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BURKHOLDER'S FUNCTION VIA MONGE–AMPÈRE EQUATION

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ABSTRACT. We will show how to get Burkholder's function from (*Ann. Probab.* **12** (1984) 647–702) by using Monge–Ampère equation. This method is quite different from those in the series of Burkholder's papers (*Ann. Probab.* **12** (1984) 647–702, *An extension of classical martingale inequality* (1986) Marcel Dekker, *Astérisque* **157–158** (1988) 75–94, In *Harmonic analysis and partial differential equations* (1989) 1–23 Springer, In *École d'Ete de Probabilités de Saint-Flour XIX* (1991) 1–66 Springer, *Ann. Probab.* **22** (1994) 995–1025, *Studia Math.* **91** (1988) 79–83).

1. Introduction

Since their creation the pioneering works of Donald Burkholder always attracted the attention of mathematical community because of their novelty and their power. Recently his ideas got used in numerous subtle questions of Harmonic Analysis, especially related to weighted estimates of Singular Integrals. Beautiful interrelations between Harmonic Analysis and Stochastic Optimal Control became apparent and extremely useful. This paper is a tribute to Donald Burkholder, which concerns a very particular (but also very important) part of his contribution: Burkholder's function.

Bellman function method in Harmonic Analysis was introduced by Burkholder for finding the norm in L^p of the Martingale transform. Later it became clear that the scope of the method is quite wide.

The technique, originated in Burkholder's papers [Bu1], [Bu2], [Bu3], [Bu4], [Bu5], [Bu6], [Bu7], can be credited for helping to solve several old Harmonic

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Analysis problems and for unifying approach to many others. In the first category, one would name the (sharp weighted) estimates of such classical operators as the Ahlfors–Beurling transform (Banuelos–Wang [BaWa1], Banuelos–Janakiraman [BaJa1], Banuelos–Mendez [BaMH], Nazarov–Volberg [NV1], Petermichl–Volberg [PV], Dragicevic–Volberg [DV2]) and the Hilbert and Riesz transforms (Petermichl [P1], [P2]). In the second category, one can name all kind of dimension free estimates of weighted and unweighted Riesz transforms (see a vast literature in [DV1], [DV2], [DV3]). Roughly, Bellman function method makes apparent the hidden scaling properties of a given Harmonic Analysis problem. Conversely, given a Harmonic Analysis problem with certain scaling properties one can (formally) associate with is a non-linear PDE, the so-called Bellman equation of the problem.

Let us recall to the reader that in the series of papers [Bu], [Bu1], [Bu2], [Bu3], [Bu4], [Bu5], [Bu6], [Bu7] Donald Burkholder investigated Martingale transform and gave the sharp bounds on this operator in various settings—but by similar methods. The methods were so novel and powerful that the influence of these articles will be felt for many years to come. The novelty was a key. One of the leading mathematician working in the domain of Harmonic Analysis told the second author that these papers of Burkholder "spin his head." In the book of Daniel Strook [Str], many pages are devoted to the technique developed by Burkholder in the above mentioned series of papers, and the reader can sense the same feeling. It is explained in [Str] that the simplest way to understand the sharp estimates of Martingale transform obtained by Burkholder is to operate with one of the so-called Burkholder's function:

$$u_p(x,y) = p\left(1-\frac{1}{p^*}\right)^{p-1}\big(|y|-\big(p^*-1\big)|x|\big)(|x|+|y|)^{p-1}, \tag{1}$$

here $p^* := \max(p, \frac{p}{p-1})$, $1 < p < \infty$.

However, the main question is of course how to get this function? Where did it come from? These questions are asked in [Str] as well. Of course, Burkholder explains in many details the way this function (and several of its relatives) are obtained. It is almost (but not quite) the least bi-concave majorant of function

$$|y|^{p-1} - \big(p^*-1\big)^p|x|^p. \tag{2}$$

It is obtained by solving a certain PDE and performing certain manipulations with the solution after that. The reader will find much more about u_p after reading this article, in particular in Section 6.

But it seems like the same questions persist even after this explanation. And a new question can appear: how wide is the applicability of the technique that Burkholder elaborated in [Bu1], [Bu2], [Bu3], [Bu4], [Bu5], [Bu6], [Bu7]? There is a vague feeling that the area of applicability is quite wide. To make this feeling more precise one should look at the function above closer and see that it is a creature from another universe, which, initially, does not have too much in common with Harmonic Analysis. Burkholder function is a natural dweller of the area called Stochastic Optimal Control. It is a solution of a corresponding Bellman equation (or a dynamic programming equation) but in the setting, when the differential equations subject to control are not the usual ones. They are stochastic differential equations. The reader can find some notes on this in [VoEcole], [NTV2], [VaVo], [VaVo2], [SlSt]. These notes explain why Stochastic Optimal Control is the right tool to work with a certain class of Harmonic Analysis problems. On the other hand, Stochastic Optimal Control problems generically can be reduced to solving a so-called Bellman PDE (and proving the so-called "verification theorems," but this is a second task). Bellman PDEs belong to the class of fully non-linear PDEs. Often they are PDEs of Monge–Ampère type. In the present article, we would like to show the reader how to obtain Burkhloder functions (the one above and others from [Bu1], [Bu2], [Bu3], [Bu4], [Bu5], [Bu6], [Bu7]) by reducing the search for them to solving certain Monge–Ampère equations. The scope of the application of the methods of Stochastic Optimal Control to Harmonic Analysis proved to be quite large. After Burkholder the first systematic application of this technique appeared in 1995 in the first preprint version of [NTV1]. It was vastly developed in [NT] and in (now) numerous papers that followed. A small part of this literature can be found in the bibliography below.

2. Notations and definitions

We shall say that an interval I and a pair of positive numbers $\alpha_I^{\pm}$ such that $\alpha^+ + \alpha^- = 1$ generate a pair of subintervals I^+ and I^- if $|I^{\pm}| = \alpha_I^{\pm}|I|$ ($|I|$ means the length of I) and $I = I^- \cup I^+$. For a given interval J, the symbol $\mathcal{J} = \mathcal{J}(\alpha)$ will denote the families of subintervals of J such that

- $J \in \mathcal{J}$;
- if $I \in \mathcal{J}$ then $I^{\pm} \in \mathcal{J}$.

For a special choice if all $\alpha_I^{\pm} = \frac{1}{2}$, we get the dyadic family $\mathcal{J} = \mathcal{D}$. Every family $\mathcal{J}$ has its own set of Haar functions:

$$\forall I \in \mathcal{J} \quad h_I(t) = \begin{cases} \sqrt{\frac{\alpha_I^+}{\alpha_I^-|I|}} & \text{if } t \in I_-, \\ -\sqrt{\frac{\alpha_I^-}{\alpha_I^+|I|}} & \text{if } t \in I_+. \end{cases}$$

If the family $\mathcal{J}$ is such that that the maximal length of the interval of n-th generation (i.e., after splitting the initial interval J into 2^n parts) tends to 0 as $n \to \infty$, the Haar family forms an orthonormal basis in the space $L^2(J) \ominus \{\text{const}\}$.

For a function $f \in L^1(I)$ the symbol $\langle f\rangle_I$ means the average of f over the interval I:

$$\langle f\rangle_I = \frac{1}{|I|}\int_I f(t)\,dt.$$

DEFINITION. Fix a real p, $1<p<\infty$, and let $p'=\frac{p}{p-1}$, $p^*=\max\{p,p'\}$. Introduce the following domain in $\mathbb{R}^3$:

$$\Omega=\Omega(p)=\big\{x=(x_1,x_2,x_3): x_3\ge 0, |x_1|^p\le x_3\big\}.$$

For a fixed partition $\mathcal{J}$ of an interval J, we define two function on this domain

$$\mathbf{B}_{\max}(x)=\mathbf{B}_{\max}(x;p)=\sup_{f,g}\big\{\big\langle |g|^p\big\rangle_J\big\},$$
$$\mathbf{B}_{\min}(x)=\mathbf{B}_{\min}(x;p)=\inf_{f,g}\big\{\big\langle |g|^p\big\rangle_J\big\},$$

where the supremum is taken over all functions f, g from $L^p(J)$ such that $\langle f\rangle_J=x_1$, $\langle g\rangle_J=x_2$, $\langle |f|^p\rangle_J=x_3$, and $|(f,h_I)|=|(g,h_I)|$. We shall refer to any such pair of functions f, g as to an admissible pair. When $|(f,h_I)|=|(g,h_I)|$ happens for all dyadic intervals inside J, *we call g a Martingale transform of f*. We shall call $\mathbf{B}_{\max}(x)$ (and $\mathbf{B}_{\min}(x)$) the Bellman functions of the problem of finding the best constant for the Martingale transform inequality:

$$\big|\langle g\rangle_J\big|\le\big|\langle f\rangle_J\big| \quad\Rightarrow\quad \big\langle |g|^p\big\rangle_J\le C(p)\big\langle |f|^p\big\rangle_J. \tag{3}$$

This best constant was found by Burkholder:

$$C(p)=\big(p^*-1\big)^p,\quad p^*:=\max\left(p,\frac{p}{p-1}\right).$$

REMARK 1. It is amazing that there is no proof that would find this $C(p)$ without finding the function of 3 variables $\mathbf{B}_{\max}(x)$ or some of its relatives (like, for example, u_p from (1)).

REMARK 2. Burkholder proved that the functions $\mathbf{B}$ do not depend on the initial interval J and on a specific choice of its partition. Below we work only with dyadic partitions.

REMARK 3. In the case $p=2$ the Bellman function are evident:

$$\mathbf{B}_{\max}(x)=\mathbf{B}_{\min}(x)=x_2^2+x_3-x_1^2.$$

Indeed, since

$$\|f\|_2^2=|J|x_3=|J|x_1^2+\sum_{I\in\mathcal{J}}\big|(f,h_I)\big|^2,$$

we have

$$\big\langle |g|^2\big\rangle_J=\frac{1}{|J|}\|g\|_2^2=x_2^2+\frac{1}{|J|}\sum_{I\in\mathcal{J}}\big|(g,h_I)\big|^2$$
$$=x_2^2+\frac{1}{|J|}\sum_{I\in\mathcal{J}}\big|(f,h_I)\big|^2=x_2^2+x_3-x_1^2.$$

Define the following function on $\mathbb{R}^2_+ = \{z = (z_1, z_2) : z_i > 0\}$:

$$F_p(z_1,z_2)=\begin{cases}\big[z_1^p-\big(p^*-1\big)^p z_2^p\big],\\ \quad \text{if } z_1\le \big(p^*-1\big)z_2,\\ p\big(1-\frac{1}{p^*}\big)^{p-1}(z_1+z_2)^{p-1}\big[z_1-\big(p^*-1\big)z_2\big],\\ \quad \text{if } z_1\ge \big(p^*-1\big)z_2.\end{cases}\tag{4}$$

Note for for $p=2$ the expressions above are reduced to $F_2(z_1,z_2)=z_1^2-z_2^2$.

3. The main result

Now we are ready to state the main result.

THEOREM 1. *The equation* $F_p(|x_1|,|x_2|)=F_p(x_3^{\frac1p},\mathbf{B}^{\frac1p})$ *determines implicitly the function* $\mathbf{B}=\mathbf{B}_{\min}(x;p)$ *and the equation* $F_p(|x_2|,|x_1|)=F_p(\mathbf{B}^{\frac1p},x_3^{\frac1p})$ *determines implicitly the function* $\mathbf{B}=\mathbf{B}_{\max}(x;p)$.

REMARK. The reader can take a look at formulae (5.23)–(5.27) on p. 660 of [Bu1] and recognize that this is how Burkholder describes $\mathbf{B}_{\max}$. The same is true for $\mathbf{B}_{\min}$.

4. How to find Bellman functions

We start from deducing the main inequality for Bellman functions. Introduce new variables $y_1=\frac12(x_2+x_1)$, $y_2=\frac12(x_2-x_1)$, and $y_3=x_3$. In terms of the new variables we define a function $\mathbf{M}$,

$$\mathbf{M}(y_1,y_2,y_3)=\mathbf{B}(x_1,x_2,x_3)=\mathbf{B}(y_1-y_2,y_1+y_2,y_3),$$

on the domain

$$\Xi=\big\{y=(y_1,y_2,y_3): y_3\ge 0, |y_1-y_2|^p\le y_3\big\}.$$

Since the point of the boundary $x_3=|x_1|^p$ ($y_3=|y_1-y_2|^p$) occurs for the only constant test function $f=x_1$ (and therefore then $g=x_2$ is a constant function as well), we have

$$\mathbf{B}\big(x_1,x_2,|x_1|^p\big)=|x_2|^p,$$

or

$$\mathbf{M}\big(y_1,y_2,|y_1-y_2|^p\big)=|y_1+y_2|^p.\tag{5}$$

Note that the function $\mathbf{B}$ is even with respect of x_1 and x_2, that is,

$$\mathbf{B}(x_1,x_2,x_3)=\mathbf{B}(-x_1,x_2,x_3)=\mathbf{B}(x_1,-x_2,x_3).$$

It follows from the definition of $\mathbf{B}$ if we consider the test functions $\tilde f=-f$ for the first equality and $\tilde g=-g$ for the second one. For the function $\mathbf{M}$, this means that we have the symmetry with respect to the lines $y_1=\pm y_2$

$$\mathbf{M}(y_1,y_2,y_3)=\mathbf{M}(y_2,y_1,y_3)=\mathbf{M}(-y_1,-y_2,y_3).\tag{6}$$

Therefore, it is sufficient to find the function $\mathbf{B}$ in the domain

$$\Omega_+ = \Omega_+(p) = \{x = (x_1, x_2, x_3) : x_i \ge 0, |x_1|^p \le x_3\}, \tag{7}$$

or the function $\mathbf{M}$ in the domain

$$\Xi_+ = \{y = (y_1, y_2, y_3) : y_1 \ge 0, -y_1 \le y_2 \le y_1, (y_1 - y_2)^p \le y_3\}. \tag{8}$$

Then we get the solution in the whole domain by putting

$$\mathbf{B}(x_1, x_2, x_3) = \mathbf{B}(|x_1|, |x_2|, x_3).$$

Due to the symmetry (6) we have the following boundary conditions on the "new part" of the boundary $\partial\Xi_+$:

$$\begin{aligned} &\frac{\partial \mathbf{M}}{\partial y_1} = \frac{\partial \mathbf{M}}{\partial y_2} \quad \text{on the hyperplane } y_2 = y_1, \\ &\frac{\partial \mathbf{M}}{\partial y_1} = -\frac{\partial \mathbf{M}}{\partial y_2} \quad \text{on the hyperplane } y_2 = -y_1. \end{aligned} \tag{9}$$

If we consider the family of test functions $\tilde{f} = \tau f$, $\tilde{g} = \tau g$ together with f and g we come to the following homogeneity condition

$$\mathbf{B}(\tau x_1, \tau x_2, \tau^p x_3) = \tau^p \mathbf{B}(x_1, x_2, x_3),$$

or

$$\mathbf{M}(\tau y_1, \tau y_2, \tau^p y_3) = \tau^p \mathbf{M}(y_1, y_2, y_3).$$

We shall use this property in the following form: take derivative with respect to τ and put $\tau = 1$

$$y_1 \frac{\partial \mathbf{M}}{\partial y_1} + y_2 \frac{\partial \mathbf{M}}{\partial y_2} + p y_3 \frac{\partial \mathbf{M}}{\partial y_3} = p\mathbf{M}(y_1, y_2, y_3). \tag{10}$$

Let us fix two points $x^\pm \in \Omega$ such that $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$, for the corresponding points $y^\pm \in \Xi$ this means that either $y_1^+ = y_1^-$, or $y_2^+ = y_2^-$. Then for an arbitrarily small number $\varepsilon > 0$ by the definition of the Bellman function $\mathbf{B} = \mathbf{B}_{\max}$ there exist two couples of test functions $f^\pm$ and $g^\pm$ on the intervals $I^\pm$ such that $\langle f^\pm \rangle_{I^\pm} = x_1^\pm$, $\langle g^\pm \rangle_{I^\pm} = x_2^\pm$, $\langle |f^\pm|^p \rangle_{I^\pm} = x_3^\pm$, and $\langle |g^\pm|^p \rangle_{I^\pm} \ge \mathbf{B}(x^\pm) - \varepsilon$. On the interval $I = I^+ \cup I^-$ we define a pair of test functions f and g as follows $f|I^\pm = f^\pm$, $g|I^\pm = g^\pm$. This is a pair of test functions that corresponds to the point $x = \alpha^+ x^+ + \alpha^- x^-$, where $\alpha^\pm = |I^\pm|/|I|$, because the property $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ means $|(f, h_I)| = |(g, h_I)|$. This yields

$$\mathbf{B}(x) \ge \langle |g|^p \rangle_I = \alpha^+ \langle |g^+|^p \rangle_I^+ + \alpha^- \langle |g^-|^p \rangle_I^- \ge \alpha^+ \mathbf{B}(x^+) + \alpha^- \mathbf{B}(x^-) - \varepsilon.$$

Since ε is arbitrary, we conclude

$$\mathbf{B}(x) \ge \alpha^+ \mathbf{B}(x^+) + \alpha^- \mathbf{B}(x^-). \tag{11}$$

For the function $\mathbf{B} = \mathbf{B}_{\min}$, we can get in a similar way

$$\mathbf{B}(x) \le \alpha^+ \mathbf{B}(x^+) + \alpha^- \mathbf{B}(x^-). \tag{12}$$

Recall that this is not quite concavity (convexity) condition, because we have the restriction $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$. But in terms of the function $\mathbf{M}$

$$\mathbf{M}_{\max}(y) \geq \alpha^+ \mathbf{M}_{\max}(y^+) + \alpha^- \mathbf{M}_{\max}(y^-),$$
$$\mathbf{M}_{\min}(y) \leq \alpha^+ \mathbf{M}_{\min}(y^+) + \alpha^- \mathbf{M}_{\min}(y^-),$$

when either $y_1 = y_1^+ = y_1^-$, or $y_2 = y_2^+ = y_2^-$, we indeed have the concavity (convexity) of the function $\mathbf{M}$ with respect to y_2, y_3 under a fixed y_1, and with respect to y_1, y_3 under a fixed y_2.

Since the domain is convex, under the assumption that the function $\mathbf{B}$ are sufficiently smooth these conditions of concavity (convexity) are equivalent to the differential inequalities

$$\begin{pmatrix} \mathbf{M}_{y_1y_1} & \mathbf{M}_{y_1y_3} \\ \mathbf{M}_{y_3y_1} & \mathbf{M}_{y_3y_3} \end{pmatrix} \leq 0, \qquad \begin{pmatrix} \mathbf{M}_{y_2y_2} & \mathbf{M}_{y_2y_3} \\ \mathbf{M}_{y_3y_2} & \mathbf{M}_{y_3y_3} \end{pmatrix} \leq 0 \quad \forall y \in \Xi \tag{13}$$

for $\mathbf{M} = \mathbf{M}_{\max}$ (here $\mathbf{M}_{y_iy_j}$ stand for the partial derivatives $\frac{\partial^2 \mathbf{M}}{\partial y_i\, \partial y_j}$) and

$$\begin{pmatrix} \mathbf{M}_{y_1y_1} & \mathbf{M}_{y_1y_3} \\ \mathbf{M}_{y_3y_1} & \mathbf{M}_{y_3y_3} \end{pmatrix} \geq 0, \qquad \begin{pmatrix} \mathbf{M}_{y_2y_2} & \mathbf{M}_{y_2y_3} \\ \mathbf{M}_{y_3y_2} & \mathbf{M}_{y_3y_3} \end{pmatrix} \geq 0 \quad \forall y \in \Xi \tag{14}$$

for $\mathbf{M} = \mathbf{M}_{\min}$.

Extremal properties of the Bellman function requires for one of matrices in (13) and (14) to be degenerated. So we arrive at the Monge–Ampère equation:

$$\mathbf{M}_{y_iy_i}\mathbf{M}_{y_3y_3} = (\mathbf{M}_{y_iy_3})^2 \tag{15}$$

either for $i = 1$ or for $i = 2$. To find a candidate M for the role of the true Bellman function $\mathbf{M}$, we shall solve this equation. After finding this solution, we shall prove that $M = \mathbf{M}$.

The method of solving homogeneous Monge–Ampère equation is described, for example, in [VaVo], [VaVo2], [SlSt]. In particular, we know that the solution of the Monge–Ampère equation has to be of the form

$$M = t_i y_i + t_3 y_3 + t_0, \tag{16}$$

where $t_k = M_{y_k}$, $k = 1, 2, 3$. The solution M is linear along the lines (let us call them *extremal trajectories*)

$$y_i\, dt_i + y_3\, dt_3 + dt_0 = 0. \tag{17}$$

One of the ends of the extremal trajectory has to be a point on the boundary $y_3 = |y_1 - y_2|^p$, where constant functions are the only test functions corresponding to these points. Denote this point by $U = (y_1, u, (y_1 - u)^p)$. Note that we write $(y_1 - u)^p$ instead of $|y_1 - u|^p$ because the domain Ξ_+ is under consideration. For the second end of the extremal trajectory, we have four possibilities

(1) it belongs to the same boundary $y_3 = (y_1 - y_2)^p$;

(2) it is at infinity $(y_1, y_2, +\infty)$, that is, the extremal lines goes parallel to the y_3-axis;
(3) it belongs to the boundary $y_2 = y_1$;
(4) it belongs to the boundary $y_2 = -y_1$.

The first possibility gives us no solution. Namely, we have the following proposition.

PROPOSITION. *If $p \neq 2$, then the function $\mathbf{B}_{\max}$ cannot be equal to $B(x) = M(y)$, where M is the solution of the Monge–Ampère equation* (15) *such that one of its extremal trajectory is of type* (1) *above. The same claim holds for $\mathbf{B}_{\min}$.*

Proof. To check this it is sufficient to verify that the test functions of the type $\alpha + \beta h_I(t)$ cannot be an extremal function of our problem with the only exception of $p = 2$, when the situation is trivial: $\mathbf{B}_{\max}(x) = \mathbf{B}_{\min}(x) = x_3 + x_2^2 - x_1^2$, and any pair of test function is extremal. We will show that the Bellman functions being solution of the homogeneous Monge–Ampère equation cannot be linear on a chord $[x^-, x^+]$ connecting two points $x^\pm$ on the boundary $\partial\Omega$, that is, such a chord cannot be an extremal trajectory of our Monge–Ampère equation.

We assume now that two points $x^\pm \in \Omega_+$ such that $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ are on the boundary $x_3^\pm = (x_1^\pm)^p$ and $x = \frac{1}{2}(x^+ + x^-)$. We need to show that $\frac{1}{2}((x_2^+)^p + (x_2^-)^p)$ can be the value neither of $\mathbf{B}_{\max}(x)$ nor of $\mathbf{B}_{\min}(x)$.

Without lost of generality, we may assume that $x_1^+ > x_1^-$. Let us denote $a := \frac{1}{2}(x_1^+ - x_1^-)$ then

$$x_1^\pm = x_1 \pm a, \qquad x_2^\pm = x_2 \pm \sigma a,$$

where $\sigma = \pm 1$ depending on the direction of our chord: it can be either in the plane $x_1 - x_2 = \text{const}$ (and then $\sigma = 1$) or in the plane $x_1 + x_2 = \text{const}$ (and then $\sigma = -1$). The pair of the test functions f, g on $I = [0, 1]$ that gives the value

$$A(a) =: \langle |g|^p \rangle_I = \frac{1}{2}\big(|x_2^+|^p + |x_2^-|^p\big) = \frac{1}{2}\big(|x_2 + a|^p + |x_2 - a|^p\big)$$

is

$$f = x_1 + a h_I, \qquad g = x_2 + \sigma a h_I.$$

First of all, we assume that $x^\pm \in \Omega_+$, but $x_i^\pm \neq 0$ that is, $x_1 \pm a > 0$, $x_2 \pm a > 0$, because if one of $x_i^\pm$ is zero we are in the cases either (3) or (4) listed before the proposition. Our aim will be to find another pair of test functions $\tilde f, \tilde g$ corresponding to the same point x, but with $\langle |\tilde g|^p \rangle_I$ either bigger than $A(a)$ (and then $A(a)$ cannot be the value of $\mathbf{B}_{\max}(x)$) or less than $A(a)$ (and then $A(a)$ cannot be the value of $\mathbf{B}_{\min}(x)$).

Let as make here two remarks. First, we see that the expression $A(a)$ does not depend on the direction σ. Therefore, the construction of the desired $\tilde f, \tilde g$

ensures us that the point x cannot be the center of an extremal trajectory with two ends on $\partial\Omega$ in any direction $\sigma = \pm 1$. Secondly, we note that it is not obligatory to look for $\tilde f, \tilde g$ for all $a \in (0, \max\{x_1, x_2\})$, it sufficient to do this for small values of a/x_2.

Indeed, suppose that the chord $L = [x^-, x^+]$ represents an extremal trajectory of the corresponding Monge–Ampère equation. Let us consider the "crescent" between the chord L and the boundary $\partial\Omega$. It should be filled in by chords on which $\mathbf{B}(x)$ are linear (this is the property of the solutions of the homogeneous Monge–Ampère equation expressed in Pogorelov's theorem, see [Pog]). Among these chords, we can take one (say $\tilde L = [\tilde x^-, \tilde x^+]$) of arbitrarily small length (arbitrarily small value of $\tilde a/\tilde x_2$). Therefore, we can work with a new point $\tilde x$ and new chord $\tilde L$: would we show that $\tilde L$ cannot be an extremal trajectory, the chord L could not be one either.

To construct the desired pair $\tilde f, \tilde g$, we use the following family of functions ϕ_s equal to 1 on $[0, 1/2 - s] \cup [1 - s, 1]$ and to -1 on $(1/2 - s, 1 - s)$. Let us note that all ϕ_s have the same distribution function as h_I, and

$$(\phi_s, h_I) = 1 - 4s.$$

We are interested in $\phi := \phi_{1/8}$. Then

$$(\phi, h_I) = \frac{1}{2},$$

and $\psi = \phi - h_I$ has

$$(\psi, h_I) = -\frac{1}{2}, \qquad (\psi, h_J) = (\phi, h_J)$$

for all other dyadic J. So ψ is a martingale transform of ϕ (it is equal to 0 on $[0, 3/8] \cup [1/2, 7/8]$ and to ± 2 on two intervals $(7/8, 1)$, $(3/8, 1/2)$) and we can examine the pair

$$\tilde f = x_1 + a\phi, \qquad \tilde g = x_2 + a\psi.$$

Since $\tilde f$ and f have the same distribution function, we have $\langle |\tilde f|^p \rangle_I = \langle |f|^p \rangle_I = x_3$, that is, $\tilde f, \tilde g$ is a pair of test functions corresponding to the same point x.

To investigate the difference $\langle |\tilde g|^p \rangle_I - \langle |g|^p \rangle_I$, we use the function

$$\lambda_p(\alpha) := \frac{1}{8}\big((1 + 2\alpha)^p + (1 - 2\alpha)^p\big) + \frac{3}{4} - \big((1 + \alpha)^p + (1 - \alpha)^p\big).$$

Since

$$\lambda_p(\alpha) = \frac{1}{8} p(p-1)(p-2)(p-3)\alpha^4 + O\big(\alpha^6\big),$$

we have $\lambda_p(\alpha)>0$ for small α if $1<p<2$ or $p>3$ and $\lambda_p(\alpha)<0$ for small α if $2<p<3$. Recall that

$$\tilde g(t)=\begin{cases} x_2, & 0<t<\frac38,\\ x_2-2a, & \frac38<t<\frac12,\\ x_2, & \frac12<t<\frac78,\\ x_2+2a, & \frac78<t<1,\end{cases}$$

therefore, $\langle|\tilde g|^p\rangle_I=\frac18((x_2+2a)^p+(x_2-2a)^p)+\frac34x_2^p$ and

$$\big\langle|\tilde g|^p\big\rangle_I-\big\langle|g|^p\big\rangle_I=x_2^p\lambda\Big(\frac{a}{x_2}\Big).$$

For small α, we have a desired example for $\mathbf{B}_{\max}$ if $p\in(1,2)\cup(3,\infty)$ (because $\lambda_p>0$) and for $\mathbf{B}_{\min}$ if $p\in(2,3)$ (because $\lambda_p<0$).

Now we interchange in a sense the roles of $\tilde f$ and $\tilde g$: instead of a we take a new parameter, say $\tilde a$, and put

$$\tilde f=x_1+\tilde a\psi,\qquad \tilde g=x_2+\tilde a\phi.$$

Since we have now

$$\big\langle|\tilde f|^p\big\rangle_I-\big\langle|f|^p\big\rangle_I=x_2^p\lambda\Big(\frac{\tilde a}{x_1}\Big)$$

and the function

$$t\mapsto\big\langle|x+t\psi|^p\big\rangle_I=\frac18\big(|x-2t|^p+|x+2t|^p\big)+\frac34|x|^p$$

is increasing in $t>0$ from $|x|^p$ till infinity, we can find $\tilde a$, $\tilde a>a$ for $p\in(2,3)$ and $\tilde a<a$ for $p\in(1,2)\cup(3,\infty)$, such that

$$\big\langle|\tilde f|^p\big\rangle_I=\big\langle|f|^p\big\rangle_I=x_3.$$

For this $\tilde a$, we get the desired pair of test function, because the function

$$t\mapsto\big\langle|x+t\phi|^p\big\rangle_I=\frac12\big(|x-t|^p+|x+t|^p\big)$$

is also increasing in $t>0$, and therefore, we have

$$\big\langle|\tilde g|^p\big\rangle_I=\big\langle|x_2+\tilde a\phi|^p\big\rangle_I>\big\langle|x_2+\tilde a\phi|^p\big\rangle_I=\big\langle|x_2+\tilde a h_I|^p\big\rangle_I=\big\langle|g|^p\big\rangle_I$$

if $p\in(2,3)$ and the opposite inequality if $p\in(1,2)\cup(3,\infty)$ (and $\tilde a<a$).

This construction failed for $p=3$ because $\lambda_3(\alpha)=0$ for all $\alpha\in(0,\frac12)$. To avoid this difficulty, we modify the function ψ, namely, we take $\psi=\phi+h_{I^+}/\sqrt2$, that is,

$$\psi(t)=\begin{cases} 1, & 0<t<\frac38,\\ -1, & \frac38<t<\frac12,\\ 0, & \frac12<t<\frac34,\\ -2, & \frac34<t<\frac78,\\ 0, & \frac78<t<1.\end{cases}$$

The function ψ is a martingale transform of ϕ, since

$$(\phi, h_{I^+}) = -\frac{1}{2\sqrt{2}}, \qquad (\psi, h_{I^+}) = \frac{1}{2\sqrt{2}}.$$

Now we put

$$\tilde{f} = x_1 + a\phi, \qquad \tilde{g} = x_2 \pm a\psi.$$

As before we have $\langle|\tilde{f}|^3\rangle_I = \frac{1}{2}((x-a)^3 + (x+a)^3) = \langle|f|^3\rangle_I$, but

$$\big\langle|\tilde{g}|^3\big\rangle_I = x_2^3 + 6x_2a^2 \mp \frac{3}{4}a^3 = \big\langle|g|^3\big\rangle_I \mp \frac{3}{4}a^3,$$

therefore, by choosing the sign in the definition of $\tilde{g}$ we are able to increase as well to decrease the value $\langle|g|^3\rangle_I$, hence it is neither the value of $\mathbf{B}_{\max}(x)$ nor $\mathbf{B}_{\min}(x)$. The proposition is completely proved. □

Now we check the second possibility among the possibilities (1)–(4) listed right before the proposition. Since the extremal line is parallel to the y_3-axis, the Bellman function has to be of the form

$$M(y) = A(y_1, y_2) + C(y_1, y_2)y_3.$$

Any pair of inequalities both (13) and (14) implies $M_{y_iy_i}M_{y_3y_3} - (M_{y_iy_3})^2 \geq 0$. Since $M_{y_3y_3} = 0$, this yields $M_{y_iy_3} = \frac{\partial C}{\partial y_i} = 0$, that is, C is a constant. From the boundary condition (5), we get

$$A(y_1, y_2) + C(y_1 - y_2)^p = (y_1 + y_2)^p,$$

whence

$$A(y_1, y_2) = (y_1 + y_2)^p - C(y_1 - y_2)^p,$$

and

$$M(y) = (y_1 + y_2)^p + C\big(y_3 - (y_1 - y_2)^p\big), \tag{18}$$

or

$$B(x) = x_2^p + C\big(x_3 - x_1^p\big). \tag{19}$$

Let us note that this solution cannot satisfy necessary conditions in the whole domain Ξ_+ except the case $p = 2$. The constant C must be positive (otherwise the extremal lines cannot tend to infinity along y_3-axes, because M must be a nonnegative function). Therefore, the straight line

$$y_1 + y_2 = C^{\frac{1}{p-2}}(y_1 - y_2), \quad \text{or} \quad x_2 = C^{\frac{1}{p-2}}y_1$$

splits Ξ_+ in two subdomains, in one of which the derivatives

$$\frac{\partial^2 M}{\partial y_1^2} = \frac{\partial^2 M}{\partial y_2^2} = p(p-1)\big((y_1 + y_2)^{p-2} - C(y_1 - y_2)^{p-2}\big)$$

is positive (that is, it could be a candidate for $\mathbf{B}_{\min}$), and in another one is negative (that is, it could be a candidate for $\mathbf{B}_{\max}$).

Thus, this simple solution cannot give us the whole Bellman function and we need to continue the consideration of the possibilities (3) and (4) (listed right before the proposition). Till now, we have not fixed which of two matrices in (13) or in (14) is degenerated, that is, what is i in the Monge–Ampère equation (15), because for the vertical extremal lines both these equations are fulfilled. Now, when considering possibility (3) or (4), we need to investigate separately both Monge–Ampère equations (15). We shall refer to these cases as (3_i) and (4_i).

Let us start with simultaneous consideration of the cases (3_1) and (4_1) (we recall that this means that y_2 is fixed). We look for a function

$$M = t_1 y_1 + t_3 y_3 + t_0$$

on the domain Ξ_+, which is linear along the extremal lines

$$y_1\, dt_1 + y_3\, dt_3 + dt_0 = 0.$$

Now one end point of our extremal line $V = (v, y_2, (v - y_2)^p)$ belongs to the boundary $y_3 = |y_1 - y_2|^p$ and the second end point $W = (|y_2|, y_2, w)$ is on the boundary $y_1 = |y_2|$, where we have boundary condition (9). Due to the symmetry (6), on the boundary $y_1 = y_2$ (this means that our fixed $y_2 \geq 0$) we have

$$\frac{\partial M}{\partial y_2} = \frac{\partial M}{\partial y_1} = t_1,$$

and

$$\frac{\partial M}{\partial y_2} = -\frac{\partial M}{\partial y_1} = -t_1,$$

on the boundary $y_1 = -y_2$ (this means that our fixed $y_2 < 0$). In both cases

$$y_2 \frac{\partial M}{\partial y_2} = y_1 \frac{\partial M}{\partial y_1} = |y_2| t_1,$$

and therefore (10) and (16) imply

$$2t_1|y_2| + pwt_3 = pM(W) = pt_1|y_2| + pwt_3 + pt_0,$$

whence

$$t_0 = \left(\frac{2}{p} - 1\right) t_1 |y_2|.$$

This gives the formula for $t_0(t_1)$ (remember that y_2 is fixed as we consider the cases (3_1), (4_1) now). Thus, we get

$$M(y) = \left[y_1 + \left(\frac{2}{p} - 1\right)|y_2| \right] t_1 + y_3 t_3. \tag{20}$$

Since $dt_0 = (\frac{2}{p} - 1)|y_2|\, dt_1$, the equation of the extremal trajectories (17) takes the form

$$\left[y_1 + \left(\frac{2}{p} - 1\right)|y_2| \right] dt_1 + y_3\, dt_3 = 0, \tag{21}$$

and we can rewrite (20) as follows

$$M(y) = \left(t_3 - t_1 \frac{dt_3}{dt_1}\right) y_3.$$

We see that the expression $M(y)/y_3$ is constant along the trajectory and we can find it evaluating at the point V, where the boundary condition (5) is known:

$$M(y) = \left(\frac{v + y_2}{v - y_2}\right)^p y_3, \tag{22}$$

where $v = v(y_1, y_2, y_3)$ satisfies the following equation:

$$\frac{y_1 + (\frac{2}{p} - 1)|y_2|}{y_3} = \frac{v + (\frac{2}{p} - 1)|y_2|}{(v - y_2)^p}, \tag{23}$$

because the point $V = (v, y_2, (v - y_2)^p)$ is on the extremal line (21). We even shall not check under what conditions equation (23) has a solution and when it is unique. Later we show that in any case the function M we have found cannot be the Bellman function we are interested in, because neither condition (13) nor (14) can be fulfilled: the matrix $\{M_{y_i y_j}\}_{i,j=2,3}$ is neither negative definite nor positive definite. We postpone this verification, because the calculation of the sign of the Hessian matrices is the same for this solution and another solution of the Monge–Ampère equation that supplies us with the true Bellman function. And these calculations will be made simultaneously a bit later. And now we only rewrite our solution in an implicit form more convenient for calculation.

We introduce

$$\omega := \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}, \tag{24}$$

then (22) yields

$$v = \frac{\omega + 1}{\omega - 1} y_2. \tag{25}$$

Since $v \geq 0$ (in fact, recall that we consider now only y: $y_1 \geq |y_2|$ domain now, and that v is just the first coordinate of the point $V = (v, y_2, (v - y_2)^p)$ in this domain), we have

$$\operatorname{sign} y_2 = \operatorname{sign}(\omega - 1). \tag{26}$$

After substitution of (25) in (23), we get

$$\left(\frac{2y_2}{\omega - 1}\right)^p \left[y_1 + \left(\frac{2}{p} - 1\right)|y_2|\right] = y_3 \left[\frac{\omega + 1}{\omega - 1} y_2 + \left(\frac{2}{p} - 1\right)|y_2|\right]$$

or

$$2^p |y_2|^{p-1} \big[p y_1 + (2 - p)|y_2|\big] = y_3 |\omega - 1|^{p-1} \big[(\omega + 1)p + (2 - p)|\omega - 1|\big].$$

For the case (3_1) we have $y_2>0$ (i.e., $x_2>x_1$, we look for $\omega>1$ or $B>y_3$) and the latter equation can be rewritten in the initial coordinates as follows

$$(x_2-x_1)^{p-1}\big[x_2+(p-1)x_1\big]=\big(B^{\frac1p}-x_3^{\frac1p}\big)^{p-1}\big[B^{\frac1p}+(p-1)x_3^{\frac1p}\big].$$

For the case (4_1), we have $y_2<0$ (i.e., $x_2<x_1$, we look for $\omega<1$ or $B<y_3$) and the equation takes the form

$$(x_1-x_2)^{p-1}\big[x_1+(p-1)x_2\big]=\big(x_3^{\frac1p}-B^{\frac1p}\big)^{p-1}\big[x_3^{\frac1p}+(p-1)B^{\frac1p}\big].$$

Introduce the following function

$$G(z_1,z_2)=(z_1+z_2)^{p-1}\big[z_1-(p-1)z_2\big]$$

defined on the half-plane $z_1+z_2\ge 0$. Then in the case (3_1), we have the relation

$$G(x_2,-x_1)=G\big(B^{\frac1p},-x_3^{\frac1p}\big), \tag{27}$$

or

$$G(y_2+y_1,y_2-y_1)=y_3G(\omega,-1).$$

In the case (4_1), we have

$$G(x_1,-x_2)=G\big(x_3^{\frac1p},-B^{\frac1p}\big), \tag{28}$$

or

$$G(y_1-y_2,-y_1-y_2)=y_3G(1,-\omega).$$

Now we have to consider the Monge–Ampère equation (15) in the cases (3_2) and (4_2). This means that we fix y_1 now. Let us begin with the cases (3_2), when an extremal line starts at a point $U=(y_1,u,(y_1-u)^p)$ on our parabola and ends at a point $W=(y_1,y_1,w)$. Again, the symmetry condition at the point W is

$$\frac{\partial M}{\partial y_1}=\frac{\partial M}{\partial y_2}=t_2,$$

and the homogeneity condition (10) plus condition (16) at W yield

$$2y_1t_2+pwt_3=pM(W)=py_1t_2+pwt_3+pt_0,$$

whence

$$t_0=\left(\frac2p-1\right)y_1t_2,$$

and therefore

$$M(y)=\left[y_2+\left(\frac2p-1\right)y_1\right]t_2+y_3t_3. \tag{29}$$

Since $dt_0=(\frac2p-1)y_1\,dt_2$, the equation of the extremal trajectories takes the form

$$\left[y_2+\left(\frac2p-1\right)y_1\right]dt_2+y_3\,dt_3=0, \tag{30}$$

and we can rewrite (29) as follows

$$M(y)=\left(t_3-t_2\frac{dt_3}{dt_2}\right)y_3.$$

Again, from here the expression $M(y)/y_3$ is constant along the trajectory and we can find it evaluating at the point U, where we the boundary condition (5) is known:

$$M(y)=\left(\frac{y_1+u}{y_1-u}\right)^p y_3, \tag{31}$$

where $u=u(y_1,y_2,y_3)$ can be found from (30):

$$\frac{y_2+(\frac{2}{p}-1)y_1}{y_3}=\frac{u+(\frac{2}{p}-1)y_1}{(y_1-u)^p}. \tag{32}$$

We see that if our extremal line starts at point $U=(y_1,u,(y_1-u)^p)$ on our parabola $u=-(\frac{2}{p}-1)y_1$, then $y_2=-(\frac{2}{p}-1)y_1=u=\text{const}$, that is, it is a line parallel to the x_3-axes. This means that no extremal line that ends at the points of the boundary $y_1=y_2$ can intersect the plane $y_2=-(\frac{2}{p}-1)y_1$. This follows from the property that extremal trajectories do not intersect. Therefore, the starting points U with $u\le-(\frac{2}{p}-1)y_1$ cannot be acceptable for the case under consideration (since these trajectories do not intersect the plane $y_2=-(\frac{2}{p}-1)y_1$, they cannot have the second end point on $y_2=y_1$, see Figure 1).

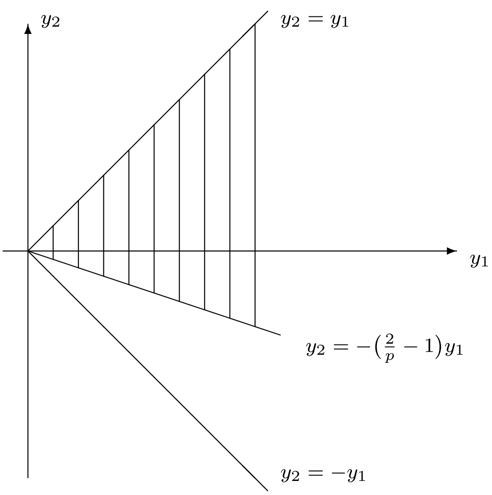

FIGURE 1. Acceptable sector for the case (3_2).

Let us check that equation (32) has exactly one solution $u = u(y_1, y_2, y_3)$ in the sector $-(\frac{2}{p} - 1)y_1 < y_2 < y_1$. Indeed, the function

$$u \mapsto y_3 \left[u + \left(\frac{2}{p} - 1 \right) y_1 \right] - (y_1 - u)^p \left[y_2 + \left(\frac{2}{p} - 1 \right) y_1 \right]$$

is monotonously increasing for $u < y_1$ and it has the negative value $-(\frac{2}{p}y_1)^p \times [y_2 + (\frac{2}{p} - 1)y_1]$ at the point $u = -(\frac{2}{p} - 1)y_1$ and the positive value $\frac{2}{p}y_1 y_3$ at the point $u = y_1$.

Now we rewrite the solution (31) in an implicit form using notations (24): $\omega := (\frac{M(y)}{y_3})^{\frac{1}{p}}$. From (31), we have

$$u = \frac{\omega - 1}{\omega + 1} y_1, \tag{33}$$

therefore, from (32) we obtain

$$2^{-p} y_3 (\omega + 1)^{p-1} \big[p(\omega - 1) + (2 - p)(\omega + 1) \big] = y_1^{p-1} \big[p y_2 + (2 - p) y_1 \big]$$

or

$$2^{-p+1} y_3 (\omega + 1)^{p-1} (\omega - p + 1) = y_1^{p-1} [p y_2 + (2 - p) y_1],$$

which is (using again notations (24): $\omega := (\frac{M(y)}{y_3})^{\frac{1}{p}}$)

$$\big(B^{\frac{1}{p}} + x_3^{\frac{1}{p}} \big)^{p-1} \big[B^{\frac{1}{p}} - (p - 1) x_3^{\frac{1}{p}} \big] = (x_1 + x_2)^{p-1} \big[x_2 - (p - 1) x_1 \big].$$

In terms of function G this can be rewritten as follows

$$G(x_2, x_1) = G\big(B^{\frac{1}{p}}, x_3^{\frac{1}{p}} \big),$$

or

$$G(y_1 + y_2, y_1 - y_2) = y_3 G(\omega, 1).$$

It remains to examine the possibility (4_2). Assume that an extremal line starts at a point $U = (y_1, u, (y_1 - u)^p)$ and ends at a point $W = (y_1, -y_1, w)$. Again, the homogeneity property (10) at the point W and the symmetry $\frac{\partial M}{\partial y_1} = -\frac{\partial M}{\partial y_2} = -t_2$ yield

$$-2 y_1 t_2 + p w t_3 = p M(W) = -p y_1 t_2 + p w t_3 + p t_0,$$

whence

$$t_0 = \left(1 - \frac{2}{p} \right) y_1 t_2,$$

and therefore

$$M(y) = \left[y_2 + \left(1 - \frac{2}{p} \right) y_1 \right] t_2 + y_3 t_3. \tag{34}$$

Since $dt_0 = (1 - \frac{2}{p}) y_1 \, dt_2$, the equation of the extremal trajectories takes the form

$$\left[y_2 + \left(1 - \frac{2}{p} \right) y_1 \right] dt_2 + y_3 \, dt_3 = 0, \tag{35}$$

and we can rewrite (34) as follows

$$M(y)=\left(t_3-t_2\frac{dt_3}{dt_2}\right)y_3.$$

Again, the expression $M(y)/y_3$ is constant along the trajectory and from the boundary condition (5) we get the same expression

$$M(y)=\left(\frac{y_1+u}{y_1-u}\right)^p y_3. \tag{36}$$

Now $u=u(y_1,y_2,y_3)$ is a solution of the equation

$$\frac{y_2-(\frac{2}{p}-1)y_1}{y_3}=\frac{u-(\frac{2}{p}-1)y_1}{(y_1-u)^p} \tag{37}$$

that we get from (35). As before, we get trajectories ending at the plane $y_2=-y_1$ not in the whole domain Ξ_+, but only in the sector $-y_1<y_2<(\frac{2}{p}-1)y_1$ (see Figure 2), and equation (37) has a unique solution for every point from this sector. As before, relation (33) allows us to rewrite the equation of extremal trajectories (37) as an implicit expression for ω (and hence for M):

$$G(x_1,x_2)=G\big(x_3^{\frac{1}{p}},B^{\frac{1}{p}}\big),$$

or

$$G(y_1-y_2,y_1+y_2)=y_3G(1,\omega).$$

Now we start the verification which of the obtained solutions satisfies conditions (13) or (14). We need to calculate $D_i:=M_{y_iy_i}M_{y_3y_3}-M^2_{y_iy_3}$, $i=1,2,$

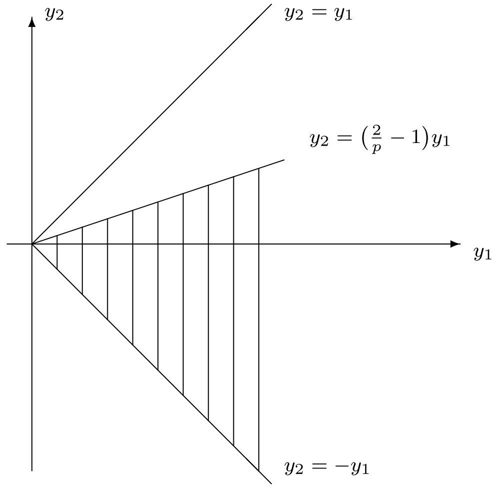

FIGURE 2. Acceptable sector for the case (4_2).

in four cases

$$(3_1)\quad G(y_1+y_2,-y_1+y_2)=y_3G(\omega,-1);$$

$$(4_1)\quad G(y_1-y_2,-y_1-y_2)=y_3G(1,-\omega);$$

$$(3_2)\quad G(y_1+y_2,y_1-y_2)=y_3G(\omega,1); \tag{38}$$

$$(4_2)\quad G(y_1-y_2,y_1+y_2)=y_3G(1,\omega), \tag{39}$$

where $M=y_3\omega^p$. In all situations, we have a relation of the form

$$\Phi(\omega)=\frac{H(y_1,y_2)}{y_3}.$$

Till some moment in the future, we will not specify the expression for Φ and H, as well as for their derivatives, and plug in the specific expression only in the final result after numerous cancellation. In particular, we introduce

$$R_1=R_1(\omega):=\frac{1}{\Phi'}\quad\text{ and }\quad R_2=R_2(\omega):=R_1'=-\frac{\Phi''}{\Phi'^2}.$$

We would like to mention here that this idea, allowing us to make calculation shorter, is taken from the original paper of Burkholder [Bu1].

First of all, we calculate the partial derivatives of ω:

$$\begin{aligned}\Phi'\omega_{y_3}&=-\frac{H}{y_3^2}\quad\Longrightarrow\quad \omega_{y_3}=-\frac{R_1H}{y_3^2},\\ \Phi'\omega_{y_i}&=\frac{H_{y_i}}{y_3}\quad\Longrightarrow\quad \omega_{y_i}=\frac{R_1H_{y_i}}{y_3}=\frac{R_1H'}{y_3},\quad i=1,2.\end{aligned}$$

Here and further we shall use notation H' for any partial derivative H_{y_i}, $i=1,2$. This cannot cause misunderstanding because only one i participate in calculation of Hessian determinants D_i. Moreover, we shall not mention anymore that the index i can take two values either $i=1$ or $i=2$.

$$\begin{aligned}\omega_{y_3y_3}&=-\frac{R_2\omega_{y_3}H}{y_3^2}+2\frac{R_1H}{y_3^3}=\frac{R_1H}{y_3^4}(R_2H+2y_3),\\ \omega_{y_3y_i}&=-\frac{R_2\omega_{y_i}H}{y_3^2}-\frac{R_1H'}{y_3^2}=-\frac{R_1H'}{y_3^3}(R_2H+y_3),\\ \omega_{y_iy_i}&=\frac{R_2\omega_{y_i}H'}{y_3}+\frac{R_1H''}{y_3}=\frac{R_1}{y_3^2}\big(R_2\big(H'\big)^2+y_3H''\big).\end{aligned}$$

Now we pass to the calculation of derivatives of $M=y_3\omega^p$:

$$\begin{aligned}M_{y_3}&=py_3\omega^{p-1}\omega_{y_3}+\omega^p,\\ M_{y_i}&=py_3\omega^{p-1}\omega_{y_i};\\ M_{y_3y_3}&=py_3\omega^{p-1}\omega_{y_3y_3}+2p\omega^{p-1}\omega_{y_3}+p(p-1)y_3\omega^{p-2}\omega_{y_3}^2\end{aligned}$$

$$=\frac{p\omega^{p-2}R_1H^2}{y_3^3}\big[\omega R_2+(p-1)R_1\big], \tag{40}$$

$$\begin{aligned}
M_{y_3y_i} &= py_3\omega^{p-1}\omega_{y_3y_i} + p\omega^{p-1}\omega_{y_i} + p(p-1)y_3\omega^{p-2}\omega_{y_3}\omega_{y_i}\\
&= -\frac{p\omega^{p-2}R_1HH'}{y_3^2}\big[\omega R_2+(p-1)R_1\big],\\
M_{y_iy_i} &= py_3\omega^{p-1}\omega_{y_iy_i} + p(p-1)y_3\omega^{p-2}\omega_{y_i}^2\\
&= \frac{p\omega^{p-2}R_1}{y_3}\big(\big[\omega R_2+(p-1)R_1\big]\big(H'\big)^2+\omega y_3H''\big).
\end{aligned}$$

This yields

$$D_i = M_{y_3y_3}M_{y_iy_i} - M_{y_3y_i}^2 = \frac{p^2\omega^{2p-3}R_1^2H^2H''}{y_3^3}\big[\omega R_2+(p-1)R_1\big]. \tag{41}$$

Notice, that H' disappeared completely.

Now we need to calculate second derivatives of

$$H(y_1,y_2) = G(\alpha_1y_1+\alpha_2y_2, \beta_1y_1+\beta_2y_2),$$

where $\alpha_i,\beta_i=\pm1$. And

$$\begin{aligned}
H'' &= \frac{\partial^2}{\partial y_i^2}G(\alpha_1y_1+\alpha_2y_2,\beta_1y_1+\beta_2y_2)\\
&= \alpha_i^2G_{z_1z_1} + 2\alpha_i\beta_iG_{z_1z_2} + \beta_i^2G_{z_2z_2}\\
&= G_{z_1z_1} + G_{z_2z_2} \pm 2G_{z_1z_2},
\end{aligned}$$

where the "+" sign has to be taken if the coefficients in front of y_i are equal and the "−" sign in the opposite case.

The derivatives of G are simple:

$$\begin{aligned}
G_{z_1} &= p(z_1+z_2)^{p-2}\big[z_1-(p-2)z_2\big],\\
G_{z_2} &= -p(p-1)z_2(z_1+z_2)^{p-2};\\
G_{z_1z_2} &= p(p-1)(z_1+z_2)^{p-3}\big[z_1-(p-3)z_2\big],\\
G_{z_1z_2} &= -p(p-1)(p-2)z_2(z_1+z_2)^{p-3},\\
G_{z_2z_2} &= -p(p-1)(z_1+z_2)^{p-3}\big[z_1+(p-1)z_2\big].
\end{aligned}$$

Note that $G_{z_1z_1}+G_{z_2z_2}=2G_{z_1z_2}$, and therefore, $H''=4G_{z_1z_2}$ if $\alpha_i=\beta_i$ and $H''=0$ if $\alpha_i=-\beta_i$. The first case occurs for $H_{y_2y_2}$ in cases (3_1), (4_1) and for $H_{y_1y_1}$ in cases (3_2), (4_2). The second case occurs for $H_{y_1y_1}$ in cases (3_1), (4_1) and for $H_{y_2y_2}$ in cases (3_2), (4_2). In fact, we know that the equality $D_i=0$ has to be fulfilled in the cases (3_i) and (4_i), because it is just the Monge–Ampère equation we have been solving.

So we have

$$\begin{aligned}
&(3_1)\qquad z_1 = y_1+y_2,\\
&\qquad\qquad z_2 = -y_1+y_2,\quad G_{z_1z_2} = p(p-1)(p-2)(y_1-y_2)(2y_2)^{p-3},\\
&(4_1)\qquad z_1 = y_1-y_2,
\end{aligned}$$

$$
\begin{aligned}
& z_2 = -y_1 - y_2, \quad G_{z_1 z_2} = p(p-1)(p-2)(y_1+y_2)(-2y_2)^{p-3},\\
(3_2)\quad & z_1 = y_1 + y_2,\\
& z_2 = y_1 - y_2, \quad G_{z_1 z_2} = -p(p-1)(p-2)(y_1-y_2)(2y_1)^{p-3},\\
(4_2)\quad & z_1 = y_1 - y_2,\\
& z_2 = y_1 + y_2, \quad G_{z_1 z_2} = -p(p-1)(p-2)(y_1+y_2)(2y_1)^{p-3}.
\end{aligned}
$$

In the first pair of cases, we have $\operatorname{sign} G_{z_1z_2} = \operatorname{sign} H'' = \operatorname{sign}(p-2)$ and the opposite sign in the second pair of cases. In the first pair of cases, we have that this sign is the sign of the Hessian determinant D_2 up to the $\operatorname{sign}[wR_2(p-1)R_1]$ (and $D_1 = 0$ identically); in the second pair of cases we have this sign is the sign of the Hessian determinant D_1 up to the $\operatorname{sign}[wR_2(p-1)R_1]$ (and $D_2 = 0$ identically).

By the way, we call the attention of the reader to the fact, that, for example, in (4_1) above we have necessarily $y_2 < 0$ (here y_2 is fixed and our extremal trajectories in the plane (y_1, y_3) here hit $y_1 = -y_2 = |y_2|$ as we are always under restrictions $-y_1 \le y_2 \le y_1$, that is $y_1 \ge |y_2|$), so $(-2y_2)^{p-3}$ makes a perfect sense. The same type of observation holds for all other cases.

To complete the investigation of $\operatorname{sign} D_i$, we need to calculate the sign of the expression in the brackets in (41):

$$
\begin{aligned}
& \omega R_2 + (p-1)R_1 = R_1^2\big[(p-1)\Phi' - \omega\Phi''\big];\\
(3_1)\quad & \Phi(\omega) = G(\omega,-1),\\
& \Phi'(\omega) = G_{z_1}(\omega,-1) = p(\omega-1)^{p-2}(\omega+p-2),\\
& \Phi''(\omega) = G_{z_1z_1}(\omega,-1) = p(p-1)(\omega-1)^{p-3}(\omega+p-3),\\
& (p-1)\Phi' - \omega\Phi'' = -p(p-1)(p-2)(\omega-1)^{p-3};\\
(4_1)\quad & \Phi(\omega) = G(1,-\omega),\\
& \Phi'(\omega) = -G_{z_2}(1,-\omega) = -p(p-1)\omega(1-\omega-1)^{p-2},\\
& \Phi''(\omega) = G_{z_2z_2}(1,-\omega) = p(p-1)(1-\omega)^{p-3}\big[1-(p-1)\omega\big],\\
& (p-1)\Phi' - \omega\Phi'' = -p(p-1)(p-2)\omega(1-\omega)^{p-3}; \qquad (42)\\
(3_2)\quad & \Phi(\omega) = G(\omega,1),\\
& \Phi'(\omega) = G_{z_1}(\omega,1) = p(\omega+1)^{p-2}(\omega-p+2),\\
& \Phi''(\omega) = G_{z_1z_1}(\omega,1) = p(p-1)(\omega+1)^{p-3}(\omega-p+3),\\
& \Phi' - \omega\Phi'' = -p(p-1)(p-2)(\omega+1)^{p-3};\\
(4_2)\quad & \Phi(\omega) = G(1,\omega),\\
& \Phi'(\omega) = G_{z_1}(1,\omega) = -p(p-1)\omega(\omega+1)^{p-2},\\
& \Phi''(\omega) = G_{z_1z_1}(\omega,-1) = -p(p-1)(\omega-1)^{p-3}\big[1+(p-1)\omega\big],\\
& (p-1)\Phi' - \omega\Phi'' = -p(p-1)(p-2)\omega(\omega+1)^{p-3}.
\end{aligned}
$$

By the way, we call the attention of the reader to the fact, that, for example, in (4_1) above we have necessarily $y_2<0$, so by (26) $\omega<1$, so $(1-\omega)^{p-3}$ is fine there. The same type of observation works for other cases above. We see that in all cases $\operatorname{sign}[(p-1)\Phi'-\omega\Phi'']=-\operatorname{sign}(p-2)$. Therefore in the first two cases, we have $D_2<0$ and this solution satisfies neither requirement (13) nor requirement (14). In the second two cases we have $D_1>0$, and the function M can be a candidate either for $\mathbf{M}_{\max}$ or for $\mathbf{M}_{\min}$ depending on the sign of the second derivative $M_{y_3y_3}$.

Recall that (see (40))

$$M_{y_3y_3}=\frac{p\omega^{p-2}R_1H^2}{y_3^3}\big[\omega R_2+(p-1)R_1\big].$$

In the case (3_2), we have $\operatorname{sign}[\omega R_2+(p-1)R_1]=-\operatorname{sign}(p-2)$, and therefore we need only to know $\operatorname{sign}R_1=\operatorname{sign}\Phi'=\operatorname{sign}\frac{d}{d\omega}G(\omega,1)$. Since this solution is considered only in the sector $\frac{p-2}{p}y_1<y_2<y_1$ (see Figure 1), we have

$$G(y_1+y_2,y_1-y_2)=(2y_1)^{p-1}\big[py_2-(p-2)y_1\big]>0, \tag{43}$$

and ω, being the unique positive solution of the equation

$$G(\omega,1)=(\omega+1)^{p-1}[\omega-p+1]=\frac{1}{y_3}G(y_1+y_2,y_1-y_2), \tag{44}$$

satisfies the condition $\omega>p-1$. Therefore, $\operatorname{sign}R_1=\operatorname{sign}\frac{d}{d\omega}G(\omega,1)=\operatorname{sign}p(\omega+1)^{p-2}(\omega-p+2)>0$, and so $\operatorname{sign}M_{y_3y_3}=-\operatorname{sign}(p-2)$, i.e., for $p>2$ this is candidate for $\mathbf{M}_{\max}$ and for $p<2$ this is candidate for $\mathbf{M}_{\min}$.

We are still considering the case (3_2). Recall that this function is defined not in the whole domain Ξ_+, but only in the sector $\frac{p-2}{p}y_1<y_2<y_1$. To get a solution everywhere, we need to "glue" this solution with that we obtained considering the case (2) (see (18)):

$$M(y)=(y_1+y_2)^p+C\big(y_3-(y_1-y_2)^p\big). \tag{45}$$

To glue this solution along the plane $y_2=\frac{p-2}{p}y_1$ with that we just obtained, let us require from the resulting function to be continuous everywhere. From (44) and (43) we see that $G(\omega,1)=0$ on this plane. Therefore, $\omega=p-1$ and $M=\omega^py_3=(p-1)^py_3$. The same value has solution (45) on this plane for $C=(p-1)^p$.

Now we need to check that we get correct continuation in the sense that if the solution satisfies (13), then its continuation satisfies the same condition as well, if the solution satisfies (14), then the same is true for its continuation. The Hessian determinants will have the right sign automatically (actually $D_2=0$ identically). We need only to check the sign of

$$M_{y_1y_1}=M_{y_2y_2}=p(p-1)\big((y_1+y_2)^{p-2}-(p-1)^p(y_1-y_2)^{p-2}\big)$$

in the domain $-y_1 < y_2 < \frac{p-2}{p} y_1$, or in the initial coordinates $0 < x_2 < (p-1)x_1$.

For $p > 2$, we have

$$\begin{aligned}(y_1+y_2)^{p-2} &= x_2^{p-2} < (p-1)^{p-2} x_1^{p-2} \\ &< (p-1)^p x_1^{p-2} = (p-1)^p (y_1-y_2)^{p-2},\end{aligned}$$

and for $p < 2$ we have

$$\begin{aligned}(y_1+y_2)^{p-2} &= x_2^{p-2} > (p-1)^{p-2} x_1^{p-2} \\ &> (p-1)^p x_1^{p-2} = (p-1)^p (y_1-y_2)^{p-2}.\end{aligned}$$

This means that M is a candidate for $\mathbf{M}_{\max}$ if $p > 2$ and a candidate for $\mathbf{M}_{\min}$ if $p < 2$, as it has to be.

Let us rewrite expression (45) in the same form, as it was made in (44).

$$M - Cy_3 = (y_1+y_2)^p - C(y_1-y_2)^p = x_2^p - Cx_1^p. \tag{46}$$

Therefore, if we change a bit the definition of G defining it on the quadrant $z_i \geq 0$ as follows

$$G_p(z_1,z_2) = \begin{cases} z_1^p - (p-1)^p z_2^p, & \text{if } z_1 \leq (p-1)z_2, \\ (z_1+z_2)^{p-1}\big[z_1 - (p-1)z_2\big], & \text{if } z_1 \geq (p-1)z_2, \end{cases} \tag{47}$$

then we can write two our solutions M on Ξ_+ in an implicit form as before:

$$G(y_1+y_2, y_1-y_2) = y_3 G(\omega, 1)$$

or solutions B on Ω_+

$$G(x_2,x_1) = G\big(B^{\frac{1}{p}}, x_3^{\frac{1}{p}}\big). \tag{48}$$

In the case (4_2), we again consider exactly the same G_p from (47). In a similar way, we can glue continuously the solution in case (4_2) found in the sector $-y_1 < y_2 < \frac{2-p}{p} y_1$

$$G(1,\omega) = (\omega+1)^{p-1}\big[1-(p-1)\omega\big] = \frac{1}{y_3} G(y_1-y_2, y_1+y_2), \tag{49}$$

which is the same as

$$G(x_1,x_2) = G\big(x_3^{\frac{1}{p}}, B^{\frac{1}{p}}\big), \tag{50}$$

with the solution (45) along the line $y_2 = \frac{2-p}{p} y_1$. Here, we have to take $C = (p'-1)^p$, because on the line $y_2 = \frac{2-p}{p} y_1$ we have $G(1,\omega) = 0$, that is, $\omega = p'-1$. Now, in the sector $-y_1 < y_2 < \frac{2-p}{p} y_1$ we have

$$M_{y_3 y_3} = \frac{p\omega^{p-2} R_1^2 H^2}{y_3^3} \cdot \frac{p-2}{\omega+1}.$$

Therefore, $\operatorname{sign} M_{y_3 y_3} = \operatorname{sign}(p-2)$, that is, for $p < 2$ this is candidate for $\mathbf{M}_{\max}$ and for $p > 2$ this is candidate for $\mathbf{M}_{\min}$.

In the "dual" sector $x_2 > (p'-1)x_1$ (or $y_2 > \frac{2-p}{p}y_1$) for $p > 2$ we have

$$\begin{aligned}(y_1+y_2)^{p-2} &= x_2^{p-2} > \big(p'-1\big)^{p-2}x_1^{p-2}\\ &> \big(p'-1\big)^{p}x_1^{p-2} = \big(p'-1\big)^{p}(y_1-y_2)^{p-2},\end{aligned}$$

and for $p < 2$ we have

$$\begin{aligned}(y_1+y_2)^{p-2} &= x_2^{p-2} < \big(p'-1\big)^{p-2}x_1^{p-2}\\ &< \big(p'-1\big)^{p}x_1^{p-2} = \big(p'-1\big)^{p}(y_1-y_2)^{p-2}.\end{aligned}$$

This means that M is a candidate for $\mathbf{M}_{\max}$ if $p < 2$ and a candidate for $\mathbf{M}_{\min}$ if $p > 2$.

Using the same "generalized" definition (47) of the function G, we can write our solutions M on Ξ_+ in an implicit form as before:

$$G(y_1-y_2, y_1+y_2) = y_3 G(1,\omega)$$

or solutions B on Ω_+

$$G(x_1,x_2) = G\big(x_3^{\frac{1}{p}}, B^{\frac{1}{p}}\big), \tag{51}$$

which should give, as we said above, the candidate for $\mathbf{B}_{\max}$ for $p < 2$ and $\mathbf{B}_{\min}$ for $p > 2$. Notice that for $p > 2$ the candidate for, say, $\mathbf{B}_{\max}$ is given by equation (48).

It is a bit inconvenient to use one equation for, say, $B_{\max}$ if $p > 2$ (this will be (48)), and another one (this will be (51)) for the same $\mathbf{B}_{\max}$ if $p < 2$. We note that after interchanging role of z_i and replacing p by p' we get the scalar multiple of the original expression in both lines of (47). This allows us to give one expression for $B_{\max}$ for all p using notation of $p^* = \max\{p, p'\}$. In such a way we come to formula (4) for F_p, where we introduce additional scalar coefficients to make this function not only continuous but C^1-smooth everywhere in Ω_+. This smoothness guarantee us that the solution B is C^1-smooth as well.

5. Proof of Theorem 1. Verification theorem

Exactly in the spirit of Stochastic Optimal Control theory, we wrote the PDE (15), we solved it in the previous section by building B which solves the equations of Theorem 1 (these are the same equations as (48), (50)). Now continuing in the spirit of general results of Stochastic Optimal Control theory [FR], [WF] we need to prove that these solutions in fact are equal to $\mathbf{B}_{\max}$, $\mathbf{B}_{\min}$. In Stochastic Optimal Control theory such proofs are called *verification theorems*, and they state roughly that if the solutions have a certain smoothness (often even slightly less than C^2), and if the domain is convex, then we are fine.

From now on, we denote by $B_{\max}$ the unique positive solution of the equation $F(|x_2|,|x_1|) = F(B^{\frac{1}{p}}, x_3^{\frac{1}{p}})$ and by $B_{\min}$ the unique positive solution of the equation $F(|x_1|,|x_2|)=F(x_3^{\frac{1}{p}},B^{\frac{1}{p}})$, where the function $F=F_p$ is defined in (4). Existence and uniqueness of the solution follows from the fact that $F(z_1,z_2)$ is strictly increasing in z_1 from $-p^{*(p-1)}(p^*-1)^p z_2^p$ till $+\infty$ as z_1 runs from 0 to $+\infty$ and it is strictly decreasing in z_2 from $p(p^*-1)^{p-1}z_2^p$ till $-\infty$ as z_2 runs from 0 to $+\infty$. Indeed, the first partial derivatives of F are

$$F_{z_1}=\begin{cases} pz_1^{p-1}, \quad \text{if } z_1\le \big(p^*-1\big)z_2, \\ p\big(1-\frac{1}{p^*}\big)^{p-1}(z_1+z_2)^{p-2}\big[pz_1-\big((p-1)\big(p^*-1\big)-1\big)z_2\big], \\ \qquad\qquad \text{if } z_1\ge \big(p^*-1\big)z_2; \end{cases} \tag{52}$$

$$F_{z_2}=\begin{cases} -\big(p^*-1\big)^p pz_2^{p-1}, \quad \text{if } z_1\le \big(p^*-1\big)z_2, \\ -p\big(1-\frac{1}{p^*}\big)^{p-1}(z_1+z_2)^{p-2}\big[\big(p^*-p\big)z_1+p\big(p^*-1\big)z_2\big], \\ \qquad\qquad\qquad \text{if } z_1\ge \big(p^*-1\big)z_2. \end{cases} \tag{53}$$

Note that both derivatives are continuous everywhere (even at the origin, where they vanish). Moreover, $F_{z_1}>0$ if $z_1>0$ and $F_{z_2}<0$ if $z_2>0$, that is, F is strictly increasing in z_1 and strictly decreasing in z_2.

In the case of $B_{\max}$, we look for a solution of the equation

$$F\big(B^{\frac{1}{p}},x_3^{\frac{1}{p}}\big)=F\big(|x_2|,|x_1|\big)$$

or

$$F(\omega,1)=\frac{1}{x_3}F\big(|x_2|,|x_1|\big).$$

Thus, we get a continuous solution $\omega(x)$ everywhere except the plane $x_3=0$, where ω is not defined. But we can easily estimate the behavior of ω nearly the line $x_3=x_1=0$. Since F is decreasing in z_2 and $0\le|x_1|\le x_3^{\frac{1}{p}}$, we have

$$F\bigg(\frac{|x_2|}{x_3^{1/p}},1\bigg)\le F(\omega,1)=F\bigg(\frac{|x_2|}{x_3^{1/p}},\frac{|x_1|}{x_3^{1/p}}\bigg)\le F\bigg(\frac{|x_2|}{x_3^{1/p}},0\bigg).$$

Since F is increasing in z_1, we get

$$\frac{|x_2|}{x_3^{1/p}}\le\omega\le\omega_0,$$

where ω_0 is the solution of the equation

$$(\omega_0+1)^{p-1}\big(\omega_0-p^*+1\big)=\frac{|x_2|^p}{x_3}.$$

Whence $\omega_0\ge p^*-1$ and

$$\big(\omega_0-p^*+1\big)^p\le(\omega_0+1)^{p-1}\big(\omega_0-p^*+1\big)=\frac{|x_2|^p}{x_3},$$

that is,

$$\omega_0\le p^*-1+\frac{|x_2|}{x_3^{1/p}}.$$

Therefore, for $B = \omega^p x_3$ we have the following estimate

$$|x_2|^p \le B \le \big(|x_2| + \big(p^* - 1\big) x_3^{1/p}\big)^p,$$

which gives the continuity near $x_3 = 0$. Thus, the solution $B_{\max}$ is continuous in the closed domain Ω.

Similar considerations gives us the continuity of $B_{\min}$. In that case, we have the equation

$$F(1,\omega) = \frac{1}{x_3} F\big(|x_1|, |x_2|\big),$$

and hence

$$F\left(0, \frac{|x_2|}{x_3^{1/p}}\right) \le F(1,\omega) = F\left(\frac{|x_1|}{x_3^{1/p}}, \frac{|x_2|}{x_3^{1/p}}\right) \le F\left(1, \frac{|x_2|}{x_3^{1/p}}\right).$$

Now $F(1,\omega)$ is decreasing in ω, therefore,

$$\frac{|x_2|}{x_3^{1/p}} \le \omega \le \omega_0,$$

where ω_0 is the solution of the equation

$$1 - \big(p^* - 1\big)^p \omega_0^p = -\big(p^* - 1\big)^p \frac{|x_2|^p}{x_3},$$

that is, $\omega_0^p = (p^* - 1)^{-p} + |x_2|^p / x_3$ and for $B = \omega^p x_3$ we have the following estimate

$$|x_2|^p \le B \le |x_2|^p + \big(p^* - 1\big)^{-p} x_3,$$

which gives the continuity near $x_3 = 0$. Thus, the solution $B_{\min}$ is continuous in the closed domain Ω as well.

First step of the proof is to check that the the main inequality (concavity (11) for the candidate $B_{\max}$ and convexity (12) for the candidate $B_{\min}$) is fulfilled if the points x^+, x^- satisfy the extra condition on their coordinates:

$$|x_1^+ - x_1^-| = |x_2^+ - x_2^-|. \tag{54}$$

This was almost done in the preceding section, when constructing these candidates. We know that the Hessians of our candidates have the required signs everywhere in our convex domain Ω except, possibly, the planes $x_1 = 0$, $x_2 = 0$, and, either $|x_2| = (p^* - 1)|x_1|$ for $B_{\max}$ or $|x_1| = (p^* - 1)|x_2|$ for $B_{\min}$. On these hyperplanes our solutions are not C^2-smooth, but this does not prevent them from being correctly convex (for the (3_2), $p > 2$ and (4_2), $p < 2$ cases) and correctly concave for the rest of the cases (namely, for the (3_2), $p < 2$ and (4_2), $p > 2$ cases). This one checks just by calculating directly the sign of the jump of the derivative. Namely, one fixes the line $L_t = a + bt$ in the direction of the vector $b = (b_1, b_2, b_3)$ such that $|b_1| = |b_2|$. We need to prove the concavity of B, the candidate for $\mathbf{B}_{\max}$, and the convexity of B, the candidate for $\mathbf{B}_{\min}$ on L_t. At any point of L_t, which is *not* the intersection of L_t with the aforementioned hyperplanes, this concavity (convexity) follows from the previous section, this is how the candidates for $\mathbf{B}_{\max}$, $\mathbf{B}_{\min}$ were built in (48), (50). At the points of intersections of L_t with the hyperplanes, one can check the sign of the jump of the derivative of $B(a+tb)$. We leave this as an exercise for the reader.

Let the triple of points x, x^+, x^- satisfies the following relations

$$\begin{gathered} |x_1^+ - x_1^-| = |x_2^+ - x_2^-|, \qquad x_3^+ \geq |x_1^+|^p, \qquad x_3^- \geq |x_1^-|^p, \\ x = \alpha^- x^- + \alpha^+ x^+, \qquad \alpha^- + \alpha^+ = 1, \qquad \alpha^\pm > 0. \end{gathered} \tag{55}$$

Now we have our solution $B_{\max}$ the following main inequality (biconcavity)

$$B(x) - \alpha^- B\big(x^-\big) - \alpha^+ B\big(x^+\big) \geq 0, \tag{56}$$

and the opposite main inequality (biconvexity)

$$B(x) - \alpha^- B\big(x^-\big) - \alpha^+ B\big(x^+\big) \leq 0 \tag{57}$$

is true for the solution $B_{\min}$.

Lemma 2. *If a continuous in Ω function B satisfies the main inequality* (56) *and the boundary restriction $B(x_1, x_2, |x_1|^p) \geq |x_2|^p$, then $B \geq \mathbf{B}_{\max}$. If it satisfies* (57) *and $B(x_1, x_2, |x_1|^p) \leq |x_2|^p$, then $B \leq \mathbf{B}_{\min}$.*

Proof. Let $I = [0,1]$ and J denote an arbitrary its dyadic subinterval. As always J_+, J_- are two sons of J. Let us fix two bounded measurable test functions f, g on I such that $|(g, h_J)| = |(f, h_J)|$ for any J. Put

$$x_J = \big(\langle f\rangle_J, \langle g\rangle_J, \big\langle |f|^p\big\rangle_J\big).$$

The fact that $|(g, h_J)| = |(f, h_J)|$ exactly guarantees that x^+, x^- satisfy the assumptions of (55) and we can rewrite inequalities (56) and (57) with $x = x_J$ in the form

$$|J|B(x) - \alpha^-|J_-|B\big(x^-\big) - |J_+|\alpha^+ B\big(x^+\big) \geq 0, \tag{58}$$

$$|J|B(x) - \alpha^-|J_-|B\big(x^-\big) - |J_+|\alpha^+ B\big(x^+\big) \leq 0. \tag{59}$$

Let $\mathcal{J}_n$ denotes the set of dyadic subintervals of nth generation, that is, $\mathcal{J}_0 = \{I\}$, and $\mathcal{J}_n$ is the set of suns of elements from $\mathcal{J}_{n-1}$. So, adding up all our inequalities (56) with $x = x_J$ for $J \in \mathcal{J}_{n-1}$ we get

$$\sum_{J \in \mathcal{J}_{n-1}} |J| B(x_J) \geq \sum_{J \in \mathcal{J}_n} |J| B(x_J).$$

Adding up these inequality over n from 1 to N we get

$$B(x) \geq \sum_{J \in \mathcal{J}_N} |J| B(x_J) = \int_0^1 B\big(x^N(t)\big)\, dt,$$

where $x^N(t)$ is the step function equal to x_J if $t \in J$ for every J from $\mathcal{J}_N$.

Notice that $\langle\varphi\rangle_J \to \varphi(t)$ almost everywhere when runs over a family of nested intervals shrinking to the point t for an arbitrary summable function φ. Therefore, almost everywhere

$$x^N(t) = \big(\langle f\rangle_J, \langle g\rangle_J, \big\langle |f|^p\big\rangle_J\big) \to \big(f(t), g(t), |f(t)|^p\big) \quad \text{as } N \to \infty$$

and since B is continuous, we have

$$B\big(x^N(t)\big) \to B\big(f(t), g(t), |f(t)|^p\big) \ge |g(t)|^p.$$

Now using Lebesgue dominant convergence theorem, we come to the estimate

$$\big\langle |g|^p\big\rangle_I \le B(x)$$

for every pair of bounded measurable functions f, g. And finally approximating arbitrary $f, g \in L^p(I)$ by its cut-off functions and using monotone convergence theorem we can extend this inequality to the set of arbitrary possible test functions f and g what means exactly that $\mathbf{B}_{\max}(x) \le B(x)$.

For the case of $\mathbf{B}_{\min}$ in all these considerations, we need to change the sign of inequalities only, and we will get $\mathbf{B}_{\min}(x) \ge B(x)$ for B satisfying (57). $\square$

We are left to prove the opposite inequalities

$$\mathbf{B}_{\max} \ge B_{\max}(x) \quad \text{and} \quad \mathbf{B}_{\min} \le B_{\min}(x).$$

This can be done by reversing the reasoning in the lemma above. Using the fact that domain $\Omega = \{x = (x_1, x_2, x_3) : x_3 \ge |x_1|^p\}$ is foliated by the straight line segments (extremal trajectories) it is possible to construct the sequence of test functions f_n, g_n corresponding any given point $x \in \Omega$ and such that $\langle |g_n|^p\rangle_I \to B(x)$. This just supply us with the required inequality. The reader can see how this type of reasoning is done in [VaVo2]. The main idea is to travel along the extremal trajectories starting from $x \in \Omega$ to build a net $\mathcal{N} := \{x^+, x^-, x^{++}, x^{+-}, x^{--}, x^{-+}, \dots\}$. All points of the net should belong to Ω, and we put them on the same extremal trajectory on which x lies for a while. If one of them, say, z hits the boundary: $\partial\Omega$ (parabola) we stop building children z^+, z^-. But then one of them, say, ζ can hit the special hyperplanes $x_1 = 0$ or $x_2 = 0$. In this case, we choose ζ^+, ζ^- in such a way that they lie in different quadrants very close to ζ. Then we start anew a building of the net for ζ^+ and ζ^- separately. The closer ζ^+, ζ^- are to ζ the smaller will be difference $\langle |g_n|^p\rangle_I - B(x)$. In such a way for arbitrary ε, we obtain the inequalities

$$\mathbf{B}_{\max}(x) \ge B_{\max}(x) - \varepsilon \quad \text{and} \quad \mathbf{B}_{\min}(x) \le B_{\min}(x) + \varepsilon.$$

The reader can address to [VaVo2] to understand how the net $\mathcal{N}$ generates a required pair of functions f, g. But in the proof of the following lemma we only state the result of the described construction that supplies us with a recursive definition of f and g.

LEMMA 3. *The functions $B_{\max}$ and $B_{\min}$ satisfies the inequalities*

$$\mathbf{B}_{\max}(x) \ge B_{\max}(x) \quad \text{and} \quad \mathbf{B}_{\min}(x) \le B_{\min}(x).$$

Proof. We construct an extremal sequence of pairs f, g for the function $\mathbf{B}_{\max}(x)$ for $p>2$ and for some point x on the plane $x_1=0$. For $p<2$ the same construction works for $x_2=0$. For all other points we "glue" the extremal pairs form the known functions on the ends of the extremal trajectories. The detailed explanation how to do this can be found for example, in [SV].

Take an arbitrarily small ε and recursively define the following pair of test functions

$$f(t)=\begin{cases}-c, & 0<t<\varepsilon,\\ \gamma f\left(\frac{t-\varepsilon}{1-2\varepsilon}\right), & \varepsilon<t<1-\varepsilon,\\ c, & 1-\varepsilon<t<1\end{cases}$$

and

$$g(t)=\begin{cases}d_-, & 0<t<\varepsilon,\\ \gamma g\left(\frac{t-\varepsilon}{1-2\varepsilon}\right), & \varepsilon<t<1-\varepsilon,\\ d_+, & 1-\varepsilon<t<1,\end{cases}$$

where the constants c, $d_\pm$, and γ will be defined from the conditions that guarantee that this is an admissible pair of test functions corresponding to a given point $x=(0,x_2,x_3)$. It is not difficult to see that these formulas correctly define f and g almost everywhere on $[0,1]$.

It is evident that $x_1=\langle f\rangle_{[0,1]}=0$. Since

$$\langle g\rangle_{[0,1]}=\varepsilon(d_-+d_+)+(1-2\varepsilon)\gamma\langle g\rangle_{[0,1]},$$

the condition $\langle g\rangle_{[0,1]}=x_2$ is

$$\varepsilon(d_-+d_+)=(1-\gamma+2\varepsilon\gamma)x_2. \tag{60}$$

The condition $\langle |f|^p\rangle_{[0,1]}=x_3$ gives the relation

$$2\varepsilon c^p=\left(1-\gamma^p+2\varepsilon\gamma^p\right)x_3. \tag{61}$$

Two more relation, we obtain by using condition $|x_1^+-x_1^-|=|x_2^+-x_2^-|$. Let $t=1-\varepsilon$ be the first splitting point then the condition $x_1^+-x_1^-=x_2^--x_2^+$ is

$$c+\frac{\varepsilon c}{1-\varepsilon}=\frac{\varepsilon d_-+(1-2\varepsilon)\gamma x_2}{1-\varepsilon}-d_+. \tag{62}$$

The left point x^+ is already on the boundary $\partial\Omega$ (the functions are constants on I^+) and we have nothing to split. The left interval I^-, we naturally split at the point $t=\varepsilon$. Then the condition $x_1^+-x_1^-=x_2^+-x_2^-$

$$c=\gamma x_2-d_-. \tag{63}$$

From (63) and (62), we get

$$d_- = \gamma x_2 - c,\qquad d_+ = \gamma x_2 - \frac{c}{1-2\varepsilon}. \tag{64}$$

Now we can plug in (64) into (60) and obtain in result

$$\gamma = 1 + 2\varepsilon\frac{1-\varepsilon}{1-2\varepsilon}\cdot\frac{c}{x_2}. \tag{65}$$

Let c_0 be the limit value of c as ε tends to 0. (By the way, c_0 has clear geometrical meaning: this is the first coordinate of the end point on $\partial\Omega$ of the extremal line the second end of which is our initial x.) Then

$$\gamma \approx 1 + 2\frac{c_0}{x_2}\varepsilon,$$
$$\gamma^p \approx 1 + 2p\frac{c_0}{x_2}\varepsilon,$$
$$1-(1-2\varepsilon)\gamma^p \approx 2\Big(1-\frac{pc_0}{x_2}\Big)\varepsilon,$$

and (61) turns into equation for c_0:

$$c_0^p = \Big(1-\frac{pc_0}{x_2}\Big)x_3, \tag{66}$$

which evidently has unique solution c_0, $0<c_0<\frac{x_2}{p}$. Further we get

$$d_\pm \to x_2 - c_0 \tag{67}$$

and therefore we are able to write down in term of c_0 the average we interested in:

$$\big\langle|g|^p\big\rangle_{[0,1]} = \frac{(d_-^p+d_+^p)\varepsilon}{1-(1-2\varepsilon)\gamma^p} \to \frac{2(x_2-c_0)^p}{2(1-\frac{pc_0}{x_2})} = \frac{x_3(x_2-c_0)^p}{c_0^p}. \tag{68}$$

If we introduce

$$\omega := x_3^{-\frac1p}\lim_{\varepsilon\to0}\big\langle|g|^p\big\rangle_{[0,1]}^{\frac1p},$$

then from (68) we get $c_0 = \frac{x_2}{\omega+1}$. We can plug this expression in (66) and conclude that ω is the unique solution of the equation

$$x_2^p = (\omega+1)^{p-1}(\omega+1-p)x_3,$$

but this is just the equation $F_p(x_2,0)=F_p(\omega,1)x_3$, whose solution by definition is

$$\omega = \Big(\frac{B_{\max}(0,x_2,x_3)}{x_3}\Big)^{\frac1p}.$$

Therefore,

$$B_{\max}(0,x_2,x_3) = \lim_{\varepsilon\to0}\big\langle|g|^p\big\rangle_{[0,1]},$$

what yields the desired inequality

$$B_{\max}(0,x_2,x_3)\le \mathbf{B}(0,x_2,x_3).$$

How to obtain this inequality for arbitrary $x\in\Omega$ was explained in the beginning of the proof. In the same paper [SV], the reader can find the detail explanation how to pass from one splitting to another one (e.g., to a dyadic family of intervals). □

6. Function u_p from function B

We found Burkholder's functions $\mathbf{B}_{\max}$ and $\mathbf{B}_{\min}$ as claimed in Theorem 1. As a corollary, we immediately we get the sharp constant in Burkholder's inequality.

THEOREM 4. *Let $I=[0,1]$, $\langle f\rangle_I=x_1$, $\langle g\rangle_I=x_2$, g is a Martingale transform of f, and $|x_2|\le|x_1|$. Then*

$$\big\langle|g|^p\big\rangle_I\le\big(p^*-1\big)^p\big\langle|f|^p\big\rangle_I.$$

The constant p^*-1, where $p^*:=\max(p,\frac{p}{p-1})$ is sharp.

Proof. We just analyze the form of function $\mathbf{B}_{\max}$ from Theorem 1 and immediately see that

$$\sup_{x\in\Omega,|x_2|\le|x_1|}\frac{\mathbf{B}_{\max}(x_1,x_2,x_3)}{x_3}=\big(p^*-1\big)^p.$$ □

THEOREM 5. *Let $I=[0,1]$, $\langle f\rangle_I=x_1$, $\langle g\rangle_I=x_2$, g is a Martingale transform of f, and $|x_2|\le|x_1|$. Then*

$$\big\langle|f|^p\big\rangle_I\le\big(p^*-1\big)^p\big\langle|g|^p\big\rangle_I.$$

The constant p^*-1, where $p^*:=\max(p,\frac{p}{p-1})$ is sharp.

Proof. We just analyze the form of function $\mathbf{B}_{\min}$ from Theorem 1 and immediately see that

$$\inf_{x\in\Omega,|x_2|\ge|x_1|}\frac{\mathbf{B}_{\min}(x_1,x_2,x_3)}{x_3}=\big(p^*-1\big)^{-p}.$$ □

REMARK. The same analysis shows that $\langle|g|^p\rangle_I\le(p^*-1)^p\langle|f|^p\rangle_I$ if and only if $|x_2|\le(p^*-1)|x_1|$ in Theorem 4, and in Theorem 5 $\langle|f|^p\rangle_I\le(p^*-1)^p\langle|g|^p\rangle_I$ if and only if $|x_2|\ge(p^*-1)^{-1}|x_1|$.

NOTATION. Below we use $\beta_p:=(p^*-1)^p$. Put

$$\phi_{\max}(x_1,x_2):=\sup_{x_3:(x_1,x_2,x_3)\in\Omega}\big[\mathbf{B}_{\max}(x_1,x_2,x_3)-\beta_px_3\big],$$

$$\phi_{\min}(x_1,x_2):=\inf_{x_3:(x_1,x_2,x_3)\in\Omega}\big[\mathbf{B}_{\min}(x_1,x_2,x_3)-\beta_p^{-1}x_3\big].$$

These functions are defined on the whole $\mathbb{R}^2$.

DEFINITION. If for all pairs of points $x^\pm \in \mathbb{R}^2$ such that

$$|x_1^+ - x_1^-| = |x_2^+ - x_2^-| \quad \text{and} \quad x = \frac{x^+ + x^-}{2} \tag{69}$$

the function ϕ on $\mathbb{R}^2$ satisfies the condition

$$\phi(x) - \frac{\phi(x^-) + \phi(x^+)}{2} \geq 0, \tag{70}$$

then it is called zigzag concave. If the opposite inequality holds

$$\phi(x) - \frac{\phi(x^-) + \phi(x^+)}{2} \leq 0 \tag{71}$$

the function ϕ is called zigzag convex. The next theorem gives an independent description of $\phi_{\max}$ and $\phi_{\min}$.

THEOREM 6. *Function $\phi_{\max}$ is the least zigzag concave majorant of the function $h_{\max}(x) := |x_2|^p - \beta_p|x_1|^p$. Function $\phi_{\min}$ is the greatest zigzag convex minorant of the function $h_{\min}(x) := |x_2|^p - \beta_p^{-1}|x_1|^p$.*

REMARK. Notice that this is slightly counterintuitive: $\mathbf{B}_{\max}(x) - \beta_p x_3$ is zigzag concave for any fixed x_3, and the supremum of concave functions is *not* usually concave. The same is true about infimum of convex functions.

Proof of Theorem 6. Let $x^\pm$ and x are as in (69). It is obvious that $\phi_{\max}$ is zigzag concave. One verifies this just by definition. In fact, if for any $x^- \in \mathbb{R}^2$ we can choose x_3^- such that the supremum in the definition of $\phi_{\max}$ is almost attained, that is, $\mathbf{B}_{\max}(x^-) - \beta_p x_3^- > \phi_{\max}(x_-) - \varepsilon$ for a given ε and the same for $x^+ \in \mathbb{R}^2$. We define $x_3 = \frac{x_3^- + x_3^+}{2}$ and $\tilde{x} = (x, x_3)$. Then using (56), we can write

$$\begin{aligned}
\phi_{\max}(x) &\geq \mathbf{B}_{\max}(\tilde{x}) - \beta_p x_3 \\
&\geq \frac{\mathbf{B}_{\max}(\tilde{x}^-) + \mathbf{B}_{\max}(\tilde{x}^+)}{2} - \beta_p \frac{x_3^- + x_3^+}{2} \\
&\geq \frac{\phi_{\max}(x^-) + \phi_{\max}(x^+)}{2} - \varepsilon,
\end{aligned}$$

what yields (70). Inequality (71) is totally similar.

As sup is bigger than lim we conclude

$$\phi_{\max}(x) \geq \lim_{x_3 \to |x_1|^p} \big[\mathbf{B}_{\max}(\tilde{x}) - \beta_p x_3\big] = |x_2|^p - \beta_p|x_1|^p = h_{\max}(x).$$

As inf is smaller than lim, we get analogously

$$\phi_{\min}(x) \leq \lim_{x_3 \to |x_1|^p} \big[\mathbf{B}_{\min}(\tilde{x}) - \beta_p^{-1} x_3\big] = |x_2|^p - \beta_p^{-1}|x_1|^p = h_{\min}(x).$$

This is because the boundary values of $\mathbf{B}_{\max}$ and $\mathbf{B}_{\min}$ are $|x_2|^p$.

We are left to see that $\phi_{\max}$ is the *least* such majorant (and a symmetric claim for $\phi_{\min}$). Let ψ be a zigzag concave function such that

$$\phi_{\max} \geq \psi \geq h_{\max}. \tag{72}$$

Consider function $\Psi(\tilde{x}) := \psi(x) + \beta_p x_3$. It is immediate that Ψ satisfies (56). On the boundary of Ω, we have $\Psi(x) \geq |x_2|^p$, this is just by the right-hand side of (72). Then Lemma 2 yields

$$\Psi(\tilde{x}) \geq \mathbf{B}_{\max}(\tilde{x}).$$

Then, obviously,

$$\psi(x) = \sup_{x_3 : \tilde{x} \in \Omega} \big[\Psi(\tilde{x}) - \beta_p x_3\big] \geq \sup_{x_3 : \tilde{x} \in \Omega} \big[\mathbf{B}_{\max}(\tilde{x}) - \beta_p x_3\big] = \phi_{\max}(x).$$

So we proved that $\phi_{\max}$ is the least zigzag concave majorant of $h_{\max}$. Symmetric consideration will bring us the fact that $\phi_{\min}$ is the largest zigzag convex minorant of $h_{\min}$. □

The reader should look now at function F_p from Theorem 1. It would be interesting to obtain the formulae for $\phi_{\max}$ and $\phi_{\min}$, especially using this F_p. It would be also interesting to understand the role of function

$$u_p(x_1, x_2) := p\left(1 - \frac{1}{p^*}\right)^{p-1} \big(|x_1| + |x_2|\big)^{p-1} \big(|x_2| - (p^* - 1)|x_1|\big), \tag{73}$$

mentioned in the introduction and used repeatedly by Burkholder. May be it is equal to $\phi_{\max}$? The answer is "no," but we can prove the following theorem.

THEOREM 7.

$$\phi_{\max}(x_1, x_2) = F_p\big(|x_2|, |x_1|\big). \tag{74}$$

Proof. We shall consider only the case $p > 2$, the case $p < 2$ is similar. Due to the symmetry with respect to the change of x_1 to $-x_1$ and x_2 to $-x_2$, it is enough to check equality (74) in the quadrant $x_1 > 0$, $x_2 > 0$. If $x_2 \leq (p-1)x_1$, we get an explicit formula for $\mathbf{B}_{\max}$ from Theorem 1: $\mathbf{B}_{\max}(\tilde{x}) = x_2^p + (p-1)^p(x_3 - x_1^p)$, and therefore,

$$\phi_{\max}(x_1, x_2) = \sup_{x_3 : \tilde{x} \in \Omega} \big[\mathbf{B}_{\max}(\tilde{x}) - (p-1)^p x_3\big] = x_2^p - (p-1)^p x_1^p = F_2(x_2, x_1).$$

So in the rest of the proof, we shall consider only the domain $\{x = (x_1, x_2) : 0 \leq (p-1)x_1 < x_2\}$. Moreover, since both functions $\phi_{\max}$ and F_p are p-homogeneous, it is sufficient to check (74) on the interval $S := \{x : 0 \leq px_1 < 1, x_1 + x_2 = 1\}$. (Indeed, the condition $px_1 < 1$ on the line $x_1 + x_2 = 1$ means $x_2 > (p-1)x_1$.)

The function F_p is linear on S: $F_p(1 - x_1, x_1) = \frac{(p-1)^{p-1}}{p^{p-2}}(1 - px_1)$. Now we check that $\phi_{\max}$ is linear as well. To this end, we check inequality

$$\phi_{\max}(x) \leq \frac{\phi_{\max}(x_1 - a, x_2 + a) + \phi_{\max}(x_1 + a, x_2 - a)}{2} \tag{75}$$

for all $x \in S$ and sufficiently small a, this just means linearity of $\phi_{\max}$ on S, because the opposite inequality follows from the zigzag concavity of $\phi_{\max}$ (70).

Fix $x \in S$ and $\varepsilon > 0$. Take x_3 such that $B(\tilde{x}) - \beta_p x_3 \geq \phi_{\max}(x) - \varepsilon$. Due to condition $x_2 > (p-1)x_1$ the extremal trajectory L_x of $\mathbf{B}_{\max}$ passing through the point $\tilde{x} = (x, x_3)$ is not vertical, it hits at some point the plane $x_1 = 0$. Therefore we can take two different points $\tilde{x}^{\pm} = (x^{\pm}, x_3^{\pm})$ on L_x such that $\tilde{x} = \frac{1}{2}(\tilde{x}^+ + \tilde{x}^-)$. We know three things:

$$\begin{aligned} \mathbf{B}_{\max}(\tilde{x}) - \beta_p x_3 &\geq \phi_{\max}(x) - \varepsilon, \\ \mathbf{B}_{\max}\big(\tilde{x}^+\big) - \beta_p x_3^+ &\leq \phi_{\max}\big(x^+\big), \\ \mathbf{B}_{\max}\big(\tilde{x}^-\big) - \beta_p x_3^- &\leq \phi_{\max}\big(x^-\big). \end{aligned}$$

Since the function $\mathbf{B}_{\max}$ is linear along L_x, we can write the following chain of inequalities

$$\begin{aligned} \phi_{\max}(x) - \varepsilon &\leq \mathbf{B}_{\max}(\tilde{x}) - \beta_p x_3 \\ &= \frac{[\mathbf{B}_{\max}(\tilde{x}^+) - \beta_p x_3^+] + [\mathbf{B}_{\max}(\tilde{x}^-) - \beta_p x_3^-]}{2} \\ &\leq \frac{\phi_{\max}(x^+) + \phi_{\max}(x^-)}{2}. \end{aligned}$$

Since ε is arbitrary, we come to the desired convexity (75).

Function $F_p(x_2, x_1)$ is a concave C^1-smooth function majorazing $h_{\max}$ on S. This is immediate from its formula. Functions $\phi_{\max}(x_1, x_2)$ and $F_p(x_2, x_1)$ are linear on S and at the point $x = x_p =: (\frac{1}{p}, 1 - \frac{1}{p})$ both are equal $h_{\max}(x_p) = 0$. Therefore, to prove that they are identical it is sufficient to check that their derivatives at x_p along S are equal as well. Since $\phi_{\max}$ is a majorant of $h_{\max}$ and both functions are equal at x_p, then the left derivative of $\phi_{\max}$ at x_p is not grater than the derivative of $h_{\max}$ at this point. On the other hand, since $\phi_{\max}$ is the least majorant it is not grater then F_p, that is, its left derivative at x_p is not less that the derivative of F_p there, but latter coincides with the derivative of $h_{\max}$. Hence, all three derivatives along S are equal at the point x_p and we proved $\phi_{\max}(x_1, x_2) = F_p(x_2, x_1)$. □

THEOREM 8.

$$\phi_{\min}(x_1, x_2) = -\big(p^* - 1\big)^{-p} F_p\big(|x_1|, |x_2|\big). \tag{76}$$

The proof of this theorem is absolutely similar to the proof of Theorem 7.

Burkholder often used function u_p from (73). To demystify it, let us notice that it is also p-homogeneous and as such can be considered only on the segment $x_1 + x_2 = 1$, $x_i > 0$. On this segment function, u_p becomes linear. It is a majorant of h_p, and its graph is tangent to the graph h_p exactly at point x_p on S, where h_p vanishes. It is not the least zigzag concave function greater than h_p (of course not, $\phi_{\max}$ is such), but it is the least zigzag concave function larger than h_p and such that on all segments $\{x : x_i > 0, x_1 + x_2 = \mathrm{const}\}$

it is not only concave, but also linear. (Keeping in mind the symmetries $x \to -x_1, x_2 \to -x_2$ we can consider the first quadrant only.)

This is already proved, and we leave the detailed reasoning to the reader. One more thing we want to mention is that we could have considered a slightly more general problem. Namely, instead of majorazing the function $h_{\max}(x_1, x_2) = |x_2|^p - (p^* - 1)^p |x_1|^p$ we could have started with any function

$$h_c(x_1, x_2) := |x_2|^p - c|x_1|^p.$$

The reader can easily see that we have proved the following theorem (of course Burkholder already proved the most of it long ago).

Corollary 9. *The smallest c for which there exists a zigzag concave function ϕ_c majorazing h_c is equal to $(p^* - 1)^p$. For this c the least zigzag majorant is $F_p(|x_1|, |x_2|)$. The smallest c or which there exists a zigzag concave function ϕ_c majorazing h_c such that it is linear on the segment $\{x : x_i > 0, x_1 + x_2 = \text{const}\}$, symmetric and p-homogeneous is equal to $(p^* - 1)^p$. For this c the least zigzag majorant linear on $\{x : x_i > 0, x_1 + x_2 = \text{const}\}$, symmetric and p-homogeneous is $u_p(x_1, x_2)$.*

Remark. Notice an interesting thing which we do not know how to explain. Given function $\mathbf{B}_{\max}$ from Theorem 1 we can easily diminish the number of variables and construct $\phi_{\max}$. But amazingly, we can also find $\mathbf{B}_{\max}$ if only $\phi_{\max}$ is given. In fact, Theorem 7 gives the formula for $\phi_{\max}$ via F_p. Then F_p allows us to find $\mathbf{B}_{\max}$. If we now combine Theorem 1 and Theorems 7–8 to conclude the following corollary.

Corollary 10. *Given a point $x \in \Omega$, if we know $\phi_{\max}$ we can find $\mathbf{B}_{\max}(x)$ by solving equation*:

$$\phi_{\max}(x_1, x_2) = \phi_{\max}\big(x_3^{\frac{1}{p}}, \mathbf{B}_{\max}^{\frac{1}{p}}\big).$$

Symmetric formula allows to find $\mathbf{B}_{\min}$ if $\phi_{\min}$ is known:

$$\phi_{\min}(x_2, x_1) = \phi_{\min}\big(\mathbf{B}_{\min}^{\frac{1}{p}}, x_3^{\frac{1}{p}}\big).$$

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