\,dy \\ &\le A_1 r^n \int_{B(0,R)} \big(r+|y|\big)^{\alpha-n} f(y)\,dy,\end{aligned}$$

so that

$$1 \le A_1 \int_{B(0,R)} \big(r+|y|\big)^{\alpha-n} f(y)\,dy. \tag{2.4}$$

We show that

$$\int_{B(0,R)} \big(r+|y|\big)^{\alpha-n} f(y)\,dy \le A_2 \tilde{\varphi}_p(r)^{-1/p+1} \|f\|_{\varphi_p,B(0,R)}. \tag{2.5}$$

For this purpose, suppose $\|f\|_{\varphi_p,B(0,R)} \le 1$. Then, considering

$$k(y) = \tilde{\varphi}_p\big(r+|y|\big)^{-1/p}\big(r+|y|\big)^{-\alpha}\Big[\big(r+|y|\big)^{n-\alpha p}\varphi\Big(\big(r+|y|\big)^{-1}\Big)\Big]^{-1/(p-1)},$$

we find by (φ4), (φ6) and Remark 2.2

$$\begin{aligned}&\int_{B(0,R/2)} \big(r+|y|\big)^{\alpha-n} f(y)\,dy \\ &\quad\le \int_{B(0,R/2)} \big(r+|y|\big)^{\alpha-n} k(y)\,dy \\ &\qquad + A_3 \int_{B(0,R/2)} \big(r+|y|\big)^{\alpha-n} f(y) \left(\frac{f(y)}{k(y)}\right)^{p-1} \frac{\varphi(f(y))}{\varphi(k(y))}\,dy \\ &\quad\le A_4 \left\{ \int_r^R \tilde{\varphi}_p(t)^{-1/p}\big[t^{n-\alpha p}\varphi\big(t^{-1}\big)\big]^{-1/(p-1)}\,dt/t \right. \\ &\qquad \left. + \int_{B(0,R)} \tilde{\varphi}_p\big(r+|y|\big)^{(p-1)/p} \varphi_p\big(f(y)\big)\,dy \right\} \\ &\quad\le A_5 \left\{ \tilde{\varphi}_p(r)^{1-1/p} + \tilde{\varphi}_p(r)^{(p-1)/p} \int_{B(0,R)} \varphi_p\big(f(y)\big)\,dy \right\} \le 2A_5 \tilde{\varphi}_p(r)^{1-1/p}.\end{aligned}$$

Next, considering

$$\begin{aligned} k &= \tilde{\varphi}_p(R/2)^{-1/p}(R/2)^{-\alpha}\big[(R/2)^{n-\alpha p}\varphi\big((R/2)^{-1}\big)\big]^{-1/(p-1)} \\ &\sim \tilde{\varphi}_p(R/2)^{1-1/p}(R/2)^{-\alpha}, \end{aligned}$$

we find by (φ4), (φ6) and Remark 2.2

$$\begin{aligned} &\int_{B(0,R)\setminus B(0,R/2)} \big(r+|y|\big)^{\alpha-n} f(y)\,dy \\ &\le (R/2)^{\alpha-n}\int_{B(0,R)\setminus B(0,R/2)} f(y)\,dy \\ &\le (R/2)^{\alpha-n}\int_{B(0,R)\setminus B(0,R/2)} k\,dy \\ &\quad + A_6(R/2)^{\alpha-n}\int_{B(0,R)\setminus B(0,R/2)} f(y)\left(\frac{f(y)}{k}\right)^{p-1}\frac{\varphi(f(y))}{\varphi(k)}\,dy \\ &\le A_7\tilde{\varphi}_p(R/2)^{1-1/p}\left(1+\int_{B(0,R)}\varphi_p\big(f(y)\big)\,dy\right) \\ &\le 2A_7\tilde{\varphi}_p(R/2)^{1-1/p} \\ &\le 2A_7\tilde{\varphi}_p(r)^{1-1/p}. \end{aligned}$$

Thus,

$$\int_{B(0,R)} \big(r+|y|\big)^{\alpha-n} f(y)\,dy \le A_8\tilde{\varphi}_p(r)^{1-1/p}$$

whenever $\|f\|_{\varphi_p,B(0,R)} \le 1$, which implies (2.5).

In view of (2.4), (2.5) and the definition of capacity, we find

$$1 \le A_9\tilde{\varphi}_p(r)^{1-1/p}C_{\alpha,\varphi_p}\big(B(0,r);B(0,R)\big),$$

which gives the conclusion. □

Proof of Theorem A. Theorem A follows from Theorems 2.7 and 2.9 together with Remark 2.8. □

3. C_{α,φ_1}-capacity

In this section, we deal with the case $p=1$. For this purpose, set

$$\varphi_1(r) = r\varphi(r)$$

and

$$\tilde{\varphi}_1(r) = r^{n-\alpha}\varphi\big(r^{-1}\big).$$

Here suppose further that $\varphi(r)$ is nondecreasing on $(0,\infty)$.

THEOREM B. *For $R > 0$, there exists a constant $A > 0$ such that*

$$A^{-1}\tilde{\varphi}_1(r) \le C_{\alpha,\varphi_1}\big(B(x,r); B(x,R)\big) \le A\tilde{\varphi}_1(r)$$

whenever $0 < r < R/2$.

The proof is quite similar to that of Theorem A, and thus we omit it.

REFERENCES

[1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Springer-Verlag, Berlin, 1996. MR 1411441

[2] D. R. Adams and R. Hurri-Syrjänen, *Vanishing exponential integrability for functions whose gradients belong to $L^n(\log(e+L))^\alpha$*, J. Funct. Anal. **197** (2003), 162–178. MR 1957679

[3] D. R. Adams and R. Hurri-Syrjänen, *Capacity estimates*, Proc. Amer. Math. Soc. **131** (2003), 1159–1167. MR 1948107

[4] N. Aissaoui and A. Benkirane, *Capacités dans les espaces d'Orlicz*, Ann. Sci. Math. Québec **18** (1994), 1–23. MR 1273865

[5] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, New York, 1988. MR 0928802

[6] D. E. Edmunds and W. D. Evans, *Hardy operators, function spaces and embeddings*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004. MR 2091115

[7] J. Joensuu, *On null sets of Sobolev–Orlicz capacities*, Illinois J. Math. **52** (2008), 1195–1211. MR 2595762

[8] J. Joensuu, *Orlicz–Sobolev capacities and their null sets*, Rev. Mat. Complut. **23** (2010), 217–232. MR 2578579

[9] J. Joensuu, *Estimates for certain Orlicz–Sobolev capacities of an Euclidean ball*, Nonlinear. Anal. **72** (2010), 4316–4330. MR 2606790

[10] N. G. Meyers, *A theory of capacities for potentials in Lebesgue classes*, Math. Scand. **8** (1970), 255–292. MR 0277741

[11] Y. Mizuta, *Continuity properties of potentials and Beppo–Levi–Deny functions*, Hiroshima Math. J. **23** (1993), 79–153 . MR 1211771

[12] Y. Mizuta, *Potential theory in Euclidean spaces*, Gakkōtosyo, Tokyo, 1996. MR 1428685

[13] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, *Boundedness of fractional integral operators on Morrey spaces and Sobolev embeddings for generalized Riesz potentials*, J. Math. Soc. Japan. **62** (2010), 707–744. MR 2648060

[14] Y. Mizuta and T. Shimomura, *Differentiability and Hölder continuity of Riesz potentials of functions in Orlicz classes*, Analysis **20** (2000), 201–223. MR 1778254

[15] Y. Mizuta and T. Shimomura, *Vanishing exponential integrability for Riesz potentials of functions in Orlicz classes*, Illinois J. Math. **51** (2007), 1039–1060. MR 2417414

[16] Y. Mizuta and T. Shimomura, *Continuity properties of Riesz potentials of Orlicz functions*, Tohoku Math. J. **61** (2009), 225–240. MR 2541407

[17] W. P. Ziemer, *Weakly differentiable functions*, Springer-Verlag, New York, 1989. MR 1014685

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