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HIGHER ORDER RIESZ TRANSFORMS FOR LAGUERRE EXPANSIONS

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ABSTRACT. In this paper, we investigate L^p-boundedness properties for the one-dimensional higher order Riesz transforms associated with Laguerre operators. We also prove that the k-th Riesz transform is a principal value singular integral operator (modulus a constant times of the function when k is even). To establish our results, we exploit a new estimate connecting Riesz transforms in the Hermite and Laguerre settings in dimension one.

1. Introduction

The aim of this paper is to investigate higher order Riesz transforms associated with Laguerre function expansions in the one-dimensional case. To achieve our goal, we use a procedure that will be described below and that was developed for the first time by the authors and Torrea in [4]. Our results complete in some senses the ones obtained by Nowak and Stempak [22] about higher order Riesz transforms for Laguerre expansions.

For every $\alpha > -1$, we consider the Laguerre differential operator

$$L_\alpha = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\alpha^2 - \frac{1}{4}\right)\right), \quad x \in (0,\infty).$$

This operator can be factorized as follows

$$L_\alpha = \frac{1}{2}\mathfrak{D}_\alpha^*\mathfrak{D}_\alpha + \alpha + 1, \tag{1.1}$$

where $\mathfrak{D}_\alpha f = (-\frac{\alpha+1/2}{x} + x + \frac{d}{dx})f = x^{\alpha+\frac{1}{2}}\frac{d}{dx}(x^{-\alpha-\frac{1}{2}}f) + xf$, and $\mathfrak{D}_\alpha^*$ denotes the formal adjoint of $\mathfrak{D}_\alpha$ in $L^2((0,\infty), dx)$.

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The factorization (1.1) for L_α suggests to define (formally), for every $k \in \mathbb{N} \setminus \{0\}$, the k-th Riesz transform $R_\alpha^{(k)}$ associated with L_α by

$$R_\alpha^{(k)} = \mathfrak{D}_\alpha^k L_\alpha^{-\frac{k}{2}}.$$

Here, $L_\alpha^{-\beta}$, $\beta > 0$, denotes the $-\beta$ power of the operator L_α (see (1.2)).

Before establishing the main result of this paper, we give a strict definition of the Riesz transform $R_\alpha^{(k)}$.

For every $n \in \mathbb{N}$, we have that $L_\alpha \varphi_n^\alpha = (2n+\alpha+1)\varphi_n^\alpha$, where

$$\varphi_n^\alpha(x) = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{x^2}{2}} x^{\alpha+\frac{1}{2}} L_n^\alpha\big(x^2\big), \quad x \in (0,\infty),$$

and L_n^α denotes the n-th Laguerre polynomial of type α ([30, p. 100] and [31, p. 7]). The system $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ of Laguerre functions is an orthonormal basis for $L^2((0,\infty), dx)$.

We define, for every $\beta > 0$, the $-\beta$ power of the operator L_α as follows

$$L_\alpha^{-\beta} f = \sum_{n=0}^{\infty} \frac{c_n^\alpha(f)}{(2n+\alpha+1)^\beta} \varphi_n^\alpha, \quad f \in L^2\big((0,\infty), dx\big). \tag{1.2}$$

Here, $c_n^\alpha(f) = \int_0^\infty \varphi_n^\alpha(x) f(x)\, dx$, for every $n \in \mathbb{N}$ and $f \in L^2((0,\infty), dx)$. If $\beta > 0$, the operator $L_\alpha^{-\beta}$ is bounded from $L^2((0,\infty), dx)$ into itself. This kind of operators, that can be seen as fractional integrals associated with the Laguerre operator L_α, has been investigated by several authors ([11], [12], [16] and [26]).

Let $k \in \mathbb{N} \setminus \{0\}$. The precise definition of $R_\alpha^{(k)} f$ for $f \in L^2((0,\infty), dx)$ is the following

$$R_\alpha^{(k)}(f) = \sum_{n=0}^{\infty} \frac{c_n^\alpha(f)}{(2n+\alpha+1)^{\frac{k}{2}}} \mathfrak{D}_\alpha^k \varphi_n^\alpha. \tag{1.3}$$

Since $\varphi_n^\alpha(x) = x^{\alpha+\frac{1}{2}} \ell_n^\alpha(x)$, $n \in \mathbb{N}$, and $\mathfrak{D}_\alpha = x^{\alpha+\frac{1}{2}} \delta x^{-\alpha-\frac{1}{2}}$, where $\delta = \frac{d}{dx} + x$ and ℓ_n^α, $n \in \mathbb{N}$, are understood as in [22] (see (1.4)), according to [22, Proposition 3.5] (see also the comment after the proof of [22, Proposition 3.5]), we have that the series in (1.3) converges in $L^2((0,\infty), dx)$, for every $f \in L^2((0,\infty), dx)$, and the operator $R_\alpha^{(k)}$ defined by (1.3) is bounded from $L^2((0,\infty), dx)$ into itself.

Moreover, if $f \in C_c^\infty(0,\infty)$, the space of C^∞-functions on $(0,\infty)$ that have compact support on $(0,\infty)$, for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$\big|c_n^\alpha(f)\big| \le C_m (1+n)^{-m}, \quad n \in \mathbb{N}.$$

Then, according to [22, (2.8) and Proposition 3.5], we can see that the series in (1.2) and in (1.3) converge uniformly in every compact subset of $(0,\infty)$, and they define C^∞-functions on $(0,\infty)$, for every $f \in C_c^\infty(0,\infty)$.

Assume that $f \in C_c^\infty(0,\infty)$. We define the C^∞-function $\Phi_k^\alpha(f)$ on $(0,\infty)$ by

$$\Phi_k^\alpha(f)(x) = \sum_{n=0}^\infty \frac{c_n^\alpha(f)}{(2n+\alpha+1)^{\frac{k}{2}}} \varphi_n^\alpha(x), \quad x \in (0,\infty).$$

Then, we can write $L_\alpha^{-\frac{k}{2}} f = \Phi_k^\alpha(f)$ as $L^2((0,\infty),dx)$-functions. Also, we have $R_\alpha^{(k)}(f) = \mathfrak{D}_\alpha^k \Phi_k^\alpha(f)$ as $L^2((0,\infty),dx)$-functions, where the differential operators in the right-hand side of the equality are understood in the classical sense.

In order to represent the k-th Riesz transform $R_\alpha^{(k)}$ as a principal value integral operator, we need to use the kernel of the heat semigroup $\{W_t^\alpha\}_{t\geq 0}$ associated with the system $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$. For every $t \geq 0$, the operator W_t^α is defined by

$$W_t^\alpha(f) = \sum_{n=0}^\infty e^{-t(2n+\alpha+1)} c_n^\alpha(f)\varphi_n^\alpha, \quad f \in L^2\big((0,\infty),dx\big).$$

We can also write

$$W_t^\alpha(f)(x) = \int_0^\infty W_t^\alpha(x,y) f(y)\,dy, \quad f \in L^2\big((0,\infty),dx\big) \text{ and } t>0,$$

where (see Mehler's formula [31, p. 8])

$$\begin{aligned} W_t^\alpha(x,y) &= \sum_{n=0}^\infty e^{-t(2n+\alpha+1)} \varphi_n^\alpha(x)\varphi_n^\alpha(y) \\ &= (\sinh t)^{-1}(xy)^{\frac{1}{2}} I_\alpha\left(\frac{xy}{\sinh t}\right) \exp\left(-\frac{1}{2}\big(x^2+y^2\big)\coth t\right), \end{aligned}$$

$t,x,y \in (0,\infty)$. Here, I_α represents the modified Bessel function of the first kind and order α.

We can now establish the main result of this paper.

THEOREM 1.1. *Let $\alpha > -1$ and $k \in \mathbb{N}\setminus\{0\}$. For every $f \in C_c^\infty(0,\infty)$ it holds*

$$R_\alpha^{(k)} f(x) = w_k f(x) + \lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y) f(y)\,dy, \quad a.e.\ x \in (0,\infty),$$

where

$$R_\alpha^{(k)}(x,y) = \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty t^{\frac{k}{2}-1} \mathfrak{D}_\alpha^k W_t^\alpha(x,y)\,dt, \quad x,y \in (0,\infty), x \neq y,$$

and $w_k = 0$, when k is odd and $w_k = -2^{\frac{k}{2}}$, when k is even.

The operator $R_\alpha^{(k)}$ can be extended, defining it by

$$R_\alpha^{(k)} f(x) = w_k f(x) + \lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y) f(y)\,dy, \quad a.e.\ x \in (0,\infty),$$

as a bounded operator from $L^p((0,\infty),x^\delta\,dx)$ *into itself, for* $1<p<\infty$ *and*

(a) $-(\alpha+\frac{3}{2})p-1<\delta<(\alpha+\frac{3}{2})p-1$, *when* k *is odd*;
(b) $-(\alpha+\frac{1}{2})p-1<\delta<(\alpha+\frac{3}{2})p-1$, *when* k *is even*;

and as a bounded operator from $L^1((0,\infty),x^\delta\,dx)$ *into* $L^{1,\infty}((0,\infty),x^\delta\,dx)$ *when*

(c) $-\alpha-\frac{5}{2}\le\delta\le\alpha+\frac{1}{2}$, *when* k *is odd*;
(d) $-\alpha-\frac{3}{2}\le\delta\le\alpha+\frac{1}{2}$, *for* $\alpha\neq-\frac{1}{2}$, *and* $-1<\delta\le 0$, *for* $\alpha=-\frac{1}{2}$, *when* k *is even.*

In Section 3, where Theorem 1.1 is proved, we show that the maximal operator $R_\alpha^{(k),*}$ defined by

$$R_\alpha^{(k),*}(f)=\sup_{\varepsilon>0}\left|\int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y)f(y)\,dy\right|$$

is bounded from $L^p((0,\infty),x^\delta\,dx)$ into itself, for $1<p<\infty$ and

(a) $-(\alpha+\frac{3}{2})p-1<\delta<(\alpha+\frac{3}{2})p-1$, when k is odd;
(b) $-(\alpha+\frac{1}{2})p-1<\delta<(\alpha+\frac{3}{2})p-1$, when k is even;

and bounded from $L^1((0,\infty),x^\delta\,dx)$ into $L^{1,\infty}((0,\infty),x^\delta\,dx)$ when

(c) $-\alpha-\frac{5}{2}\le\delta\le\alpha+\frac{1}{2}$, when k is odd;
(d) $-\alpha-\frac{3}{2}\le\delta\le\alpha+\frac{1}{2}$, for $\alpha\neq-\frac{1}{2}$, and $-1<\delta\le 0$, for $\alpha=-\frac{1}{2}$, when k is even.

Therefore, the operator $\mathbb{R}_\alpha^{(k)}$ defined by

$$\mathbb{R}_\alpha^{(k)}(f)(x)=w_kf(x)+\lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y)f(y)\,dy$$

has also those L^p-boundedness properties. In particular, $\mathbb{R}_\alpha^{(k)}$ is bounded from $L^2((0,\infty),dx)$ into itself. Hence, since $\mathbb{R}_\alpha^{(k)}=R_\alpha^{(k)}$ on $C_c^\infty(0,\infty)$ and $C_c^\infty(0,\infty)$ is dense in $L^2((0,\infty),dx)$, $\mathbb{R}_\alpha^{(k)}=R_\alpha^{(k)}$ on $L^2((0,\infty),dx)$.

We also get the corresponding principal value property in the Hermite context (see Proposition 2.1) which completes, in the one dimensional case, the results in [28] about the higher order Riesz transform associated with the Hermite operator.

First order Riesz transforms in the L_α-setting were studied in [21] for $\alpha\ge-1/2$ and in [1] for $\alpha>-1$. Also, the procedure developed in [15] can be used to investigate strong, weak and restricted weak type with respect to the measure $x^\delta\,dx$ on $(0,\infty)$ for the Riesz transforms $R_\alpha^{(1)}$.

As it was mentioned, we establish boundedness properties for $R_\alpha^{(k)}$ in $L^p((0,\infty),x^\delta\,dx)$ (Theorem 1.1).

In the multidimensional Laguerre-function setting when $\alpha\in[-1/2,\infty)^d$, Nowak and Stempak in [22] studied weighted L^p-boundedness properties of the higher order Riesz transforms. They used the Calderón–Zygmund theory on certain homogeneous type spaces (in the sense of Coifman and Weiss [7]).

Our procedure here is completely different from the one used in [12] and [22]. In a first step, we split the operator $R_\alpha^{(k)}$ into two parts, namely: a local operator and a global one. These operators are integral operators defined by kernels supported close to and far from the diagonal, respectively. The global operator is upper bounded by Hardy type operators. The novelty of our method is the way followed to study the local part. We establish a pointwise estimate connecting the kernel of $R_\alpha^{(k)}$ with the corresponding one to the k-th Riesz transform associated with the Hermite operator in one dimension, for every $\alpha > -1$ (see Proposition 3.3(iii)). By using this identity, we transfer boundedness and convergence results from the k-th Riesz transform for Hermite operator in one dimension to the k-th Riesz transform in the L_α-setting.

In the literature (see, for instance [6], [21] and [26]), we can find other systems of Laguerre functions different from $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$. In particular, from the Laguerre polynomials $\{L_n^\alpha\}_{n\in\mathbb{N}}$ we can also derive the system $\{\ell_n^\alpha\}_{n\in\mathbb{N}}$, where, for every $n \in \mathbb{N}$,

$$\ell_n^\alpha(x) = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{x^2}{2}} L_n^\alpha\big(x^2\big), \quad x \in (0,\infty), \tag{1.4}$$

$\{\ell_n^\alpha\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2((0,\infty), x^{2\alpha+1}\,dx)$.

As it is shown in [1], harmonic analysis operators associated with $\{\ell_n^\alpha\}_{n\in\mathbb{N}}$ is closely connected with the corresponding operators related to the family $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$. The connection is given by a multiplication operator defined by $M_\beta f = x^\beta f$, for certain $\beta \in \mathbb{R}$. From the strong type results for $R_\alpha^{(k)}$ established in Theorem 1.1, the corresponding results for the k-th Riesz transform in the $\{\ell_n^\alpha\}_{n\in\mathbb{N}}$ setting can be deduced. Moreover, the weak type results for the k-th Riesz transform associated with $\{\ell_n^\alpha\}_{n\in\mathbb{N}}$ can be obtained by proceeding as in the $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ case in Theorem 1.1. In particular, our results in Theorem 1.1 lead to, when the dimension is one, that the higher order Riesz operator $\mathfrak{R}_k^\alpha$, associated with $\{\ell_n^\alpha\}_{n\in\mathbb{N}}$ and considered in [22], can be extended to $L^p((0,\infty), x^\delta\,dx)$ as a bounded operator from $L^p((0,\infty), x^\delta\,dx)$ into itself provided that $1 < p < \infty$, $\alpha > -1$, k is odd and $-p-1 < \delta < (2\alpha+2)p-1$ or when k is even and $-1 < \delta < (2\alpha+2)p-1$, and from $L^1((0,\infty), x^\delta\,dx)$ into $L^{1,\infty}((0,\infty), x^\delta\,dx)$ when $\alpha > -1$, k is odd and $-2 \le \delta \le 2\alpha+1$ or when $\alpha \neq -1/2$, k is even and $-1 \le \delta \le 2\alpha+1$. From [22, Theorem 3.8], it can be inferred for power weights only that $\mathfrak{R}_k^\alpha$ is bounded in $L^p((0,\infty), x^\delta\,dx)$ when $1 < p < \infty$, $\alpha \ge -1/2$, $k \in \mathbb{N}$ and $-1 < \delta < (2\alpha+2)p-1$, and that $\mathfrak{R}_k^\alpha$ is bounded from $L^1((0,\infty), x^\delta\,dx)$ into $L^{1,\infty}((0,\infty), x^\delta\,dx)$, when $\alpha \ge -1/2$, $k \in \mathbb{N}$ and $-1 < \delta \le 0$.

The organization of the paper is the following. Section 2 contains some basic facts needed in the sequel. Section 3 is devoted to prove the main result of this paper (Theorem 1.1) where we establish that the higher order Riesz transforms are principal value singular integral operators (modulus a constant times of the function, when k is even) and we show $L^p((0,\infty), x^\delta\, dx)$-boundedness properties for them.

Throughout this paper, $C_c^\infty(I)$ denotes the space of functions in $C^\infty(I)$ having compact support on I. By C and c we always represent positive constants that can change from one line to the other one, and $E[r]$, $r \in \mathbb{R}$, stands for the integer part of r.

2. Preliminaries

In this section, we recall some definitions and properties that will be useful in the sequel. By H, we denote the Hermite differential operator

$$
\begin{aligned}
H &= \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) \\
&= -\frac{1}{4}\left[\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} - x\right) + \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right)\right].
\end{aligned} \tag{2.1}
$$

Note that $\frac{d}{dx} + x$ and $-\frac{d}{dx} + x$ are formal adjoint operators in $L^2(\mathbb{R}, dx)$. Moreover, if $n \in \mathbb{N}$, H_n represents the n-th Hermite polynomial [30, p. 104] and h_n is the Hermite function given by $h_n(x) = (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x)$, $x \in \mathbb{R}$, then it follows that

$$
H h_n = \left(n + \frac{1}{2}\right) h_n, \quad n \in \mathbb{N}.
$$

Moreover, the system $\{h_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathbb{R}, dx)$.

The investigations of harmonic analysis in the Hermite setting were begun by Muckenhoupt [18]. This author considered Hermite polynomial expansions instead of Hermite function expansions. In the last decades several authors have studied harmonic analysis operators in the Hermite (polynomial or function) context (see, for instance, [8], [9], [10], [13], [14], [23], [25], [27], [28] and [32]).

The heat semigroup $\{W_t\}_{t>0}$ associated with the family $\{h_n\}_{n\in\mathbb{N}}$ is defined by

$$
W_t(f) = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})t} c_n(f) h_n, \quad f \in L^2(\mathbb{R}, dx) \text{ and } t > 0,
$$

where $c_n(f) = \int_{-\infty}^{+\infty} h_n(x) f(x)\, dx$, $n \in \mathbb{N}$ and $f \in L^2(\mathbb{R}, dx)$.

For every $t > 0$, the operator W_t can be described by the integral

$$
W_t(f)(x) = \int_{-\infty}^{+\infty} W_t(x, y) f(y)\, dy, \quad f \in L^2(\mathbb{R}, dx),
$$

where, according to Mehler's formula we have that (see [29, (1.4)]), for each $x, y \in \mathbb{R}$ and $t > 0$,

$$\begin{aligned} W_t(x,y) &= \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})t} h_n(x) h_n(y) \\ &= (2\pi \sinh t)^{-\frac{1}{2}} \exp\left[-\frac{1}{4}\left(\tanh\left(\frac{t}{2}\right)(x+y)^2 + \coth\left(\frac{t}{2}\right)(x-y)^2\right)\right] \end{aligned}$$

(see [29, (1.4)]).

Let $\beta > 0$. The negative power $H^{-\beta}$ of H is given by

$$H^{-\beta} f = \sum_{n=0}^{\infty} \frac{c_n(f)}{(n+\frac{1}{2})^\beta} h_n, \quad f \in L^2(\mathbb{R}, dx). \tag{2.2}$$

Thus, $H^{-\beta}$ is a bounded operator from $L^2(\mathbb{R}, dx)$ into itself.

We also define the operators T_β and S_β as follows:

$$T_\beta(f)(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} W_t(f)(x)\, dt, \quad f \in L^2(\mathbb{R}, dx), \tag{2.3}$$

and

$$S_\beta(f)(x) = \int_{-\infty}^{+\infty} K_{2\beta}(x,y) f(y)\, dy, \quad f \in L^2(\mathbb{R}, dx), \tag{2.4}$$

where, for every $\gamma > 0$,

$$K_\gamma(x,y) = \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^\infty t^{\frac{\gamma}{2}-1} W_t(x,y)\, dt, \quad x, y \in \mathbb{R}, x \neq y.$$

We have that $T_\beta = S_\beta = H^{-\beta}$ on $L^2(\mathbb{R}, dx)$ (see Proposition 3.1).

Suppose now that $f \in C_c^\infty(\mathbb{R})$. Then, according to [27, (2.1)] the series in (2.2) converges uniformly in $\mathbb{R}$ and it defines the function

$$\Psi_\beta(f)(x) = \sum_{n=0}^{\infty} \frac{c_n(f)}{(n+\frac{1}{2})^\beta} h_n(x), \quad x \in \mathbb{R},$$

that is continuous on $\mathbb{R}$. Moreover, the function Λ_β defined by

$$\Lambda_\beta(f)(x) = \int_{-\infty}^{+\infty} f(y) K_{2\beta}(x,y)\, dy, \quad x \in \mathbb{R},$$

is also continuous on $\mathbb{R}$ (see Proposition 3.1). Hence, $\Psi_\beta(f)(x) = \Lambda_\beta(f)(x)$, $x \in \mathbb{R}$.

The factorization in (2.1) suggests to define formally the Riesz transform R associated with H by

$$R = \left(\frac{d}{dx} + x\right) H^{-\frac{1}{2}}.$$

Since $(\frac{d}{dx}+x)h_n=\sqrt{2n}h_{n-1}$, $n\in\mathbb{N}$ (here, $h_{-1}=0$), as in [27, (3.3)], we define the Riesz transform R on $L^2(\mathbb{R},dx)$ by

$$Rf=\sum_{n=0}^{\infty}\left(\frac{2n}{n+\frac{1}{2}}\right)^{\frac{1}{2}}c_n(f)h_{n-1},\quad f\in L^2(\mathbb{R},dx).$$

L^p-boundedness properties of the Riesz transform (even in the n-dimensional case) were established in [27].

In [28], higher order Riesz transforms in the Hermite function setting on $\mathbb{R}^n$ were investigated. Assume that $k\in\mathbb{N}\setminus\{0\}$. The k-th Riesz transform $R^{(k)}$ associated with H is defined formally by

$$R^{(k)}=\left(\frac{d}{dx}+x\right)^k H^{-\frac{k}{2}}.$$

On $L^2((0,\infty),dx)$ the Riesz transform $R^{(k)}$ is defined in a precise way as follows

$$R^{(k)}f=\sum_{n=k}^{\infty}\frac{2^{\frac{k}{2}}(n(n-1)\cdots(n-k+1))^{\frac{1}{2}}}{(n+\frac{1}{2})^{\frac{k}{2}}}c_n(f)h_{n-k},\quad f\in L^2(\mathbb{R},dx).\tag{2.5}$$

It is clear that $R^{(k)}$ is a bounded operator from $L^2(\mathbb{R},dx)$ into itself. Moreover, $R^{(k)}$ admits the integral representation

$$R^{(k)}f(x)=\int_{-\infty}^{+\infty}R^{(k)}(x,y)f(y)\,dy,\quad x\in\mathbb{R}\setminus\operatorname{supp}f, f\in L^2(\mathbb{R},dx),$$

where

$$R^{(k)}(x,y)=\frac{1}{\Gamma(\frac{k}{2})}\int_0^{\infty}t^{\frac{k}{2}-1}\left(\frac{\partial}{\partial x}+x\right)^k W_t(x,y)\,dt,\quad x,y\in\mathbb{R},x\neq y.$$

L^p-boundedness properties of the Riesz transform $R^{(k)}$ were established in [28, Theorem 2.3] by invoking the Calderón–Zygmund singular integral theory. For every $k\in\mathbb{N}\setminus\{0\}$, $R^{(k)}$ can be extended to $L^p(\mathbb{R},dx)$ as a bounded operator from $L^p(\mathbb{R},dx)$ into itself, when $1<p<\infty$, and from $L^1(\mathbb{R},dx)$ into $L^{1,\infty}(\mathbb{R},dx)$. It is remarkable to note that the L^p-mapping properties for the higher order Riesz transform in the Hermite polynomial setting are essentially different to the corresponding ones in the Hermite function context ([9] and [10]).

Assume now $f\in C_c^{\infty}(0,\infty)$. According to [27, Lemma 1.2], for every $m\in\mathbb{N}$ there exists $C_m>0$ such that

$$\left|c_n(f)\right|\leq C_m(1+n)^{-m},\quad n\in\mathbb{N}.$$

Hence, from [27, (2.2)] we deduce that the series in (2.2) and (2.5) converge also uniformly in $\mathbb{R}$. By defining as above the function $\Psi_{\frac{k}{2}}(f)$ on $\mathbb{R}$ by

$$\Psi_{\frac{k}{2}}(f)(x) = \sum_{n=0}^{\infty} \frac{c_n(f)}{(n+\frac{1}{2})^{\frac{k}{2}}} h_n(x), \quad x \in \mathbb{R},$$

$\Psi_{\frac{k}{2}}(f) = H^{-\frac{k}{2}} f$ as $L^2(\mathbb{R}, dx)$-functions. Moreover, by [27, (2.2), (3.2) and Lemma 1.2], $\Psi_{\frac{k}{2}}(f)$ is a C^∞-function on $\mathbb{R}$, the series in (2.5) converges uniformly in $\mathbb{R}$, and for $x \in \mathbb{R}$,

$$\left(\frac{d}{dx} + x\right)^k \Psi_{\frac{k}{2}}(f)(x) = \sum_{n=k}^{\infty} \frac{2^{\frac{k}{2}}(n(n-1)\cdots(n-k+1))^{\frac{1}{2}}}{(n+\frac{1}{2})^{\frac{k}{2}}} c_n(f) h_{n-k}(x).$$

We have also that $(\frac{d}{dx} + x)^k \Psi_{\frac{k}{2}}(f)(x) = R^{(k)} f$ as $L^2(\mathbb{R}, dx)$-functions.

In order to investigate the higher order Riesz transforms associated with the Laguerre operator, as it was mentioned, we shall exploit a connection between higher order Riesz transforms in the Hermite and Laguerre settings.

The following new property that will be proved in Section 3 is needed in the proof of Theorem 1.1. It states that the higher order Riesz transform for the Hermite operator is actually a principal value integral operator.

PROPOSITION 2.1. *Let $k \in \mathbb{N} \setminus \{0\}$. Then, for every $f \in C_c^\infty(\mathbb{R})$, $1 \le p < \infty$,*

$$R^{(k)} f(x) = w_k f(x) + \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R^{(k)}(x,y) f(y)\, dy, \quad \text{a.e. } x \in \mathbb{R}, \tag{2.6}$$

where

$$R^{(k)}(x,y) = \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty t^{\frac{k}{2}-1} \left(\frac{\partial}{\partial x} + x\right)^k W_t(x,y)\, dt, \quad x, y \in \mathbb{R}, x \neq y,$$

and $w_k = 0$, when k is odd, and $w_k = -2^{\frac{k}{2}}$, when k is even.

Since $R^{(k)}(x,y)$, $x, y \in \mathbb{R}$, is a Calderón–Zygmund kernel [28] by using standard density arguments we deduce, from Proposition 2.1, that the operator $R^{(k)}$ can be extended by (2.6) to $L^p(\mathbb{R}, dx)$, $1 \le p < \infty$, as a bounded operator from $L^p(\mathbb{R}, dx)$ into itself, $1 < p < \infty$, and from $L^1(\mathbb{R}, dx)$ into $L^{1,\infty}(\mathbb{R}, dx)$.

As it was indicated, the modified Bessel function I_α of the first kind and order α appears in the kernel of the heat semigroup associated to the system $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$. The following properties of the function I_α will be repeatedly used in the sequel (see [17] and [33]):

(P1) $I_\alpha(z) \sim z^\alpha$, $z \to 0$.

(P2) $\sqrt{z} I_\alpha(z) = \frac{e^z}{\sqrt{2\pi}} (\sum_{r=0}^n (-1)^r [\alpha, r](2z)^{-r} + O(z^{-n-1}))$, $n = 0, 1, 2, \ldots,$ where $[\alpha, 0] = 1$ and

$$[\alpha, r] = \frac{(4\alpha^2 - 1)(4\alpha^2 - 3^2)\cdots(4\alpha^2 - (2r-1)^2)}{2^{2r}\Gamma(r+1)}, \quad r = 1, 2, \ldots.$$

(P3) $\frac{d}{dz}(z^{-\alpha}I_\alpha(z)) = z^{-\alpha}I_{\alpha+1}(z)$, $z \in (0,\infty)$.

On the other hand, in our study of the global part of the operators involved, it will be useful to consider the Hardy type operators defined by

$$H_0^\eta(f)(x) = x^{-\eta-1}\int_0^x y^\eta f(y)\,dy, \quad x \in (0,\infty),$$

and

$$H_\infty^\eta(f)(x) = x^\eta \int_x^\infty y^{-\eta-1} f(y)\,dy, \quad x \in (0,\infty),$$

where $\eta > -1$. L^p-boundedness properties of the operators H_0^η and H_∞^η were established by Muckenhoupt [19] and Andersen and Muckenhoupt [2]. In particular, mappings properties for H_0^η and H_∞^η on $L^p((0,\infty), x^\delta\,dx)$ can be encountered in [6, Lemmas 3.1 and 3.2].

The following formula established in [12, Lemma 4.3, (4.6)] will be frequently used in the paper. For every $N \in \mathbb{N}$, and a sufficiently smooth function $g : (0,\infty) \longrightarrow \mathbb{R}$, it holds

$$\frac{d^N}{dx^N}\big[g\big(x^2\big)\big] = \sum_{l=0}^{E[\frac{N}{2}]} E_{N,l}x^{N-2l}\bigg(\frac{d^{N-l}}{dx^{N-l}}g\bigg)\big(x^2\big), \tag{2.7}$$

where

$$E_{N,l} = 2^{N-2l}\frac{N!}{l!(N-2l)!}, \quad 0 \le l \le E\left[\frac{N}{2}\right].$$

We finish this section establishing the following technical lemma that is needed in the proof of Proposition 3.3.

LEMMA 2.1. *Let $\alpha > -1$ and $j \in \mathbb{N}\setminus\{0\}$. For every $m = 0,1,\dots,E[\frac{j}{2}]$, we have*

$$\sum_{n=0}^{m}\sum_{l=2n}^{j}(-1)^{l+n}\binom{j}{l}\frac{E_{l,n}}{2^{l-2n}}[\alpha+l-n, m-n] = 0. \tag{2.8}$$

Proof. For every $s = 0,\dots,j$ we denote by $A_{j,s}$ the values

$$A_{j,s} = \sum_{l=0}^{j}(-1)^l\binom{j}{l}l^s,$$

where we take the convention $0^0 = 1$.

In [3, (43)], it was established that

$$A_{j,s} = 0, \quad s = 0,1,\dots,j-1. \tag{2.9}$$

On the other hand, since $\binom{m}{n} = \frac{m}{n}\binom{m-1}{n-1}$, for $m \ge n \ge 1$, by using (2.9) we obtain that $A_{j,j} = -jA_{j-1,j-1}$, $j \in \mathbb{N}$, $j \ge 1$ and so $A_{j,j} = (-1)^j j!$, $j \in \mathbb{N}$.

Consider $m=0,1,\dots,E[\frac{j}{2}]$ and $n=0,1,\dots,m$. We can write

$$\begin{aligned}
&\sum_{l=2n}^{j}(-1)^l\binom{j}{l}\frac{E_{l,n}}{2^{l-2n}}[\alpha+l-n,m-n]\\
&=\sum_{s=0}^{j-2n}(-1)^s\binom{j}{s+2n}\frac{E_{s+2n,n}}{2^s}[\alpha+s+n,m-n]\\
&=\frac{j!}{n!}\sum_{s=0}^{j-2n}\frac{(-1)^s}{s!(j-2n-s)!}[\alpha+s+n,m-n]\\
&=\frac{j!}{n!(j-2n)!}\sum_{s=0}^{j-2n}(-1)^s\binom{j-2n}{s}[\alpha+s+n,m-n].
\end{aligned}$$

We observe that $[\alpha+s+n,m-n]$ is a polynomial in s which has degree $2(m-n)$. Besides, if j is odd, $2(m-n)\le j-2n-1$. Hence, (2.9) allows us to conclude (2.8) in this case.

Assume now that j is even. Then $2(m-n)\le j-2n$ and (2.9) leads to

$$\sum_{n=0}^{m}\sum_{l=2n}^{j}(-1)^{l+n}\binom{j}{l}\frac{E_{l,n}}{2^{l-2n}}[\alpha+l-n,m-n]=0,$$

when $m=0,1,\dots,\frac{j}{2}-1$.

For $m=\frac{j}{2}$ and again by (2.9), we can write

$$\begin{aligned}
&\sum_{n=0}^{m}\sum_{l=2n}^{j}(-1)^{l+n}\binom{j}{l}\frac{E_{l,n}}{2^{l-2n}}[\alpha+l-n,m-n]\\
&=\sum_{n=0}^{\frac{j}{2}}\frac{(-1)^n j!}{n!(j-2n)!}\sum_{s=0}^{j-2n}(-1)^s\binom{j-2n}{s}\frac{s^{j-2n}}{(\frac{j}{2}-n)!}\\
&=j!\sum_{n=0}^{\frac{j}{2}}\frac{(-1)^n}{n!(j-2n)!(\frac{j}{2}-n)!}A_{j-2n,j-2n}=\frac{j!}{(\frac{j}{2})!}\sum_{n=0}^{\frac{j}{2}}(-1)^n\binom{\frac{j}{2}}{n}=0.
\end{aligned}$$

Thus, (2.8) is established. □

3. Higher order Riesz transforms associated with Laguerre expansions

In this section, we prove our main result (Theorem 1.1) concerning to higher order Riesz transforms associated with the sequence $\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$ of Laguerre functions. As it was mentioned, our procedure is based on certain connection between higher order Riesz transforms in the Laguerre and Hermite settings.

We start proving the representations (2.3) and (2.4) for the negative powers $H^{-\beta}$, $\beta>0$.

Proposition 3.1. *Let $\beta > 0$. Then, for every $f \in L^2(\mathbb{R}, dx)$,*

$$H^{-\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} W_t(f)(x)\, dt, \tag{3.1}$$

as $L^2(\mathbb{R}, dx)$-functions. Moreover, if $f \in C_c^\infty(\mathbb{R})$ then, the function

$$\Psi_\beta(f)(x) = \sum_{n=0}^\infty \frac{c_n(f)}{(n+1/2)^\beta} h_n(x), \quad x \in \mathbb{R}, \tag{3.2}$$

is continuous on $\mathbb{R}$, and

$$\Psi_\beta(f)(x) = \int_{-\infty}^{+\infty} K_{2\beta}(x, y) f(y)\, dy, \quad x \in \mathbb{R},$$

where, for every $\gamma > 0$,

$$K_\gamma(x, y) = \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^\infty t^{\frac{\gamma}{2}-1} W_t(x, y)\, dt, \quad x, y \in \mathbb{R}, x \neq y.$$

Proof. We define the operator T_β as follows

$$T_\beta(f)(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} W_t(f)(x)\, dt, \quad f \in L^2(\mathbb{R}, dx).$$

According to [27, Remark 2.10] and by using Minkowski's inequality we get

$$\begin{aligned} \|T_\beta f\|_2 &\leq \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} \|W_t f\|_2\, dt \\ &\leq \frac{\|f\|_2}{\Gamma(\beta)} \int_0^\infty \frac{t^{\beta-1}}{\sqrt{\cosh t}}\, dt \leq \|f\|_2, \quad f \in L^2(\mathbb{R}, dx). \end{aligned}$$

Hence, T_β is a bounded operator from $L^2(\mathbb{R}, dx)$ into itself. Moreover, it is not hard to see that $T_\beta f = H^{-\beta} f$, for every $f \in \operatorname{span}\{h_n\}_{n\in\mathbb{N}}$. Since $\operatorname{span}\{h_n\}_{n\in\mathbb{N}}$ is a dense subspace of $L^2(\mathbb{R}, dx)$ [27, Lemma 2.3], we conclude that $H^{-\beta} f = T_\beta f$, for every $f \in L^2(\mathbb{R}, dx)$.

Assume that $f \in C_c^\infty(\mathbb{R})$. By [27, (2.1)], the series in (3.2) converges uniformly on $\mathbb{R}$. Hence, $\Psi_\beta(f)$ is a continuous function on $\mathbb{R}$. Moreover,

$$\Psi_\beta(f)(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} W_t(f)(x)\, dt, \quad \text{a.e. } x \in \mathbb{R}.$$

On the other hand, we have that, for certain $-\infty < a < b < +\infty$, and $x \in \mathbb{R}$,

$$\begin{aligned} &\int_0^\infty t^{\beta-1} \int_{-\infty}^{+\infty} W_t(x, y) |f(y)|\, dy\, dt \\ &\quad \leq C \int_a^b |f(y)| \left(\int_0^1 t^{\beta-\frac{3}{2}} e^{-c\frac{(x-y)^2}{t}}\, dt + \int_1^\infty e^{-\frac{t}{2}}\, dt \right) dy < \infty. \end{aligned}$$

By Fubini's theorem, we obtain

$$\Psi_\beta(f)(x) = \int_{-\infty}^{+\infty} K_{2\beta}(x,y) f(y)\, dy, \quad \text{a.e. } x \in \mathbb{R}.$$

We now define the function $\Lambda_\beta(f)$ by

$$\Lambda_\beta(f)(x) = \int_{-\infty}^{+\infty} K_{2\beta}(x,y) f(y)\, dy, \quad x \in \mathbb{R}.$$

The function $\Lambda_\beta(f)$ is continuous on $\mathbb{R}$.

In order to see this, we split the inner integral in $\Lambda_\beta(f)$ into two parts and write

$$\begin{aligned}\Lambda_\beta(f)(x) &= \frac{1}{\Gamma(\beta)} \int_{-\infty}^{+\infty} f(y) \left(\int_0^1 + \int_1^\infty \right) t^{\beta-1} W_t(x,y)\, dt\, dy \\ &= \Lambda_{\beta,1}(f)(x) + \Lambda_{\beta,2}(f)(x), \quad x \in \mathbb{R}.\end{aligned}$$

Firstly, we analyze the function $\Lambda_{\beta,2}(f)$. We can write

$$\left| \int_1^\infty t^{\beta-1} W_t(x,y)\, dt \right| \le C \int_1^\infty t^{\beta-1} e^{-\frac{t}{2}}\, dt \le C, \quad x,y \in \mathbb{R}.$$

Then, by applying the dominated convergence theorem we can prove that the function $\Lambda_{\beta,2}(f)$ is continuous on $\mathbb{R}$.

The analysis of $\Lambda_{\beta,1}(f)$ is more dedicated. If $\beta > \frac{1}{2}$, it follows that

$$\left| \int_0^1 t^{\beta-1} W_t(x,y)\, dt \right| \le C \int_0^1 t^{\beta-\frac{3}{2}}\, dt \le C, \quad x,y \in \mathbb{R},$$

and by using again dominated convergence theorem we conclude that the function $\Lambda_{\beta,1}(f)$ is continuous on $\mathbb{R}$. Assume now that $0 < \beta \le \frac{1}{2}$. By making the change of variables $s = \tanh(\frac{t}{2})$ (due to Meda), we get

$$\begin{aligned}&\int_0^1 t^{\beta-1} W_t(x,y)\, dt \\ &\quad = \frac{1}{\sqrt{2\pi}} \int_0^a \left(\frac{2s}{1-s^2} \right)^{-\frac{1}{2}} e^{-\frac{1}{4}(s(x+y)^2 + \frac{1}{s}(x-y)^2)} \frac{2}{1-s^2} \left(\log \frac{1+s}{1-s} \right)^{\beta-1} ds \\ &\quad = \frac{1}{\sqrt{2\pi}} \int_0^a \left[\left(\frac{2s}{1-s^2} \right)^{-\frac{1}{2}} e^{-\frac{1}{4}(s(x+y)^2 + \frac{1}{s}(x-y)^2)} \frac{2}{1-s^2} \left(\log \frac{1+s}{1-s} \right)^{\beta-1} \right. \\ &\qquad \left. - \frac{2}{(2s)^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4s}} (2s)^{\beta-1} \right] ds + \frac{2^{\beta-\frac{1}{2}}}{\sqrt{2\pi}} \int_0^a e^{-\frac{(x-y)^2}{4s}} s^{\beta-\frac{3}{2}}\, ds,\end{aligned}$$

$x,y \in \mathbb{R}, x \neq y$, where $a = \tanh(\frac{1}{2})$.

By using the mean value theorem, it holds

$$\begin{aligned}
&\Bigg|\left(\frac{2s}{1-s^2}\right)^{-\frac{1}{2}} e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\frac{2}{1-s^2}\left(\log\frac{1+s}{1-s}\right)^{\beta-1}\\
&\qquad -\frac{2}{(2s)^{\frac{1}{2}}}e^{-\frac{(x-y)^2}{4s}}(2s)^{\beta-1}\Bigg|\\
&\le C\Bigg(\left|\left(\frac{1-s^2}{2s}\right)^{\frac{1}{2}}-\frac{1}{(2s)^{\frac{1}{2}}}\right|s^{\beta-1}+s^{\beta-\frac{3}{2}}\left|\frac{1}{1-s^2}-1\right|\\
&\qquad +\frac{1}{s^{\frac{1}{2}}}\left|\left(\log\frac{1+s}{1-s}\right)^{\beta-1}-(2s)^{\beta-1}\right|+s^{\beta-\frac{3}{2}}e^{-\frac{(x-y)^2}{4s}}\left|e^{-\frac{(x+y)^2}{4s}}-1\right|\Bigg)\\
&\le Cs^{\beta-\frac{1}{2}}\big(1+(x+y)^2\big),\quad x,y\in\mathbb{R}\text{ and }s\in(0,a).
\end{aligned}$$

Since f has compact support, the dominated convergence theorem allows us to prove that the function $\Lambda_{\beta,1,1}(f)$ defined by

$$\begin{aligned}
&\Lambda_{\beta,1,1}(f)(x)\\
&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\int_0^a\Bigg[\left(\frac{2s}{1-s^2}\right)^{-\frac{1}{2}}e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\\
&\quad\times\frac{2}{1-s^2}\left(\log\frac{1+s}{1-s}\right)^{\beta-1}-\frac{2}{(2s)^{\frac{1}{2}}}e^{-\frac{(x-y)^2}{4s}}(2s)^{\beta-1}\Bigg]\,ds\,dy,\quad x\in\mathbb{R},
\end{aligned}$$

is continuous on $\mathbb{R}$.

Finally, we note that the function $\Lambda_{\beta,1,2}(f)=\Lambda_{\beta,1}(f)-\Lambda_{\beta,1,1}(f)$ can be written

$$\Lambda_{\beta,1,2}(f)(x)=(f*\mathcal{K}_\beta)(x),\quad x\in\mathbb{R},$$

where $\mathcal{K}_\beta(z)=\frac{2^{\beta-1}}{\sqrt{\pi}}\int_0^a e^{-\frac{|z|^2}{4s}}s^{\beta-\frac{3}{2}}\,ds,\ z\in\mathbb{R}$. Since $|\mathcal{K}_\beta(z)|\le C|z|^{2\beta-1}$, $z\in\mathbb{R}\setminus\{0\}$ (see, for instance, [27, Lemma 1.1]), we conclude that $\Lambda_{\beta,1,2}(f)$ is continuous on $\mathbb{R}$.

Putting together the arguments above, we establish that Λ_β is continuous on $\mathbb{R}$. Hence, $\Psi_\beta(f)(x)=\Lambda_\beta(f)(x)$, $x\in\mathbb{R}$. □

In the sequel, if $f\in C_c^\infty(\mathbb{R})$ and $\beta>0$, we write $H^{-\beta}f$ to refer the continuous function $\Psi_\beta(f)$ on $\mathbb{R}$.

We now prove that, for every $f\in C_c^\infty(\mathbb{R})$ and $k\in\mathbb{N}\setminus\{0\}$, $(\frac{d}{dx}+x)^kH^{-\frac{k}{2}}f$ is given, for almost all $x\in\mathbb{R}$, by a principal value integral plus, when k is even, a multiple of $f(x)$.

PROPOSITION 3.2. *Let $f\in C_c^\infty(\mathbb{R})$ and $k\in\mathbb{N}\setminus\{0\}$. Then,*

$$\left(\frac{d}{dx}+x\right)^kH^{-\frac{k}{2}}f(x)=w_kf(x)+\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon}R^{(k)}(x,y)f(y)\,dy,\tag{3.3}$$

a.e. $x\in\mathbb{R}$,

where $w_k = 0$, *if* k *is odd, and* $w_k = -2^{\frac{k}{2}}$, *when* k *is even.*

REMARK 3.1. Note that if $x \notin \operatorname{supp} f$, then the limit in (3.3) coincides with the absolutely convergent integral

$$\int_{-\infty}^{+\infty} R^{(k)}(x,y) f(y)\, dy$$

and (3.3) reduces to

$$\left(\frac{d}{dx} + x\right)^k H^{-\frac{k}{2}} f(x) = \int_{-\infty}^{+\infty} R^{(k)}(x,y) f(y)\, dy.$$

Proof of Proposition 3.2. By making the change of variables $s = \tanh(\frac{t}{2})$, $t \in (0,\infty)$, we obtain

$$H^{-\frac{k}{2}} f(x) = \frac{1}{\Gamma(\frac{k}{2})} \int_{-\infty}^{+\infty} f(y) \int_0^1 \left(\log \frac{1+s}{1-s}\right)^{\frac{k}{2}-1} \left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}} \\ \times e^{-\frac{1}{4}(s(x+y)^2 + \frac{1}{s}(x-y)^2)} \frac{2}{1-s^2}\, ds\, dy$$

for every $x \in \mathbb{R}$. Since, for every $m \in \mathbb{N}$,

$$\left(\frac{d}{dx} + x\right)^m g(x) = \sum_{0 \le \rho+\sigma \le m} c^m_{\rho,\sigma} x^\rho \frac{d^\sigma}{dx^\sigma} g(x), \tag{3.4}$$

where $c^m_{\rho,\sigma} \in \mathbb{R}$, $\rho, \sigma \in \mathbb{N}$, $0 \le \rho + \sigma \le m$ and $c^m_{0,m} = 1$, in order to prove this proposition it is sufficient to see that

$$\frac{d^r}{dx^r} H^{-\frac{k}{2}} f(x) = \frac{1}{\Gamma(\frac{k}{2})} \int_{-\infty}^{+\infty} f(y) \int_0^1 \left(\log \frac{1+s}{1-s}\right)^{\frac{k}{2}-1} \left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}} \\ \times \frac{\partial^r}{\partial x^r}\left(e^{-\frac{1}{4}(s(x+y)^2 + \frac{1}{s}(x-y)^2)}\right) \frac{2}{1-s^2}\, ds\, dy$$

for every $x \in \mathbb{R}$ and $r \in \mathbb{N}$, $1 \le r \le k-1$, and

$$\frac{d^k}{dx^k} H^{-\frac{k}{2}} f(x) = w_k f(x) \\ + \lim_{\varepsilon \to 0^+} \frac{1}{\Gamma(\frac{k}{2})} \int_{|x-y|>\varepsilon} f(y) \int_0^1 \left(\log \frac{1+s}{1-s}\right)^{\frac{k}{2}-1} \\ \times \left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}} \frac{\partial^r}{\partial x^r}\left(e^{-\frac{1}{4}(s(x+y)^2 + \frac{1}{s}(x-y)^2)}\right) \\ \times \frac{2}{1-s^2}\, ds\, dy, \quad \text{a.e. } x \in \mathbb{R}.$$

Let $r \in \mathbb{N}$. According to [28, p. 50], $\frac{\partial^r}{\partial x^r}(e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)})$ is a linear combination of terms of the form

$$\left(s+\frac{1}{s}\right)^{b_1} e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\left(s(x+y)^2+\frac{1}{s}(x-y)^2\right)^{b_2},$$

where $b_1, b_2 \in \mathbb{N}$ and $2b_1 + b_2 \leq r$. Hence, we have that

$$\begin{aligned}
&\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}}\frac{\partial^r}{\partial x^r}(e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)})\frac{1}{1-s^2}\\
&\leq C\left(\frac{1-s^2}{s}\right)^{\frac{1}{2}} e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\frac{1}{1-s^2}\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\\
&\quad\times\sum_{b_1,b_2\in\mathbb{N},2b_1+b_2\leq r}\left(s+\frac{1}{s}\right)^{b_1}\left(s(x+y)^2+\frac{1}{s}(x-y)^2\right)^{b_2}\\
&\leq C\frac{(-\log(1-s))^{\frac{k}{2}-1}}{\sqrt{1-s}},\quad s\in(0,1), x,y\in\mathbb{R}.
\end{aligned}$$

By taking into account the mean value and dominated convergence theorems, we get

$$\begin{aligned}
&\frac{d^r}{dx^r}\int_{-\infty}^{+\infty} f(y)\int_{\frac{1}{2}}^{1}\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\frac{2}{1-s^2}W_{\log\frac{1+s}{1-s}}(x,y)\,ds\,dy\\
&=\int_{-\infty}^{+\infty} f(y)\int_{\frac{1}{2}}^{1}\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\frac{2}{1-s^2}\frac{\partial^r}{\partial x^r}W_{\log\frac{1+s}{1-s}}(x,y)\,ds\,dy,\quad x\in\mathbb{R}.
\end{aligned}$$

By using (2.7), we can write, for every $s \in (0,1)$ and $x, y \in \mathbb{R}$,

$$\begin{aligned}
(3.5)\quad &\frac{\partial^r}{\partial x^r}W_{\log\frac{1+s}{1-s}}(x,y)\\
&=\left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}}\frac{\partial^r}{\partial x^r}\left[e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}\right]\\
&=\left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}}\sum_{j=0}^{r}\binom{r}{j}\frac{\partial^j}{\partial x^j}(e^{-\frac{s}{4}(x+y)^2})\frac{\partial^{r-j}}{\partial x^{r-j}}(e^{-\frac{1}{4s}(x-y)^2})\\
&=\left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}} e^{-\frac{s}{4}(x+y)^2}\frac{\partial^r}{\partial x^r}(e^{-\frac{1}{4s}(x-y)^2})\\
&\quad+\left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}}\sum_{j=1}^{r}\binom{r}{j}\left(\sum_{l=0}^{E[\frac{j}{2}]}E_{j,l}(x+y)^{j-2l}\left(-\frac{s}{4}\right)^{j-l}e^{-\frac{s}{4}(x+y)^2}\right)\\
&\quad\times\left(\sum_{m=0}^{E[\frac{r-j}{2}]}E_{r-j,m}(x-y)^{r-j-2m}\left(-\frac{1}{4s}\right)^{r-j-m}e^{-\frac{1}{4s}(x-y)^2}\right).
\end{aligned}$$

Assume that Ω is a compact subset of $\mathbb{R}$ and $r \in \mathbb{N}$, $1 \leq r \leq k$.
We define, for every $j = 1, \dots, r$, $0 \leq l \leq E[\frac{j}{2}]$ and $0 \leq m \leq E[\frac{r-j}{2}]$,

$$F^j_{l,m}(x,y,s) = \frac{(x+y)^{j-2l}(x-y)^{r-j-2m}}{\sqrt{1-s^2}} \left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1} \frac{e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}}{s^{r-2j+l-m+\frac{1}{2}}}$$

for each $x, y \in \mathbb{R}$ and $s \in (0, \frac{1}{2})$. Since $\log\frac{1+s}{1-s} \sim 2s$, as $s \to 0^+$, and $j \geq 1$, it follows that

$$\begin{aligned} \left|F^j_{l,m}(x,y,s)\right| &\leq C|x+y|^{j-2l}|x-y|^{r-j-2m} s^{\frac{k-2r}{2}-\frac{3}{2}+2j-l+m} \\ &\quad \times e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)} \\ &\leq Cs^{j-\frac{3}{2}+\frac{k-r}{2}} \leq Cs^{-\frac{1}{2}}, \quad x,y\in\mathbb{R}, s \in \left(0,\frac{1}{2}\right). \end{aligned} \tag{3.6}$$

We now observe that the mean value theorem leads to

$$\begin{aligned} &\left| e^{-\frac{s}{4}(x+y)^2} \left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1} \frac{1}{\sqrt{1-s^2}} - (2s)^{\frac{k}{2}-1} \right| \\ &\leq \left| \left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1} - (2s)^{\frac{k}{2}-1} \right| + \left[\left| \frac{1}{\sqrt{1-s^2}} - 1 \right| e^{-\frac{s}{4}(x+y)^2} \right. \\ &\quad \left. + \left| e^{-\frac{s}{4}(x+y)^2} - 1 \right| \right] \left| \log\frac{1+s}{1-s} \right|^{\frac{k}{2}-1} \\ &\leq C\left(s^{\frac{k}{2}+1} + \left(s^2 e^{-\frac{s}{4}(x+y)^2} + s(x+y)^2\right) s^{\frac{k}{2}-1}\right) \\ &\leq Cs^{\frac{k}{2}}, \quad x,y\in\mathbb{R}, s\in\left(0,\frac{1}{2}\right). \end{aligned} \tag{3.7}$$

By using (2.7) and (3.7), we obtain

$$\begin{aligned} &\left| \left(\frac{1-s^2}{4\pi s}\right)^{\frac{1}{2}} e^{-\frac{s}{4}(x+y)^2} \left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1} \frac{2}{1-s^2} - \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} \right| \\ &\quad \times \left| \frac{\partial^r}{\partial x^r} \left(e^{-\frac{(x-y)^2}{4s}}\right) \right| \\ &\leq Cs^{\frac{k-1}{2}} \left| \frac{\partial^r}{\partial x^r} \left(e^{-\frac{(x-y)^2}{4s}}\right) \right| \\ &\leq Cs^{\frac{k-1}{2}} \sum_{n=0}^{E[\frac{r}{2}]} |x-y|^{r-2n} s^{n-r} e^{-\frac{(x-y)^2}{4s}} \\ &\leq Cs^{\frac{k-r-1}{2}}, \end{aligned} \tag{3.8}$$

$s \in (0, \frac{1}{2})$, $x \in \Omega$ and $y \in \operatorname{supp} f$.

By combining (3.5), (3.6) and (3.8), we get

$$\left|\frac{\partial^r}{\partial x^r}\left[\frac{2}{1-s^2}\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}W_{\log\frac{1+s}{1-s}}(x,y)-\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}e^{-\frac{(x-y)^2}{4s}}\right]\right|\leq\frac{C}{\sqrt{s}},$$

when $s\in(0,\frac{1}{2})$, $x\in\Omega$ and $y\in\operatorname{supp}f$.

Then, the dominated convergence theorem allows us to show that

$$\begin{aligned}(3.9)\quad \Gamma\left(\frac{k}{2}\right)\frac{d^r}{dx^r}H^{-\frac{k}{2}}f(x)-\frac{d^r}{dx^r}\int_{-\infty}^{+\infty}f(y)\int_0^{\frac{1}{2}}\frac{1}{\sqrt{\pi s}}e^{-\frac{(x-y)^2}{4s}}(2s)^{\frac{k}{2}-1}\,ds\,dy\\ =\int_{-\infty}^{+\infty}f(y)\int_0^{\frac{1}{2}}\frac{\partial^r}{\partial x^r}\left(\frac{2}{1-s^2}\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\right.\\ \left.\times W_{\log\frac{1+s}{1-s}}(x,y)-\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}e^{-\frac{(x-y)^2}{4s}}\right)ds\,dy\end{aligned}$$

for every $x\in\mathbb{R}$. In view of properties (3.5) and (3.9), to establish the desired property (3.3) it is sufficient to prove that

$$\begin{aligned}&\frac{d^k}{dx^k}\int_{-\infty}^{+\infty}f(y)\frac{1}{\Gamma(\frac{k}{2})}\int_0^{\frac{1}{2}}\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}e^{-\frac{1}{4s}(x-y)^2}\,ds\,dy\\ &=w_kf(x)+\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon}f(y)\frac{1}{\Gamma(\frac{k}{2})}\int_0^{\frac{1}{2}}\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}\frac{\partial^k}{\partial x^k}\left(e^{-\frac{1}{4s}(x-y)^2}\right)ds\,dy,\end{aligned}$$

$x\in\mathbb{R}$, where $w_k=0$, if k is odd, and $w_k=-2^{\frac{k}{2}}$, when k is even.

Assume that $k\in\mathbb{N}$, $k\geq2$. As earlier, we can see that

$$\begin{aligned}&\frac{d^{k-2}}{dx^{k-2}}\int_{-\infty}^{+\infty}f(y)\int_0^{\frac{1}{2}}\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}e^{-\frac{1}{4s}(x-y)^2}\,ds\,dy\\ &=\int_{-\infty}^{+\infty}f(y)\int_0^{\frac{1}{2}}\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}\frac{\partial^{k-2}}{\partial x^{k-2}}\left(e^{-\frac{1}{4s}(x-y)^2}\right)ds\,dy,\quad x\in\mathbb{R}.\end{aligned}$$

Let us represent by Υ_m, $m=k-2,k-1$, the following function

$$\Upsilon_m(x)=\frac{1}{\Gamma(\frac{k}{2})}\int_0^{\frac{1}{2}}\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}\frac{\partial^m}{\partial x^m}\left(e^{-\frac{x^2}{4s}}\right)ds,\quad x\in\mathbb{R}.$$

By proceeding as above, we can see that $\Upsilon_m\in L^1(\mathbb{R})$. Indeed, for $m=k-2,k-1$, (2.7) leads to

$$\begin{aligned}\Upsilon_m(x)=\frac{1}{\Gamma(\frac{k}{2})}\sum_{l=0}^{E[\frac{m}{2}]}(-1)^{m-l}E_{m,l}x^{m-2l}\\ \times\int_0^{\frac{1}{2}}\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}\left(\frac{1}{4s}\right)^{m-l}e^{-\frac{x^2}{4s}}\,ds,\quad x\in\mathbb{R},\end{aligned}$$

and then, according to [28, Lemma 1.1],

$$
\begin{aligned}
(3.10) \qquad \left|\Upsilon_m(x)\right| &\leq C \sum_{l=0}^{E[\frac{m}{2}]} |x|^{m-2l} \int_0^{\frac{1}{2}} s^{\frac{k}{2}-\frac{3}{2}-m+l} e^{-\frac{x^2}{4s}}\, ds \\
&\leq C \begin{cases} e^{-\frac{x^2}{4}} \int_0^{\frac{1}{2}} \frac{e^{-\frac{x^2}{8s}}}{s}\, ds \leq C \frac{e^{-\frac{x^2}{4}}}{\sqrt{|x|}}, \\ \quad x \in \mathbb{R} \setminus \{0\}, \text{ if } m = k-1, \\ \sqrt{|x|} e^{-\frac{x^2}{4}} \int_0^{\frac{1}{2}} \frac{e^{-\frac{x^2}{8s}}}{s^{\frac{3}{4}}}\, ds \leq C \sqrt{|x|} e^{-\frac{x^2}{4}}, \\ \quad x \in \mathbb{R}, \text{ if } m = k-2. \end{cases}
\end{aligned}
$$

We can write, by taking into account (3.10),

$$
\begin{aligned}
&\frac{d^{k-1}}{dx^{k-1}} \int_{-\infty}^{+\infty} f(y) \int_0^{\frac{1}{2}} \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} e^{-\frac{1}{4s}(x-y)^2}\, ds\, dy \\
&\quad = \frac{d}{dx} \int_{-\infty}^{+\infty} f(y) \int_0^{\frac{1}{2}} \frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}} \frac{\partial^{k-2}}{\partial x^{k-2}} \left(e^{-\frac{1}{4s}(x-y)^2}\right) ds\, dy \\
&\quad = \frac{d}{dx} \int_{-\infty}^{+\infty} f(x-u) \Upsilon_{k-2}(u)\, du \\
&\quad = \int_{-\infty}^{+\infty} f'(x-u) \Upsilon_{k-2}(u)\, du \\
&\quad = -\lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \frac{d}{du} W_{\log \frac{1+s}{1-s}}(x,y)\, ds\, dy (x-u) \Upsilon_{k-2}(u)\, du \\
&\quad = -\lim_{\varepsilon \to 0^+} \Bigg(f(x+\varepsilon) \Upsilon_{k-2}(-\varepsilon) - f(x-\varepsilon) \Upsilon_{k-2}(\varepsilon) \\
&\qquad\quad - \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) f(x-u) \Upsilon'_{k-2}(u)\, du \Bigg) \\
&\quad = -\lim_{\varepsilon \to 0^+} \Bigg(f(x-\varepsilon) \Upsilon_{k-2}(\varepsilon) - f(x+\varepsilon) \Upsilon_{k-2}(-\varepsilon) \\
&\qquad\quad + \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) f(x-u) \Upsilon_{k-1}(u)\, du \Bigg) \\
&\quad = \lim_{\varepsilon \to 0^+} \big(f(x-\varepsilon) - f(x+\varepsilon) \big) \Upsilon_{k-2}(\varepsilon) \\
&\qquad\quad + \lim_{\varepsilon \to 0^+} f(x+\varepsilon) \big(\Upsilon_{k-2}(\varepsilon) - \Upsilon_{k-2}(-\varepsilon) \big) \\
&\qquad\quad + \int_{-\infty}^{+\infty} f(x-u) \Upsilon_{k-1}(u)\, du, \quad x \in \mathbb{R}.
\end{aligned}
$$

Hence, by (3.10) we conclude that

$$\begin{aligned}\frac{d^{k-1}}{dx^{k-1}}\int_{-\infty}^{+\infty} f(y)\int_0^{\frac12}\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}e^{-\frac{1}{4s}(x-y)^2}\,ds\,dy\\ =\int_{-\infty}^{+\infty} f(y)\int_0^{\frac12}\frac{(2s)^{\frac{k}{2}-1}}{\sqrt{\pi s}}\frac{\partial^{k-1}}{\partial x^{k-1}}e^{-\frac{1}{4s}(x-y)^2}\,ds\,dy,\quad x\in\mathbb{R}.\end{aligned}$$

Moreover, if m is even, then Υ_m is even, and if m is odd, then Υ_m is odd. When k is even, we can also see that

$$\lim_{\varepsilon\to 0}\Upsilon_{k-1}(\varepsilon)=-2^{\frac{k}{2}-1}. \tag{3.11}$$

Indeed, if k is even we can write

$$\Upsilon_{k-1}(\varepsilon)=-\frac{1}{\Gamma(\frac{k}{2})\sqrt{\pi}}\sum_{l=0}^{\frac{k}{2}-1}(-1)^l E_{k-1,l}\frac{\varepsilon^{k-1-2l}}{2^{\frac{3k}{2}-2l-1}}\int_0^{\frac12}\frac{e^{-\frac{\varepsilon^2}{4s}}}{s^{\frac{k}{2}+\frac12-l}}\,ds,\quad \varepsilon\in\mathbb{R}.$$

Hence, the duplication formula [17, (1.2.3)] allows us to write

$$\begin{aligned}\lim_{\varepsilon\to 0}\Upsilon_{k-1}(\varepsilon)&=-\frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})\sqrt{\pi}}\lim_{\varepsilon\to 0}\sum_{l=0}^{\frac{k}{2}-1}(-1)^l E_{k-1,l}\int_{\frac{\varepsilon^2}{2}}^{\infty} e^{-u}u^{\frac{k}{2}-\frac32-l}\,du\\ &=\frac{-1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})\sqrt{\pi}}\sum_{l=0}^{\frac{k}{2}-1}(-1)^l E_{k-1,l}\Gamma\left(\frac{k-1}{2}-l\right)\\ &=\frac{-(k-1)!}{2^{\frac{k}{2}-1}(\Gamma(\frac{k}{2}))^2}\sum_{l=0}^{\frac{k}{2}-1}(-1)^l\binom{\frac{k}{2}-1}{l}\frac{1}{k-1-2l}\\ &=\frac{-(k-1)!}{2^{\frac{k}{2}-1}(\Gamma(\frac{k}{2}))^2}\int_0^1\left(1-t^2\right)^{\frac{k}{2}-1}dt\\ &=\frac{-(k-1)!}{2^{\frac{k}{2}}(\Gamma(\frac{k}{2}))^2}\frac{\Gamma(\frac{k}{2})\Gamma(\frac12)}{\Gamma(\frac{k+1}{2})}=-2^{\frac{k}{2}-1},\end{aligned}$$

and (3.11) is thus established.

Note that $\Upsilon_{k-1}\in L^1(\mathbb{R})$ and $\Upsilon_{k-1}\in C^\infty(\mathbb{R}\setminus\{0\})$. By proceeding as above we obtain that, for each $x\in\mathbb{R}$,

$$\begin{aligned}\frac{d}{dx}\int_{-\infty}^{+\infty} f(y)\Upsilon_{k-1}(x-y)\,dy=\lim_{\varepsilon\to 0^+}\Bigg[\left(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{+\infty}\right)f(y)\Upsilon'_{k-1}(x-y)\,dy\\ +f(x-\varepsilon)\Upsilon_{k-1}(\varepsilon)-f(x+\varepsilon)\Upsilon_{k-1}(-\varepsilon)\Bigg].\end{aligned}$$

Suppose now that k is odd. Then Υ_{k-1} is an even function and from (3.10) we obtain, for every $x\in\mathbb{R}$,

$$\left|f(x-\varepsilon)\Upsilon_{k-1}(\varepsilon)-f(x+\varepsilon)\Upsilon_{k-1}(-\varepsilon)\right|\le C\varepsilon\left|\Upsilon_{k-1}(\varepsilon)\right|\longrightarrow 0,\quad \text{as } \varepsilon\to 0^+.$$

On the other hand, assuming that k is even, (3.11) leads to

$$\lim_{\varepsilon\to 0^+} f(x-\varepsilon)\Upsilon_{k-1}(\varepsilon) - f(x+\varepsilon)\Upsilon_{k-1}(-\varepsilon)$$
$$= \lim_{\varepsilon\to 0^+} \big(f(x+\varepsilon)+f(x-\varepsilon)\big)\Upsilon_{k-1}(\varepsilon) = -2^{\frac{k}{2}} f(x)$$

for every $x\in\mathbb{R}$.

Hence,

$$\frac{d}{dx}\int_{-\infty}^{+\infty} f(y)\Upsilon_{k-1}(x-y)\,dy$$
$$= w_k f(x) + \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} f(y)\frac{\partial}{\partial x}\big(\Upsilon_{k-1}(x-y)\big)\,dy,$$

$x\in\mathbb{R}$, where $w_k=0$, if k is odd, and $w_k=-2^{\frac{k}{2}}$, when k is even. Thus, the proof is finished. □

The following relation between the kernels $R_\alpha^{(k)}(x,y)$ and $R^{(k)}(x,y)$, $x,y\in(0,\infty)$, $x\neq y$, is the key of our procedure in order to establish that the k-order Riesz transform associated with the Laguerre operator is a principal value integral operator.

Proposition 3.3. *Let $\alpha>-1$ and $k\in\mathbb{N}\setminus\{0\}$. For every $M>1$ we have that*

(i) $|R_\alpha^{(k)}(x,y)|\leq C\frac{y^{\alpha+\frac12}}{x^{\alpha+\frac32}}$, $0<y<\frac{x}{M}$.

(ii) $|R_\alpha^{(k)}(x,y)|\leq C\frac{x^{\alpha+\frac12}}{y^{\alpha+\frac32}}$, *$y>Mx$ and k even, and* $|R_\alpha^{(k)}(x,y)|\leq C\frac{x^{\alpha+\frac32}}{y^{\alpha+\frac52}}$, *$y>Mx$ and k odd.*

(iii) $|R_\alpha^{(k)}(x,y)-R^{(k)}(x,y)|\leq C\frac{1}{x}\big(1+(\frac{x}{|x-y|})^{\frac12}\big)$, $\frac{x}{M}<y<Mx$, $x\neq y$.

Proof. We prove the property for $M=2$. We can proceed in the same way for every $M>1$. A short calculation using the induction procedure, property (P3) and combinatorial properties of the coefficients $E_{j,n}$ shows that, for every $j\in\mathbb{N}$ and $t,x,y\in(0,\infty)$,

$$\frac{\partial^j}{\partial x^j}\left[\left(\frac{xy}{\sinh t}\right)^{-\alpha} I_\alpha\left(\frac{xy}{\sinh t}\right)\right] \tag{3.12}$$
$$=\sum_{n=0}^{E[\frac{j}{2}]} E_{j,n}\frac{x^{j-2n}}{2^{j-n}}\left(\frac{y}{\sinh t}\right)^{2(j-n)}$$
$$\times\left(\frac{xy}{\sinh t}\right)^{-\alpha+n-j} I_{\alpha-n+j}\left(\frac{xy}{\sinh t}\right).$$

Let us now prove (i) and (ii). Since $(\frac{d}{dx}+x)g = e^{-\frac{x^2}{2}}\frac{d}{dx}(e^{\frac{x^2}{2}}g)$, for every differentiable function g, we can write, for $t,x,y\in(0,\infty)$,

$$\begin{aligned}\mathfrak{D}_\alpha^k W_t^\alpha(x,y) &= x^{\alpha+\frac12}e^{-\frac{x^2}{2}}\frac{\partial^k}{\partial x^k}\big(e^{\frac{x^2}{2}}x^{-\alpha-\frac12}W_t^\alpha(x,y)\big)\\ &= (\sinh t)^{-\frac12}\left(\frac{xy}{\sinh t}\right)^{\alpha+\frac12}e^{-\frac{x^2}{2}-\frac{y^2}{2}\coth t}\\ &\quad\times\frac{\partial^k}{\partial x^k}\left[\left(\frac{xy}{\sinh t}\right)^{-\alpha}I_\alpha\left(\frac{xy}{\sinh t}\right)e^{-\frac{x^2}{2}(\coth t-1)}\right].\end{aligned}$$

By taking into account formulas (2.7) and (3.12), we get

$$\begin{aligned}&\frac{\partial^k}{\partial x^k}\left[\left(\frac{xy}{\sinh t}\right)^{-\alpha}I_\alpha\left(\frac{xy}{\sinh t}\right)e^{-\frac{x^2}{2}(\coth t-1)}\right]\\ &\quad=\sum_{j=0}^k\binom{k}{j}\frac{\partial^j}{\partial x^j}\left[\left(\frac{xy}{\sinh t}\right)^{-\alpha}I_\alpha\left(\frac{xy}{\sinh t}\right)\right]\frac{\partial^{k-j}}{\partial x^{k-j}}\left[e^{-\frac{x^2}{2}(\coth t-1)}\right]\\ &\quad=e^{-\frac{x^2}{2}(\coth t-1)}\sum_{j=0}^k\sum_{n=0}^{E[\frac j2]}\sum_{m=0}^{E[\frac{k-j}{2}]}\binom{k}{j}\frac{E_{j,n}E_{k-j,m}}{2^{j-n}}\\ &\qquad\times\left(\frac{y}{\sinh t}\right)^{2(j-n)}\left(\frac{1-\coth t}{2}\right)^{k-j-m}\\ &\qquad\times x^{k-2m-2n}\left(\frac{xy}{\sinh t}\right)^{-\alpha-j+n}\\ &\qquad\times I_{\alpha+j-n}\left(\frac{xy}{\sinh t}\right),\quad t,x,y\in(0,\infty).\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}&\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\\ &\quad=(\sinh t)^{-\frac12}\left(\frac{xy}{\sinh t}\right)^{\alpha+\frac12}e^{-\frac12(x^2+y^2)\coth t}\\ &\qquad\times\sum_{j=0}^k\sum_{n=0}^{E[\frac j2]}\sum_{m=0}^{E[\frac{k-j}{2}]}\binom{k}{j}\frac{E_{j,n}E_{k-j,m}}{2^{j-n}}\\ &\qquad\times\left(\frac{y}{\sinh t}\right)^{2(j-n)}\left(\frac{1-\coth t}{2}\right)^{k-j-m}\\ &\qquad\times x^{k-2m-2n}\left(\frac{xy}{\sinh t}\right)^{-\alpha-j+n}I_{\alpha+j-n}\left(\frac{xy}{\sinh t}\right),\quad t,x,y\in(0,\infty).\end{aligned}\tag{3.13}$$

By using property (P1), it follows that

$$\begin{aligned}
&\left|\int_{0,\frac{xy}{\sinh t}\le 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\,dt\right| \\
&\quad\le C(xy)^{\alpha+\frac12}\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]} x^{k-2m-2n}y^{2(j-n)} \\
&\qquad\times\int_{0,\frac{xy}{\sinh t}\le 1}^{\infty} t^{\frac{k}{2}-1}e^{-\frac12(x^2+y^2)\coth t}(\sinh t)^{2n+m-k-j-\alpha-1}\,dt \\
&\quad\le C(xy)^{\alpha+\frac12}\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]} x^{k-2m-2n}y^{2(j-n)} \\
&\qquad\times\Bigg(\int_0^1 t^{-\frac{k}{2}-\alpha-2-j+2n+m}e^{-c\frac{x^2+y^2}{t}}\,dt \\
&\qquad+e^{-c(x^2+y^2)}\int_1^\infty t^{\frac{k}{2}-1}e^{-(\alpha+1)t}\,dt\Bigg).
\end{aligned}$$

Hence, by taking into account [28, Lemma 1.1] we conclude that

$$\begin{aligned}
&\left|\int_{0,\frac{xy}{\sinh t}\le 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\,dt\right| \\
&\quad\le C\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]}\frac{(xy)^{\alpha+\frac12}x^{k-2m-2n}y^{2(j-n)}}{(x^2+y^2)^{\frac{k}{2}+\alpha+1+j-2n-m}} \\
&\quad\le C\frac{(xy)^{\alpha+\frac12}}{(x^2+y^2)^{\alpha+1}}\le C\begin{cases}\dfrac{y^{\alpha+\frac12}}{x^{\alpha+\frac32}}, & 0<y<x,\\[2mm] \dfrac{x^{\alpha+\frac12}}{y^{\alpha+\frac32}}, & y>x>0.\end{cases}
\end{aligned}\tag{3.14}$$

Note that if k is odd we can improve the estimate when $y>x>0$ as follows

$$\left|\int_{0,\frac{xy}{\sinh t}\le 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\,dt\right|\le C\frac{(xy)^{\alpha+\frac12}x}{(x^2+y^2)^{\alpha+\frac32}}\le\frac{x^{\alpha+\frac32}}{y^{\alpha+\frac52}}.\tag{3.15}$$

Assume now that $\frac{xy}{\sinh t}\ge 1$. From (3.13) and property (P2), we get

$$\begin{aligned}
\left|\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\right| &\le C\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]} e^{-\frac12(x^2+y^2)\coth t+\frac{xy}{\sinh t}} \\
&\quad\times x^{k-2m-2n}y^{2(j-n)}(\sinh t)^{2n+m-k-j-\frac12},\quad t,x,y\in(0,\infty).
\end{aligned}$$

We also observe that

$$-\frac12\left(x^2+y^2\right)\coth t+\frac{xy}{\sinh t}=-\frac{(x-ye^{-t})^2+(y-xe^{-t})^2}{2(1-e^{-2t})}.$$

Thus, if $0<y<\frac{x}{2}$, we can write

$$\begin{aligned}\left|\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\right| &\leq C\sum_{j=0}^{k}\sum_{n=0}^{E[\frac{j}{2}]}\sum_{m=0}^{E[\frac{k-j}{2}]} e^{-\frac{x^2(1+\coth t)}{16}} x^{k-2m-2n+2(j-n)} \\ &\quad\times(\sinh t)^{2n+m-k-j-\frac{1}{2}} \\ &\leq Ce^{-cx^2(1+\coth t)}(\sinh t)^{-\frac{k}{2}-\frac{1}{2}}, \quad t\in(0,\infty).\end{aligned}$$

Hence, if $-1<\alpha<-\frac{1}{2}$, [28, Lemma 1.1] leads to

$$\begin{aligned}\left|\int_{0,\frac{xy}{\sinh t}\geq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\,dt\right| &\leq C\left(\int_0^1 \frac{e^{-c\frac{x^2}{t}}}{t^{\frac{3}{2}}}\,dt+e^{-cx^2}\right) \\ &\leq C\frac{1}{x}\leq C\frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}, \quad 0<y<\frac{x}{2}.\end{aligned}$$

For $\alpha>-\frac{1}{2}$, we can proceed as follows.

$$\begin{aligned}&\left|\int_{0,\frac{xy}{\sinh t}\geq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\,dt\right| \\ &\leq C(xy)^{\alpha+\frac{1}{2}}\int_0^\infty t^{\frac{k}{2}-1}e^{-cx^2(1+\coth t)} \\ &\quad\times(\sinh t)^{-\frac{k}{2}-\alpha-1}\,dt \\ &\leq C(xy)^{\alpha+\frac{1}{2}}\left(\int_0^1 \frac{e^{-c\frac{x^2}{t}}}{t^{\alpha+2}}\,dt+e^{-cx^2}\right) \\ &\leq C\frac{(xy)^{\alpha+\frac{1}{2}}}{x^{2\alpha+2}}\leq C\frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}, \quad 0<y<\frac{x}{2}.\end{aligned}$$

In a similar way, if $0<2x<y$, we can write

$$\begin{aligned}&\left|\int_{0,\frac{xy}{\sinh t}\geq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\,dt\right| \\ &\leq C(xy)^{\alpha+\frac{3}{2}}\int_0^\infty t^{\frac{k}{2}-1}e^{-cy^2(1+\coth t)} \\ &\quad\times(\sinh t)^{-\frac{k}{2}-\alpha-2}\,dt \\ &\leq C(xy)^{\alpha+\frac{3}{2}}\left(\int_0^1 \frac{e^{-c\frac{y^2}{t}}}{t^{\alpha+3}}\,dt+e^{-cy^2}\right) \\ &\leq C\frac{(xy)^{\alpha+\frac{3}{2}}}{y^{2\alpha+4}}\leq C\frac{x^{\alpha+\frac{3}{2}}}{y^{\alpha+\frac{5}{2}}}.\end{aligned}$$

These estimations allow us to get

$$\left|\int_{0,\frac{xy}{\sinh t}\geq 1}^{\infty} t^{\frac{k}{2}-1}\mathfrak{D}_\alpha^k W_t^\alpha(x,y)\,dt\right| \leq C\begin{cases}\dfrac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}, & 0<y<\frac{x}{2},\\ \dfrac{x^{\alpha+\frac{3}{2}}}{y^{\alpha+\frac{5}{2}}}, & y>2x>0.\end{cases} \tag{3.16}$$

Hence, by (3.14), (3.15) and (3.16), (i) and (ii) are proved.

Next, we establish statement (iii). Observe first that, since $(\frac{d}{dx}+x)g = e^{-\frac{x^2}{2}}\frac{d}{dx}(e^{\frac{x^2}{2}}g)$, when g is a differentiable function,

$$\begin{aligned}
\mathfrak{D}_\alpha^k W_t^\alpha(x,y) &= x^{\alpha+\frac{1}{2}}\left(\frac{\partial}{\partial x}+x\right)^k\left[x^{-\alpha-\frac{1}{2}}W_t^\alpha(x,y)\right]\\
&= \sqrt{2\pi}x^{\alpha+\frac{1}{2}}\left(\frac{\partial}{\partial x}+x\right)^k\Bigg[x^{-\alpha-\frac{1}{2}}e^{-\frac{xy}{\sinh t}}\left(\frac{xy}{\sinh t}\right)^{\frac{1}{2}}\\
&\quad\times I_\alpha\left(\frac{xy}{\sinh t}\right)W_t(x,y)\Bigg]\\
&= \sqrt{2\pi}\left(\frac{xy}{\sinh t}\right)^{\alpha+\frac{1}{2}}\sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j}{\partial x^j}\Bigg[e^{-\frac{xy}{\sinh t}}\left(\frac{xy}{\sinh t}\right)^{-\alpha}\\
&\quad\times I_\alpha\left(\frac{xy}{\sinh t}\right)\Bigg]\left(\frac{\partial}{\partial x}+x\right)^{k-j}W_t(x,y)\\
&= \sqrt{2\pi}\left(\frac{xy}{\sinh t}\right)^{\alpha+\frac{1}{2}}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{\partial}{\partial x}+x\right)^{k-j}W_t(x,y)\\
&\quad\times\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}\left(\frac{y}{\sinh t}\right)^{j-l}e^{-\frac{xy}{\sinh t}}\\
&\quad\times\frac{\partial^l}{\partial x^l}\left(\left(\frac{xy}{\sinh t}\right)^{-\alpha}I_\alpha\left(\frac{xy}{\sinh t}\right)\right),\quad t,x,y\in(0,\infty).
\end{aligned}$$

Hence, by using formula (3.12) we obtain that, for every $t,x,y\in(0,\infty)$,

$$\begin{aligned}
\mathfrak{D}_\alpha^k W_t^\alpha(x,y) &= \sqrt{2\pi}e^{-\frac{xy}{\sinh t}}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{\partial}{\partial x}+x\right)^{k-j}\\
&\quad\times\big(W_t(x,y)\big)\left(\frac{y}{\sinh t}\right)^j\\
&\quad\times\sum_{n=0}^{E[\frac{j}{2}]}\sum_{l=2n}^{j}(-1)^l\binom{j}{l}\frac{E_{l,n}}{2^{l-n}}\\
&\quad\times\left(\frac{xy}{\sinh t}\right)^{-n}\left(\frac{xy}{\sinh t}\right)^{\frac{1}{2}}I_{\alpha-n+l}\left(\frac{xy}{\sinh t}\right).
\end{aligned} \tag{3.17}$$

Let us consider now $x,y,t\in(0,\infty)$ such that $\frac{xy}{\sinh t}\geq 1$. By taking into account property (P2) and (3.17), we can write

$$
\begin{aligned}
\mathfrak{D}_\alpha^k W_t^\alpha(x,y) &= \Big(\frac{\partial}{\partial x}+x\Big)^k \big(W_t(x,y)\big)\Big(1+O\Big(\frac{\sinh t}{xy}\Big)\Big)\\
&\quad+\sum_{j=1}^k(-1)^j\binom{k}{j}\Big(\frac{\partial}{\partial x}+x\Big)^{k-j}\big(W_t(x,y)\big)\Big(\frac{y}{\sinh t}\Big)^j\\
&\quad\times\sum_{n=0}^{E[\frac{j}{2}]}\sum_{l=2n}^{j}(-1)^l\binom{j}{l}\frac{E_{l,n}}{2^{l-n}}\Big(\frac{\sinh t}{xy}\Big)^n\\
&\quad\times\Bigg(\sum_{r=0}^{E[\frac{j}{2}]}\frac{(-1)^r[\alpha+l-n,r]}{2^r}\Big(\frac{\sinh t}{xy}\Big)^r\\
&\quad+O\Big(\Big(\frac{\sinh t}{xy}\Big)^{E[\frac{j}{2}]+1}\Big)\Bigg)\\
&=\Big(\frac{\partial}{\partial x}+x\Big)^k \big(W_t(x,y)\big)+\sum_{j=1}^k(-1)^j\binom{k}{j}\\
&\quad\times\Big(\frac{\partial}{\partial x}+x\Big)^{k-j}\big(W_t(x,y)\big)\Big(\frac{y}{\sinh t}\Big)^j\\
&\quad\times\sum_{n=0}^{E[\frac{j}{2}]}\sum_{l=2n}^{j}\sum_{r=0}^{E[\frac{j}{2}]}(-1)^{l+r}\binom{j}{l}\frac{E_{l,n}}{2^{l-n}}\frac{[\alpha+l-n,r]}{2^r}\Big(\frac{\sinh t}{xy}\Big)^{n+r}\\
&\quad+\sum_{j=0}^k(-1)^j\binom{k}{j}\Big(\frac{\partial}{\partial x}+x\Big)^{k-j}\big(W_t(x,y)\big)\\
&\quad\times O\Big(\Big(\frac{y}{\sinh t}\Big)^{j-E[\frac{j}{2}]-1}\frac{1}{x^{E[\frac{j}{2}]+1}}\Big).
\end{aligned}
$$

Lemma 2.1 allows us to see that, for every $j\in\mathbb{N}$, $j=1,\dots,k$,

$$
\begin{aligned}
&\sum_{n=0}^{E[\frac{j}{2}]}\sum_{l=2n}^{j}\sum_{r=0}^{E[\frac{j}{2}]}(-1)^{l+r}\binom{j}{l}\frac{E_{l,n}}{2^{l-n}}\frac{[\alpha+l-n,r]}{2^r}\Big(\frac{\sinh t}{xy}\Big)^{n+r}\\
&\quad=\sum_{n=0}^{E[\frac{j}{2}]}\sum_{l=2n}^{j}\sum_{m=n}^{E[\frac{j}{2}]+n}(-1)^{l+m-n}\binom{j}{l}\frac{E_{l,n}}{2^{l-n}}\frac{[\alpha+l-n,m-n]}{2^{m-n}}\Big(\frac{\sinh t}{xy}\Big)^{m}\\
&\quad=\sum_{m=0}^{E[\frac{j}{2}]}\Big(\frac{\sinh t}{2xy}\Big)^m\sum_{n=0}^{m}\sum_{l=2n}^{j}(-1)^{l+m-n}\binom{j}{l}\frac{E_{l,n}}{2^{l-2n}}[\alpha+l-n,m-n]
\end{aligned}
$$

$$+\sum_{m=E[\frac{j}{2}]+1}^{2E[\frac{j}{2}]}\left(\frac{\sinh t}{2xy}\right)^m \sum_{n=m-E[\frac{j}{2}]}^{E[\frac{j}{2}]}\sum_{l=2n}^{j}(-1)^{l+m-n}\binom{j}{l}$$
$$\times\frac{E_{l,n}}{2^{l-2n}}[\alpha+l-n,m-n]=O\left(\left(\frac{\sinh t}{xy}\right)^{E[\frac{j}{2}]+1}\right).$$

Hence, it follows that

$$\mathfrak{D}_\alpha^k W_t^\alpha(x,y)=\left(\frac{\partial}{\partial x}+x\right)^k W_t(x,y)+\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{\partial}{\partial x}+x\right)^{k-j}$$
$$\times\big(W_t(x,y)\big)O\left(\left(\frac{y}{\sinh t}\right)^{j-E[\frac{j}{2}]-1}\frac{1}{x^{E[\frac{j}{2}]+1}}\right).$$

Assume that $0<\frac{x}{2}<y<2x$, $x\neq y$. In order to establish (iii), we now proceed as in the proof of Proposition 3.2. First, note that by formula (3.4)

$$\left|\mathfrak{D}_\alpha^k W_t^\alpha(x,y)-\left(\frac{\partial}{\partial x}+x\right)^k W_t(x,y)\right|$$
$$\leq C\sum_{j=0}^{k}\sum_{0\leq\rho+\sigma\leq k-j}x^\rho\left|\frac{\partial^\sigma}{\partial x^\sigma}W_t(x,y)\right|\left(\frac{y}{\sinh t}\right)^{j-E[\frac{j}{2}]-1}\frac{1}{x^{E[\frac{j}{2}]+1}}.$$

Assume that $j,\rho,\sigma,b_1,b_2\in\mathbb{N}$, $0\leq j\leq k$, $0\leq\rho+\sigma\leq k-j$ and $2b_1+b_2\leq\sigma$. According to [28, p. 50] and by making the change of variable $s=\tanh(\frac{t}{2})$, we must analyze the following integral

$$I_{\rho,\sigma,j}^{b_1,b_2}(x,y)=\frac{x^\rho y^j}{(xy)^{1+E[\frac{j}{2}]}}\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{1}\left(\log\frac{1+s}{1-s}\right)^{\frac{k}{2}-1}\left(\frac{1-s^2}{s}\right)^{j-E[\frac{j}{2}]-\frac{1}{2}}$$
$$\times\left(s+\frac{1}{s}\right)^{b_1}e^{-\frac{1}{4}(s(x+y)^2+\frac{1}{s}(x-y)^2)}$$
$$\times\left(s(x+y)+\frac{1}{s}(x-y)\right)^{b_2}\frac{ds}{1-s^2}$$
$$=J_{\rho,\sigma,j}^{b_1,b_2}(x,y)+H_{\rho,\sigma,j}^{b_1,b_2}(x,y),\quad x,y\in(0,\infty),$$

where J and H are defined as I but replacing the integral over $(0,1)$ by the integral over $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$, respectively.

Since $\log\frac{1+s}{1-s}\sim 2s$, as $s\to 0^+$, it follows that

$$J_{\rho,\sigma,j}^{b_1,b_2}(x,y)\leq C\frac{x^\rho y^j}{(xy)^{1+E[\frac{j}{2}]}}\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{\frac{1}{2}}s^{\frac{k}{2}-\frac{1}{2}-j+E[\frac{j}{2}]-b_1}e^{-\frac{1}{4}(s(x+y)^2+\frac{(x-y)^2}{s})}$$
$$\times\left|s(x+y)+\frac{(x-y)}{s}\right|^{b_2}ds$$

$$\leq C\frac{y^j}{(xy)^{1+E[\frac{j}{2}]}}$$
$$\times \int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{\frac{1}{2}} s^{\frac{1}{2}(k-j-2b_1-b_2-\rho)+E[\frac{j}{2}]-\frac{j}{2}-\frac{1}{2}} e^{-c\frac{(x-y)^2}{s}}\, ds$$
$$\leq C\frac{y^j}{(xy)^{1+E[\frac{j}{2}]}}\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{\frac{1}{2}} s^{E[\frac{j}{2}]-\frac{j}{2}-\frac{1}{2}} e^{-c\frac{(x-y)^2}{s}}\, ds.$$

By taking into account that $0<\frac{x}{2}<y<2x$ and using [28, Lemma 1.1], we get

$$J_{\rho,\sigma,j}^{b_1,b_2}(x,y)\leq C\frac{x^{j-2E[\frac{j}{2}]-\frac{3}{2}}}{\sqrt{x}}\int_{0,\frac{(1-s^2)xy}{2s}\geq 1}^{\frac{1}{2}} s^{E[\frac{j}{2}]-\frac{j}{2}-\frac{1}{2}} e^{-c\frac{(x-y)^2}{s}}\, ds$$
$$\leq C\frac{1}{\sqrt{x}}\int_0^{\frac{1}{2}}\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{\frac{5}{4}}}\, ds\leq C\frac{1}{x}\left(\frac{x}{|x-y|}\right)^{\frac{1}{2}}.$$

On the other hand, since that $\log\frac{1+s}{1-s}\sim -\log(1-s)$, as $s\to 1^-$, we have that

$$H_{\rho,\sigma,j}^{b_1,b_2}(x,y)\leq C\frac{x^\rho y^j}{(xy)^{1+E[\frac{j}{2}]}}\int_{\frac{1}{2},\frac{(1-s^2)xy}{2s}\geq 1}^{1}\big(-\log(1-s)\big)^{\frac{k}{2}-1}$$
$$\times (1-s)^{j-E[\frac{j}{2}]-\frac{3}{2}} e^{-cs(x+y)^2}\, ds$$
$$\leq Ce^{-c(x+y)^2}\int_{\frac{1}{2}}^{1}\big(-\log(1-s)\big)^{\frac{k}{2}-1}(1-s)^{j-\frac{1}{2}}\, ds$$
$$\leq Ce^{-c(x+y)^2},\quad x,y\in(0,\infty).$$

Hence, we conclude that, if $0<\frac{x}{2}<y<2x$, $x\neq y$,

$$\left|\int_{0,\frac{xy}{\sinh t}\geq 1}^{\infty} t^{\frac{k}{2}-1}\left(\mathfrak{D}_\alpha^k W_t^\alpha(x,y)-\left(\frac{\partial}{\partial x}+x\right)^k W_t(x,y)\right) dt\right|$$
$$\leq C\frac{1}{x}\left(\frac{x}{|x-y|}\right)^{\frac{1}{2}}. \tag{3.18}$$

Also, by using again (2.7) we obtain, for each $t,x,y\in(0,\infty)$,

$$\left(\frac{\partial}{\partial x}+x\right)^k W_t(x,y)$$
$$=e^{-\frac{x^2}{2}}\frac{\partial^k}{\partial x^k}\left[e^{\frac{x^2}{2}}W_t(x,y)\right]$$
$$=W_t(x,y)\sum_{j=0}^{k}\sum_{l=0}^{E[\frac{j}{2}]}\binom{k}{j}E_{j,l}x^{j-2l}\left(\frac{y}{\sinh t}\right)^{k-j}\left(\frac{1-\coth t}{2}\right)^{j-l}.$$

Hence it follows that, when $0 < \frac{x}{2} < y < 2x$,

$$\left|\int_{0,\frac{xy}{\sinh t}\leq 1}^{\infty} t^{\frac{k}{2}-1}\left(\frac{\partial}{\partial x}+x\right)^k W_t(x,y)\,dt\right| \tag{3.19}$$
$$\leq C\sum_{l=0}^{E[\frac{k}{2}]} x^{k-2l}\left(\int_0^1 t^{-\frac{k}{2}-\frac{3}{2}+l}e^{-c\frac{x^2}{t}}\,dt + e^{-cx^2}\int_1^\infty t^{\frac{k}{2}-1}e^{-\frac{t}{2}}\,dt\right)$$
$$\leq C\frac{1}{x}.$$

The estimations (3.14), (3.18) and (3.19) allow us to finish the proof of (iii). □

As it was mentioned in the Introduction, for every $\beta > 0$, the $-\beta$-power $L_\alpha^{-\beta}$ of the Laguerre operator L_α defined by

$$L_\alpha^{-\beta} f = \sum_{n=0}^{\infty} \frac{c_n^\alpha(f)}{(2n+\alpha+1)^\beta}\varphi_n^\alpha, \tag{3.20}$$

is bounded from $L^2((0,\infty),dx)$ into itself. Moreover, if $f \in C_c^\infty(0,\infty)$, the series in (3.20) converges uniformly in every compact subset of $(0,\infty)$ and it defines a function

$$\Phi_{\alpha,\beta}(f)(x) = \sum_{n=0}^{\infty} \frac{c_n^\alpha(f)}{(2n+\alpha+1)^\beta}\varphi_n^\alpha(x), \quad x\in(0,\infty),$$

that belongs to $C^\infty(0,\infty)$ (see [22, (2.8)]).

We now prove the useful integral representation (3.21) for $L_\alpha^{-\beta}$.

Proposition 3.4. *Let $\beta > 0$ and $\alpha > -1$. Then, for every $f \in L^2((0,\infty),$ $dx)$,*

$$L_\alpha^{-\beta} f(x) = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}W_t^\alpha(f)(x)\,dt, \tag{3.21}$$

as $L^2((0,\infty),dx)$-functions.

Proof. It is not hard to see that (3.21) holds for every $f \in \operatorname{span}\{\varphi_n^\alpha\}_{n\in\mathbb{N}}$. Then, to show (3.21) for every $f\in L^2((0,\infty),dx)$ it is enough to prove that the operator defined by

$$T_{\alpha,\beta}(f)(x) = \int_0^\infty t^{\beta-1}W_t^\alpha(f)(x)\,dt, \quad f\in L^2\big((0,\infty),dx\big),$$

is bounded from $L^2((0,\infty),dx)$ into itself.

Suppose that $f \in L^2((0,\infty),dx)$, $f \geq 0$. We can write

$$\begin{aligned}\int_0^\infty t^{\beta-1} W_t^\alpha(f)(x)\,dt &= \int_0^\infty f(y) \int_0^\infty t^{\beta-1} W_t^\alpha(x,y)\,dt\,dy \\ &= \left(\int_0^{\frac{x}{2}} + \int_{\frac{x}{2}}^{2x} + \int_{2x}^\infty\right) f(y) \int_0^\infty t^{\beta-1} W_t^\alpha(x,y)\,dt\,dy \\ &= \sum_{j=1}^3 T^j_{\alpha,\beta}(f)(x), \quad x \in (0,\infty).\end{aligned}$$

We analyze the operators $T^j_{\alpha,\beta}$, $j = 1,2,3$. Assume firstly that $0 < 2x < y < \infty$. According to (P2) we have that

$$\begin{aligned}\int_{0,\frac{xy}{\sinh t}\geq 1}^1 t^{\beta-1} W_t^\alpha(x,y)\,dt &\leq C \int_{0,\frac{xy}{\sinh t}\geq 1}^1 (\sinh t)^{-\frac{1}{2}} t^{\beta-1} \\ &\qquad \times \exp\left(-\frac{|x-e^{-t}y|^2 + |y-e^{-t}x|^2}{2(1-e^{-2t})}\right) dt \\ &\leq C \int_0^1 \left(\frac{xy}{\sinh t}\right)^{\alpha+1} t^{\beta-\frac{3}{2}} e^{-\frac{cy^2}{t}}\,dt \\ &\leq C(xy)^{\alpha+1} \int_0^1 t^{\beta-\alpha-5/2} e^{-\frac{cy^2}{t}}\,dt \\ &\leq C\frac{(xy)^{\alpha+1}}{y^{2\alpha+3}} \leq C\frac{x^{\alpha+1}}{y^{\alpha+2}},\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\int_{1,\frac{xy}{\sinh t}\geq 1}^\infty t^{\beta-1} W_t^\alpha(x,y)\,dt &\leq C(xy)^{\alpha+1} e^{-cy^2} \int_1^\infty e^{-t(\alpha+\frac{3}{2})}\,dt \\ &\leq C\frac{(xy)^{\alpha+1}}{y^{2\alpha+3}} \leq C\frac{x^{\alpha+1}}{y^{\alpha+2}}.\end{aligned} \tag{3.23}$$

Also, by (P1) it follows that

$$\begin{aligned}\int_{0,\frac{xy}{\sinh t}\leq 1}^1 t^{\beta-1} W_t^\alpha(x,y)\,dt &\leq C(xy)^{\alpha+\frac{1}{2}} \int_0^1 t^{\beta-\alpha-2} e^{-c\frac{x^2+y^2}{t}}\,dt \\ &\leq C\frac{(xy)^{\alpha+\frac{1}{2}}}{(x^2+y^2)^{\alpha+1}} \leq C\frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}},\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\int_{1,\frac{xy}{\sinh t}\leq 1}^\infty t^{\beta-1} W_t^\alpha(x,y)\,dt &\leq C(xy)^{\alpha+\frac{1}{2}} e^{-c(x^2+y^2)} \int_1^\infty e^{-t(\alpha+1)}\,dt \\ &\leq C\frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}}.\end{aligned} \tag{3.25}$$

From (3.22), (3.23), (3.24) and (3.25), we deduce that

$$\int_0^\infty t^{\beta-1} W_t^\alpha(x,y)\,dt \le C\frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{3}{2}}}. \tag{3.26}$$

From (3.26), we obtain that

$$\left|T^3_{\alpha,\beta}(f)(x)\right| \le Cx^{\alpha+\frac{1}{2}}\int_{2x}^\infty \frac{f(y)}{y^{\alpha+\frac{3}{2}}}\,dy, \quad x\in(0,\infty).$$

Then, since $\mathcal{H}_\infty^{\alpha+\frac{1}{2}}$ is bounded from $L^2((0,\infty),dx)$ into itself ([6]), $T^3_{\alpha,\beta}$ has the same boundedness property.

By taking into account that $W_t^\alpha(x,y)=W_t^\alpha(y,x)$, $x,y\in(0,\infty)$, from (3.26) it infers

$$\int_0^\infty t^{\beta-1} W_t^\alpha(x,y)\,dt \le C\frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{3}{2}}}, \quad 0<y<\frac{x}{2}.$$

Therefore,

$$\left|T^1_{\alpha,\beta}(f)(x)\right| \le C\mathcal{H}_0^{\alpha+\frac{1}{2}}(f)(x), \quad x\in(0,\infty).$$

Thus, $T^1_{\alpha,\beta}$ is bounded from $L^2((0,\infty),dx)$ into itself, because the operator $\mathcal{H}_0^{\alpha+\frac{1}{2}}$ is also bounded from $L^2((0,\infty),dx)$ into itself.

On the other hand, we can write

$$W_t^\alpha(x,y)-\sqrt{2}W_t(x,y)=\left(\sqrt{2\pi}\left(\frac{xy}{\sinh t}\right)^{\frac{1}{2}} I_\alpha\left(\frac{xy}{\sinh t}\right)e^{-\frac{xy}{\sinh t}}-1\right)W_t(x,y)$$

for every $t,x,y\in(0,\infty)$.

By (P2), we get

$$\begin{aligned}
&\int_{1,\frac{xy}{\sinh t}\ge 1}^\infty t^{\beta-1}\left|W_t^\alpha(x,y)-\sqrt{2}W_t(x,y)\right|dt \\
&\quad\le C\int_{1,\frac{xy}{\sinh t}\ge 1}^\infty t^{\beta-1}\left(\frac{\sinh t}{xy}\right)(\sinh t)^{-\frac{1}{2}}\,dt \\
&\qquad\times e^{-\frac{1}{4}(\tanh(\frac{t}{2})(x+y)^2+\coth(\frac{t}{2})(x-y)^2)} \\
&\quad\le Ce^{-cx^2}\int_1^\infty t^{\beta-1}e^{-\frac{t}{2}}\,dt\le\frac{C}{x}, \quad 0<\frac{x}{2}<y<2x<\infty, x\ne y
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
&\int_{0,\frac{xy}{\sinh t}\ge 1}^1 t^{\beta-1}\left|W_t^\alpha(x,y)-\sqrt{2}W_t(x,y)\right|dt \\
&\quad\le C\int_{0,\frac{xy}{\sinh t}\ge 1}^1 t^{\beta-1}\left(\frac{\sinh t}{xy}\right)^{\frac{1}{4}}(\sinh t)^{-\frac{1}{2}} \\
&\qquad\times e^{-\frac{1}{4}(\tanh(\frac{t}{2})(x+y)^2+\coth(\frac{t}{2})(x-y)^2)}\,dt
\end{aligned} \tag{3.28}$$

$$\leq \frac{C}{(xy)^{\frac{1}{4}}}\int_0^1 t^{\beta-\frac{5}{4}}e^{-\frac{1}{4}(\tanh(\frac{t}{2})(x+y)^2+\coth(\frac{t}{2})(x-y)^2)}\,dt$$
$$\leq \frac{C}{(xy)^{\frac{1}{4}}}\int_0^1 \frac{e^{-c\frac{(x-y)^2}{t}}}{t^{\frac{5}{4}}}\,dt$$
$$\leq \frac{C}{x}\sqrt{\frac{x}{|x-y|}}, \quad 0<\frac{x}{2}<y<2x<\infty.$$

Moreover, according to (P1) it follows

$$\begin{aligned}(3.29)\qquad &\int_{1,\frac{xy}{\sinh t}\leq 1}^{\infty} t^{\beta-1}\big|W_t^\alpha(x,y)-\sqrt{2}W_t(x,y)\big|\,dt\\ &\leq \int_{1,\frac{xy}{\sinh t}\leq 1}^{\infty} t^{\beta-1}\big(W_t^\alpha(x,y)+\sqrt{2}W_t(x,y)\big)\,dt\\ &\leq C\bigg((xy)^{\alpha+\frac{1}{2}}e^{-c(x^2+y^2)}\int_1^\infty e^{-t(\alpha+1)}\,dt\\ &\qquad + e^{-c(x^2+y^2)}\int_1^\infty e^{-\frac{t}{2}}\,dt\bigg)\\ &\leq \frac{C}{x}, \quad 0<\frac{x}{2}<y<2x<\infty,\end{aligned}$$

and

$$\begin{aligned}(3.30)\qquad &\int_{0,\frac{xy}{\sinh t}\leq 1}^{1} t^{\beta-1}\big|W_t^\alpha(x,y)-\sqrt{2}W_t(x,y)\big|\,dt\\ &\leq C\bigg((xy)^{\alpha+\frac{1}{2}}\int_0^1\frac{e^{-c\frac{x^2+y^2}{t}}}{t^{\alpha+2}}\,dt+\int_0^1\frac{e^{-c\frac{x^2+y^2}{t}}}{t^{\frac{3}{2}}}\,dt\bigg)\\ &\leq C\bigg(\frac{(xy)^{\alpha+\frac{1}{2}}}{(x^2+y^2)^{\alpha+1}}+\frac{1}{(x^2+y^2)^{\frac{1}{2}}}\bigg)\\ &\leq \frac{C}{x}, \quad 0<\frac{x}{2}<y<2x<\infty.\end{aligned}$$

By combining (3.27), (3.28), (3.29) and (3.30), we obtain that

$$(3.31)\quad \big|T^2_{\alpha,\beta}(f)(x)\big|\leq C\bigg(\int_{\frac{x}{2}}^{2x}\frac{1}{x}\bigg(1+\Big(\frac{x}{|x-y|}\Big)^{\frac{1}{2}}\bigg)\big|f(y)\big|\,dy+T^2_\beta\big(|f|\big)(x)\bigg),$$

$x\in(0,\infty)$, where

$$T^2_\beta(f)(g)(x)=\int_{\frac{x}{2}}^{2x} g(y)\int_0^\infty t^{\beta-1}W_t(x,y)\,dt\,dy, \quad x\in(0,\infty).$$

By using Jensen's inequality, we can see that the operator

$$T(g) = \int_{\frac{x}{2}}^{2x} \frac{1}{x}\left(1 + \left(\frac{x}{|x-y|}\right)^{\frac{1}{2}}\right) g(y)\, dy, \quad x \in (0,\infty),$$

is bounded from $L^2((0,\infty), dx)$ into itself.

If g is a measurable function on $(0,\infty)$, we denote by g_0 the function defined by

$$g_0(x) = \begin{cases} g(x), & x \in (0,\infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

We have that

$$T^2_\beta(g) = H^{-\beta}(g_0) - M^\beta_0(g)(x) - M^\beta_\infty(g), \tag{3.32}$$

where

$$M^\beta_0(g)(x) = \int_0^{\frac{x}{2}} g(y) \int_0^\infty t^{\beta-1} W_t(x,y)\, dt\, dy, \quad x \in (0,\infty),$$

and

$$M^\beta_\infty(g)(x) = \int_{2x}^\infty g(y) \int_0^\infty t^{\beta-1} W_t(x,y)\, dt\, dy, \quad x \in (0,\infty).$$

It is not hard to see that

$$W_t(x,y) \leq C \begin{cases} e^{-\frac{t}{2}} e^{-c(x-y)^2}, & t \geq 1, \\ \frac{1}{\sqrt{t}} e^{-c\frac{(x-y)^2}{t}}, & 0 < t < 1, \end{cases} \qquad x, y \in (0,\infty).$$

Then,

$$\begin{aligned} \int_0^\infty t^{\beta-1} W_t(x,y)\, dt &\leq C\left(\int_0^1 \frac{e^{-c\frac{(x-y)^2}{t}}}{t^{\frac{3}{2}}}\, dt + e^{-c(x-y)^2} \int_1^\infty t^{\beta-1} e^{-\frac{t}{2}}\, dt\right) \\ &\leq \frac{C}{|x-y|}, \quad x, y \in (0,\infty), x \neq y. \end{aligned}$$

Hence, we get

$$\left|M^\beta_0(g)(x)\right| \leq \frac{C}{x} \int_0^{\frac{x}{2}} \left|g(y)\right| dy, \quad x \in (0,\infty),$$

and

$$\left|M^\beta_\infty(g)(x)\right| \leq C \int_{2x}^\infty \frac{1}{y} \left|g(y)\right| dy, \quad x \in (0,\infty).$$

By using well-known Hardy's inequalities, we conclude that the operators M^β_0 and M^β_∞ are bounded from $L^2((0,\infty), dx)$ into itself.

Since $H^{-\beta}$ is bounded from $L^2((0,\infty), dx)$ into itself, from (3.31) and (3.32) it deduces that $T^2_{\alpha,\beta}$ is bounded from $L^2((0,\infty), dx)$ into itself.

By combining the results above, we get that the operator $T_{\alpha,\beta}$ is bounded from $L^2((0,\infty), dx)$ into itself. □

Assume now that $f\in C_c^\infty((0,\infty))$. We fix $x\in(0,\infty)$. There exist $0<t_1<1<t_2<+\infty$ such that $\frac{xy}{\sinh t}<1$, if $t>t_2$, and $\frac{xy}{\sinh t}>1$, when $t\in(0,t_1)$, for every $y\in\operatorname{supp} f$. According to (P1), we have that

$$\begin{aligned}\int_{t_2}^\infty\int_{\operatorname{supp} f} W_t^\alpha(x,y)\big|f(y)\big|\,dy\,dt &\le C\int_{t_2}^\infty t^{\beta-1}\int_{\operatorname{supp} f}(\sinh t)^{-1}\\ &\qquad\times\left(\frac{xy}{\sinh t}\right)^\alpha\sqrt{xy}\,dy\,dt\\ &\le C\int_{t_2}^\infty t^{\beta-1}e^{-(\alpha+1)t}\,dt<\infty.\end{aligned}$$

By (P2) and [27, Lemma 1.1], it follows that

$$\begin{aligned}&\int_0^{t_1} t^{\beta-1}\int_{\operatorname{supp} f} W_t^\alpha(x,y)\big|f(y)\big|\,dy\,dt\\ &\quad\le C\int_0^{t_1} t^{\beta-1}\int_{\operatorname{supp} f}(\sinh t)^{-\frac12}e^{-c\frac{x^2+y^2}{t}}\,dy\,dt\\ &\quad\le C\int_{\operatorname{supp} f}\int_0^{t_1} t^{\beta-\frac32}e^{-c\frac{x^2}{t}}\,dt\,dy<\infty.\end{aligned}$$

Finally, it holds

$$\int_{t_1}^{t_2} t^{\beta-1}\int_{\operatorname{supp} f} W_t^\alpha(x,y)\big|f(y)\big|\,dy\,dt<\infty.$$

Then, we conclude that

$$\int_0^\infty t^{\beta-1}\int_0^\infty W_t^\alpha(x,y)\big|f(y)\big|\,dy\,dt<\infty.$$

Hence,

$$\int_0^\infty t^{\beta-1}W_t^\alpha(f)(x)\,dt=\int_0^\infty f(y)\int_0^\infty t^{\beta-1}W_t^\alpha(x,y)\,dt\,dy.$$

Moreover, we have the following result.

PROPOSITION 3.5. *Let $\beta>0$, $\alpha>-1$ and $f\in C_c^\infty(0,\infty)$. Then,*

$$\Phi_{\alpha,\beta}(f)(x)=\frac{1}{\Gamma(\beta)}\int_0^\infty f(y)\int_0^\infty t^{\beta-1}W_t^\alpha(x,y)\,dt\,dy,\quad x\in(0,\infty).$$

Proof. It is sufficient to see that the function

$$\Psi_{\alpha,\beta}(f)(x)=\int_0^\infty f(y)\int_0^\infty t^{\beta-1}W_t^\alpha(x,y)\,dt\,dy,\quad x\in(0,\infty),$$

is continuous on $(0,\infty)$.

According to Proposition 3.1, the function

$$\Lambda_\beta(f)(x)=\int_0^\infty f(y)\int_0^\infty t^{\beta-1}W_t(x,y)\,dt\,dy,\quad x\in\mathbb{R},$$

is continuous on $\mathbb{R}$. Hence, our proof will be finished when we establish that the function $G_{\alpha,\beta}(f) = \Psi_{\alpha,\beta}(f) - \sqrt{2}\Lambda_\beta(f)$ is continuous on $(0,\infty)$. In order to see this, according to the dominated convergence theorem, it is enough to show that for every compact subset Ω of $(0,\infty)$ there exists a function $g_\Omega \in L^1(0,\infty)$, $g_\Omega \geq 0$, such that

$$t^{\beta-1}\left|W_t^\alpha(x,y) - \sqrt{2}W_t(x,y)\right| \leq g_\Omega(t), \quad x \in \Omega, y \in \operatorname{supp} f \text{ and } t \in (0,\infty).$$

Let Ω be a compact subset of $(0,\infty)$. There exist $0 < t_1 < 1 < t_2 < +\infty$ such that $\frac{xy}{\sinh t} < 1$, if $t > t_2$, and $\frac{xy}{\sinh t} > 1$, when $t \in (0,t_1)$, for every $x \in \Omega$ and $y \in \operatorname{supp} f$. According to (P1), we have that

$$\begin{aligned} t^{\beta-1}\left|W_t^\alpha(x,y) - \sqrt{2}W_t(x,y)\right| &\leq Ct^{\beta-1}\big(W_t(x,y) + W_t^\alpha(x,y)\big) \\ &\leq Ct^{\beta-1}\big(e^{-t(\alpha+1)} + e^{-t}\big), \end{aligned}$$

$t > t_2, x \in \Omega$ and $y \in \operatorname{supp} f$.

Also, by (P2) it follows that

$$t^{\beta-1}\left|W_t^\alpha(x,y) - \sqrt{2}W_t(x,y)\right| \leq Ct^{\beta-1}\frac{(\sinh t)^{\frac{1}{2}}}{xy} \leq Ct^{\beta-\frac{1}{2}},$$

$0 < t < t_1, x \in \Omega$ and $y \in \operatorname{supp} f$.

Finally,

$$t^{\beta-1}\left|W_t^\alpha(x,y) - \sqrt{2}W_t(x,y)\right| \leq C, \quad t_1 \leq t \leq t_2, x \in \Omega \text{ and } y \in \operatorname{supp} f.$$

Then, by defining

$$g_\Omega(t) = \begin{cases} t^{\beta-1}\big(e^{-t(\alpha+1)} + e^{-t}\big), & t > t_2, \\ 1, & t_1 \leq t \leq t_2, \\ t^{\beta-\frac{1}{2}}, & 0 < t < t_1, \end{cases}$$

we have that

$$t^{\beta-1}\left|W_t^\alpha(x,y) - \sqrt{2}W_t(x,y)\right| \leq Cg_\Omega(t), \quad x \in \Omega, y \in \operatorname{supp} f \text{ and } t \in (0,\infty).$$

Thus, the proof is completed. □

In the sequel, when $f \in C_c^\infty(0,\infty)$ and $\beta > 0$ we define $L_\alpha^{-\beta}f$ as the $C^\infty(0,\infty)$-function $\Phi_{\alpha,\beta}(f)$.

We now obtain a representation of the higher order Riesz transform in the Laguerre setting on $C_c^\infty(0,\infty)$ as a principal value integral operator.

We previously give conditions on a function f defined on $\mathbb{R} \times \mathbb{R}$ in order that the formula

$$\frac{\partial}{\partial x}\int_{\mathbb{R}} f(x,y)\,dy = \int_{\mathbb{R}} \frac{\partial}{\partial x} f(x,y)\,dy, \quad \text{a.e. } x \in \mathbb{R},$$

holds. We think that this result is known but we have not found an exact reference and we present a proof for the sake of completeness (see also, [5]).

LEMMA 3.1. *Suppose that f is a measurable function defined on $\mathbb{R}\times\mathbb{R}$ that satisfies the following conditions*:

(i) *for every compact subset K of $\mathbb{R}$, $\int_K\int_{\mathbb{R}}|f(x,y)|\,dy\,dx<\infty$, and*

(ii) *there exists a measurable function g on $\mathbb{R}\times\mathbb{R}$ such that*

$$\int_K\int_{\mathbb{R}}\big|g(x,y)\big|\,dy\,dx<\infty$$

for every compact subset K of $\mathbb{R}$, and that the distributional derivative $D_x f(\cdot, y)$ is represented by $g(\cdot,y)$, for every $y\in\mathbb{R}$.

Then,

$$\frac{\partial}{\partial x}\int_{\mathbb{R}} f(x,y)\,dy=\int_{\mathbb{R}}\frac{\partial}{\partial x}f(x,y)\,dy,\quad \text{a.e. } x\in\mathbb{R},$$

where the derivatives are understood in the classical sense.

Proof. We define the function $h(x)=\int_{\mathbb{R}} f(x,y)\,dy$, $x\in\mathbb{R}$. By (i) h defines a regular distribution that we continue denoting by h. According to [24, Chap. 2, §5, Theorem V], we have that

$$\frac{\partial}{\partial x}f(x,y)=g(x,y),\quad \text{a.e. } (x,y)\in\mathbb{R}\times\mathbb{R},$$

where the derivative is understood in the classical sense.

Moreover, if $F\in C_c^\infty(\mathbb{R})$, then

$$\begin{aligned}\langle D_x h,F\rangle &= -\int_{\mathbb{R}} F'(x)h(x)\,dx=-\int_{\mathbb{R}} F'(x)\int_{\mathbb{R}} f(x,y)\,dy\,dx\\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)F'(x)\,dx\,dy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial f}{\partial x}(x,y)F(x)\,dx\,dy\\ &= \int_{\mathbb{R}} F(x)\int_{\mathbb{R}}\frac{\partial f}{\partial x}(x,y)\,dy\,dx.\end{aligned}$$

Hence, $D_x h(x)=\int_{\mathbb{R}}\frac{\partial f}{\partial x}(x,y)\,dy$ in the distributional sense. By using again [24, Chap. 2, §5, Theorem V], we conclude that

$$\frac{\partial}{\partial x}h(x)=\int_{\mathbb{R}}\frac{\partial}{\partial x}f(x,y)\,dy,\quad \text{a.e. } x\in\mathbb{R}.$$

Thus, the proof is completed. □

A useful result in the sequel is the following one.

LEMMA 3.2. *Let $-\infty\le a<b\le+\infty$. Assume that f is a continuous function on $I\times I$, where $I=(a,b)$, such that*

(i) *For every $y\in I$, the function $\frac{\partial}{\partial x}f(x,y)\,dy$ is continuous on $I\setminus\{y\}$, where the derivative is understood in the classical sense.*

(ii) *For every* $y \in I$ *and every compact subset* K *of* I, $\int_K |f(x,y)|\,dx < +\infty$, *and* $\int_K |\frac{\partial f}{\partial x}(x,y)|\,dx < +\infty$.

Then, $D_x f(x,y) = \frac{\partial}{\partial x} f(x,y)$, *for every* $y \in I$. *Here, as above,* $D_x f(x,y)$ *denotes the distributional derivative respect to* x *of* f.

Proof. Let $g \in C_c^\infty(I)$. We can write

$$\begin{aligned}
\big\langle D_x f(x,y), g(x)\big\rangle &= -\int_I g'(x) f(x,y)\,dx \\
&= -\lim_{\varepsilon \to 0^+} \left(\int_a^{y-\varepsilon} + \int_{y+\varepsilon}^b\right) g'(x) f(x,y)\,dx \\
&= \lim_{\varepsilon \to 0^+} \Bigg[-g(y-\varepsilon) f(y-\varepsilon, y) + g(y+\varepsilon) f(y+\varepsilon, y) \\
&\quad + \left(\int_a^{y-\varepsilon} + \int_{y+\varepsilon}^b\right) g(x) \frac{\partial f}{\partial x}(x,y)\,dx\Bigg] \\
&= \int_a^b g(x) \frac{\partial f}{\partial x}(x,y)\,dx, \quad y \in I.
\end{aligned}$$

Then, $D_x f(x,y) = \frac{\partial f}{\partial x}(x,y)$, $y \in I$. $\square$

Proposition 3.6. *Let* $\alpha > -1$, $k \in \mathbb{N} \setminus \{0\}$ *and* $f \in C_c^\infty(0,\infty)$. *Then*

$$\mathfrak{D}_\alpha^k L_\alpha^{-\frac{k}{2}} f(x) = w_k f(x) + \lim_{\varepsilon \to 0^+} \int_{0, |x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y) f(y)\,dy, \quad a.e.\ x \in (0,\infty),$$

where $w_k = 0$, *when* k *is odd, and* $w_k = -2^{\frac{k}{2}}$, *when* k *is even.*

Proof. Assume that Ω is a compact subset of $(0,\infty)$. There exists $M > 0$ such that $\frac{Mx}{2} < y < Mx$, $x \in \Omega$ and $y \in \operatorname{supp} f$. By proceeding as in the proof of (iii) in Proposition 3.3, we can see that, for every $m = 1, \dots, k-1$, there exists a function $g_m \in L^1(0,\infty)$, $g_m \geq 0$, such that

$$\left|\mathfrak{D}_\alpha^m W_t^\alpha(x,y) - \left(\frac{\partial}{\partial x} + x\right)^m W_t(x,y)\right| \leq C g_m(t),$$

$t \in (0,\infty)$, $x \in \Omega$ and $y \in \operatorname{supp} f$. Then, the dominated convergence theorem implies that

$$\begin{aligned}
&\mathfrak{D}_\alpha^m L_\alpha^{-\frac{k}{2}} f(x) - \left(\frac{d}{dx} + x\right)^m H^{-\frac{k}{2}} f(x) \\
&\quad = \int_0^\infty \big(R_\alpha^{(k,m)}(x,y) - R^{(k,m)}(x,y)\big) f(y)\,dy, \quad x \in (0,\infty),
\end{aligned} \tag{3.33}$$

where $m = 1, \dots, k-1$, and

$$R_\alpha^{(k,m)}(x,y) = \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty t^{\frac{k}{2}-1} \mathfrak{D}_\alpha^m W_t^\alpha(x,y)\,dt, \quad x, y \in (0,\infty).$$

Moreover, according to Lemma 3.1 and Proposition 3.2, we can write

$$\begin{aligned}
&\mathfrak{D}_\alpha^k L_\alpha^{-\frac{k}{2}} f(x) - \left(\frac{d}{dx}+x\right)^k H^{-\frac{k}{2}} f(x) \\
&\quad = \int_0^\infty \big(R_\alpha^{(k)}(x,y) - R^{(k)}(x,y)\big) f(y)\,dy, \quad \text{a.e. } x \in (0,\infty),
\end{aligned}$$

and the last integral is absolutely convergent.

Hence, for almost $x \in (0,\infty)$, we have that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y) f(y)\,dy \\
&\quad = \lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^\infty \big(R_\alpha^{(k)}(x,y) - R^{(k)}(x,y)\big) f(y)\,dy \\
&\qquad + \lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^\infty R^{(k)}(x,y) f(y)\,dy \\
&\quad = \int_0^\infty \left(\mathfrak{D}_\alpha^k K_{\alpha,k}(x,y) - \left(\frac{\partial}{\partial x}+x\right)^k K_k(x,y)\right) f(y)\,dy \\
&\qquad + \lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^\infty R^{(k)}(x,y) f(y)\,dy \\
&\quad = \frac{d}{dx}\left(\int_0^\infty \left[\mathfrak{D}_\alpha^{k-1} K_{\alpha,k}(x,y) - \left(\frac{\partial}{\partial x}+x\right)^{k-1} K_k(x,y)\right] f(y)\,dy\right) \\
&\qquad + \left(x - \frac{\alpha+\frac{1}{2}}{x}\right) \int_0^\infty \mathfrak{D}_\alpha^{k-1}\big(K_{\alpha,k}(x,y)\big) f(y)\,dy \\
&\qquad - x \int_0^\infty \left(\frac{\partial}{\partial x}+x\right)^{k-1} \big(K_k(x,y)\big) f(y)\,dy \\
&\qquad + \lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^\infty R^{(k)}(x,y) f(y)\,dy.
\end{aligned}$$

By taking into account Proposition 3.2, we can conclude that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \int_{0,|x-y|>\varepsilon}^\infty R_\alpha^{(k)}(x,y) f(y)\,dy \\
&\quad = \frac{d}{dx}\left(\int_0^\infty \left[\mathfrak{D}_\alpha^{k-1} K_{\alpha,k}(x,y) - \left(\frac{\partial}{\partial x}+x\right)^{k-1} K_k(x,y)\right] f(y)\,dy\right) \\
&\qquad + \left(x - \frac{\alpha+\frac{1}{2}}{x}\right) \int_0^\infty \mathfrak{D}_\alpha^{k-1} K_{\alpha,k}(x,y) f(y)\,dy - w_k f(x) \\
&\qquad + \frac{d}{dx} \int_0^\infty \left(\frac{\partial}{\partial x}+x\right)^{k-1} K_k(x,y) f(y)\,dy \\
&\quad = \mathfrak{D}_\alpha^k L_\alpha^{-\frac{k}{2}} f(x) - w_k f(x), \quad \text{a.e. } x \in (0,\infty),
\end{aligned}$$

where $w_k = 0$, for k odd, and $w_k = -2^{\frac{k}{2}}$, when k is even. Thus, the proof is finished. $\square$

We now prove the main result of the paper.

Proof of Theorem 1.1. We consider the maximal operator associated with $R_\alpha^{(k)}$ defined by

$$R_{\alpha,*}^{(k)} f(x) = \sup_{\varepsilon>0} \left| \int_{0,|x-y|>\varepsilon}^{\infty} R_\alpha^{(k)}(x,y) f(y)\, dy \right|.$$

According to Proposition 3.3, we get

$$R_{\alpha,*}^{(k)} f(x) \le C\big(H_0^{\alpha+\frac{1}{2}}(|f|)(x) + H_\infty^{\alpha+\frac{1}{2}+\delta_k}(|f|)(x) + R_{\mathrm{loc},*}^{(k)}(f)(x) + N(f)(x)\big),$$

where $\delta_k = 1$, when k is odd, $\delta_k = 0$, when k is even,

$$R_{\mathrm{loc},*}^{(k)}(f)(x) = \sup_{\varepsilon>0} \left| \int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} \left(\frac{\partial}{\partial x} + x\right)^k K_k(x,y) f(y)\, dy \right|,$$

and

$$N(f)(x) = \int_{\frac{x}{2}}^{2x} |f(y)| \frac{1}{y} \left(1 + \left(\frac{x}{|x-y|}\right)^{\frac{1}{2}}\right) dy.$$

By [6, Lemma 3.1] $H_0^{\alpha+\frac{1}{2}}$ is of strong type (p,p) with respect to $x^\delta\, dx$, when $1 < p < \infty$ and $\delta < (\alpha + \frac{3}{2})p - 1$, and of weak type $(1,1)$ when $\delta \le \alpha + \frac{1}{2}$. Also from [6, Lemma 3.2] the operator $H_\infty^{\alpha+\frac{1}{2}+\delta_k}$ is of strong type (p,p) for $x^\delta\, dx$, when $1 < p < \infty$ and $-(\alpha + \frac{1}{2})p - 1 < \delta$, and of weak type $(1,1)$ with respect to $x^\delta\, dx$ when $-\alpha - \frac{5}{2} \le \delta$, if k is odd; and, in the case that k is even, when $\delta \ge -\alpha - \frac{3}{2}$, and $\alpha \ne -\frac{1}{2}$ and when $\delta > -1$ and $\alpha = -\frac{1}{2}$.

On the other hand, by using Jensen's inequality, we can see that the operator N is bounded from $L^p((0,\infty), x^\delta\, dx)$ into itself, for every $1 \le p < \infty$ and $\delta \in \mathbb{R}$.

In [28] it was established that the kernel $R^{(k)}(x,y)$, $x,y \in \mathbb{R}$, $x \ne y$, is a Calderón–Zygmund kernel. Then, according to [20, Theorem 4.3], the operator $R_{\mathrm{loc},*}^{(k)}$ is of strong type (p,p), $1 < p < \infty$, and of weak type $(1,1)$ with respect to $x^\delta\, dx$, for every $\delta \in \mathbb{R}$.

Then we conclude that $R_{\alpha,*}^{(k)}$ defines an operator of strong type (p,p) for $x^\delta\, dx$ when $1 < p < \infty$ and $-(\alpha + \frac{1}{2} + \delta_k)p - 1 < \delta < (\alpha + \frac{3}{2})p - 1$. We have also that $R_\alpha^{(k)}$ is of weak type $(1,1)$ for $x^\delta\, dx$ when $-\alpha - \frac{5}{2} \le \delta \le \alpha + \frac{1}{2}$, if k is odd. When k is even the maximal operator $R_{\alpha,*}^{(k)}$ is of weak type (1,1) with respect to $x^\delta\, dx$, for $-\alpha - \frac{3}{2} \le \delta \le \alpha + \frac{1}{2}$ and $\alpha \ne -\frac{1}{2}$, and for $-1 < \delta \le 0$, when $\alpha = -\frac{1}{2}$.

By using standard arguments, since $C_c^\infty(0,\infty)$ is dense in $L^p((0,\infty), x^\delta\, dx)$, we can deduce from Proposition 3.6 that there exists the limit

$$\lim_{\varepsilon\to 0^+}\int_{0,|x-y|>\varepsilon}^{\infty} R_\alpha^{(k)}(x,y) f(y)\, dy, \quad \text{a.e. } x\in(0,\infty),$$

provided that $f\in L^p((0,\infty), x^\delta\, dx)$ and one of the three conditions is satisfied

(i) $1<p<\infty$ and $-(\alpha+\frac{1}{2}+\delta_k)p-1<\delta<(\alpha+\frac{3}{2})p-1$,

(ii) k is odd, $p=1$ and $-\alpha-\frac{5}{2}\le\delta\le\alpha+\frac{1}{2}$,

(iii) k is even, $p=1$, and $-\alpha-\frac{3}{2}\le\delta\le\alpha+\frac{1}{2}$ when $\alpha\neq-\frac{1}{2}$, and $-1<\delta\le 0$, when $\alpha=-\frac{1}{2}$.

Also, the operator $\mathbb{R}_\alpha^{(k)}$ defined by

$$\mathbb{R}_\alpha^{(k)}(f)(x) = w_k f(x) + \lim_{\varepsilon\to 0^+}\int_{0,|x-y|>\varepsilon}^{\infty} R_\alpha^{(k)}(x,y) f(y)\, dy, \quad \text{a.e. } x\in(0,\infty),$$

is of strong type (p,p) for $x^\delta\, dx$ when $1<p<\infty$ and $-(\alpha+\frac{1}{2}+\delta_k)p-1<\delta<(\alpha+\frac{3}{2})p-1$, and of weak type $(1,1)$ for $x^\delta\, dx$ when $-\alpha-\frac{5}{2}\le\delta\le\alpha+\frac{1}{2}$, if k is odd; and for $-\alpha-\frac{3}{2}\le\delta\le\alpha+\frac{1}{2}$ and $\alpha\neq-\frac{1}{2}$, and for $-1<\delta\le 0$, when $\alpha=-\frac{1}{2}$, if k is even.

Note that $0\in(-2(\alpha+\frac{1}{2}+\delta_k)-1, 2\alpha+2)$. Hence, $\mathbb{R}_\alpha^{(k)}$ is a bounded operator from $L^2((0,\infty),dx)$ into itself. Since $R_\alpha^{(k)}$ defined by (1.3) is also bounded from $L^2((0,\infty),dx)$ into itself and $C_c^\infty(0,\infty)$ is dense in $L^2((0,\infty),dx)$, Proposition 3.6 implies that $R_\alpha^{(k)}(f)=\mathbb{R}_\alpha^{(k)}(f)$, $f\in L^2((0,\infty),dx)$.

Thus, the proof of this theorem is finished. □

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References

[1] I. Abu-Falahah, R. Macías, C. Segovia and J. L. Torrea, *Transferring strong boundedness among Laguerre orthogonal systems*, Proc. Indian Acad. Sci. Math. Sci. **119** (2009), 203–220. MR 2526424

[2] K. F. Andersen and B. Muckenhoupt, *Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions*, Studia Math. **72** (1982), 9–26. MR 0665888

[3] J. J. Betancor, J. C. Fariña, T. Martínez and L. Rodríguez-Mesa, *Higher order Riesz transforms associated with Bessel operators*, Ark. Mat. **46** (2008), 219–250. MR 2430725

[4] J. J. Betancor, J. C. Fariña, L. Rodríguez-Mesa, A. Sanabria-García and J. L. Torrea, *Transference between Laguerre and Hermite settings*, J. Funct. Anal. **254** (2008), 826–850. MR 2381163

[5] S. Cheng, *Differentiation under the integral sign using weak derivatives*, unpublished manuscript, available at: http://www.gold-saucer.org/math.

[6] A. Chicco Ruiz and E. Harboure, *Weighted norm inequalities for the heat-diffusion Laguerre's semigroups*, Math. Z. **257** (2007), 329–354. MR 2324805

[7] R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Étude de certaines intégrales singulières, Lecture Notes in Mathematics, vol. 242, Springer-Verlag, Berlin, 1971. MR 0499948

[8] E. Fabes, C. Gutiérrez and R. Scotto, *Weak-type estimates for the Riesz transforms associated with the Gaussian measure*, Rev. Mat. Iberoam. **10** (1994), 229–281. MR 1286476

[9] L. Forzani and R. Scotto, *The higher order Riesz transform for Gaussian measure need not be of weak type* $(1,1)$, Studia Math. **131** (1998), 205–214. MR 1644460

[10] J. García-Cuerva, G. Mauceri, P. Sjögren and J. L. Torrea, *Higher-order Riesz operators for the Ornstein–Uhlenbeck semigroup*, Potential Anal. **10** (1999), 379–407. MR 1698617

[11] G. Gasper, K. Stempak and W. Trebels, *Fractional integration for Laguerre expansions*, Methods Appl. Anal. **2** (1995), 67–75. MR 1337453

[12] P. Graczyk, J.-L. Loeb, I. A. López, A. Nowak and W. Urbina, *Higher order Riesz transforms, fractional derivatives and Sobolev spaces for Laguerre expansions*, J. Math. Pures Appl. (9) **84** (2005), 375–405. MR 2121578

[13] C. Gutiérrez, C. Segovia and J. L. Torrea, *On higher Riesz transforms for Gaussian measures*, J. Fourier Anal. Appl. **2** (1996), 583–596. MR 1423529

[14] E. Harboure, L. de Rosa, C. Segovia and J. L. Torrea, *L^p-dimension free boundedness for Riesz transforms associated to Hermite functions*, Math. Ann. **328** (2004), 653–682. MR 2047645

[15] E. Harboure, C. Segovia, J. L. Torrea and B. Viviani, *Power weighted L^p-inequalities for Laguerre–Riesz transforms*, Ark. Mat. **46** (2008), 285–313. MR 2430728

[16] Y. Kanjin and E. Sato, *The Hardy–Littlewood theorem on fractional integration for Laguerre series*, Proc. Amer. Math. Soc. **123** (1995), 2165–2171. MR 1257113

[17] N. N. Lebedev, *Special functions and their applications*, Dover, New York, 1972. MR 0350075

[18] B. Muckenhoupt, *Poisson integrals for Hermite and Laguerre expansions*, Trans. Amer. Math. Soc. **139** (1969), 231–242. MR 0249917

[19] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. **44** (1972), 31–38. MR 0311856

[20] A. Nowak and K. Stempak, *Weighted estimates for the Hankel transform transplantation operator*, Tohoku Math. J. (2) **58** (2006), 277–301. MR 2248434

[21] A. Nowak and K. Stempak, *Riesz transforms and conjugacy for Laguerre function expansions of Hermite type*, J. Funct. Anal. **244** (2007), 399–443. MR 2297030

[22] A. Nowak and K. Stempak, *Riesz transforms for multi-dimensional Laguerre function expansions*, Adv. Math. **215** (2007), 642–678. MR 2355604

[23] S. Pérez and F. Soria, *Operators associated with the Ornstein–Uhlenbeck semigroup*, J. London Math. Soc. (2) **61** (2000), 857–871. MR 1766110

[24] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1973. MR 0209834

[25] P. Sjögren, *On the maximal function for the Mehler kernel*, Lecture Notes in Math., vol. 992, Springer, Berlin, 1983. MR 0729346

[26] K. Stempak, *Heat-diffusion and Poisson integrals for Laguerre expansions*, Tohoku Math. J. (2) **46** (1994), 83–104. MR 1256729

[27] K. Stempak and J. L. Torrea, *Poisson integrals and Riesz transforms for Hermite function expansions with weights*, J. Funct. Anal. **202** (2003), 443–472. MR 1990533

[28] K. Stempak and J. L. Torrea, *Higher Riesz transforms and imaginary powers associated to the harmonic oscillator*, Acta Math. Hungar. **111** (2006), 43–64. MR 2188971

[29] K. Stempak and J. L. Torrea, *BMO results for operators associated to Hermite expansions*, Illinois J. Math. **49** (2005), 1111–1131. MR 2210354

[30] G. Szegö, *Orthogonal polynomials*, Colloquium Publications, vol. 23, American Math. Soc., Providence, RI, 1975. MR 0372517

[31] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes, vol. 42, Princeton Univ. Press, Princeton, 1993. MR 1215939

[32] W. Urbina, *On singular integrals with respect to the Gaussian measure*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **17** (1990), 531–567. MR 1093708

[33] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, Cambridge, 1995. MR 1349110

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