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HOMOGENEOUS PARACONTACT METRIC THREE-MANIFOLDS

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Dedicated to Professor Domenico Perrone in the occasion of his sixtieth birthday

Abstract. The complete classification of three-dimensional homogeneous paracontact metric manifolds is obtained. In the symmetric case, such a manifold is either flat or of constant sectional curvature −1. In the non-symmetric case, it is a Lie group equipped with a left-invariant paracontact metric structure.

1. Introduction

An *almost paracontact structure* on a $(2n+1)$-dimensional smooth manifold M is a triplet (φ,ξ,η), where φ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\text{(i)}\quad \varphi(\xi)=0,\quad \eta\circ\varphi=0,$$
$$\text{(ii)}\quad \eta(\xi)=1,\quad \varphi^2=Id-\eta\otimes\xi \tag{1.1}$$

and the restriction J of φ on the horizontal distribution $\operatorname{Ker}\eta$ is an almost paracomplex structure (that is, the eigensubbundles T^+,T^- corresponding to the eigenvalues $1,-1$ of J have equal dimension n). A pseudo-Riemannian metric g on M is said to be *compatible* with the almost paracontact structure (φ,ξ,η) if and only if

$$g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y). \tag{1.2}$$

Remark that, by (1.1) and (1.2), $\eta(X)=g(\xi,X)$ for any compatible metric. Any almost paracontact structure admits a compatible metric. Moreover,

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compatible metrics necessarily have signature $(n+1,n)$ [14]. If

$$g(X,\varphi Y) = (d\eta)(X,Y), \tag{1.3}$$

then the manifold (M,η,g) (or (M,φ,ξ,η,g)) is called a *paracontact metric manifold* and g the *associated metric*.

Almost paracontact structures were introduced in [11]. Since then, paracontact and almost paracontact metric manifolds have been studied by several authors, even if most of the results focused on the very special case of *paraSasakian manifolds*. A remarkable exception is given by [10], where harmonicity of "natural" maps between almost contact and paracontact metric manifolds is discussed in a unified way. Another is the recent paper [14], where a systemathic study of paracontact metric manifolds was undertaken, introducing all the technical apparatus which is needed for further investigations.

The aim of this paper is to obtain the complete classification of homogeneous paracontact metric manifolds in dimension three. This will also provide some interesting explicit examples of paracontact metric manifolds which in general are not paraSasakian. Similarly to the contact case, a paracontact manifold (M,η) is said to be *homogeneous* if there exists a connected Lie group G of diffeomorphisms acting transitively on M and leaving η invariant. If g satisfies (1.3) and G is a group of isometries, then (M,η,g) is said to be a *homogeneous paracontact metric manifold*.

In dimension three, a metric g compatible with a paracontact structure (φ,ξ,η) has signature $(2,1)$, that is, g is Lorentzian. The author proved in [3] that a simply connected complete homogeneous Lorentzian three-manifold is either symmetric or locally isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric.

We shall prove that a symmetric paracontact metric three-space is either flat or of constant sectional curvature -1. This result is the paracontact analogue of the contact result proved in [2], and completes the classification of paracontact manifolds of constant sectional curvature, which in dimension $2n+1 \geq 5$ was given in [14]. The classification of homogeneous paracontact metric three-manifolds is resumed in the following.

THEOREM 1.1. *A simply connected complete homogeneous paracontact metric three-manifold is isometric to a Lie group G equipped with a left-invariant paracontact metric structure (φ,ξ,η,g). More precisely, one of the following cases occurs*:

(i) *If G is unimodular, then there exists a pseudo-orthonormal frame field $\{e_1,e_2,e_3\}$, with e_3 time-like, such that the Lie algebra of G is one of the following*:

(1) $[e_1,e_2] = \gamma e_2 - \beta e_3, [e_1,e_3] = -\beta e_2 + \gamma e_3, [e_2,e_3] = 2e_1$, *with* $\gamma \neq 0$.

Then, G is either the identity component of $O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

(2) $[e_1,e_2] = -\gamma e_3, [e_1,e_3] = -\beta e_2, [e_2,e_3] = 2e_1$.

In this case, G is

(2a) *the identity component of* $O(1,2)$ *or* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *if* $\beta,\gamma>0$ *or* $\beta,\gamma<0$;
(2b) $\widetilde{E}(2)$ *if* $\beta>0=\gamma$ *or* $\beta=0>\gamma$;
(2c) $E(1,1)$ *if* $\beta<0=\gamma$ *or* $\beta=0<\gamma$;
(2d) *either* $\mathrm{SO}(3)$ *or* $\mathrm{SU}(2)$ *if* $\beta>0$ *and* $\gamma<0$;
(2e) *the Heisenberg group* H_3 *if* $\beta=\gamma=0$.

(3) $[e_1,e_2]=-e_2+(2\varepsilon-\beta)e_3, [e_1,e_3]=-\beta e_2+e_3, [e_2,e_3]=2e_1$, *with* $\varepsilon=\pm 1$.

In this case, G is

(3a) *the identity component of* $O(1,2)$ *or* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *if* $\beta\neq\varepsilon$;
(3b) $\widetilde{E}(2)$ *if* $\beta=\varepsilon=1$;
(3c) $E(1,1)$ *if* $\beta=\varepsilon=-1$.

(ii) *if G is non-unimodular, then there exists a pseudo-orthonormal frame field $\{e_1,e_2,e_3\}$, with e_3 time-like, such that the Lie algebra of G is one of the following*:

(4) $[e_1,e_2]=[e_1,e_3]=0, [e_2,e_3]=2e_1+\delta e_2$, *with* $\delta\neq 0$.

(5) $[e_1,e_2]=-[e_1,e_3]=-\beta(e_2+e_3), [e_2,e_3]=2e_1+\delta(e_2+e_3)$, *with* $\delta\neq 0$.

Paracontact three-manifolds of constant sectional curvature equal to either 0 or -1 are included in the classification given in Theorem 1.1 above. In particular, in case (2a) with $\alpha=\beta=\gamma=2$, unimodular Lie groups $O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ have constant sectional curvature -1, while in case (2b) with $\alpha=\beta=2$, unimodular Lie group $\widetilde{E}(2)$ is flat ([3], [4]).

Theorem 1.1 is the paracontact counterpart of the classification of homogeneous contact Riemannian three-manifolds obtained in [12]. Comparing Theorem 1.1 with Theorem 3.1 of [12], one can see how Lorentzian settings and paracontact structures allow more cases than their Riemannian and contact analogues.

The paper is organized in the following way. In Section 2, we will report and prove some basic facts about paracontact metric manifolds and homogeneous Lorentzian manifolds. The classification of three-dimensional symmetric paracontact metric spaces will be given in Section 3. In Section 4, we shall prove Theorem 1.1, classifying left-invariant paracontact metric structures on three-dimensional Lorentzian Lie groups.

2. Preliminaries

Let (M,η,g) be a paracontact metric manifold. By ∇ and R, we shall denote the Levi–Civita connection and the curvature tensor of M, respectively, the latter taken with the sign convention $R(X,Y)=\nabla_{[X,Y]}-[\nabla_X,\nabla_Y]$ (note that this convention is opposite to the one used in [14]). Taking into account

(1.1) and (1.3), tensors

$$h = \frac{1}{2}\mathcal{L}_\xi \varphi, \qquad \ell = R(\xi,\cdot)\xi, \tag{2.1}$$

$\mathcal{L}$ being the Lie derivative, are defined on (M,η,g) and play an important role in describing its geometry. In particular, as proved in [14], h is self-adjoint, $h\varphi = -\varphi h$ and $h\xi = \operatorname{tr} h = 0$. Moreover, the covariant derivative and the curvature satisfy the following properties:

$$\nabla_\xi \xi = 0, \qquad \nabla_\xi \varphi = 0, \tag{2.2}$$

$$\nabla_\xi X = -\varphi X + \varphi h X, \tag{2.3}$$

$$(\nabla_\xi h)X = -\varphi X + h^2\varphi X - \varphi \ell X, \tag{2.4}$$

$$\ell X + \varphi\ell\varphi X = 2h^2 X - 2\varphi^2 X. \tag{2.5}$$

We recall the following definition.

DEFINITION 2.1. A paracontact metric manifold (M,η,g) is said to be

(i) *paraSasakian* if it is normal, that is, equivalently,

$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{2.6}$$

(ii) *K-paracontact* if $h=0$, that is, equivalently, ξ is a Killing vector field.

We explicitly remark that, contrary to the contact Riemannian case, $|h|^2 = 0$ is a necessary *but not sufficient* condition in order to have a K-paracontact manifold, since in pseudo-Riemannian settings it holds whenever hX is light-like for any tangent vector X. Moreover, every paraSasakian manifold is K-paracontact (Theorem 2.8 of [14]). Even if the the converse does not hold in general, we can prove the following.

THEOREM 2.2. *A three-dimensional K-paracontact metric manifold (M,η,g) is paraSasakian.*

Proof. Up to some changes of sign, the argument is very similar to the one used in the contact Riemannian case (see, for example, Chapter 6 of [1]). For this reason, the details will be omitted. For any three-dimensional paracontact metric manifold (M,η,g), one can prove that

$$(\nabla_X\varphi)Y = -g(X-hX,Y)\xi + \eta(Y)(X-hX). \tag{2.7}$$

In particular, if (M,η,g) is K-paracontact, then $h=0$ and (2.7) becomes (2.6). Hence, (M,η,g) is paraSasakian. □

We also recall that any (almost) paracontact metric manifold (M^{2n+1},η,g) admits a special kind of local pseudo-orthonormal basis, called a *φ-basis* [14]. Such a basis is of the form $\{\xi,E_1,\dots,E_n,\varphi E_1,\dots,\varphi E_n\}$, where $\xi,E_1,\dots,E_n$ are space-like vector fields and so, by (1.2), vector fields $\varphi E_1,\dots,\varphi E_n$ are time-like.

We now turn our attention to homogeneous paracontact metric manifolds, seeking some restrictions, due to homogeneity, on the characteristic vector field ξ. For this reason, we report here the definitions of homogeneous geodesics and geodesic vectors in a homogeneous space.

Let $(M=K/H,g)$ be a pseudo-Riemannian reductive homogeneous manifold, $\mathfrak{k}=\mathfrak{m}\oplus\mathfrak{h}$ the corresponding reductive split of the Lie algebra of K. A geodesic Γ through the origin $o\in M=K/H$ is called *homogeneous* if it is the orbit of a one-parameter subgroup. The following characterization holds.

Proposition 2.3 ([9]). *A geodesic* $\Gamma(t)$ *of* $M=K/H$, *with* $\Gamma(0)=o$ *and* $\Gamma'(0)=X_{\mathfrak{m}}\in\mathfrak{m}$ $(\equiv T_o(K/H))$, *is homogeneous if and only if there exists* $X_{\mathfrak{h}}\in\mathfrak{h}$ *such that* $X=X_{\mathfrak{m}}+X_{\mathfrak{h}}\in\mathfrak{k}$ *satisfies*

$$\big\langle [X,Y]_{\mathfrak{m}},X_{\mathfrak{m}}\big\rangle = k\langle X_{\mathfrak{m}},Y\rangle \tag{2.8}$$

for all $Y\in\mathfrak{m}$ *and some* $k\in\mathbb{R}$ *depending on* $X_{\mathfrak{m}}$.

A vector $X\in\mathfrak{k}$ satisfying (2.8) is called a *geodesic vector*. When $X_{\mathfrak{m}}$ is either space-like or time-like, applying (2.8) we get at once $k=0$, while for a light-like vector $X_{\mathfrak{m}}$, k may be any real constant. We are now ready to prove the following theorem.

Theorem 2.4. *If* $(M=K/H,\eta,g)$ *is a reductive homogeneous paracontact metric manifold*, *then* $\xi\in\mathfrak{m}$ *is a geodesic vector.*

Proof. The characteristic vector field ξ belongs to $\mathfrak{m}$, because K leaves both η and g invariant. For the same reason, $\operatorname{Ker}\eta\subset\mathfrak{m}$. So, we can fix a basis $\{\xi,e_1,\dots,e_{2n}\}$ of $\mathfrak{m}$ with $\{e_1,\dots,e_{2n}\}$ spanning $\operatorname{Ker}\eta$ and it suffices to check (2.8) for these vectors. For any $i=1,\dots,2n$, taking into account (1.1) and (1.3), we have

$$\big\langle [\xi,e_i]_{\mathfrak{m}},\xi\big\rangle=\eta[\xi,e_i]=-2(d\eta)(\xi,e_i)=-2g(\xi,\varphi e_i)=0$$

and so, ξ is a geodesic vector field. □

We now consider the special case of a paracontact Lie group. If G is a Lie group, equipped with a homogeneous paracontact metric structure (φ,ξ,η,g), the same argument illustrated in [12] for the contact case shows that this structure is invariant under left translations. Hence, denoting by $\mathfrak{g}$ the Lie algebra of G, we have $\xi\in\mathfrak{g}$, η is a 1-form over $\mathfrak{g}$, $\operatorname{Ker}\eta\subset\mathfrak{g}$. Moreover, starting from a φ-basis of tangent vectors at $e\in G$, we use left translations to build a φ-basis $\{\xi,E_1,\dots,E_n,\varphi E_1,\dots,\varphi E_n\}$ of the Lie algebra $\mathfrak{g}$. We now prove the following theorem.

Theorem 2.5. *If* (φ,ξ,η,g) *is a left-invariant paracontact metric structure on a pseudo-Rieman-nian Lie group* G, *then* $\operatorname{tr} ad_\xi=0$, *that is*, ξ *belongs to the unimodular kernel* $\mathfrak{u}$ *of* $\mathfrak{g}$.

Proof. With respect to a φ-basis $\{\xi, E_1, \dots, E_n, \varphi E_1, \dots, \varphi E_n\}$ of $\mathfrak{g}$, taking into account (1.1) and (1.3), we get

$$\begin{aligned}
\operatorname{tr} ad_\xi &= \sum_i \big\langle [\xi, E_i], E_i \big\rangle - \sum_i \big\langle [\xi, \varphi E_i], \varphi E_i \big\rangle \\
&= -\sum_i \langle \nabla_{E_i} \xi, E_i \rangle + \sum_i \langle \nabla_{\varphi E_i} \xi, \varphi E_i \rangle \\
&= \sum_i \langle \varphi E_i - \varphi h E_i, E_i \rangle - \sum_i \big\langle \varphi^2 E_i - \varphi h \varphi E_i, \varphi E_i \big\rangle \\
&= -\sum_i \langle \varphi h E_i, E_i \rangle - \sum_i \big\langle \varphi^2 h E_i, \varphi E_i \big\rangle \\
&= -\sum_i \langle \varphi h E_i, E_i \rangle + \sum_i \langle \varphi h E_i, E_i \rangle = 0.
\end{aligned}$$

□

REMARK 2.6. Arguments similar to the ones we used to prove Theorems 2.4 and 2.5 also apply to the contact Riemannian case. Thus,

(i) *the characteristic vector field of a homogeneous contact Riemannian manifold is a geodesic vector;*

(ii) *the characteristic vector field of a left-invariant contact metric structure on a Riemannian Lie group belongs to its unimodular kernel.*

Property (ii) was proved in [12] in dimension three. Apart from this, up to our knowledge, these basic properties were not pointed out in so far.

We end this section reporting the classification of homogeneous Lorentzian three-manifolds.

THEOREM 2.7 ([3]). *A three-dimensional simply connected complete homogeneous Lorentzian manifold (M, g) is either symmetric, or $M = G$ is a three-dimensional Lie group and g is left-invariant. Precisely, one of the following cases occurs:*

- *If G is unimodular, then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with e_3 time-like, such that the Lie algebra of G is one of the following:*

  (a)

$$(2.9) \qquad (\mathfrak{g}_1): \quad \begin{aligned} [e_1, e_2] &= \alpha e_1 - \beta e_3, \\ [e_1, e_3] &= -\alpha e_1 - \beta e_2, \\ [e_2, e_3] &= \beta e_1 + \alpha e_2 + \alpha e_3, \quad \alpha \neq 0. \end{aligned}$$

  If $\beta \neq 0$, G is the identity component of the pseudo-orthogonal group $O(1,2)$ or the universal covering of the special linear group $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, while if $\beta = 0$, $G = E(1,1)$ is the group of rigid motions of the Minkowski two-space.

(b)

$$(2.10)\qquad (\mathfrak{g}_2): \quad \begin{aligned} &[e_1,e_2] = \gamma e_2 - \beta e_3, \\ &[e_1,e_3] = -\beta e_2 + \gamma e_3, \quad \gamma \neq 0, \\ &[e_2,e_3] = \alpha e_1. \end{aligned}$$

In this case, $G = O(1,2)$ *or* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *if* $\alpha \neq 0$*, while* $G = E(1,1)$ *if* $\alpha = 0$*.*

(c)

$$(2.11)\qquad (\mathfrak{g}_3): \quad \begin{aligned} &[e_1,e_2] = -\gamma e_3, \\ &[e_1,e_3] = -\beta e_2, \\ &[e_2,e_3] = \alpha e_1. \end{aligned}$$

The following Table 1 (where $\widetilde{E}(2)$ *and* H_3*, respectively denote the universal covering of the group of rigid motions in the Euclidean two-space and the Heisenberg group) lists all the Lie groups* G *which admit a Lie algebra* g_3*, according to the different possibilities for* α*,* β *and* γ*.*

(d)

$$(2.12)\qquad (\mathfrak{g}_4): \quad \begin{aligned} &[e_1,e_2] = -e_2 + (2\varepsilon - \beta) e_3, \quad \varepsilon = \pm 1, \\ &[e_1,e_3] = -\beta e_2 + e_3, \\ &[e_2,e_3] = \alpha e_1. \end{aligned}$$

Table 2 describes all Lie groups G *admitting a Lie algebra* g_4*.*

- *If* G *is non-unimodular, then there exists a pseudo-orthonormal frame field* $\{e_1, e_2, e_3\}$*, with* e_3 *time-like, such that the Lie algebra of* G *is one of the*

Table 1. Lie groups with a Lie algebra g_3

G	α	β	γ
$O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$	+	+	+
$O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$	+	−	−
SO(3) or SU(2)	+	+	−
$\widetilde{E}(2)$	+	+	0
$\widetilde{E}(2)$	+	0	−
$E(1,1)$	+	−	0
$E(1,1)$	+	0	+
H_3	+	0	0
H_3	0	0	−
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	0	0	0

TABLE 2. Lie groups with a Lie algebra g_4

G ($\varepsilon=1$)	α	β	G ($\varepsilon=-1$)	α	β
$O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$	$\neq 0$	$\neq 1$	$O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$	$\neq 0$	$\neq -1$
$E(1,1)$	0	$\neq 1$	$E(1,1)$	0	$\neq -1$
$E(1,1)$	<0	1	$E(1,1)$	>0	-1
$\widetilde{E}(2)$	>0	1	$\widetilde{E}(2)$	<0	-1
H_3	0	1	H_3	0	-1

following:

(e)

$$(\mathfrak{g}_5): \quad \begin{aligned} &[e_1,e_2]=0,\\ &[e_1,e_3]=\alpha e_1+\beta e_2,\\ &[e_2,e_3]=\gamma e_1+\delta e_2, \quad \alpha+\delta\neq 0,\ \alpha\gamma+\beta\delta=0. \end{aligned} \tag{2.13}$$

(f)

$$(\mathfrak{g}_6): \quad \begin{aligned} &[e_1,e_2]=\alpha e_2+\beta e_3,\\ &[e_1,e_3]=\gamma e_2+\delta e_3,\\ &[e_2,e_3]=0, \quad \alpha+\delta\neq 0, \alpha\gamma-\beta\delta=0. \end{aligned} \tag{2.14}$$

(g)

$$(\mathfrak{g}_7): \quad \begin{aligned} &[e_1,e_2]=-\alpha e_1-\beta e_2-\beta e_3,\\ &[e_1,e_3]=\alpha e_1+\beta e_2+\beta e_3,\\ &[e_2,e_3]=\gamma e_1+\delta e_2+\delta e_3, \quad \alpha+\delta\neq 0, \alpha\gamma=0. \end{aligned} \tag{2.15}$$

3. Three-dimensional paracontact symmetric spaces

We first recall the classification of Lorentzian symmetric three-spaces.

THEOREM 3.1 ([4]). *A three-dimensional Lorentzian locally symmetric space is locally isometric to either*

(i) *a Lorentzian space form* $\mathbb{S}^3_1$, $\mathbb{R}^3_1$ *or* $\mathbb{H}^3_1$;
(ii) *a direct product* $\mathbb{R}\times\mathbb{S}^2_1$, $\mathbb{R}\times\mathbb{H}^2_1$, $\mathbb{S}^2\times\mathbb{R}$ *or* $\mathbb{H}^2\times\mathbb{R}$; *or*
(iii) *a symmetric space admitting a parallel null* (*that is*, *light-like*) *vector field.*

In order to classify locally symmetric paracontact metric three-manifolds, we start with the following.

LEMMA 3.2. *If a paracontact metric manifold is locally symmetric*, *then* $\nabla_\xi h=0$.

Lemma 3.2 easily follows from (2.3) and (2.5) and can be proved exactly as its contact Riemannian analogue obtained in [2].

Note that for any paracontact metric manifold (M,η,g) with $\nabla_\xi h = 0$, by (2.4) and (1.1) it follows at once

$$R(\xi,X)\xi = -X + \eta(X)\xi + h^2 X \tag{3.1}$$

for any tangent vector X. If M is three-dimensional, consider a pseudo-orthonormal basis $\{E_1,E_2\}$ of $\operatorname{Ker}\eta$, with E_2 time-like. Using (1.1) and (1.3) we can easily conclude that $\{E_1,\varphi E_1 = \pm E_2\}$ is a φ-basis. Being self-adjoint, the restriction of h to $\operatorname{Ker}\eta$ with respect to $\{E_1,E_2\}$ has the general form

$$hE_1 = a_{11}E_1 - a_{12}E_2, \qquad hE_2 = a_{12}E_1 - a_{11}E_2$$

for some smooth functions a_{11}, a_{12}. Hence,

$$h^2 = \left(a_{11}^2 - a_{12}^2\right) I_2 \tag{3.2}$$

and so, h^2 is diagonalizable even if, contrary to the contact Riemannian case, h itself may be not diagonalizable. In particular, using (3.2) into (3.1), we also have

$$g\big(R(\xi,E_1)\xi,E_2\big) = g\big(R(\xi,E_2)\xi,E_1\big) = 0. \tag{3.3}$$

We now prove that Lorentzian locally symmetric three-spaces corresponding to cases (ii), (iii) of Theorem 3.1 do not admit a paracontact metric structure, unless they are flat. We start with the reducible case.

PROPOSITION 3.3. *If a reducible Lorentzian locally symmetric three-space (M,g) admits a paracontact metric structure, then (M,g) is flat.*

Proof. By Theorem 3.1, (M,g) is locally isometric to the Lorentzian product of a real line with a surface of constant curvature k. Suppose such a space admits a paracontact metric structure (φ,ξ,η) associated to its symmetric Lorentzian metric g. We treat the different cases separately.

(a) *Products* $\mathbb{R}_1 \times M^2(k)$. Since ξ is space-like, it cannot be tangent to the unidimensional factor. We denote by $e_3 = \frac{\partial}{\partial t}$ the local unit vector field tangent to $\mathbb{R}_1$, by e_1 the unit vector field in the direction of the projection of ξ on M^2 and by e_2 a local unit vector field tangent to M^2 and orthogonal to e_1. Then, we have $\xi = ae_1 + ce_3$, with $a^2 - c^2 = 1$. Moreover, $\operatorname{Ker}\eta = \operatorname{Span}\{E_1,E_2\}$, where we put $E_1 = e_2, E_2 = ce_1 + ae_3$. Since the manifold is reducible, we have at once

$$R(\xi,E_1)\xi = ka^2E_1, \qquad R(\xi,E_2)\xi = 0.$$

Equation (3.1) then yields $h^2E_1 = (1+ka^2)E_1$ and $h^2E_2 = E_2$ and so (3.2) implies $ka^2 = 0$. If $k \neq 0$, then $a = 0$ and $-c^2 = 1$, which cannot occur. Hence, $k = 0$ and the manifold is flat.

(b) *Products* $\mathbb{R}\times M_1^2(k)$. The characteristic vector field ξ cannot be everywhere tangent to the unimodular factor. In fact, in this case a local pseudo-orthonormal basis $\{e_2, e_3=\varphi e_2\}$ of M_1^2 spans $\operatorname{Ker}\eta$ and so, the paracontact condition (1.3) gives

$$1=g(e_2,e_2)=(d\eta)(e_2,\varphi e_2)=-\frac{1}{2}\eta[e_2,e_3]=0,$$

which cannot occur. Denote by $e_1=\frac{\partial}{\partial t}$ the local unit vector field tangent to $\mathbb{R}$ and by w the the projection of ξ on M^2. If w is either space-like or time-like, we can proceed exactly as in the previous case to conclude that $k=0$ and the manifold is flat.

Thus, we are left with the case where the projection of ξ on M_1^2 is a light-like vector field, that is (locally) $\xi=e_1+w$, with $w\neq 0$ and $|w|^2=0$.

Choose a local pseudo-orthonormal basis $\{e_2,e_3\}$ of vector fields tangent to M_1^2 with e_3 time-like, so that $w=b(e_2+e_3)$. It is easy to check that $\operatorname{Ker}\eta$ is (locally) spanned by the pseudo-orthonormal frame field $\{E_1,E_2\}$, where we put

$$E_1=\frac{1}{\sqrt{2}}\left(-e_1+\frac{b^2+1}{2b}e_2+\frac{b^2-1}{2b}e_3\right),$$
$$E_2=\frac{1}{\sqrt{2}}\left(e_1+\frac{3b^2-1}{2b}e_2+\frac{3b^2+1}{2b}e_3\right).$$

The reducibility of the manifold easily implies

$$R(\xi,E_1)\xi=-\frac{kb}{\sqrt{2}}(e_2+e_3)=-\frac{k}{2}(E_1+E_2)$$

and applying (3.3), we then have $0=g(R(\xi,E_1)\xi,E_2)=\frac{k}{2}$. Thus, the manifold is flat. □

With regard to Lorentzian three-spaces admitting a parallel null vector field, we prove the following more general result.

THEOREM 3.4. *If a Lorentzian three-manifold (M,g) with a parallel null vector field admits a paracontact metric structure (φ,ξ,η,g) satisfying $\nabla_\xi h=0$ (in particular, a locally symmetric paracontact metric structure), then (M,g) is flat.*

Proof. A Lorentzian three-manifold (M,g) with a parallel null vector field w admits a system of canonical local coordinates (t,x,y), adapted to a parallel plane field in such a way that $w=\frac{\partial}{\partial t}$ and there exists a smooth function $f=f(x,y)$, such that

$$g=\begin{pmatrix}0&0&1\\0&\varepsilon&0\\1&0&f\end{pmatrix}, \tag{3.4}$$

where $\varepsilon = \pm 1$. From now on, we shall take $\varepsilon = 1$, so that the Lorentzian metric g has signature $(+,+,-)$. A general description of these manifolds was given in [7]. Denote by $\mathcal{U}$ the open subset of $\mathbb{R}^3$ where local coordinates (t,x,y) are defined. As shown in [7], on $\mathcal{U}$ the Levi–Civita connection ∇ of (M,g) is completely determined by

$$\nabla_{\partial_x}\partial_y = \frac{1}{2}f_x\partial_t, \qquad \nabla_{\partial_y}\partial_y = \frac{1}{2}f_y\partial_t - \frac{1}{2}f_x\partial_x, \tag{3.5}$$

where we put $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$. Next, the only non-vanishing local components of the curvature tensor are

$$R(\partial_x,\partial_y)\partial_x = -\frac{1}{2}f_{xx}\partial_t, \qquad R(\partial_x,\partial_y)\partial_y = \frac{1}{2}f_{xx}\partial_x. \tag{3.6}$$

We now put $\mathcal{V}_1 = \{p \in \mathcal{U} : f(p) > 0\}$, $\mathcal{V}_2 = \{p \in \mathcal{U} : f(p) < 0\}$ and $\mathcal{V}_3 = \{p \in \mathcal{U} : f = 0$ in a neighbourhood of $p\}$. Then, $\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ is a dense open subset of $\mathcal{U}$. If we prove that the curvature vanishes at each point of $\mathcal{V}_i$, $i = 1,2,3$, by a continuity argument we can conclude that $\mathcal{U}$ is flat and hence, so is (M,g).

If $p \in \mathcal{V}_3$, then $f_{xx}(p) = 0$ and (3.6) yields at once that the curvature vanishes at p.

Next, as $f > 0$ on $\mathcal{V}_1$, using (3.4) (with $\varepsilon = 1$), a direct calculation gives that vector fields

$$e_1 = \frac{1}{\sqrt{f}}\partial_y, \qquad e_2 = \partial_x, \qquad e_3 = \sqrt{f}\partial_t - \frac{1}{\sqrt{f}}\partial_y \tag{3.7}$$

form a pseudo-orthonormal frame field on $\mathcal{V}_1$, with e_3 time-like.

Let now (φ,ξ,η,g) be a paracontact metric structure of M satisfying $\nabla_\xi h = 0$, where g is the metric (3.4). With respect to $\{e_i\}$ described in (3.7), the characteristic vector field is (locally) given by $\xi = ae_1 + be_2 + ce_3$, for some smooth functions a,b,c such that $a^2 + b^2 - c^2 = 1$, and $\operatorname{Ker}\eta = \operatorname{Span}\{E_1,E_2\}$, with E_2 time-like, where we put

$$E_1 = -\frac{b}{\sqrt{a^2+b^2}}e_1 + \frac{a}{\sqrt{a^2+b^2}}e_2,$$
$$E_2 = \frac{ac}{\sqrt{a^2+b^2}}e_1 + \frac{bc}{\sqrt{a^2+b^2}}e_2 + \sqrt{a^2+b^2}\,e_3.$$

Using (3.6) and (3.7), a standard calculation now gives

$$\begin{aligned} R(\xi,E_1)\xi &= \frac{f_{xx}}{2f}\cdot\frac{a^2+b^2-ac}{\sqrt{a^2+b^2}}\{be_1 + (c-a)e_2 + be_3\}, \\ R(\xi,E_2)\xi &= \frac{f_{xx}}{2f}\cdot\frac{b}{\sqrt{a^2+b^2}}\{be_1 + (c-a)e_2 + be_3\}. \end{aligned} \tag{3.8}$$

Applying (3.3) to (3.8), we then find

$$\frac{f_{xx}}{2f}\cdot b\big(a^2+b^2-ac\big) = 0.$$

We now prove that $f_{xx}=0$ on $\mathcal{V}_1$ (and so, $\mathcal{V}_1$ is flat by (3.6)). In fact, if $f_{xx}\neq 0$ at some point $p\in\mathcal{V}_1$, equation above necessarily yields either $b=0$ or $a^2+b^2-ac=0$ at p and we show that both conditions lead to a contradiction.

Suppose first that $b(p)=0$. Then, using (3.8) into (3.1), we easily find

$$\left(h^2E_1\right)_p=\left(1-\frac{f_{xx}}{2f}\cdot a(c-a)^2\right)E_{1p},\qquad \left(h^2E_2\right)_p=E_{2p},$$

which, by (3.2), implies at once $f_{xx}\cdot a(c-a)^2=0$ at p. As $f_{xx}(p)\neq 0$, either $a(p)=0$ or $a(p)=c(p)$. Correspondingly, either $\xi_p=c(p)e_{3p}$ is a time-like vector or $\xi_p=a(p)(e_1+e_3)_p$ is light-like, which cannot occur.

In the same way, if $a^2+b^2-ac=0$ at p, then

$$\left(h^2E_1\right)_p=E_{1p},\qquad \left(h^2E_2\right)_p=\left(1-\frac{f_{xx}}{2|f|}\cdot\frac{b^2}{\sqrt{a^2+b^2}}\right)E_{2p}.$$

Thus, (3.2) gives again $b(p)=0$, contradicting the fact that ξ_p is space-like.

A similar argument leads to conclude that $\mathcal{V}_2$ is flat. In fact, as $f<0$ on $\mathcal{V}_2$, by (3.4) we find that vector fields

$$e_1'=\sqrt{|f|}\partial_t+\frac{1}{\sqrt{|f|}}\partial_y,\qquad e_2'=\partial_x,\qquad e_3'=\frac{1}{\sqrt{|f|}}\partial_y \tag{3.9}$$

form a pseudo-orthonormal frame field on $\mathcal{V}_2$, with e_3 time-like. Denoted by (φ,ξ,η,g) a paracontact metric structure of M satisfying $\nabla_\xi h=0$, on $\mathcal{V}_2$ we have $\xi=ae_1'+be_2'+ce_3'$, for some smooth functions a,b,c such that $a^2+b^2-c^2=1$, and $\operatorname{Ker}\eta=\operatorname{Span}\{E_1',E_2'\}$, with E_2' time-like, where

$$\begin{aligned}E_1'&=-\frac{b}{\sqrt{a^2+b^2}}e_1'+\frac{a}{\sqrt{a^2+b^2}}e_2',\\ E_2'&=\frac{ac}{\sqrt{a^2+b^2}}e_1'+\frac{bc}{\sqrt{a^2+b^2}}e_2'+\sqrt{a^2+b^2}e_3'.\end{aligned}$$

By (3.6) and (3.9) we now obtain

$$\begin{aligned}R(\xi,E_1')\xi&=\frac{f_{xx}}{2|f|}\cdot\frac{a^2+b^2+ac}{\sqrt{a^2+b^2}}\left\{be_1'-(a+c)e_2'-be_3'\right\},\\ R(\xi,E_2')\xi&=-\frac{f_{xx}}{2|f|}\cdot\frac{b}{\sqrt{a^2+b^2}}\left\{be_1'-(a+c)e_2'-be_3'\right\}\end{aligned}\tag{3.10}$$

and (3.3) yields

$$\frac{f_{xx}}{2|f|}\cdot b\left(a^2+b^2+ac\right)=0.$$

Making use of (3.1) and (3.10) and proceeding as in the case of $\mathcal{V}_1$, we conclude that if $f_{xx}\neq 0$ at some point $p\in\mathcal{V}_2$, then ξ_p is not space-like. As this cannot occur, $f_{xx}=0$ on $\mathcal{V}_2$, that is, $\mathcal{V}_2$ is flat by (3.6) and this ends the proof. □

We now complete the description of locally symmetric paracontact metric three-manifolds, proving the main result of this section.

THEOREM 3.5. *A three-dimensional locally symmetric paracontact metric manifold (M,η,g) is either flat or of constant sectional curvature -1.*

Proof. Theorem 3.1, together with Proposition 3.3 and Theorem 3.4, imply that (M,g) has necessarily constant sectional curvature k.

Since (M,g) is symmetric, $\nabla_\xi h = 0$ by Lemma 3.2 and so, (3.1) holds. Consider now a local φ-basis $\{e,\varphi e\}$ of $\operatorname{Ker}\eta$, with φe time-like. We already remarked that in general we have $he = ae - b\varphi e, h\varphi e = be - a\varphi e$ and $h^2 = (a^2-b^2)I_2$. By (3.1), it now follows $k = a^2 - b^2 - 1$.

Using (2.3), we easily obtain $\eta(\nabla_e \varphi e) = a-1$ and $\eta(\nabla_{\varphi e} e) = a+1$. Therefore,

$$\nabla_e \varphi e = (a-1)\xi + \beta e, \qquad \nabla_{\varphi e} e = (a+1)\xi + \alpha\varphi e,$$

which also easily imply

$$\nabla_e e = b\xi + \beta\varphi e, \qquad \nabla_{\varphi e}\varphi e = b\xi + \alpha e$$

for some smooth functions α,β. We can now calculate $R(e,\varphi e)e = k\varphi e = (a^2-b^2-1)\varphi e$ using formulae above and we find

$$\begin{aligned}(3.11)\quad & \big(a^2-b^2-1\big)\varphi e \\ &= -(a+1)\nabla_e\xi - e(\alpha)\varphi e - \alpha(a-1)\xi + b\nabla_{\varphi e}\xi + \varphi e(\beta)\varphi e + b\beta\xi \\ &\quad - 2\nabla_\xi e + \beta(b\xi + \beta\varphi e) - \alpha\big((a+1)\xi + \alpha\varphi e\big).\end{aligned}$$

Applying η to both sides of (3.11), we then get $2(b\beta - a\alpha) = 0$. By the same argument, calculating $R(e,\varphi e)\varphi e$, we obtain

$$\begin{aligned}(3.12)\quad & \big(a^2-b^2-1\big)e \\ &= -b\nabla_e\xi - e(\alpha)e - b\alpha\xi + (a-1)\nabla_{\varphi e}\xi + \varphi e(\beta)e + \beta(a+1)\xi \\ &\quad - 2\nabla_\xi\varphi e + \beta\big((a-1)\xi + \beta e\big) - \alpha(b\xi + \alpha e),\end{aligned}$$

from which, applying η to both sides, we find $2(a\beta - b\alpha) = 0$. Thus, $b\beta - a\alpha = a\beta - b\alpha = 0$ and we have to consider two different cases. First, if $a^2 - b^2 = 0$, then $k = -1$ and the conclusion follows. On the other hand, if $a^2 - b^2 \neq 0$, then $\alpha = \beta = 0$ and equations (3.11), (3.12) easily yield

$$(3.13)\qquad \nabla_\xi e = -\big(a^2-b^2-1\big)\varphi e, \qquad \nabla_\xi\varphi e = -\big(a^2-b^2-1\big)e.$$

Taking into account (2.3), we also have $[\xi,e] = be + (2-a^2-b^2-a)\varphi e$. Using this formula and (3.13) to calculate $(a^2-b^2-1)e = R(\xi,e)\xi$, a straightforward calculation gives

$$2\big(a^2-b^2-1\big)\{ae - b\varphi e\} = 0.$$

Since $a = b = 0$ contradicts $a^2 - b^2 \neq 0$, we necessarily have $k = a^2 - b^2 - 1 = 0$. Therefore, (M,η,g) is flat and this ends the proof. □

REMARK 3.6. If (M,η,g) is a three-dimensional paracontact metric manifold of constant sectional curvature $k=-1$, then (3.1) gives $h^2 = 0$, which in general does not ensure $h=0$. We shall prove that $h=0$ for the paracontact metric structure on $\mathbb{H}^3_1(-1)$ in the next section, realizing $\mathbb{H}^3_1(-1)$ as a Lorentzian Lie group.

As proved in [14], in dimension $2n+1\geq 5$, if a paracontact metric manifold has constant curvature k, then $k=-1$ and $|h|^2=0$. Theorem 3.5 above completes the classification of paracontact metric manifolds of constant sectional curvature.

4. Three-dimensional paracontact Lie groups

We now decide, for any admissible form of the Lie algebra of a three-dimensional Lorentzian Lie group, whether there exists a compatible left-invariant almost paracontact metric structure. We treat the unimodular and non-unimodular cases separately.

Unimodular cases. Let (φ,ξ,η) be a left-invariant almost paracontact metric structure on a unimodular Lorentzian Lie group (G,g). Then, Theorem 2.5 does not give restrictions on the characteristic vector field ξ.

In connection with Theorem 1.1, every three-dimensional simply connected complete homogeneous Lorentzian manifold admits a homogeneous Lorentzian structure and so, is reductive (see [3]). By Theorem 2.4, the characteristic vector field ξ is a space-like unit geodesic vector. Geodesic vector fields on unimodular Lorentzian Lie groups were classified in [5]. Therefore, from ([5], Section 4) we can deduce the list of possible characteristic vector fields for the different unimodular Lie algebras. The result is resumed in the Table 3, were Lie algebras $\mathfrak{g}_1-\mathfrak{g}_4$ are described by equations (2.9)–(2.15), respectively.

We now check, case by case, whether vector fields above give rise to a left-invariant paracontact metric structure.

$(\mathfrak{g}_2)$: Let (G,g) be a Lorentzian Lie group having a unimodular Lie algebra $\mathfrak{g}_2$, and suppose that (φ,ξ,η) is a left-invariant paracontact metric structure on G, having g as associated metric. By Table 3, $\xi=\pm e_1$. It suffices to consider the case when $\xi=e_1$, since for $\xi=-e_1$ we only have to replace η by $-\eta$ and we obtain isometric structures.

Table 3. Geodesic vector fields of 3D Lorentzian Lie algebras

Lie algebra	ξ as a geodesic vector
$\mathfrak{g}_1$	none
$\mathfrak{g}_2$	$\pm e_1$
$\mathfrak{g}_3$ with $\alpha\neq\gamma\neq\beta\neq\alpha$	$\pm e_1,\pm e_2$
$\mathfrak{g}_3$ with $\alpha=\beta$, $\alpha=\gamma$ or $\beta=\gamma$	all space-like unit vector fields
$\mathfrak{g}_4$ with $\alpha\neq\beta-\varepsilon$	$\pm e_1$
$\mathfrak{g}_4$ with $\alpha=\beta-\varepsilon$	all space-like unit vector fields

Since $\operatorname{Ker}\eta$ is orthogonal to ξ, the paracontact distribution is spanned by $\{e_2, e_3\}$. Being pseudo-orthonormal, $\{\xi, e_2, e_3\}$ form a φ-basis. We use (2.10) to express the paracontact metric condition (1.3) for vector fields $\xi = e_1, e = e_2, \varphi e = e_3$ and we have

$$0 = g(\xi, e) = -\frac{1}{2}\eta\big([\xi, \varphi e]\big), \qquad 0 = g(\xi, \varphi e) = -\frac{1}{2}\eta\big([\xi, e]\big),$$
$$1 = g(e, e) = -\frac{1}{2}\eta\big([e, \varphi e]\big) = \mp\frac{1}{2}\alpha.$$

Hence, $\alpha = \pm 2$ is a necessary and sufficient condition for the Lorentzian metric g of G to be associated to the paracontact metric structure (φ, ξ, η). As in [12], we can restrict to the case when $\alpha > 0$, since for $\alpha < 0$ it suffices to replace e_2 by $-e_2$ in (2.10). Thus, $\alpha = 2$ and $\{\xi = e_1, e = e_2, \varphi e = -e_3\}$. By Theorem 2.7, G is either $O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. This proves the case (1) in Theorem 1.1.

To complete this case, we shall now describe explicitly the left-invariant paracontact metric structure on a three-dimensional Lorentzian Lie group endowed with a Lie algebra $\mathfrak{g}_2$. As we proved, this left-invariant paracontact metric structure, unique up to isometries, is determined by the characteristic vector field $\xi = e_1$, the 1-form $\eta = \theta^1$ dual to e_1 with respect to the left-invariant Lorentzian metric g and the $(1,1)$-tensor φ such that $\varphi e_2 = -e_3$. With respect to the φ-basis $\{\xi, e, \varphi e\}$, the Lie algebra is described by

$$[\xi, e] = \gamma e + \beta\varphi e, \qquad [\xi, \varphi e] = \beta e + \gamma\varphi e, \qquad [e, \varphi e] = -2\xi.$$

Using Lie brackets above to calculate $h = \frac{1}{2}\mathcal{L}_\xi\varphi$, a straightforward calculation gives $h = 0$.

$(\mathfrak{g}_3)$ *with* $\alpha \neq \beta \neq \gamma \neq \alpha$: If (G, g) is a Lorentzian Lie group with unimodular Lie algebra $\mathfrak{g}_3$ and (φ, ξ, η) is a corresponding left-invariant paracontact metric structure on G, then either $\xi = \pm e_1$ or $\xi = \pm e_2$ by Table 3. The role of e_1 and e_2 in (2.11) is perfectly interchanging and different signs for ξ give isometric structures. So, it suffices to describe the case when $\xi = e_1$. As in the previous case, $\operatorname{Ker}\eta$ is spanned by $\{e_2, e_3\}$ and $\{\xi, e = e_2, \varphi e = \pm e_3\}$ is a φ-basis. Expressing (1.3) by means of (2.11), we get

$$0 = g(\xi, e) = -\frac{1}{2}\eta\big([\xi, \varphi e]\big), \qquad 0 = g(\xi, \varphi e) = -\eta\left(\frac{1}{2}[\xi, e]\right) = 0,$$
$$1 = g(e, e) = -\frac{1}{2}\eta\big([e, \varphi e]\big) = \mp\frac{1}{2}\alpha.$$

As before, it suffices to consider the case when $\alpha > 0$ (note that this simplification was already used for Table 1 in [13] and [3]) and we have $\alpha = 2$ and a φ-basis of the form $\{\xi = e_1, e = e_2, \varphi e = -e_3\}$.

$(\mathfrak{g}_3)$ *with* $\alpha = \beta$: When at least two among α, β, γ coincide, a Lorentzian Lie group (G, g) with unimodular Lie algebra $\mathfrak{g}_3$ is naturally reductive [6]. In particular, all vectors in $\mathfrak{g}_3$ are geodesic.

Suppose now that (φ, ξ, η) is a corresponding left-invariant paracontact metric structure on G. Then, $\xi = ae_1 + be_2 + ce_3$ may be any space-like unit vector in the Lie algebra. However, since $\alpha = \beta$ and both e_1, e_2 are space-like, we can replace $\{e_1, e_2\}$ by $\{\tilde{e}_1 = \frac{1}{\sqrt{a^2+b^2}}(ae_1 + be_2), \tilde{e}_2 = \frac{1}{\sqrt{a^2+b^2}}(be_1 - ae_2)\}$. The Lie algebra maintains the same form (2.11) with respect to the new basis $\{\tilde{e}_1, \tilde{e}_2, e_3\}$ (for some new constants $\tilde{\alpha} = \tilde{\beta}$), but now ξ is a linear combination of $\tilde{e}_1, e_3$ only. Thus, without loss of generality, we can write $\xi = ae_1 + ce_3$, with $a^2 - c^2 = 1$.

Put $E_1 = e_2, E_2 = \pm(ce_1 + ae_3)$, so that $\{E_1, E_2\}$ forms a pseudo-orthonormal frame field of the paracontact distribution $\operatorname{Ker}\eta$. Without loss of generality, we can assume $\varphi E_1 = E_2 = ce_1 + ae_3$. By (2.11), we have $[\xi, E_1] = -\alpha ce_1 - \gamma ae_3$. So, applying (1.3), we get

$$0 = g(\xi, E_2) = -\frac{1}{2}\eta\big([\xi, E_1]\big) = -\frac{1}{2}\big(\alpha c\eta(e_1) + \gamma a\eta(e_3)\big),$$

which, taking into account $\eta(E_2) = 0$, gives $ac(\gamma - \alpha) = 0$. Hence, either $ac = 0$ or $\alpha (= \beta) = \gamma$. Note that $a \neq 0$, since ξ is space-like. If $c = 0$, then $\xi = (\pm)e_1$ and $\{\xi, e_2, \pm e_3\}$ is a φ-basis. Then, we have that (1.3) holds if and only if $\alpha = \beta = 2$, and putting $e = e_2$ we get

$$he = (2 - \gamma)e, \qquad h\varphi e = -(2 - \gamma)\varphi e.$$

Hence, h is diagonalizable. Unless $\alpha = \beta = \gamma = 2$, $h \neq 0$ and so, these paracontact Lie groups are not paraSasakian.

If $\alpha = \beta = \gamma$, then (G, g) has constant sectional curvature $k = -\frac{\alpha^2}{4} \neq 0$ [4]. Being G paracontact, Theorem 3.5 implies that $k = -1$ and so, $\alpha = \beta = \gamma = (\pm)2$ and G is isometric to $\mathbb{H}^3_1(-1)$. Calculating the Lie brackets of the φ-basis $\{\xi, e = E_1, \varphi e = -E_2\}$ by (2.11), we now have

$$[\xi, e] = 2\varphi e, \qquad [\xi, \varphi e] = 2e, \qquad [e, \varphi e] = -2\xi, \tag{4.1}$$

from which it follows at once that (1.3) holds. Hence, (4.1) is the general form of the unimodular Lie algebra (2.11) of a Lorentzian Lie group G of constant sectional curvature -1, admitting a left-invariant paracontact metric structure. It easily follows from (4.1) that $h = 0$, that is, because of Theorem 2.2, such a structure is paraSasakian.

($\mathfrak{g}_3$) *with either* $\alpha = \gamma$ *or* $\beta = \gamma$: clearly, it suffices to consider the case when $\beta = \gamma$, since one can interchange space-like vectors e_1 and e_2 in (2.11).

Suppose that (φ, ξ, η) is a corresponding left-invariant paracontact metric structure on such a Lorentzian Lie group (G, g). Being G naturally reductive [6], $\xi = ae_1 + be_2 + ce_3$ may be any space-like vector with $a^2 + b^2 - c^2 = 1$.

If $b^2-c^2>0$, we replace e_2, e_3 by $\{\tilde{e}_2=\frac{1}{\sqrt{b^2-c^2}}(be_2+ce_3), \tilde{e}_3=\frac{1}{\sqrt{b^2-c^2}}(ce_2+be_3)\}$. The Lie algebra maintains the same form (2.11) and ξ is a linear combination of $e_1, \tilde{e}_2$. Proceeding as in the previous case (and skipping the constant sectional curvature case, which we already treated), we conclude that $\xi=(\pm)e_1$, $\{\xi, e_2, \pm e_3\}$ is a φ-basis and (1.3) is satisfied if and only if $\alpha=2$.

In the same way, if $b^2-c^2<0$, we replace e_2, e_3 by $\{\tilde{e}_2=\frac{1}{\sqrt{c^2-b^2}}(ce_2+be_3), \tilde{e}_3=\frac{1}{\sqrt{c^2-b^2}}(be_2+ce_3)\}$ and ξ is a linear combination of $e_1, \tilde{e}_3$ only. The same argument shows that $\xi=(\pm)e_1$, $\{\xi, e_2, \pm e_3\}$ is a φ-basis and (1.3) is equivalent to requiring that $\alpha=2$.

Now, we are left with the case $b^2-c^2=0$, that is, $c=\varepsilon b$, with $\varepsilon=\pm 1$. If $b=c=0$, then $\xi=(\pm)e_1$ and once again we conclude that $\alpha=2$ is a necessary and sufficient condition for (1.3). On the other hand, if $b\neq 0$, changing the sign of e_1, e_3 if needed, we have $\xi=e_1+b(e_2+e_3)$ and an orthonormal basis of $\operatorname{Ker}\varphi$ is given by

$$E_1=\frac{1}{\sqrt{2}}\left(-e_1+\frac{b^2+1}{2b}e_2+\frac{b^2-1}{2b}e_3\right),$$
$$E_2=\frac{1}{\sqrt{2}}\left(e_1+\frac{3b^2-1}{2b}e_2+\frac{3b^2+1}{2b}e_3\right).$$

Hence, $\{\xi, e=E_1, \varphi e=\pm E_2\}$ is a φ-basis of the Lie algebra. Using (2.11) and the fact that b is a constant, we then have

$$[\xi, e]=-\frac{1}{\sqrt{2}}\left(\alpha e_1+\frac{3b^2-1}{2b}\gamma e_2+\frac{3b^2+1}{2b}\gamma e_3\right)$$

and condition (1.3) gives

$$0=g(\xi, \varphi e)=(d\eta)(\xi, e)=-\frac{1}{2}\eta\big([\xi, e]\big)=\sqrt{2}(\alpha-\gamma).$$

Thus, $\alpha=\beta=\gamma$ and G has constant sectional curvature $-\frac{\alpha^2}{4}$. If $\alpha=0$, then (1.3) yields a contradiction, since $1=g(e,e)=-\frac{1}{2}\eta[e, \varphi e]=0$. By Theorem 3.5, we then necessarily have $-\frac{\alpha^2}{4}=-1$, that is, $\alpha=(\pm)2$.

Summarizing, $\alpha=2$ is (up to isometries) a necessary and sufficient condition on the Lie algebra (2.11) to admit a corresponding left-invariant paracontact metric structure. From Table 1 we then deduce all and the ones Lorentzian Lie groups with Lie algebra (2.11) for which we can have $\alpha=2$. They correspond to case (2) of Theorem 1.1.

As we proved, a unimodular Lie group having Lie algebra (2.11) with $\alpha=2$, admits a left-invariant paracontact metric structure, unique up to isomorphism, which permits to express the Lie algebra in the form

$$[\xi, e]=\gamma\varphi e, \qquad [\xi, \varphi e]=\beta e, \qquad [e, \varphi e]=-2\xi.$$

Calculating h from Lie brackets above, we then obtain

$$h\xi = 0, \qquad he = (\beta-\gamma)e, \qquad h\varphi e = -(\beta-\gamma)\varphi e.$$

Hence, h is diagonal. In particular, $h=0$ if and only if $\beta=\gamma$. As special cases, when $\beta=\gamma=2$ we get the paraSasakian contact metric structure on $\mathbb{H}^3_1(-1)$, and when $\beta-2=0=\gamma$ a left-invariant paracontact metric structure on $E(1,1)$, whose associated metric is flat ([3], [13]).

($\mathfrak{g}_4$) *with* $\alpha\neq\beta-\varepsilon$: Proceeding as before, if (G,g) is a Lorentzian Lie group with unimodular Lie algebra $\mathfrak{g}_4$ and (φ,ξ,η) is a corresponding left-invariant paracontact metric structure on G, then $\xi=e_1$ by Table 3. So, $\operatorname{Ker}\eta$ is spanned by $\{e_2,e_3\}$ and $\{\xi, e=e_2, \varphi e=\pm e_3\}$ is a φ-basis. We express (1.3) using (2.12) and we obtain

$$0 = g(\xi,e) = -\frac{1}{2}\eta\big([\xi,\varphi e]\big), \qquad 0 = g(\xi,\varphi e) = -\frac{1}{2}\eta\big([\xi,e]\big),$$
$$1 = g(e,e) = -\frac{1}{2}\eta\big([e,\varphi e]\big) = \mp\frac{1}{2}\alpha.$$

We again restrict to the case $\alpha>0$, without loss of generality. So, $\alpha=2$ is a necessary and sufficient condition for (1.3).

($\mathfrak{g}_4$) *with* $\alpha=\beta-\varepsilon$: In this case, (G,g) is naturally reductive [6]. Hence, any vector in its Lie algebra is geodesic. Let now (φ,ξ,η) be a left-invariant paracontact metric structure (φ,ξ,η) on G. Then, $\xi = ae_1+be_2+ce_3\in\mathfrak{g}_4$, with $a^2+b^2-c^2=1$. Correspondingly, the paracontact distribution $\operatorname{Ker}\varphi$ admits the pseudo-orthonormal basis $\{E_1,E_2\}$, with E_2 time-like, where we put

$$E_1 = -\frac{b}{\sqrt{a^2+b^2}}e_1 + \frac{a}{\sqrt{a^2+b^2}}e_2,$$
$$E_2 = \frac{ac}{\sqrt{a^2+b^2}}e_1 + \frac{bc}{\sqrt{a^2+b^2}}e_2 - \sqrt{a^2+b^2}\,e_3,$$

and so, the Lie algebra has a φ-basis of the form $\{\xi, e=E_1, \varphi e=\pm E_2\}$. Using (2.12), we find

$$[\xi,E_1] = -\frac{1}{\sqrt{a^2+b^2}}\big\{\alpha ac e_1 + \big(a^2+b^2+(\alpha+\varepsilon)bc\big)e_2 + \big((\alpha-\varepsilon)\big(a^2+b^2\big)-bc\big)e_3\big\}, \tag{4.2}$$

$$[\xi,E_2] = \frac{1}{\sqrt{a^2+b^2}}\big\{\alpha be_1 - (\alpha+\varepsilon)ae_2 + ae_3\big\}. \tag{4.3}$$

By (4.2), the compatibility condition (1.3) implies $0=g(\xi,E_1)=\pm\frac{1}{2}\eta([\xi,E_2])$, which holds if and only if either $a=0$ or $c+\varepsilon b=0$. We treat these cases separately.

If $a=0$ then $\xi=be_2+ce_3$, $e=-e_1$ and $\varphi e=\pm(ce_2+be_3)$. From (1.3) we have $0=g(\xi,\varphi e)=\pm\frac{1}{2}\eta([\xi,e])$, which by (4.3) now gives $\varepsilon b+c=0$. But then, $|\xi|^2=b^2-c^2=0$, which cannot occur.

Assume now $\varepsilon b + c = 0$. Then, $|\xi|^2 = a^2 = 1$ and we can take $a = 1$. So, $\xi = e_1 + b(e_2 - \varepsilon e_3)$, $e = E_1 = -\frac{b}{\sqrt{1+b^2}} e_1 + \frac{1}{\sqrt{1+b^2}} e_2$ and $\varphi e = \pm E_2 = \pm \frac{\varepsilon}{\sqrt{1+b^2}} (b e_1 + b^2 e_2 - \varepsilon(1+b^2) e_3)$. By (2.12), a direct calculation now gives

$$
\begin{aligned}
&[\xi, E_1] = -\frac{1}{1+b^2} E_1 + \frac{1}{1+b^2} \big(\varepsilon - \alpha(1+b^2)\big) E_2, \\
&[\xi, E_2] = \frac{1}{1+b^2} \big(\varepsilon b^2 - (\alpha + \varepsilon)(1+b^2)\big) E_1 + \frac{1}{1+b^2} E_2, \\
&[E_1, E_2] = \alpha \xi.
\end{aligned}
\tag{4.4}
$$

Thus, $\eta([\xi, e]) = 0$ and compatibility condition (1.3) reduces to $1 = g(e,e) = (d\eta)(e, \varphi e) = -\frac{1}{2}\eta([e, \varphi e]) = \pm\frac{1}{2}\alpha$. Restricting once again to the case when $\alpha > 0$, we then have that $\alpha = 2$ is a necessary and sufficient condition for the existence of a left-invariant paracontact metric structure on a Lie group with unimodular Lie algebra $\mathfrak{g}_4$, which is described in terms of the φ-basis $\{\xi, e = E_1, \varphi e = -E_2\}$.

Unimodular Lie groups with Lie algebra $\mathfrak{g}_4$ satisfying $\alpha = 2$ can be deduced at once from Table 2. They correspond to case (3) of Theorem 1.1.

We proved that a unimodular Lie group, whose Lie algebra is (2.12) with $\alpha = 2$, admits a left-invariant paracontact metric structure, unique up to isometries. This structure is described in terms of a φ-basis $\{\xi, e, \varphi e\}$ of the Lie algebra, such that

$$[\xi, e] = -e + (\beta - 2\varepsilon)\varphi e, \qquad [\xi, \varphi e] = \beta e + \varphi e, \qquad [e, \varphi e] = -2\xi.$$

Hence, tensor h is determined by

$$h\xi = 0, \qquad he = \varepsilon e + \varphi e, \qquad h\varphi e = -e - \varepsilon \varphi e.$$

Thus, $h \neq 0$. Moreover, it is easy to check that $\lambda = 0$ is the only eigenvalue of h, associated to a two-dimensional eigenspace. Therefore, h is not diagonalizable and two-step nilpotent.

Non-unimodular cases. Geodesic vectors on non-unimodular Lorentzian Lie groups were classified in [6]. However, for non-unimodular Lorentzian Lie groups, it will suffice to make use of Theorem 2.5, that is, to take into account the fact that ξ belongs to the unimodular kernel $\mathfrak{u}$ of the Lie algebra. We treat non-unimodular Lie groups with Lie algebras (2.13)–(2.15) separately.

($\mathfrak{g}_5$): By [8], the unimodular kernel $\mathfrak{u}$ of the Lie algebra $\mathfrak{g}_5$ is the space-like plane spanned by $\{e_1, e_2\}$. Hence, $\xi = a e_1 + b e_2$, with $a^2 + b^2 = 1$. An easy calculation shows that $\mathfrak{g}_5$ maintains the same form with respect to the pseudo-orthonormal basis $\{\tilde{e}_1 = a e_1 + b e_2, \tilde{e}_2 = b e_1 - a e_2, e_3\}$. Thus, without loss of generality, we can take $\xi = e_1$. The paracontact distribution $\operatorname{Ker}\eta$ then admits $\{e_2, e_3\}$ as a pseudo-orthonormal basis, and $\{\xi, e = e_2, \varphi e = \pm e_3\}$ forms a φ-basis. By (2.13) and the paracontact condition (1.3), we have

$$0 = g(\xi, e) = -\frac{1}{2}\eta[\xi, \varphi e] = \pm\frac{1}{2}\alpha$$

and so, $\alpha = 0$. Moreover, by (2.13) also we have $\alpha\gamma + \beta\delta = 0$ and $\alpha + \delta \neq 0$. Hence, $\alpha = \beta = 0 \neq \delta$. Paracontact condition (1.3) now reduces to

$$1 = g(e,e) = -\frac{1}{2}[e, \varphi e] = \pm\frac{1}{2}\gamma,$$

that is, $\gamma = \pm 2$. Changing the sign of e_3 if needed, we can restrict to the case when $\gamma > 0$ and so, $\gamma = 2$. Thus, a non-unimodular Lie group with Lie algebra $\mathfrak{g}_5$ admits a left-invariant paracontact metric structure if and only if $\alpha = \beta = 0 \neq \delta$ and $\gamma = 2$. This corresponds to case (4) in Theorem 1.1.

The left-invariant paracontact metric structure on a Lie group with non-unimodular Lie algebra $\mathfrak{g}_5$, is determined by $\xi = e_1$, $\eta = \theta^1$ and $\varphi e_2 = -e_3$. With respect to the φ-basis $\{\xi, e, \varphi e\}$, the Lie algebra is described by

$$[\xi, e] = [\xi, \varphi e] = 0, \qquad [e, \varphi e] = -2\xi - \delta e,$$

from which it follows at once that $h = 0$.

Remark 4.1. Notice that a non-unimodular Lorentzian Lie group with Lie algebra $\mathfrak{g}_5$ satisfying $\alpha = \beta = 0 \neq \delta$, is not symmetric [4].

($\mathfrak{g}_6$): This case is quite similar to the previous one, and we shall omit some details. The unimodular kernel $\mathfrak{u}$ of $\mathfrak{g}_6$ is the space-time plane spanned by $\{e_2, e_3\}$ [8]. Making a suitable choice for the basis of $\mathfrak{u}$, without loss of generality we can take $\xi = e_2$. Hence, $\operatorname{Ker}\eta$ admits $\{e_1, e_3\}$ as a pseudo-orthonormal basis, and $\{\xi, e = e_1, \varphi e = \pm e_3\}$ will be a φ-basis. Using (2.14) to calculate the paracontact condition (1.3), we conclude that $\alpha = \beta = 0 \neq \delta$ and $\gamma = (\pm)2$ are necessary and sufficient conditions for the existence of a left-invariant paracontact metric structure.

It is evident that left-invariant paracontact metric structures on Lie groups with Lie algebras $\mathfrak{g}_5$ and $\mathfrak{g}_6$ are isometric to one another. Thus, they both correspond to case (4) in Theorem 1.1.

($\mathfrak{g}_7$): We know by [8] that the unimodular kernel $\mathfrak{u}$ of the Lie algebra $\mathfrak{g}_7$ is the degenerate plane spanned by $\{e_1, u = e_2 + e_3\}$. Being ξ a unit vector belonging to $\mathfrak{u}$, we then have $\xi = e_1 + bu$ (changing the sign of e_1 if needed), for some real constant b. It is easily seen that $\mathfrak{g}_7$ maintains the same form (2.15) if expressed with respect to the pseudo-orthonormal basis $e_1 + bu, e_2, e_3$. Thus, without loss of generality, we can take $\xi = e_1$ and $\operatorname{Ker}\eta$ will have $\{e_2, e_3\}$ as a pseudo-orthonormal basis. Consequently, $\{\xi, e = e_2, \varphi e = \pm e_3\}$ will be a φ-basis. By (2.15), we now have

$$[\xi, e_2] = -[\xi, e_3] = -\alpha\xi - \beta(e_2 + e_3), \qquad [e_2, e_3] = \gamma\xi + \delta(e_2 + e_3).$$

Using Lie brackets above to calculate the paracontact condition (1.3), we then have

$$0 = g(\xi, e) = -\frac{1}{2}\eta[\xi, \varphi e] = \pm\frac{1}{2}\alpha, \qquad 1 = g(e,e) = -\frac{1}{2}\eta[e, \varphi e] = \mp\frac{1}{2}\gamma.$$

Therefore, $\alpha = 0$ and $\gamma = (\pm)2$ are necessary and sufficient conditions for the existence of left-invariant paracontact metric structures on a Lorentzian Lie group with Lie algebra $\mathfrak{g}_7$. Without loss of generality, we restrict to the case $\gamma > 0$. Thus, $\gamma = 2$ and we have a φ-basis $\{\xi, e = e_2, \varphi e = -e_3\}$ of $\mathfrak{g}_7$. This is case (5) in Theorem 1.1.

For the non-unimodular lie algebra $\mathfrak{g}_7$, we showed that the left-invariant paracontact metric structure is determined, up to isometries, by $\xi = e_1$, $\eta = \theta^1$ and $\varphi e_2 = -e_3$. Expressing the Lie brackets with respect to $\{\xi, e, \varphi e\}$, we get

$$[\xi, e] = -[\xi, \varphi e] = -\beta e + \beta \varphi e, \qquad [e, \varphi e] = -2\xi - \delta e + \delta \varphi e,$$

from which it is easily seen that $h = 0$.

Remark 4.2. By Theorem 2.2, $h = 0$ is a necessary and sufficient condition for a three-dimensional paracontact metric manifold to be paraSasakian. Thus, from the study above we can deduce at once the following classification of homogeneous paraSasakian metric three-manifolds.

Theorem 4.3. *A simply connected and complete homogeneous para-Sasakian three-manifold is isometric to one of the following Lie groups G equipped with a left-invariant paracontact metric structure*:

(a) *the identity component of $O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, having unimodular Lie algebra $\mathfrak{g}_2$ with $\alpha = 2$;*
(b) *the identity component of $O(1,2)$ or $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, having unimodular Lie algebra $\mathfrak{g}_3$ with $\alpha = 2$ and $\beta = \gamma \neq 0$.*
(c) *the Heisenberg group H_3, having unimodular Lie algebra $\mathfrak{g}_3$ with $\alpha = 2$ and $\beta = \gamma = 0$;*
(d) *a non-unimodular Lie group, having Lie algebra $\mathfrak{g}_5$ (or $\mathfrak{g}_6$) with $\alpha = \beta = 0 \neq \delta$ and $\gamma = 2$;*
(e) *a non-unimodular Lie group, having Lie algebra $\mathfrak{g}_7$ with $\alpha = \beta = 0 \neq \delta$ and $\gamma = 2$.*

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