Illinois Journal of Mathematics
Volume 54, Number 3, Fall 2010, Pages 1025–1067
S 0019-2082

OPTIMAL STOPPING FOR DYNAMIC CONVEX RISK MEASURES

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ABSTRACT. We use martingale and stochastic analysis techniques to study a continuous-time optimal stopping problem, in which the decision maker uses a dynamic convex risk measure to evaluate future rewards. We also find a saddle point for an equivalent zero-sum game of control and stopping, between an agent (the "stopper") who chooses the termination time of the game, and an agent (the "controller," or "nature") who selects the probability measure.

1. Introduction

Let us consider a complete, filtered probability space $(\Omega, \mathcal{F}, P)$, $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$, and on it a bounded, adapted process Y that satisfies certain regularity conditions. Given an arbitrary stopping time ν of the filtration $\mathbf{F}$, our goal is to find a stopping time $\tau_*(\nu) \in \mathcal{S}_{\nu,T}$ which satisfies

$$\operatorname*{ess\,inf}_{\gamma\in\mathcal{S}_{\nu,T}} \rho_{\nu,\gamma}(Y_\gamma) = \rho_{\nu,\tau_*(\nu)}\big(Y_{\tau_*(\nu)}\big), \quad P\text{-a.s.} \tag{1.1}$$

Here $\mathcal{S}_{\nu,T}$ is the set of stopping times γ satisfying $\nu \leq \gamma \leq T$, P-a.s., and the collection of functionals $\{\rho_{\nu,\gamma} : \mathbb{L}^\infty(\mathcal{F}_\gamma) \to \mathbb{L}^\infty(\mathcal{F}_\nu)\}_{\nu\in\mathcal{S}_{0,T},\gamma\in\mathcal{S}_{\nu,T}}$ is a "dynamic convex risk measure" in the sense of [7]. Our motivation is to solve the optimal stopping problem of a decision maker who evaluates future rewards/risks using dynamic convex risk measures rather than statistical expectations. This question can also be cast as a *robust optimal stopping* problem,

Received January 5, 2010; received in final form September 4, 2010.

E. Bayraktar is supported in part by the National Science Foundation under an Applied Mathematics Research Grant and a Career Grant, DMS-09-06257 and DMS-09-55463, respectively, and in part by the Susan M. Smith Professorship.

I. Karatzas is supported in part by the National Science Foundation under Grant DMS-09-05754.

2010 *Mathematics Subject Classification.* 60G40, 60H30, 91A15.

in which the decision maker has to act in the presence of so-called "Knightian uncertainty" regarding the underlying probability measure.

When the filtration $\mathbf{F}$ is generated by a Brownian motion, the dynamic convex risk measure admits the following representation: There exists a suitable nonnegative function f, convex in its spatial argument, such that the representation

$$\rho_{\nu,\gamma}(\xi) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_\nu} E_Q\left[-\xi - \int_\nu^\gamma f(s,\theta_s^Q)\,ds \middle| \mathcal{F}_\nu\right], \quad P\text{-a.s.}$$

holds for all $\xi \in \mathbb{L}^\infty(\mathcal{F}_\gamma)$. Here, $\mathcal{Q}_\nu$ is the collection of probability measures Q which are equivalent to P on $\mathcal{F}$, equal to P on $\mathcal{F}_\nu$, and satisfy a certain integrability condition; whereas θ^Q is the predictable process whose stochastic exponential gives the density of Q with respect to P. In this setting, we establish a minimax result, namely

$$\begin{aligned}(1.2)\qquad V(\nu) &\triangleq \operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}}\left(\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} E_Q\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^Q)\,ds \middle| \mathcal{F}_\nu\right]\right)\\ &= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}\left(\operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}} E_Q\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^Q)\,ds \middle| \mathcal{F}_\nu\right]\right),\end{aligned}$$

and construct an optimal stopping time $\tau(\nu)$ as the limit of stopping times which are optimal under expectation criteria—see Theorem 3.9. We show that the process $\{\mathbf{1}_{\{t\geq\nu\}} V(\tau(\nu)\wedge t)\}_{t\in[0,T]}$ admits an RCLL modification $V^{0,\nu}$ with the property that for any $\gamma\in\mathcal{S}_{0,T}$, we have $V_\gamma^{0,\nu} = \mathbf{1}_{\{\gamma\geq\nu\}} V(\tau(\nu)\wedge\gamma)$, P-a.s. We show that the stopping time $\tau_V(\nu) \triangleq \inf\{t\in[\nu,T] : V_t^{0,\nu} = Y_t\}$ attains the infimum in (1.1). Finally, we construct a saddle point for the stochastic game in (1.2).

The discrete-time optimal stopping problem for coherent risk measures was studied by [11, Section 6.5] and [5, Sections 5.2 and 5.3]. The papers [6] and [14], on the other hand, considered continuous-time optimal stopping problems in which the essential infimum over the stopping times in (1.1) is replaced by an essential supremum. The controller-and-stopper problem of [20] and [15], and the optimal stopping for nonlinear expectations in [1] and [2], are the closest in spirit to our work. However, since our assumptions concerning the random function f and the set $\mathcal{Q}_\nu$ are dictated by the representation theorem for dynamic convex risk measures, the results in these papers cannot be directly applied. In particular, because of the integrability assumption that appears in the definition of $\mathcal{Q}_\nu$ (Section 1.1), this set may not be closed under *pasting*; see Remark 3.8. Moreover, the extant results on controller-and-stopper games would require that f and the θ^Q's be bounded. We overcome these technical difficulties by using approximation arguments which rely on *truncation* and *localization* techniques. On the other hand, in finding a saddle point, the authors of [15] used the weak compactness of the collection of probability measures, in particular the boundedness of θ^Q's. We avoid making this assumption by using techniques from Reflected Backward Stochastic Differential Equations (RBSDEs). In particular, using a comparison theorem and the fact that V can be approximated by solutions of RBSDEs with Lipschitz generators, we show that V solves a quadratic RBSDE (QRBSDE). The relationship between the solutions of QRBSDEs and the BMO martingales helps us construct a saddle point. We should point out that the convexity of f is not needed to derive our results; cf. Remark 3.1.

The layout of the paper is simple. In Section 2, we recall the definition of the dynamic convex risk measures and a representation theorem. We solve the optimal stopping problem in Section 3. In Section 4, we find a saddle point for the stochastic controller-and-stopper game in (1.2). The proofs of our results are given in Section 5.

1.1. Notation and preliminaries. Throughout this paper, we let B be a d-dimensional Brownian Motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and consider the augmented filtration generated by it, that is,

$$\mathbf{F} = \big\{\mathcal{F}_t \stackrel{\triangle}{=} \sigma\big(\sigma(B_s; s \in [0,t]) \cup \mathscr{N}\big)\big\}_{t \geq 0},$$

where $\mathscr{N}$ is the collection of all P-null sets in $\mathcal{F}$.

We fix a finite time horizon $T > 0$, denote by $\mathscr{P}$ (resp. $\widehat{\mathscr{P}}$) the predictably (resp. progressively) measurable σ-field on $\Omega \times [0,T]$, and let $\mathcal{S}_{0,T}$ be the set of all $\mathbf{F}$-stopping times ν such that $0 \leq \nu \leq T$, P-a.s. From now on, when writing $\nu \leq \gamma$, we always mean two stopping times $\nu, \gamma \in \mathcal{S}_{0,T}$ such that $\nu \leq \gamma$, P-a.s. For any $\nu \leq \gamma$, we define $\mathcal{S}_{\nu,\gamma} \stackrel{\triangle}{=} \{\sigma \in \mathcal{S}_{0,T} | \nu \leq \sigma \leq \gamma, P\text{-a.s.}\}$ and let $\mathcal{S}^\star_{\nu,\gamma}$ denote all finite-valued stopping times in $\mathcal{S}_{\nu,\gamma}$.

The following spaces of functions will be used in the sequel:

- Let $\mathcal{G}$ be a generic sub-σ-field of $\mathcal{F}$. $\mathbb{L}^0(\mathcal{G})$ denotes the space of all real-valued, $\mathcal{G}$-measurable random variables.
- $\mathbb{L}^\infty(\mathcal{G}) \stackrel{\triangle}{=} \{\xi \in \mathbb{L}^0(\mathcal{G}) : \|\xi\|_\infty \stackrel{\triangle}{=} \operatorname{ess\,sup}_{\omega \in \Omega} |\xi(\omega)| < \infty\}$.
- $\mathbb{L}^0_{\mathbf{F}}[0,T]$ denotes the space of all real-valued, $\mathbf{F}$-adapted processes.
- $\mathbb{L}^\infty_{\mathbf{F}}[0,T] \stackrel{\triangle}{=} \{X \in \mathbb{L}^0_{\mathbf{F}}[0,T] : \|X\|_\infty \stackrel{\triangle}{=} \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |X_t(\omega)| < \infty\}$.
- $\mathbb{C}^p_{\mathbf{F}}[0,T] \stackrel{\triangle}{=} \{X \in \mathbb{L}^p_{\mathbf{F}}[0,T] : X \text{ has continuous paths}\}$, $p = 0, \infty$.
- $\mathbb{C}^2_{\mathbf{F}}[0,T] \stackrel{\triangle}{=} \{X \in \mathbb{C}^0_{\mathbf{F}}[0,T] : E(\sup_{t \in [0,T]} |X_t|^2) < \infty\}$.
- $\mathbb{H}^2_{\mathbf{F}}([0,T]; \mathbb{R}^d)$ (resp. $\widehat{\mathbb{H}}^2_{\mathbf{F}}([0,T]; \mathbb{R}^d)$) denotes the space of all $\mathbb{R}^d$-valued, $\mathbf{F}$-adapted predictably (resp. progressively) measurable processes X with $E\int_0^T |X_t|^2 \, dt < \infty$.
- $\mathbb{H}^\infty_{\mathbf{F}}([0,T]; \mathbb{R}^d)$ denotes the space of all $\mathbb{R}^d$-valued, $\mathbf{F}$-adapted predictably measurable processes X with $\operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |X_t(\omega)| < \infty$.
- $\mathbb{K}_{\mathbf{F}}[0,T]$ denotes the space of all real-valued, $\mathbf{F}$-adapted continuous increasing processes K with $K_0 = 0$.

Let us consider the set $\mathcal{M}^e$ of all probability measures on $(\Omega,\mathcal{F})$ which are equivalent to P. For any $Q\in\mathcal{M}^e$, it is well-known that there is an $\mathbb{R}^d$-valued predictable process θ^Q with $\int_0^T|\theta_t^Q|^2\,dt<\infty$, P-a.s., such that the density process Z^Q of Q with respect to P is the stochastic exponential of θ^Q, namely,

$$Z_t^Q=\mathscr{E}(\theta^Q\bullet B)_t=\exp\biggl\{\int_0^t\theta_s^Q\,dB_s-\frac{1}{2}\int_0^t|\theta_s^Q|^2\,ds\biggr\},\quad 0\le t\le T.$$

We denote $Z_{\nu,\gamma}^Q\triangleq Z_\gamma^Q/Z_\nu^Q=\exp\{\int_\nu^\gamma\theta_s^Q\,dB_s-\frac{1}{2}\int_\nu^\gamma|\theta_s^Q|^2\,ds\}$ for any $\nu\le\gamma$. Moreover, for any $\nu\in\mathcal{S}_{0,T}$ and with the notation $[\![0,\nu[\![\triangleq\{(t,\omega)\in[0,T]\times\Omega: t<\nu(\omega)\}$ for the stochastic interval, we define

$$\begin{aligned}\mathcal{P}_\nu&\triangleq\{Q\in\mathcal{M}^e:Q=P\text{ on }\mathcal{F}_\nu\}\\&=\{Q\in\mathcal{M}^e:\theta_t^Q(\omega)=0,dt\otimes dP\text{-a.e. on }[\![0,\nu[\![\},\\ \mathcal{Q}_\nu&\triangleq\biggl\{Q\in\mathcal{P}_\nu:E_Q\int_\nu^T f(s,\theta_s^Q)\,ds<\infty\biggr\}.\end{aligned}$$

Moreover, we use the convention $\inf\{\emptyset\}\triangleq\infty$.

2. Dynamic convex risk measures

DEFINITION 2.1. A dynamic convex risk measure is a family of functionals $\{\rho_{\nu,\gamma}:\mathbb{L}^\infty(\mathcal{F}_\gamma)\to\mathbb{L}^\infty(\mathcal{F}_\nu)\}_{\nu\le\gamma}$ which satisfy the following properties: For any stopping times $\nu\le\gamma$ and any $\mathbb{L}^\infty(\mathcal{F}_\gamma)$-measurable random variables ξ,η, we have

- *"Monotonicity":* $\rho_{\nu,\gamma}(\xi)\le\rho_{\nu,\gamma}(\eta)$, P-a.s. if $\xi\ge\eta$, P-a.s.
- *"Translation Invariance":* $\rho_{\nu,\gamma}(\xi+\eta)=\rho_{\nu,\gamma}(\xi)-\eta$, P-a.s. if $\eta\in\mathbb{L}^\infty(\mathcal{F}_\nu)$.
- *"Convexity":* $\rho_{\nu,\gamma}(\lambda\xi+(1-\lambda)\eta)\le\lambda\rho_{\nu,\gamma}(\xi)+(1-\lambda)\rho_{\nu,\gamma}(\eta)$, P-a.s. for any $\lambda\in(0,1)$.
- *"Normalization":* $\rho_{\nu,\gamma}(0)=0$, P-a.s.

The paper [7] provides a representation result, Proposition 2.2 below, for dynamic convex risk measures $\{\rho_{\nu,\gamma}\}_{\nu\le\gamma}$ that satisfy the following properties:

(A1) *"Continuity from above":* For any decreasing sequence $\{\xi_n\}\subset\mathbb{L}^\infty(\mathcal{F}_\gamma)$ with $\xi\triangleq\lim_{n\to\infty}\downarrow\xi_n\in\mathbb{L}^\infty(\mathcal{F}_\gamma)$, it holds P-a.s. that $\lim_{n\to\infty}\uparrow\rho_{\nu,\gamma}(\xi_n)=\rho_{\nu,\gamma}(\xi)$.

(A2) *"Time Consistency":* For any $\sigma\in\mathcal{S}_{\nu,\gamma}$, we have: $\rho_{\nu,\sigma}(-\rho_{\sigma,\gamma}(\xi))=\rho_{\nu,\gamma}(\xi)$, P-a.s.

(A3) *"Zero–One Law":* For any $A\in\mathcal{F}_\nu$, we have: $\rho_{\nu,\gamma}(\mathbf{1}_A\xi)=\mathbf{1}_A\rho_{\nu,\gamma}(\xi)$, P-a.s.

(A4) $\operatorname{ess\,inf}_{\xi\in\mathcal{A}_t} E_P[\xi|\mathcal{F}_t]=0$ $\mathbb{P}$-a.s., where $\mathcal{A}_t \stackrel{\triangle}{=} \{\xi\in\mathbb{L}^\infty(\mathcal{F}_T):\rho_{t,T}(\xi)\le 0\}$.

We can think of $\rho_\nu(\xi)$ as a measure of the risk associated with assuming at time ν a liability $\xi\in\mathbb{L}^0(\mathcal{F}_\gamma)$, whose true size gets revealed only at time $\gamma\ge\nu$.

PROPOSITION 2.2. *Let $\{\rho_{\nu,\gamma}\}_{\nu\le\gamma}$ be a dynamic convex risk measure satisfying* (A1)–(A4). *Then for any $\nu\le\gamma$ and $\xi\in\mathbb{L}^\infty(\mathcal{F}_\gamma)$, we have*

$$\rho_{\nu,\gamma}(\xi)=\operatorname*{ess\,sup}_{Q\in\mathcal{Q}_\nu} E_Q\left[-\xi-\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right],\quad P\text{-a.s.} \tag{2.1}$$

Here $f:[0,T]\times\Omega\times\mathbb{R}^d\to[0,\infty]$ is a suitable measurable function, such that

(𝔣1) *$f(\cdot,\cdot,z)$ is predictable for any $z\in\mathbb{R}^d$;*

(𝔣2) *$f(t,\omega,\cdot)$ is proper convex, and lower semi-continuous for $dt\otimes dP$-a.e. $(t,\omega)\in[0,T]\times\Omega$;*

(𝔣3) *$f(t,\omega,0)=0$, $dt\otimes dP$-a.e.*

We refer to [22], page 24 for the notion of "proper convex function," and review some basic properties of the essential extrema as in [21, Proposition VI-**1**-1] or [11, Theorem A.32].

LEMMA 2.3. *Let $\{\xi_i\}_{i\in\mathcal{I}}$ and $\{\eta_i\}_{i\in\mathcal{I}}$ be two classes of $\mathcal{F}$-measurable random variables with the same index set $\mathcal{I}$.*

(1) *If $\xi_i\le(=)\,\eta_i$, P-a.s. holds for all $i\in\mathcal{I}$, then $\operatorname{ess\,sup}_{i\in\mathcal{I}}\xi_i\le(=)\operatorname{ess\,sup}_{i\in\mathcal{I}}\eta_i$, P-a.s.*

(2) *For any $A\in\mathcal{F}$, it holds P-a.s. that $\operatorname{ess\,sup}_{i\in\mathcal{I}}(\mathbf{1}_A\xi_i+\mathbf{1}_{A^c}\eta_i)=\mathbf{1}_A\operatorname{ess\,sup}_{i\in\mathcal{I}}\xi_i+\mathbf{1}_{A^c}\operatorname{ess\,sup}_{i\in\mathcal{I}}\eta_i$. In particular, $\operatorname{ess\,sup}_{i\in\mathcal{I}}(\mathbf{1}_A\xi_i)=\mathbf{1}_A\times\operatorname{ess\,sup}_{i\in\mathcal{I}}\xi_i$, P-a.s.*

(3) *For any $\mathcal{F}$-measurable random variable γ and any $\lambda>0$, we have*

$$\operatorname*{ess\,sup}_{i\in\mathcal{I}}(\lambda\xi_i+\gamma)=\lambda\operatorname*{ess\,sup}_{i\in\mathcal{I}}\xi_i+\gamma,\quad P\text{-a.s.}$$

Moreover, (1)–(3) *hold when we replace* $\operatorname{ess\,sup}_{i\in\mathcal{I}}$ *by* $\operatorname{ess\,inf}_{i\in\mathcal{I}}$.

3. The optimal stopping problem

In this section, we study the optimal stopping problem for dynamic convex risk measures. More precisely, given $\nu\in\mathcal{S}_{0,T}$, we seek an optimal stopping time $\tau_*(\nu)\in\mathcal{S}_{\nu,T}$ that satisfies (1.1). We shall assume throughout that the reward process $Y\in\mathbb{L}^\infty_{\mathbf{F}}[0,T]$ is right-continuous and $\mathcal{Q}_0$*-quasi-left-continuous:* to wit, for any increasing sequence $\{\nu_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}_{0,T}$ with $\nu\stackrel{\triangle}{=}\lim_{n\to\infty}\uparrow\nu_n\in\mathcal{S}_{0,T}$, and any $Q\in\mathcal{Q}_0$, we have

$$\varlimsup_{n\to\infty} E_Q[Y_{\nu_n}|\mathcal{F}_{\nu_1}]\le E_Q[Y_\nu|\mathcal{F}_{\nu_1}],\quad P\text{-a.s.}$$

In light of the representation (2.1), we can alternatively express (1.1) as a *robust optimal stopping problem*, in the following sense:

$$\operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}}\left(\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\right) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_{\tau_*(\nu)}+\int_\nu^{\tau_*(\nu)} f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]. \tag{3.1}$$

REMARK 3.1. We will study the robust optimal stopping problem (3.1) in a setting more general than alluded to heretofore: From now on, we only assume that $f:[0,T]\times\Omega\times\mathbb{R}^d\to[0,\infty]$ is a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)/\mathscr{B}([0,\infty])$-measurable function which satisfies ($\mathfrak{f}$3); that is, the convexity property ($\mathfrak{f}$2) is not necessary for solving (3.1).

In order to find a stopping time which is optimal, that is, attains the essential supremum in (3.1), we introduce the lower- and upper-value, respectively, of the stochastic game suggested by (3.1), to wit, for every $\nu\in\mathcal{S}_{0,T}$:

$$\underline{V}(\nu)\stackrel{\triangle}{=}\operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}}\left(\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\right),$$
$$\overline{V}(\nu)\stackrel{\triangle}{=}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}\left(\operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}}E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\right).$$

In Theorem 3.9, we shall show that the quantities $\underline{V}(\nu)$ and $\overline{V}(\nu)$ coincide at any $\nu\in\mathcal{S}_{0,T}$, that is, a min–max theorem holds; we shall also identify two optimal stopping times in Theorems 3.9 and 3.13, respectively.

Given any probability measure $Q\in\mathcal{Q}_0$, let us introduce for each fixed $\nu\in\mathcal{S}_{0,T}$ the quantity

$$R^Q(\nu)\stackrel{\triangle}{=}\operatorname*{ess\,sup}_{\zeta\in\mathcal{S}_{\nu,T}}E_Q\left[Y_\zeta+\int_\nu^\zeta f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right] = \operatorname*{ess\,sup}_{\sigma\in\mathcal{S}_{0,T}}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu} f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\ge Y_\nu, \tag{3.2}$$

and recall from the classical theory of optimal stopping [see [8] or [16, Appendix D]] the following result.

PROPOSITION 3.2. *Fix a probability measure $Q\in\mathcal{Q}_0$.*

(1) *The process $\{R^Q(t)\}_{t\in[0,T]}$ admits an RCLL (right-continuous, with limits from the left) modification $R^{Q,0}$ such that, for any $\nu\in\mathcal{S}_{0,T}$, we have*

$$R_\nu^{Q,0}=R^Q(\nu),\quad P\text{-a.s.} \tag{3.3}$$

(2) *For every* $\nu \in \mathcal{S}_{0,T}$, *the stopping time* $\tau^Q(\nu) \triangleq \inf\{t \in [\nu, T] : R_t^{Q,0} = Y_t\} \in \mathcal{S}_{\nu,T}$ *satisfies for any* $\gamma \in \mathcal{S}_{\nu,\tau^Q(\nu)}$:

$$
\begin{aligned}
R^Q(\nu) &= E_Q\left[Y_{\tau^Q(\nu)} + \int_\nu^{\tau^Q(\nu)} f(s, \theta_s^Q)\, ds \Big| \mathcal{F}_\nu\right] \\
&= E_Q\left[R^Q(\tau^Q(\nu)) + \int_\nu^{\tau^Q(\nu)} f(s, \theta_s^Q)\, ds \Big| \mathcal{F}_\nu\right] \\
&= E_Q\left[R^Q(\gamma) + \int_\nu^{\gamma} f(s, \theta_s^Q)\, ds \Big| \mathcal{F}_\nu\right], \quad P\text{-a.s.}
\end{aligned} \tag{3.4}
$$

Therefore, $\tau^Q(\nu)$ *is an optimal stopping time for maximizing the quantity* $E_Q[Y_\zeta + \int_\nu^\zeta f(s, \theta_s^Q)\, ds | \mathcal{F}_\nu]$ *over* $\zeta \in \mathcal{S}_{\nu,T}$.

For any $\nu \in \mathcal{S}_{0,T}$ and $k \in \mathbb{N}$, we introduce the collection of probability measures

$$\mathcal{Q}_\nu^k \triangleq \{Q \in \mathcal{P}_\nu : |\theta_t^Q(\omega)| \vee f(t, \omega, \theta_t^Q(\omega)) \le k, dt \otimes dP\text{-a.e. on }]\!]\nu, T]\!]\}.$$

REMARK 3.3. It is clear that $\mathcal{Q}_\nu^k \subset \mathcal{Q}_\nu$; and from (f3) one can deduce that for any $\nu \le \gamma$ we have

$$\mathcal{Q}_\gamma \subset \mathcal{Q}_\nu \quad \text{and} \quad \mathcal{Q}_\gamma^k \subset \mathcal{Q}_\nu^k, \quad \forall k \in \mathbb{N}.$$

Given a $Q \in \mathcal{Q}_\nu$ for some $\nu \in \mathcal{S}_{0,T}$, we *truncate* it in the following way: The predictability of the process θ^Q and Proposition 2.2 imply that $\{f(t, \theta_t^Q)\}_{t \in [0,T]}$ is also a predictable process. Therefore, for any given $k \in \mathbb{N}$, the set

$$A_{\nu,k}^Q \triangleq \{(t, \omega) \in]\!]\nu, T]\!] : |\theta_t^Q(\omega)| \vee f(t, \omega, \theta_t^Q(\omega)) \le k\} \in \mathscr{P} \tag{3.5}$$

is predictable. Then the predictable process $\theta^{Q^{\nu,k}} \triangleq \mathbf{1}_{A_{\nu,k}^Q} \theta^Q$ gives rise to a probability measure $Q^{\nu,k} \in \mathcal{Q}_\nu^k$ via the recipe $dQ^{\nu,k} \triangleq \mathscr{E}(\theta^{Q^{\nu,k}} \bullet B)_T\, dP$. Let us define the stopping times

$$\sigma_m^Q \triangleq \inf\left\{t \in [0, T] : \int_0^t |\theta_s^Q|^2\, ds > m\right\} \wedge T, \quad m \in \mathbb{N}.$$

There exists a null set N such that, for any $\omega \in \Omega \setminus N$, we have $\sigma_m^Q(\omega) = T$ for some $m = m(\omega) \in \mathbb{N}$. Since $E \int_0^{\sigma_m^Q} |\theta_t^Q|^2\, dt \le m$ holds for each $m \in \mathbb{N}$, we have $|\theta_t^Q(\omega)| < \infty, dt \otimes dP$-a.e. on $[\![0, \sigma_m^Q]\!]$.

As $(\bigcup_{m \in \mathbb{N}} [\![0, \sigma_m^Q]\!]) \cup ([0, T] \times N) = [0, T] \times \Omega$, it follows that $|\theta_t^Q(\omega)| < \infty$ holds $dt \otimes dP$-a.e. on $[0, T] \times \Omega$. On the other hand, since $Q \in \mathcal{Q}_\nu$ we have $E_Q \int_\nu^T f(s, \theta_s^Q)\, ds < \infty$, which implies $\mathbf{1}_{]\!]\nu, T]\!]}(t, \omega) f(t, \omega, \theta_t^Q(\omega)) < \infty$ holds $dt \otimes dQ$-a.s., or equivalently $dt \otimes dP$-a.e. Therefore, we see that

$$\lim_{k \to \infty} \uparrow \mathbf{1}_{A_{\nu,k}^Q} = 1_{]\!]\nu, T]\!]}, \quad dt \otimes dP\text{-a.e.} \tag{3.6}$$

For any $\nu \in \mathcal{S}_{0,T}$, the upper value $\overline{V}(\nu)$ can be approximated from above in two steps, presented in the next two lemmas.

LEMMA 3.4. *Let* $\nu \in \mathcal{S}_{0,T}$. (1) *For any* $\gamma \in \mathcal{S}_{\nu,T}$ *we have*

$$\begin{aligned}(3.7)\qquad &\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} E_Q\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right] \\ &\quad = \lim_{k\to\infty}\downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} E_Q\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right], \quad P\text{-a.s.}\end{aligned}$$

(2) *It holds* P-*a.s. that*

$$(3.8)\qquad \overline{V}(\nu) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} R^Q(\nu) = \lim_{k\to\infty}\downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu).$$

LEMMA 3.5. *Let* $k \in \mathbb{N}$ *and* $\nu \in \mathcal{S}_{0,T}$.

(1) *For any* $\gamma \in \mathcal{S}_{\nu,T}$ *there exists a sequence* $\{Q_n^{\gamma,k}\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu^k$ *such that*

$$\begin{aligned}(3.9)\qquad &\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} E_Q\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right] \\ &\quad = \lim_{n\to\infty}\downarrow E_{Q_n^{\gamma,k}}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_n^{\gamma,k}})\,ds\Big|\mathcal{F}_\nu\right], \quad P\text{-a.s.}\end{aligned}$$

(2) *There exists a sequence* $\{Q_n^{(k)}\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu^k$ *such that*

$$(3.10)\qquad \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu) = \lim_{n\to\infty}\downarrow R^{Q_n^{(k)}}(\nu), \quad P\text{-a.s.}$$

Let us fix $\nu \in \mathcal{S}_{0,T}$. For any $k \in \mathbb{N}$, the infimum of the family $\{\tau^Q(\nu)\}_{Q\in\mathcal{Q}_\nu^k}$ of optimal stopping times in Proposition 3.2 can be approached by a decreasing sequence in this family. As a result, the infimum is also a stopping time.

LEMMA 3.6. *Let* $\nu \in \mathcal{S}_{0,T}$ *and* $k \in \mathbb{N}$. *There exists a sequence* $\{Q_n^{(k)}\}_{n\in\mathbb{N}} \subseteq \mathcal{Q}_\nu^k$ *such that*

$$\tau_k(\nu) \stackrel{\triangle}{=} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} \tau^Q(\nu) = \lim_{n\to\infty}\downarrow \tau^{Q_n^{(k)}}(\nu), \quad P\text{-a.s.}$$

in the notation of Proposition 3.2, *thus* $\tau_k(\nu) \in \mathcal{S}_{\nu,T}$.

Since $\{\mathcal{Q}_\nu^k\}_{k\in\mathbb{N}}$ is an increasing sequence, $\{\tau_k(\nu)\}_{k\in\mathbb{N}}$ is in turn a decreasing sequence. Hence,

$$(3.11)\qquad \tau(\nu) \stackrel{\triangle}{=} \lim_{k\to\infty}\downarrow \tau_k(\nu)$$

defines a stopping time in $\mathcal{S}_{\nu,T}$. The family of stopping times $\{\tau(\nu)\}_{\nu\in\mathcal{S}_{0,T}}$ will play a crucial role in this section.

The next lemma is concerned with the *pasting* of two probability measures.

LEMMA 3.7. *Given* $\nu \in \mathcal{S}_{0,T}$, *let* $\widetilde{Q} \in \mathcal{Q}_\nu^k$ *for some* $k \in \mathbb{N}$. *For any* $Q \in \mathcal{Q}_\nu$ *and* $\gamma \in \mathcal{S}_{\nu,T}$, *the predictable process*

$$\theta_t^{Q'} \stackrel{\triangle}{=} \mathbf{1}_{\{t \le \gamma\}} \theta_t^Q + \mathbf{1}_{\{t > \gamma\}} \theta_t^{\widetilde{Q}}, \quad t \in [0,T] \tag{3.12}$$

induces a probability measure $Q' \in \mathcal{Q}_\nu$ *by* $dQ' \stackrel{\triangle}{=} \mathscr{E}(\theta^{Q'} \bullet B)_T \, dP$. *If* Q *belongs to* $\mathcal{Q}_\nu^k$, *so does* Q'. *Moreover, for any* $\sigma \in \mathcal{S}_{\gamma,T}$, *we have*

$$R_\sigma^{Q',0} = R^{Q'}(\sigma) = R^{\widetilde{Q}}(\sigma) = R_\sigma^{\widetilde{Q},0}, \quad P\text{-a.s.} \tag{3.13}$$

REMARK 3.8. The probability measure Q' in Lemma 3.7 is called the pasting of Q and $\widetilde{Q}$; see, for example, Section 6.7 of [11]. In general, $\mathcal{Q}_\nu$ is not closed under such "pasting."

The proofs of the following results use schemes similar to the ones in [15]. The main technical difficulty in our case is mentioned in Remark 3.8. Moreover, in order to use the results of [15] directly, we would have to assume that f and the θ^Q's are all bounded. We overcome these difficulties by using approximation arguments that rely on *truncation* and *localization* techniques.

First, we shall show that at any $\nu \in \mathcal{S}_{0,T}$ we have $\underline{V}(\nu) = \overline{V}(\nu)$, P-a.s.

THEOREM 3.9 (Existence of value). *For any* $\nu \in \mathcal{S}_{0,T}$, *we have*

$$\begin{aligned} \underline{V}(\nu) &= \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_\nu} E_Q\left[Y_{\tau(\nu)} + \int_\nu^{\tau(\nu)} f(s, \theta_s^Q)\, ds \Big| \mathcal{F}_\nu \right] \\ &= \overline{V}(\nu) \ge Y_\nu, \quad P\text{-a.s.} \end{aligned} \tag{3.14}$$

Therefore, the stopping time $\tau(\nu)$ *of* (3.11) *is optimal for the robust optimal stopping problem* (3.1) (*i.e.*, *attains the essential infimum there*).

We shall denote the common value in (3.14) by $V(\nu)$ $(= \underline{V}(\nu) = \overline{V}(\nu))$.

PROPOSITION 3.10. *For any* $\nu \in \mathcal{S}_{0,T}$, *we have* $V(\tau(\nu)) = Y_{\tau(\nu)}$, P*-a.s.*

Note that $\tau(\nu)$ may not be the first time after ν when the value process coincides with the reward process. Actually, since the value process $\{V(t)\}_{t \in [0,T]}$ is not necessarily right-continuous, the random time $\inf\{t \in [\nu, T] : V(t) = Y_t\}$ may not even be a stopping time. We address this issue in the next three results.

PROPOSITION 3.11. *Given* $\nu \in \mathcal{S}_{0,T}$, $Q \in \mathcal{Q}_\nu$, *and* $\gamma \in \mathcal{S}_{\nu,\tau(\nu)}$, *we have*

$$E_Q\left[V(\gamma) + \int_\nu^\gamma f(s, \theta_s^Q)\, ds \Big| \mathcal{F}_\nu \right] \ge V(\nu), \quad P\text{-a.s.} \tag{3.15}$$

LEMMA 3.12. *For any $\nu,\gamma,\sigma\in\mathcal{S}_{0,T}$, we have the P-a.s. equalities*

$$\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]$$
$$=\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma}E_Q\left[Y_{\sigma\vee\gamma}+\int_\gamma^{\sigma\vee\gamma}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\gamma\right] \tag{3.16}$$

and

$$\mathbf{1}_{\{\nu=\gamma\}}V(\nu)=\mathbf{1}_{\{\nu=\gamma\}}V(\gamma). \tag{3.17}$$

Next, we show that for any given $\nu\in\mathcal{S}_{0,T}$, the process $\{\mathbf{1}_{\{t\ge\nu\}}V(\tau(\nu)\wedge t)\}_{t\in[0,T]}$ admits an RCLL modification $V^{0,\nu}$. As a consequence, the first time after ν when the process $V^{0,\nu}$ coincides with the process Y, is an optimal stopping time for the robust optimal stopping problem (3.1).

THEOREM 3.13 (Regularity of the value). *Let us fix a stopping time $\nu\in\mathcal{S}_{0,T}$.*

(1) *The process $\{\mathbf{1}_{\{t\ge\nu\}}V(\tau(\nu)\wedge t)\}_{t\in[0,T]}$ admits an RCLL modification $V^{0,\nu}$ such that, for any $\gamma\in\mathcal{S}_{0,T}$:*

$$V_\gamma^{0,\nu}=\mathbf{1}_{\{\gamma\ge\nu\}}V\big(\tau(\nu)\wedge\gamma\big),\quad P\text{-a.s.} \tag{3.18}$$

(2) *Consequently,*

$$\tau_V(\nu)\triangleq\inf\{t\in[\nu,T]:V_t^{0,\nu}=Y_t\} \tag{3.19}$$

is a stopping time which, in fact, attains the essential infimum in (3.1).

We should point out that, in order to determine the optimal stopping time in (1.1), knowledge of the function f in the representation (2.1) is not necessary. Indeed, let the ρ-Snell envelope be the RCLL modification of $\operatorname{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}}(-\rho_{\nu,\gamma}(Y_\gamma))$, $\nu\in\mathcal{S}_{0,T}$. From our results above, the first time after ν that the ρ-Snell envelope touches the reward process Y is an optimal stopping time; this is consistent with the classical theory of optimal stopping.

4. The saddle point problem

In this section, we will contruct a saddle point of the stochastic game in (1.2). As in the previous section, we shall assume here that $f:[0,T]\times\Omega\times\mathbb{R}^d\to[0,\infty]$ is a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)/\mathscr{B}([0,\infty])$-measurable function which satisfies ($\mathfrak{f}$3). For any given $Q\in\mathcal{Q}_0$ and $\nu\in\mathcal{S}_{0,T}$, let us denote

$$Y_\nu^Q\triangleq Y_\nu+\int_0^\nu f(s,\theta_s^Q)\,ds\quad\text{and}\quad V^Q(\nu)\triangleq V(\nu)+\int_0^\nu f(s,\theta_s^Q)\,ds.$$

DEFINITION 4.1. A pair $(Q^*,\sigma_*)\in\mathcal{Q}_0\times\mathcal{S}_{0,T}$ is called a saddle point for the stochastic game suggested by (3.1), if for every $Q\in\mathcal{Q}_0$ and $\nu\in\mathcal{S}_{0,T}$ we have

$$E_{Q^*}(Y_\nu^{Q^*})\le E_{Q^*}(Y_{\sigma_*}^{Q^*})\le E_Q(Y_{\sigma_*}^Q). \tag{4.1}$$

THEOREM 4.2 (Sufficient conditions for a saddle point). *A pair* $(Q^*, \sigma_*) \in \mathcal{Q}_0 \times \mathcal{S}_{0,T}$ *is a saddle point for the stochastic game suggested by* (3.1), *if the following conditions are satisfied*:

(i) $Y_{\sigma_*} = R^{Q^*}(\sigma_*)$, P-*a.s.*;
(ii) *for any* $Q \in \mathcal{Q}_0$, *we have* $V(0) \le E_Q[V^Q(\sigma_*)]$;
(iii) *for any* $\nu \in \mathcal{S}_{0,\sigma_*}$, *we have* $V^{Q^*}(\nu) = E_{Q^*}[V^{Q^*}(\sigma_*)|\mathcal{F}_\nu]$, P-*a.s.*

To construct a saddle point, we need the following two notions.

DEFINITION 4.3 (Bounded mean oscillation). We call $\mathcal{Z} \in \widehat{\mathbb{H}}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ a BMO (short for Bounded Mean Oscillation) process if

$$\|\mathcal{Z}\|_{\mathrm{BMO}} \stackrel{\triangle}{=} \sup_{\tau \in \mathcal{S}_{0,T}} \left\| E\left[\int_\tau^T |\mathcal{Z}_s|^2\,ds \Big| \mathcal{F}_\tau\right]^{1/2} \right\|_\infty < \infty.$$

When $\mathcal{Z}$ is a BMO process, $\mathcal{Z} \bullet B$ is a BMO martingale; see, for example, [17].

DEFINITION 4.4 (BSDE with reflection). Let $h : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a $\widehat{\mathscr{P}} \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function. Given $S \in \mathbb{C}^0_{\mathbf{F}}[0,T]$ and $\xi \in \mathbb{L}^0(\mathcal{F}_T)$ with $\xi \ge S_T$, P-a.s., a triple $(\Gamma, \mathcal{Z}, K) \in \mathbb{C}^0_{\mathbf{F}}[0,T] \times \widehat{\mathbb{H}}^2_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ is called a solution to the reflected backward stochastic differential equation with terminal condition ξ, generator h, and obstacle S (RBSDE (ξ, h, S) for short), if P-a.s., we have the comparison

$$S_t \le \Gamma_t = \xi + \int_t^T h(s, \Gamma_s, \mathcal{Z}_s)\,ds + K_T - K_t - \int_t^T \mathcal{Z}_s\,dB_s, \quad t \in [0,T],$$

and the so-called *flat-off condition*

$$\int_0^T \mathbf{1}_{\{\Gamma_s > S_s\}}\,dK_s = 0, \quad P\text{-a.s.}$$

In the rest of this section, we shall assume that the reward process $Y \in \mathbb{L}^\infty_{\mathbf{F}}[0,T]$ is continuous and that the function $f : [0,T] \times \Omega \times \mathbb{R}^d \to [0,\infty]$ satisfies the following additional conditions:

(H1) For every $(t,\omega) \in [0,T] \times \Omega$, the mapping $z \mapsto f(t,\omega,z)$ is continuous.

(H2) It holds $dt \otimes dP$-a.e. that

$$f(t,\omega,z) \ge \varepsilon |z - \Upsilon_t(\omega)|^2 - \ell \quad \forall z \in \mathbb{R}^d.$$

Here, $\varepsilon > 0$ is a real constant, Υ is an $\mathbb{R}^d$-valued process which satisfies $\|\Upsilon\|_\infty \stackrel{\triangle}{=} \operatorname{ess\,sup}_{(t,\omega) \in [0,T] \times \Omega} |\Upsilon_t(\omega)| < \infty$, and $\ell \ge \varepsilon \|\Upsilon\|^2_\infty$.

(H3) For any $(t,\omega,u) \in [0,T] \times \Omega \times \mathbb{R}^d$, the mapping $z \mapsto f(t,\omega,z) + \langle u, z \rangle$ attains its infimum over $\mathbb{R}^d$ at some $z^* = z^*(t,\omega,u) \in \mathbb{R}^d$, namely,

$$\widetilde{f}(t,\omega,u) \stackrel{\triangle}{=} \inf_{z \in \mathbb{R}^d} \big(f(t,\omega,z) + \langle u, z \rangle\big) = f(t,\omega,z^*(t,\omega,u)) + \langle u, z^*(t,\omega,u) \rangle. \tag{4.2}$$

Without loss of generality, we can assume that the mapping $z^* : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R}^d)$-measurable thanks to the Measurable Selection theorem [see, e.g., Lemma 1 of [3], or Lemma 16.34 of [10]]. We further assume that there exist a nonnegative BMO process ψ and a $M > 0$ such that for $dt \otimes dP$-a.e. $(t,\omega) \in [0,T] \times \Omega$

$$|z^*(t,\omega,u)| \le \psi_t(\omega) + M|u| \quad \forall u \in \mathbb{R}^d.$$

EXAMPLE 4.5. Let $\lambda \ge 0$ and let $\Lambda, \Upsilon \in \mathbb{H}^\infty_{\mathbf{F}}([0,T];\mathbb{R}^d)$ with $\Lambda_t(\omega) \ge \varepsilon > 0$, $dt \otimes dP$-a.e. Define

$$f(t,\omega,z) \stackrel{\triangle}{=} \Lambda_t(\omega)\big(|z - \Upsilon_t(\omega)|^{2+\lambda} - |\Upsilon_t(\omega)|^{2+\lambda}\big) \quad \forall (t,\omega,z) \in [0,T] \times \Omega \times \mathbb{R}^d.$$

Clearly, $f^+ = f \vee 0$ is a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}([0,\infty])$-measurable function that satisfies ($\mathfrak{f}$3) and (H1). It turns out that f^+ satisfies (H2), since $dt \otimes dP$-a.e. we have that

$$\begin{aligned} f^+(t,\omega,z) \ge f(t,\omega,z) &\ge \Lambda_t(\omega)\big(|z - \Upsilon_t(\omega)|^2 - 1\big) - \Lambda_t(\omega)|\Upsilon_t(\omega)|^{2+\lambda} \\ &\ge \varepsilon|z - \Upsilon_t(\omega)|^2 - \|\Lambda\|_\infty(1 + \|\Upsilon\|_\infty^{2+\lambda}) \quad \forall z \in \mathbb{R}^d. \end{aligned}$$

For any $(t,\omega,u) \in [0,T] \times \Omega \times \mathbb{R}^d$ the gradient

$$\nabla_z\big(f(t,\omega,z) + \langle u,z\rangle\big) = (2+\lambda)\Lambda_t(\omega)|z - \Upsilon_t(\omega)|^\lambda\big(z - \Upsilon_t(\omega)\big) + u \quad \forall z \in \mathbb{R}^d,$$

is null only at $\hat{z}(t,\omega,u) = -[(2+\lambda)\Lambda_t(\omega)]^{-\frac{1}{1+\lambda}}|u|^{-\frac{\lambda}{1+\lambda}}u + \Upsilon_t(\omega)$, where the mapping $z \mapsto f(t,\omega,z) + \langle u,z\rangle$ attains its infimum over $\mathbb{R}^d$. When $|u| \ge r_t(\omega) \stackrel{\triangle}{=} (2+\lambda)\Lambda_t(\omega)|\Upsilon_t(\omega)|^{1+\lambda}$, we have

$$\hat{z}(t,\omega,u) \in A \stackrel{\triangle}{=} \{z \in \mathbb{R}^d : |z - \Upsilon_t(\omega)| \ge |\Upsilon_t(\omega)|\}.$$

It follows that

$$\begin{aligned} \inf_{z\in\mathbb{R}^d}\big(f^+(t,\omega,z) + \langle u,z\rangle\big) &\le f^+(t,\omega,\hat{z}(t,\omega,u)) + \langle u,\hat{z}(t,\omega,u)\rangle \\ &= f(t,\omega,\hat{z}(t,\omega,u)) + \langle u,\hat{z}(t,\omega,u)\rangle \\ &= \inf_{z\in\mathbb{R}^d}\big(f(t,\omega,z) + \langle u,z\rangle\big) \\ &\le \inf_{z\in\mathbb{R}^d}\big(f^+(t,\omega,z) + \langle u,z\rangle\big). \end{aligned} \tag{4.3}$$

On the other hand, when $|u| < r_t(\omega)$ or equivalently $\hat{z}(t,\omega,u) \notin A$, the gradient $\nabla_z(f(t,\omega,z) + \langle u,z\rangle) \ne 0$ for any $z \in A$, which implies that the mapping $z \mapsto f(t,\omega,z) + \langle u,z\rangle$ cannot attain its infimum over A at an interior point of A. Thus,

$$\inf_{z\in A}\big(f(t,\omega,z) + \langle u,z\rangle\big) = \inf_{z\in\partial A}\big(f(t,\omega,z) + \langle u,z\rangle\big) = \inf_{z\in\partial A}\langle u,z\rangle.$$

Then it follows that

$$\inf_{z\in\mathbb{R}^d}\big(f^+(t,\omega,z) + \langle u,z\rangle\big) = \inf_{z\in A^c}\langle u,z\rangle \wedge \inf_{z\in A}\big(f(t,\omega,z) + \langle u,z\rangle\big) = \inf_{z\in\overline{A^c}}\langle u,z\rangle.$$

The latter infimum is attained uniquely at some $\widetilde{z}(t,\omega,u)\in\overline{A^c}$, which together with (4.3) implies that

$$z^*(t,\omega,u)=\mathbf{1}_{\{|u|\ge r_t(\omega)\}}\hat{z}(t,\omega,u)+\mathbf{1}_{\{|u|<r_t(\omega)\}}\widetilde{z}(t,\omega,u).$$

Therefore, f^+ satisfies (H3), since for $dt\otimes dP$-a.e. $(t,\omega)\in[0,T]\times\Omega$ we have

$$\begin{aligned}|z^*(t,\omega,u)|&\le|\hat{z}(t,\omega,u)|+|\widetilde{z}(t,\omega,u)|\le\big((2+\lambda)\varepsilon\big)^{-\frac{1}{1+\lambda}}|u|^{\frac{1}{1+\lambda}}+3\|\Upsilon\|_\infty\\&\le\big((2+\lambda)\varepsilon\big)^{-\frac{1}{1+\lambda}}|u|+\big((2+\lambda)\varepsilon\big)^{-\frac{1}{1+\lambda}}+3\|\Upsilon\|_\infty\quad\forall u\in\mathbb{R}^d.\end{aligned}$$

REMARK 4.6. The "entropic" risk measure with risk tolerance coefficient $r>0$, namely

$$\rho^r_{\nu,\gamma}(\xi)\stackrel{\triangle}{=}r\log\big\{E\big[e^{-\frac{1}{r}\xi}|\mathcal{F}_\nu\big]\big\},\quad \xi\in\mathbb{L}^\infty(\mathcal{F}_\gamma),$$

is a typical example of a dynamic convex risk measures satisfying (A1)–(A4). The corresponding f in (2.1) is $f(z)=\frac{r}{2}|z|^2$, $z\in\mathbb{R}^d$.

EXAMPLE 4.7. Let b^1, b^2 be two real-valued processes such that $-\varpi\le b^1_t(\omega)\le 0\le b^2_t(\omega)\le\varpi$, $dt\otimes dP$-a.e. for some $\varpi>0$. Let $\varphi:[0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ be a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$-measurable function that satisfies the following two assumptions:

(i) For any $(t,\omega)\in[0,T]\times\Omega$, $\varphi(t,\omega,\cdot)$ is a bijective locally-integrable function or a continuous surjective locally-integrable function on $\mathbb{R}$.

(ii) For some $\varepsilon_1,\varepsilon_2>0$, it holds $dt\otimes dP$-a.e. that

$$\varphi(t,\omega,x)\begin{cases}\ge\big(2\varepsilon_1x+b^1_t(\omega)\big)\vee 0, & \text{if } x>0,\\ \le\big(2\varepsilon_2x+b^2_t(\omega)\big)\wedge 0, & \text{if } x<0.\end{cases}$$

Then $f(t,\omega,z)\stackrel{\triangle}{=}\int_0^z\varphi(t,\omega,x)\,dx$, $z\in\mathbb{R}$ defines a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})/\mathscr{B}([0,\infty])$-measurable nonnegative function that satisfies ($\mathfrak{f}$3) and (H1). Let $\varepsilon=\varepsilon_1\wedge\varepsilon_2$. For $dt\otimes dP$-a.e. $(t,\omega)\in[0,T]\times\Omega$, if $z>0$, then

$$\begin{aligned}f(t,\omega,z)&\ge\int_0^z\big(2\varepsilon_1x+b^1_t(\omega)\big)\,dx=\varepsilon_1z^2+b^1_t(\omega)z\\&\ge\varepsilon z^2-\varpi z=\varepsilon\left(z-\frac{\varpi}{2\varepsilon}\right)^2-\frac{\varpi^2}{4\varepsilon};\end{aligned}$$

on the other hand, if $z<0$, then

$$\begin{aligned}f(t,\omega,z)&=-\int_z^0\varphi(t,\omega,x)\,dx\ge-\int_z^0\big(2\varepsilon_2x+b^2_t(\omega)\big)\,dx\\&=\varepsilon_2z^2+b^2_t(\omega)z\ge\varepsilon z^2+\varpi z\\&=\frac{1}{2}\varepsilon\left(z-\frac{\varpi}{2\varepsilon}\right)^2+\frac{1}{2}\varepsilon\left(z+\frac{3\varpi}{2\varepsilon}\right)^2-\frac{5\varpi^2}{4\varepsilon}.\end{aligned}$$

Thus, it holds $dt \otimes dP$-a.e. that $f(t,\omega,z) \ge \frac{1}{2}\varepsilon(z-\frac{\varpi}{2\varepsilon})^2 - \frac{5\varpi^2}{4\varepsilon}$, that is, (H2) is satisfied.

For any $(t,\omega,u)\in[0,T]\times\Omega\times\mathbb{R}$, since $\frac{d}{dz}(f(t,\omega,z)+uz)=\varphi(t,\omega,z)+u$, the mapping $z\mapsto f(t,\omega,z)+uz$ attains its infimum over $\mathbb{R}$ at each $z\in\{z\in\mathbb{R}:\varphi(t,\omega,z)=x\}$. Thus $\varphi_-^{-1}(t,\omega,x)\le z^*(t,\omega,u)\le\varphi_+^{-1}(t,\omega,x)$, where

$$\varphi_-^{-1}(t,\omega,x)\stackrel{\triangle}{=}\inf\{z\in\mathbb{R}:\varphi(t,\omega,z)=x\}$$

and

$$\varphi_+^{-1}(t,\omega,x)\stackrel{\triangle}{=}\sup\{z\in\mathbb{R}:\varphi(t,\omega,z)=x\}.$$

It is clear that $\varphi(t,\omega,\varphi_-^{-1}(t,\omega,x))=x$ and $\varphi(t,\omega,\varphi_+^{-1}(t,\omega,x))=x$. For $dt\otimes dP$-a.e. $(t,\omega)\in[0,T]\times\Omega$ and $u\in\mathbb{R}$, if $\varphi_-^{-1}(t,\omega,x)>0$, then

$$-u=\varphi(t,\omega,\varphi^{-1}(t,\omega,-u))\ge 2\varepsilon_1\varphi_-^{-1}(t,\omega,x)+b_t^1(\omega),$$

which implies that $0<\varphi_-^{-1}(t,\omega,x)\le\frac{1}{2\varepsilon}(|u|+\varpi)$. On the other hand, if $\varphi_-^{-1}(t,\omega,x)<0$, one can deduce that $-\frac{1}{2\varepsilon}(|u|+\varpi)\le\varphi_-^{-1}(t,\omega,x)<0$ by a similar argument. Hence, $|\varphi_-^{-1}(t,\omega,x)|\le\frac{1}{2\varepsilon}(|u|+\varpi)$. Similarly, this inequality also holds for $\varphi_+^{-1}(t,\omega,x)$, thus for $z^*(t,\omega,u)$. As a result, (H3) is also satisfied.

One can easily deduce from (H2) and ($\mathfrak{f}$3) that $dt\otimes dP$-a.e.

$$-\frac{1+\varepsilon}{4\varepsilon}|u|^2-\|\Upsilon\|_\infty^2-\ell\le\widetilde{f}(t,\omega,u)\le 0\quad\forall u\in\mathbb{R}^d,$$

which shows that $\widetilde{f}$ has quadratic growth in u. Thanks to Theorems 1 and 3 of [19], the RBSDE $(Y_T,\widetilde{f},Y)$ admits a solution $(\widetilde{\Gamma},\widetilde{\mathcal{Z}},\widetilde{K})\in\mathbb{C}_{\mathbf{F}}^\infty[0,T]\times\mathbb{H}_{\mathbf{F}}^2([0,T];\mathbb{R}^d)\times\mathbb{K}_{\mathbf{F}}[0,T]$.

In fact, $\widetilde{\mathcal{Z}}$ is a BMO process. To see this, we set $\kappa\stackrel{\triangle}{=}\frac{1+\varepsilon}{4\varepsilon}\vee(\|\Upsilon\|^2+\ell)$. For any $\nu\in\mathcal{S}_{0,T}$, applying Itô's formula to $e^{-4\kappa\widetilde{\Gamma}_t}$ we get

$$\begin{aligned}
&e^{-4\kappa\widetilde{\Gamma}_\nu}+8\kappa^2\int_\nu^T e^{-4\kappa\widetilde{\Gamma}_s}|\widetilde{\mathcal{Z}}_s|^2\,ds\\
&\quad=e^{-4\kappa Y_T}-4\kappa\int_\nu^T e^{-4\kappa\widetilde{\Gamma}_s}\widetilde{f}(s,\widetilde{\mathcal{Z}}_s)\,ds-4\kappa\int_\nu^T e^{-4\kappa\widetilde{\Gamma}_s}\,d\widetilde{K}_s\\
&\qquad+4\kappa\int_\nu^T e^{-4\kappa\widetilde{\Gamma}_s}\widetilde{\mathcal{Z}}_s\,dB_s\\
&\quad\le e^{-4\kappa Y_T}+4\kappa^2\int_\nu^T e^{-4\kappa\widetilde{\Gamma}_s}(1+|\widetilde{\mathcal{Z}}_s|^2)\,ds+4\kappa\int_\nu^T e^{-4\kappa\widetilde{\Gamma}_s}\widetilde{\mathcal{Z}}_s\,dB_s.
\end{aligned}$$

Taking conditional expectations in the above expression, we obtain

$$\begin{aligned}
e^{-4\kappa\|\widetilde{\Gamma}\|_\infty}E\left[\int_\nu^T|\widetilde{\mathcal{Z}}_s|^2\,ds\middle|\mathcal{F}_\nu\right]&\le E\left[\int_\nu^T e^{-4\kappa\widetilde{\Gamma}_s}|\widetilde{\mathcal{Z}}_s|^2\,ds\middle|\mathcal{F}_\nu\right]\\
&\le\frac{1}{4\kappa^2}E[e^{-4\kappa Y_T}|\mathcal{F}_\nu]+e^{4\kappa\|\widetilde{\Gamma}\|_\infty}T
\end{aligned}$$

which implies that $\|\widetilde{\mathcal{Z}}\|_{\mathrm{BMO}} \le e^{4\kappa\|\widetilde{\Gamma}\|_\infty}(\frac{1}{4\kappa^2}+T)^{1/2}$.

Since the mapping $z^*:[0,T]\times\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R}^d)$-measurable (see (H3)),

$$\theta_t^*(\omega) \stackrel{\triangle}{=} z^*(t,\omega,\widetilde{\mathcal{Z}}_t(\omega)), \quad (t,\omega)\in[0,T]\times\Omega \tag{4.4}$$

is a predictable process. It follows from (H3) that for any $\nu\in[0,T]$

$$E\left[\int_\nu^T |\theta_s^*|^2\,ds\Big|\mathcal{F}_\nu\right] \le 2E\left[\int_\nu^T \psi_s^2\,ds\Big|\mathcal{F}_\nu\right] + 2M^2E\left[\int_\nu^T |\widetilde{\mathcal{Z}}_s|^2\,ds\Big|\mathcal{F}_\nu\right], \quad P\text{-a.s.},$$

which implies that θ^* is a BMO process.

Fix $\nu\in\mathcal{S}_{0,T}$. Since $\theta_t^{*,\nu}\stackrel{\triangle}{=}\mathbf{1}_{\{t>\nu\}}\theta_t^*$, $t\in[0,T]$ is also a BMO process, we know from Theorem 2.3 of [17] that the stochastic exponential $\{\mathscr{E}(\theta^{*,\nu}\bullet B)_t\}_{t\in[0,T]}$ is a uniformly integrable martingale. Therefore, $dQ^{*,\nu}\stackrel{\triangle}{=}\mathscr{E}(\theta^{*,\nu}\bullet B)_T\,dP$ defines a probability measure $Q^{*,\nu}\in\mathcal{P}_\nu$. As

$$\widetilde{f}(s,\widetilde{\mathcal{Z}}_s) = f(s,z^*(s,\widetilde{\mathcal{Z}}_s)) + \langle\widetilde{\mathcal{Z}}_s, z^*(s,\widetilde{\mathcal{Z}}_s)\rangle = f(s,\theta_s^*) + \langle\widetilde{\mathcal{Z}}_s,\theta_s^*\rangle, \quad dt\otimes dP\text{-a.e.}$$

by (4.2), (4.4), and the Girsanov theorem, we can deduce

$$\begin{aligned}\widetilde{\Gamma}_{\nu\vee t} &= Y_T + \int_{\nu\vee t}^T [f(s,\theta_s^{*,\nu}) + \langle\widetilde{\mathcal{Z}}_s,\theta_s^{*,\nu}\rangle]\,ds \\ &\quad + \widetilde{K}_T - \widetilde{K}_{\nu\vee t} - \int_{\nu\vee t}^T \widetilde{\mathcal{Z}}_s\,dB_s \\ &= Y_T + \int_{\nu\vee t}^T f(s,\theta_s^{*,\nu})\,ds + \widetilde{K}_T - \widetilde{K}_{\nu\vee t} \\ &\quad - \int_{\nu\vee t}^T \widetilde{\mathcal{Z}}_s\,dB_s^{Q^{*,\nu}}, \quad t\in[0,T],\end{aligned} \tag{4.5}$$

where $B^{Q^{*,\nu}}$ is a Brownian Motion under $Q^{*,\nu}$. Letting $t=0$ and taking the expectation $E_{Q^{*,\nu}}$ yield that

$$E_{Q^{*,\nu}}\int_\nu^T f(s,\theta_s^{*,\nu})\,ds \le E_{Q^{*,\nu}}(\widetilde{\Gamma}_\nu - Y_T) \le 2\|\widetilde{\Gamma}\|_\infty,$$

thus $Q^{*,\nu}\in\mathcal{Q}_\nu$. The lemma below shows that $\widetilde{\Gamma}$ is indistinguishable from $R^{Q^{*,\nu},0}$ on the stochastic interval $[\![\nu,T]\!]$.

LEMMA 4.8. *Given $\nu\in\mathcal{S}_{0,T}$, it holds P-a.s. that*

$$\widetilde{\Gamma}_t = R_t^{Q^{*,\nu},0} \quad \forall t\in[\nu,T]. \tag{4.6}$$

Let $k\in\mathbb{N}$ and $Q\in\mathcal{Q}_\nu^k$. It is easy to see that the function $h_Q(s,\omega,z)\stackrel{\triangle}{=} f(s,\omega,\theta_s^Q(\omega)) + \langle z,\theta_s^Q(\omega)\rangle$ is Lipschitz continuous in z: to wit, for $dt\otimes dP$-a.e. $(t,\omega)\in[0,T]\times\Omega$, it holds for any $z,z'\in\mathbb{R}^d$ that

$$|h_Q(s,\omega,z) - h_Q(s,\omega,z')| = |\langle z-z',\theta_s^Q\rangle| \le |\theta_s^Q|\cdot|z-z'| \le k|z-z'|.$$

Moreover, we have

$$E\int_0^T |h_Q(s,0)|^2\,ds = E\int_0^T |f(s,\theta_s^Q)|^2\,ds = E\int_\nu^T |f(s,\theta_s^Q)|^2\,ds \le k^2 T.$$

Theorem 5.2 of [9] assures now that there exists a unique solution $(\Gamma^Q, \mathcal{Z}^Q, K^Q) \in \mathbb{C}^2_{\mathbf{F}}[0,T] \times \mathbb{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ to the RBSDE(Y_T, h_Q, Y). Fix $t \in [0,T]$. For any $\gamma \in \mathcal{S}_{t,T}$, the Girsanov theorem implies

$$\begin{aligned}\Gamma_t^Q &= Y_T + \int_t^T h_Q(s, \mathcal{Z}_s^Q)\,ds + K_T^Q - K_t^Q - \int_t^T \mathcal{Z}_s^Q\,dB_s\\ &= \Gamma_\gamma^Q + \int_t^\gamma f(s,\theta_s^Q)\,ds + K_\gamma^Q - K_t^Q - \int_t^\gamma \mathcal{Z}_s^Q\,dB_s^Q, \quad P\text{-a.s.,}\end{aligned}$$

where B^Q is a Brownian Motion under Q. By analogy with Lemma 4.8, it holds P-a.s. that

$$\Gamma_t^Q = R_t^{Q,0} \quad \forall t \in [0,T]. \tag{4.7}$$

In particular, we see that $R^{Q,0}$ is, in fact, a continuous process.

Next, we recall a comparison theorem of RBSDEs; see Theorem 4.1 of [9]. (We restate it in a more general form.)

PROPOSITION 4.9. *Let* $(\Gamma, \mathcal{Z}, K)$ (*resp.* $(\Gamma', \mathcal{Z}', K')$) $\in \mathbb{C}^2_{\mathbf{F}}[0,T] \times \widehat{\mathbb{H}}^2_{\mathbf{F}}([0,T]; \mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}}[0,T]$ *be a solution of RBSDE* (ξ, h, S) (*resp. RBSDE* (ξ', h', S')) *in the sense of Definition* 4.4. *Additionally, assume that*

(i) *either h or h' is Lipschitz in (y,z)*;

(ii) *it holds P-a.s. that $\xi \le \xi'$ and $S_t \le S_t'$ for any $t \in [0,T]$*;

(iii) *it holds $dt \otimes dP$-a.e. that $h(t,\omega,y,z) \le h'(t,\omega,y,z)$ for any $(y,z) \in \mathbb{R} \times \mathbb{R}^d$.*

Then it holds P-a.s. that $\Gamma_t \le \Gamma_t'$ for any $t \in [0,T]$.

Since it holds $dt \otimes dP$-a.e. that

$$\begin{aligned}\widetilde{f}(t,\omega,u) &\stackrel{\triangle}{=} \inf_{z\in\mathbb{R}^d}\bigl(f(t,\omega,z) + \langle u, z\rangle\bigr)\\ &\le f(s,\omega,\theta_s^Q(\omega)) + \langle u, \theta_s^Q(\omega)\rangle = h_Q(s,\omega,u) \quad \forall u \in \mathbb{R}^d\end{aligned}$$

we see from Proposition 4.9 and (4.7) that we have P-a.s.

$$\widetilde{\Gamma}_t \le \Gamma_t^Q = R_t^{Q,0} \quad \forall t \in [0,T]. \tag{4.8}$$

Letting $t = \nu$, taking the essential infimum of the right-hand side over $Q \in \mathcal{Q}_\nu^k$, and then letting $k \to \infty$, we can deduce from Lemma 4.8, (3.8), and (3.3) that

$$\begin{aligned}R_\nu^{Q^{*,\nu},0} = \widetilde{\Gamma}_\nu &\le \lim_{k\to\infty} \downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} R_\nu^{Q,0} = \lim_{k\to\infty} \downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu)\\ &= \overline{V}(\nu) = V(\nu) \le R^{Q^{*,\nu}}(\nu) = R_\nu^{Q^{*,\nu},0}, \quad P\text{-a.s.}\end{aligned}$$

which implies that $V(\nu) = \widetilde{\Gamma}_\nu$, P-a.s. Applying Lemma 4.8 and using (3.3), we see that the value process $V(\cdot)$ of Theorem 3.9 is connected to the solution of a BSDE with Reflection:

$$V(\nu) = \widetilde{\Gamma}_\nu = R_\nu^{Q^*,0} = R^{Q^*}(\nu), \quad P\text{-a.s.} \tag{4.9}$$

where $Q^* \stackrel{\triangle}{=} Q^{*,0} \in \mathcal{Q}_0$. It is clear that $dQ^* = dQ^{*,0} = \mathscr{E}(\theta^{*,0} \bullet B)_T\, dP = \mathscr{E}(\theta^* \bullet B)_T\, dP$.

We are now ready to state the main result of this section.

THEOREM 4.10 (Existence of a saddle point). *The pair* $(Q^*, \tau^{Q^*}(0))$ *is a saddle point as in* (4.1).

5. Proofs

5.1. Proof of the results in Sections 2 and 3.

Proof of Proposition 2.2. From [4, Proposition 1], we know that

$$\rho_{\nu,\gamma}(\xi) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_{\nu,\gamma}} \big(E_Q[-\xi | \mathcal{F}_\nu] - \alpha_{\nu,\gamma}(Q)\big), \quad P\text{-a.s.} \tag{5.1}$$

Here we have set $\mathcal{Q}_{\nu,\gamma} \stackrel{\triangle}{=} \{Q \in \mathcal{P}_\nu : E_Q[\alpha_{\nu,\gamma}(Q)] < \infty\}$, and the quantity

$$\alpha_{\nu,\gamma}(Q) \stackrel{\triangle}{=} \operatorname*{ess\,sup}_{\eta \in \mathbb{L}^\infty(\mathcal{F}_\gamma)} \big(E_Q[-\eta | \mathcal{F}_\nu] - \rho_{\nu,\gamma}(\eta)\big)$$

is known as the "minimal penalty" of $\rho_{\nu,\gamma}$. [The representation (5.1) was shown for $Q \ll P$ rather than $Q \sim P$ in [4]. However, our assumption (A4) assures that (5.1) also holds. For a proof, see [12, Lemma 3.5] and [18, Theorem 3.1].]

Thanks to [7, Theorem 5(i) and the proof of Proposition 9(v)], there exists a nonnegative function $f : [0,T] \times \Omega \times \mathbb{R}^d \to [0,\infty]$ satisfying ($\mathfrak{f}$1)–($\mathfrak{f}$3), such that for each $Q \in \mathcal{Q}_{\nu,\gamma}$ we have

$$\alpha_{\nu,\gamma}(Q) = E_Q\left(\int_\nu^\gamma f(s, \theta_s^Q)\, ds \middle| \mathcal{F}_\nu\right), \quad P\text{-a.s.}$$

Hence, we can rewrite $\mathcal{Q}_{\nu,\gamma} = \{Q \in \mathcal{P}_\nu : E_Q \int_\nu^\gamma f(s, \theta_s^Q)\, ds < \infty\}$, and (5.1) becomes

$$\rho_{\nu,\gamma}(\xi) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_{\nu,\gamma}} E_Q\left[-\xi - \int_\nu^\gamma f(s, \theta_s^Q)\, ds \middle| \mathcal{F}_\nu\right], \quad P\text{-a.s.} \tag{5.2}$$

Since $\mathcal{Q}_\nu \equiv \mathcal{Q}_{\nu,T} \subset \mathcal{Q}_{\nu,\gamma}$, it follows readily that

$$\begin{aligned} &\operatorname*{ess\,inf}_{Q \in \mathcal{Q}_\nu} E_Q\left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q)\, ds \middle| \mathcal{F}_\nu\right] \\ &\qquad \geq \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_{\nu,\gamma}} E_Q\left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q)\, ds \middle| \mathcal{F}_\nu\right], \quad P\text{-a.s.} \end{aligned} \tag{5.3}$$

On the other hand, for any given $Q \in \mathcal{Q}_{\nu,\gamma}$, the predictable process $\theta_t^{\widetilde{Q}} \triangleq \mathbf{1}_{\{t \le \gamma\}} \theta_t^Q$, $t \in [0,T]$ induces a probability measure $\widetilde{Q} \in \mathcal{P}_\nu$ via $d\widetilde{Q} \triangleq \mathscr{E}(\theta^{\widetilde{Q}} \bullet B)_T \, dP$. Since $f(t, \theta_t^{\widetilde{Q}}) = \mathbf{1}_{\{t \le \gamma\}} f(t, \theta_t^Q)$, $dt \otimes dP$-a.e. from ($\mathfrak{f}$3), it follows

$$E_{\widetilde{Q}} \int_\nu^T f(s, \theta_s^{\widetilde{Q}}) \, ds = E_{\widetilde{Q}} \int_\nu^\gamma f(s, \theta_s^Q) \, ds = E_Q \int_\nu^\gamma f(s, \theta_s^Q) \, ds < \infty,$$

thus $\widetilde{Q} \in \mathcal{Q}_\nu$. Then we can deduce that P-a.s.

$$\begin{aligned} \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_\nu} E_Q \left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q) \, ds \Big| \mathcal{F}_\nu \right] &\le E_{\widetilde{Q}} \left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^{\widetilde{Q}}) \, ds \Big| \mathcal{F}_\nu \right] \\ = E_{\widetilde{Q}} \left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q) \, ds \Big| \mathcal{F}_\nu \right] &= E_Q \left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q) \, ds \Big| \mathcal{F}_\nu \right]. \end{aligned}$$

Taking the essential infimum of the right-hand side over $Q \in \mathcal{Q}_{\nu,\gamma}$ yields

$$\begin{aligned} &\operatorname*{ess\,inf}_{Q \in \mathcal{Q}_\nu} E_Q \left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q) \, ds \Big| \mathcal{F}_\nu \right] \\ &\quad \le \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_{\nu,\gamma}} E_Q \left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q) \, ds \Big| \mathcal{F}_\nu \right], \quad P\text{-a.s.}; \end{aligned}$$

this, together with (5.3) and (5.2), proves (2.1). □

Proof of Lemma 3.4. (1) Since $\{\mathcal{Q}_\nu^k\}_{k \in \mathbb{N}}$ is an increasing sequence of sets contained in $\mathcal{Q}_\nu$, it follows that

$$\begin{aligned} &\operatorname*{ess\,inf}_{Q \in \mathcal{Q}_\nu} E_Q \left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q) \, ds \Big| \mathcal{F}_\nu \right] \\ &\quad \le \lim_{k \to \infty} \downarrow \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_\nu^k} E_Q \left[Y_\gamma + \int_\nu^\gamma f(s, \theta_s^Q) \, ds \Big| \mathcal{F}_\nu \right], \quad P\text{-a.s.} \end{aligned} \tag{5.4}$$

Now let us fix a probability measure $Q \in \mathcal{Q}_\nu$, and define the stopping times

$$\delta_m^Q \triangleq \inf \left\{ t \in [\nu, T] : \int_\nu^t [f(s, \theta_s^Q) + |\theta_s^Q|^2] \, ds > m \right\} \wedge T, \quad m \in \mathbb{N}.$$

It is easy to see that $\lim_{m \to \infty} \uparrow \delta_m^Q = T$, P-a.s. For any $(m,k) \in \mathbb{N}^2$, the predictable process $\theta_t^{Q^{m,k}} \triangleq \mathbf{1}_{\{t \le \delta_m^Q\}} \mathbf{1}_{A_{\nu,k}^Q} \theta_t^Q$, $t \in [0,T]$ induces a probability measure $Q^{m,k} \in \mathcal{Q}_\nu^k$ by

$$dQ^{m,k} \triangleq \mathscr{E}(\theta^{Q^{m,k}} \bullet B)_T \cdot dP \tag{5.5}$$

[recall the notation of (3.5)]. It follows from ($\mathfrak{f}$3) that

$$f(t, \theta_t^{Q^{m,k}}) = \mathbf{1}_{\{t \le \delta_m^Q\}} \mathbf{1}_{A_{\nu,k}^Q} f(t, \theta_t^Q), \quad dt \otimes dP\text{-a.e.} \tag{5.6}$$

Then we can deduce from the Bayes Rule [see, e.g., [13, Lemma 3.5.3]] that

$$
\begin{aligned}
(5.7)\qquad & \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} E_Q\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right] \\
&\le E_{Q^{m,k}}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q^{m,k}})\,ds\Big|\mathcal{F}_\nu\right] \\
&= E\left[Z_{\nu,T}^{Q^{m,k}}\left(Y_\gamma + \int_\nu^{\gamma\wedge\delta_m^Q} \mathbf{1}_{A_{\nu,k}^Q} f(s,\theta_s^Q)\,ds\right)\Big|\mathcal{F}_\nu\right] \\
&\le E\left[Z_{\nu,T}^{Q^{m,k}}\left(Y_\gamma + \int_\nu^{\gamma\wedge\delta_m^Q} f(s,\theta_s^Q)\,ds\right)\Big|\mathcal{F}_\nu\right] \\
&= E\left[(Z_{\nu,T}^{Q^{m,k}} - Z_{\nu,\delta_m^Q}^Q)\left(Y_\gamma + \int_\nu^{\gamma\wedge\delta_m^Q} f(s,\theta_s^Q)\,ds\right)\Big|\mathcal{F}_\nu\right] \\
&\quad + E[(Z_{\nu,\delta_m^Q}^Q - Z_{\nu,T}^Q)\cdot Y_\gamma|\mathcal{F}_\nu] + E[Z_{\nu,T}^Q Y_\gamma|\mathcal{F}_\nu] \\
&\quad + E\left[Z_{\nu,\delta_m^Q}^Q \int_\nu^{\gamma\wedge\delta_m^Q} f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right] \\
&\le (\|Y\|_\infty + m)\cdot E[|Z_{\nu,T}^{Q^{m,k}} - Z_{\nu,\delta_m^Q}^Q||\mathcal{F}_\nu] \\
&\quad + \|Y\|_\infty\cdot E[|Z_{\nu,\delta_m^Q}^Q - Z_{\nu,T}^Q||\mathcal{F}_\nu] \\
&\quad + E_Q[Y_\gamma|\mathcal{F}_\nu] + E_Q\left[\int_\nu^{\gamma\wedge\delta_m^Q} f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right] \\
&\le (\|Y\|_\infty + m)\cdot E[|Z_{\nu,T}^{Q^{m,k}} - Z_{\nu,\delta_m^Q}^Q||\mathcal{F}_\nu] \\
&\quad + \|Y\|_\infty\cdot E[|Z_{\nu,\delta_m^Q}^Q - Z_{\nu,T}^Q||\mathcal{F}_\nu] \\
&\quad + E_Q\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right], \quad P\text{-a.s.}
\end{aligned}
$$

From the equation (3.6) and the Dominated Convergence theorem, we observe

$$
\begin{aligned}
&\lim_{k\to\infty} E\left(\int_\nu^{\delta_m^Q} (\mathbf{1}_{A_{\nu,k}^Q} - 1)\theta_s^Q\,dB_s\right)^2 \\
&\quad = \lim_{k\to\infty} E\int_\nu^{\delta_m^Q} (1-\mathbf{1}_{A_{\nu,k}^Q})|\theta_s^Q|^2\,ds = 0, \quad P\text{-a.s.}
\end{aligned}
$$

Thus, we can find a subsequence of $\{A_{\nu,k}^Q\}_{k\in\mathbb{N}}$ (we still denote it by $\{A_{\nu,k}^Q\}_{k\in\mathbb{N}}$) such that P-a.s.

$$
\lim_{k\to\infty}\int_\nu^{\delta_m^Q} \mathbf{1}_{A_{\nu,k}^Q}\theta_s^Q\,dB_s = \int_\nu^{\delta_m^Q} \theta_s^Q\,dB_s
$$

and

$$\lim_{k\to\infty}\int_\nu^{\delta_m^Q}\mathbf{1}_{A_{\nu,k}^Q}|\theta_s^Q|^2\,ds=\int_\nu^{\delta_m^Q}|\theta_s^Q|^2\,ds$$

and consequently, P-a.s.:

$$\begin{aligned}\lim_{k\to\infty}Z_{\nu,T}^{Q^{m,k}}&=\lim_{k\to\infty}\exp\left\{\int_\nu^{\delta_m^Q}\mathbf{1}_{A_{\nu,k}^Q}\left(\theta_s^Q\,dB_s-\frac{1}{2}|\theta_s^Q|^2\,ds\right)\right\}\\&=\exp\left\{\int_\nu^{\delta_m^Q}\left(\theta_s^Q\,dB_s-\frac{1}{2}|\theta_s^Q|^2\,ds\right)\right\}=Z_{\nu,\delta_m^Q}^Q.\end{aligned}\tag{5.8}$$

Since $E(Z_{\nu,T}^{Q^{m,k}}|\mathcal{F}_\nu)=E(Z_{\nu,\delta_m^Q}^Q|\mathcal{F}_\nu)=1$, P-a.s. for any $k\in\mathbb{N}$, it follows from Scheffé's lemma [see, e.g., [23, Section 5.10]] that

$$\lim_{k\to\infty}E[|Z_{\nu,T}^{Q^{m,k}}-Z_{\nu,\delta_m^Q}^Q||\mathcal{F}_\nu]=0,\quad P\text{-a.s.}\tag{5.9}$$

Hence, letting $k\to\infty$ in (5.7), we obtain that P-a.s.

$$\begin{aligned}&\lim_{k\to\infty}\downarrow\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&\quad\le E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]+\|Y\|_\infty\cdot E[|Z_{\nu,\delta_m^Q}^Q-Z_{\nu,T}^Q||\mathcal{F}_\nu].\end{aligned}\tag{5.10}$$

It is easy to see that $\lim_{m\to\infty}\uparrow\delta_m^Q=T$, P-a.s. The right-continuity of the process Z^Q then implies that $\lim_{m\to\infty}Z_{\nu,\delta_m^Q}^Q=Z_{\nu,T}^Q$, P-a.s. Since $E[Z_{\nu,\delta_m^Q}^Q|\mathcal{F}_\nu]=E[Z_{\nu,T}^Q|\mathcal{F}_\nu]=1$, P-a.s. for any $m\in\mathbb{N}$, using Scheffé's lemma once again we obtain

$$\lim_{m\to\infty}E[|Z_{\nu,\delta_m^Q}^Q-Z_{\nu,T}^Q||\mathcal{F}_\nu]=0,\quad P\text{-a.s.}\tag{5.11}$$

Therefore, letting $m\to\infty$ in (5.10) we obtain that P-a.s.

$$\begin{aligned}&\lim_{k\to\infty}\downarrow\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&\quad\le E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right].\end{aligned}$$

Taking the essential infimum of right-hand side over $Q\in\mathcal{Q}_\nu$ gives

$$\begin{aligned}&\lim_{k\to\infty}\downarrow\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&\quad\le\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_\gamma+\int_\nu^\gamma f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right],\quad P\text{-a.s.}\end{aligned}$$

which, together with (5.4), proves (3.7).

(2) By analogy with (5.4), we have

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} R^Q(\nu) \le \lim_{k\to\infty}\downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu), \quad P\text{-a.s.} \tag{5.12}$$

Taking the essential supremum in (5.7) over $\gamma \in \mathcal{S}_{\nu,T}$, we get

$$\begin{aligned}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu) &\le R^{Q^{m,k}}(\nu)\\ &\le R^Q(\nu) + (\|Y\|_\infty + m)\cdot E[|Z^{Q^{m,k}}_{\nu,T} - Z^Q_{\nu,\delta^Q_m}||\mathcal{F}_\nu]\\ &\quad + \|Y\|_\infty \cdot E[|Z^Q_{\nu,\delta^Q_m} - Z^Q_{\nu,T}||\mathcal{F}_\nu], \quad P\text{-a.s.}\end{aligned} \tag{5.13}$$

In light of (5.9) and (5.11), letting $k\to\infty$ and subsequently letting $m\to\infty$ in (5.13), we obtain

$$\lim_{k\to\infty}\downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu) \le R^Q(\nu), \quad P\text{-a.s.}$$

Taking the essential infimum of right-hand side over $Q\in\mathcal{Q}_\nu$ yields

$$\lim_{k\to\infty}\downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} R^Q(\nu) \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} R^Q(\nu), \quad P\text{-a.s.},$$

which, together with (5.12), proves (3.8). $\square$

Proof of Lemma 3.5. (1) We first show that the family

$$\left\{E_Q\left[Y_\gamma + \int_\nu^\gamma f(s,\theta^Q_s)\,ds\Big|\mathcal{F}_\nu\right]\right\}_{Q\in\mathcal{Q}_\nu^k}$$

is directed downwards, that is, for any $Q_1, Q_2 \in \mathcal{Q}_\nu^k$, there exists a $Q_3\in\mathcal{Q}_\nu^k$ such that P-a.s.

$$\begin{aligned}&E_{Q_3}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta^{Q_3}_s)\,ds\Big|\mathcal{F}_\nu\right]\\ &\quad\le E_{Q_1}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta^{Q_1}_s)\,ds\Big|\mathcal{F}_\nu\right] \wedge E_{Q_2}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta^{Q_2}_s)\,ds\Big|\mathcal{F}_\nu\right].\end{aligned} \tag{5.14}$$

To see this, we let $Q_1, Q_2\in\mathcal{Q}_\nu^k$ and let $A\in\mathcal{F}_\nu$. It is clear that

$$\theta^{Q_3}_t \triangleq \mathbf{1}_{\{t>\nu\}}(\mathbf{1}_A\theta^{Q_1}_t + \mathbf{1}_{A^c}\theta^{Q_2}_t), \quad t\in[0,T] \tag{5.15}$$

forms a predictable process, thus we can define a probability measure $Q_3\in\mathcal{M}^e$ via $dQ_3 \triangleq \mathscr{E}(\theta^{Q_3}\bullet B)_T\,dP$. It follows from (f3) that $dt\otimes dP$-a.e.

$$f(t,\theta^{Q_3}_t) = \mathbf{1}_{\{t>\nu\}}\big(\mathbf{1}_A f(t,\theta^{Q_1}_t) + \mathbf{1}_{A^c} f(t,\theta^{Q_2}_t)\big), \tag{5.16}$$

which together with (5.15) implies that $\theta^{Q_3} = 0\ dt\otimes dP$-a.e. on $[\![0,\nu]\!]$ and $|\theta^{Q_3}_t(\omega)| \vee f(t,\omega,\theta^{Q_3}_t(\omega)) = \mathbf{1}_A(\omega)|\theta^{Q_1}_t(\omega)| \vee f(t,\omega,\theta^{Q_1}_t(\omega)) + \mathbf{1}_{A^c}(\omega)|\theta^{Q_2}_t(\omega)| \vee$

$f(t,\omega,\theta_t^{Q_2}(\omega)) \le k$, $dt \otimes dP$-a.e. on $]\!]\nu,T]\!]$. Hence, $Q_3 \in \mathcal{Q}_\nu^k$. For any $\gamma \in \mathcal{S}_{\nu,T}$, we have

$$
\begin{aligned}
Z^{Q_3}_{\nu,\gamma} &= \exp\biggl\{\int_\nu^\gamma (\mathbf{1}_A\theta_s^{Q_1} + \mathbf{1}_{A^c}\theta_s^{Q_2})\,dB_s \\
&\qquad - \frac{1}{2}\int_\nu^\gamma (\mathbf{1}_A|\theta_s^{Q_1}|^2 + \mathbf{1}_{A^c}|\theta_s^{Q_2}|^2)\,ds\biggr\} \\
&= \exp\biggl\{\mathbf{1}_A\biggl(\int_\nu^\gamma \theta_s^{Q_1}\,dB_s - \frac{1}{2}\int_\nu^\gamma |\theta_s^{Q_1}|^2\,ds\biggr) \\
&\qquad + \mathbf{1}_{A^c}\biggl(\int_\nu^\gamma \theta_s^{Q_2}\,dB_s - \frac{1}{2}\int_\nu^\gamma |\theta_s^{Q_2}|^2\,ds\biggr)\biggr\} \\
&= \mathbf{1}_A \exp\biggl\{\int_\nu^\gamma \theta_s^{Q_1}\,dB_s - \frac{1}{2}\int_\nu^\gamma |\theta_s^{Q_1}|^2\,ds\biggr\} \\
&\qquad + \mathbf{1}_{A^c} \exp\biggl\{\int_\nu^\gamma \theta_s^{Q_2}\,dB_s - \frac{1}{2}\int_\nu^\gamma |\theta_s^{Q_2}|^2\,ds\biggr\} \\
&= \mathbf{1}_A Z^{Q_1}_{\nu,\gamma} + \mathbf{1}_{A^c} Z^{Q_2}_{\nu,\gamma}, \quad P\text{-a.s.}
\end{aligned} \tag{5.17}
$$

Then the Bayes Rule implies

$$
\begin{aligned}
&E_{Q_3}\biggl[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_3})\,ds\Big|\mathcal{F}_\nu\biggr] \\
&\quad = E\biggl[Z^{Q_3}_{\nu,T}\biggl(Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_3})\,ds\biggr)\Big|\mathcal{F}_\nu\biggr] \\
&\quad = E\biggl[\mathbf{1}_A Z^{Q_1}_{\nu,T}\biggl(Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_1})\,ds\biggr) \\
&\qquad\quad + \mathbf{1}_{A^c} Z^{Q_2}_{\nu,T}\biggl(Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_2})\,ds\biggr)\Big|\mathcal{F}_\nu\biggr] \\
&\quad = \mathbf{1}_A E_{Q_1}\biggl[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_1})\,ds\Big|\mathcal{F}_\nu\biggr] \\
&\qquad\quad + \mathbf{1}_{A^c} E_{Q_2}\biggl[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_2})\,ds\Big|\mathcal{F}_\nu\biggr], \quad P\text{-a.s.}
\end{aligned} \tag{5.18}
$$

Letting $A = \{E_{Q_1}[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_1})\,ds|\mathcal{F}_\nu] \le E_{Q_2}[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_2})\,ds|\mathcal{F}_\nu]\} \in \mathcal{F}_\nu$ above, one obtains that P-a.s.

$$
\begin{aligned}
&E_{Q_3}\biggl[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_3})\,ds\Big|\mathcal{F}_\nu\biggr] \\
&\quad = E_{Q_1}\biggl[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_1})\,ds\Big|\mathcal{F}_\nu\biggr] \wedge E_{Q_2}\biggl[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_2})\,ds\Big|\mathcal{F}_\nu\biggr],
\end{aligned}
$$

proving (5.14). Appealing to the basic properties of the essential infimum [e.g., [21, Proposition VI-**1**-1]], we can find a sequence $\{Q_n^{\gamma,k}\}_{n\in\mathbb{N}}$ in $\mathcal{Q}_\nu^k$ such that (3.9) holds.

(2) Taking essential suprema over $\gamma \in \mathcal{S}_{\nu,T}$ on both sides of (5.18), we can deduce from Lemma 2.3 that

$$\begin{aligned} R^{Q_3}(\nu) &= \operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}} E_{Q_3}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_3})\,ds\Big|\mathcal{F}_\nu\right] \\ &= \mathbf{1}_A \operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}} E_{Q_1}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_1})\,ds\Big|\mathcal{F}_\nu\right] \\ &\quad + \mathbf{1}_{A^c} \operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}} E_{Q_2}\left[Y_\gamma + \int_\nu^\gamma f(s,\theta_s^{Q_2})\,ds\Big|\mathcal{F}_\nu\right] \\ &= \mathbf{1}_A R^{Q_1}(\nu) + \mathbf{1}_{A^c} R^{Q_2}(\nu), \quad P\text{-a.s.} \end{aligned}$$

Taking $A = \{R^{Q_1}(\nu) \le R^{Q_2}(\nu)\} \in \mathcal{F}_\nu$ yields that $R^{Q_3}(\nu) = R^{Q_1}(\nu) \wedge R^{Q_2}(\nu)$, P-a.s., thus the family $\{R^Q(\nu)\}_{Q\in\mathcal{Q}_\nu^k}$ is directed downwards. Applying Proposition VI-**1**-1 of [21] once again, one can find a sequence $\{Q_n^{(k)}\}_{n\in\mathbb{N}}$ in $\mathcal{Q}_\nu^k$ such that (3.10) holds. ☐

Proof of Lemma 3.6. Let $Q_1, Q_2 \in \mathcal{Q}_\nu^k$. We define the stopping time $\gamma \stackrel{\triangle}{=} \tau^{Q_1}(\nu) \wedge \tau^{Q_2}(\nu) \in \mathcal{S}_{\nu,T}$ and the event $A \stackrel{\triangle}{=} \{R_\gamma^{Q_1,0} \le R_\gamma^{Q_2,0}\} \in \mathcal{F}_\gamma$. It is clear that

$$\theta_t^{Q_3} \stackrel{\triangle}{=} \mathbf{1}_{\{t>\gamma\}}(\mathbf{1}_A \theta_t^{Q_1} + \mathbf{1}_{A^c}\theta_t^{Q_2}), \quad t\in[0,T] \tag{5.19}$$

forms a predictable process, thus we can define a probability measure $Q_3 \in \mathcal{M}^e$ by $(dQ_3/dP) \stackrel{\triangle}{=} \mathscr{E}(\theta^{Q_3} \bullet B)_T$. By analogy with (5.16), we have

$$f(t,\theta_t^{Q_3}) = \mathbf{1}_{\{t>\gamma\}}\big(\mathbf{1}_A f(t,\theta_t^{Q_1}) + \mathbf{1}_{A^c} f(t,\theta_t^{Q_2})\big), \quad dt\otimes dP\text{-a.e.} \tag{5.20}$$

which together with (5.19) implies that $\theta^{Q_3} = 0$, $dt\otimes dP$-a.e. on $[\![0,\gamma]\!]$ and $|\theta_t^{Q_3}(\omega)| \vee f(t,\omega,\theta_t^{Q_3}(\omega)) \le k$, $dt\otimes dP$-a.e. on $]\!]\gamma,T]\!]$. Hence $Q_3 \in \mathcal{Q}_\gamma^k \subset \mathcal{Q}_\nu^k$, thanks to Remark 3.3. Moreover, by analogy with (5.17), we can deduce that for any $\zeta\in\mathcal{S}_{\gamma,T}$ we have

$$Z_{\gamma,\zeta}^{Q_3} = \mathbf{1}_A Z_{\gamma,\zeta}^{Q_1} + \mathbf{1}_{A^c} Z_{\gamma,\zeta}^{Q_2}, \quad P\text{-a.s.} \tag{5.21}$$

Now fix $t\in[0,T]$. For any $\sigma\in\mathcal{S}_{\gamma\vee t,T}$, (5.21) shows that P-a.s.

$$Z_{\gamma\vee t,\sigma}^{Q_3} = \frac{Z_{\gamma,\sigma}^{Q_3}}{Z_{\gamma,\gamma\vee t}^{Q_3}} = \mathbf{1}_A \frac{Z_{\gamma,\sigma}^{Q_1}}{Z_{\gamma,\gamma\vee t}^{Q_1}} + \mathbf{1}_{A^c}\frac{Z_{\gamma,\sigma}^{Q_2}}{Z_{\gamma,\gamma\vee t}^{Q_2}} = \mathbf{1}_A Z_{\gamma\vee t,\sigma}^{Q_1} + \mathbf{1}_{A^c} Z_{\gamma\vee t,\sigma}^{Q_2},$$

and Bayes' Rule together with (5.20) then imply that P-a.s.

$$\begin{aligned} &E_{Q_3}\left[Y_\sigma + \int_{\gamma\vee t}^\sigma f(s,\theta_s^{Q_3})\,ds\Big|\mathcal{F}_{\gamma\vee t}\right] \\ &\quad = E\left[Z_{\gamma\vee t,\sigma}^{Q_3}\left(Y_\sigma + \int_{\gamma\vee t}^\sigma f(s,\theta_s^{Q_3})\,ds\right)\Big|\mathcal{F}_{\gamma\vee t}\right] \end{aligned}$$

$$= \mathbf{1}_A E_{Q_1}\left[Y_\sigma + \int_{\gamma\vee t}^{\sigma} f(s,\theta_s^{Q_1})\,ds \Big| \mathcal{F}_{\gamma\vee t}\right] + \mathbf{1}_{A^c} E_{Q_2}\left[Y_\sigma + \int_{\gamma\vee t}^{\sigma} f(s,\theta_s^{Q_2})\,ds \Big| \mathcal{F}_{\gamma\vee t}\right].$$

Taking essential suprema over $\sigma\in\mathcal{S}_{\gamma\vee t,T}$ on both sides above, we can deduce from Lemma 2.3 as well as (3.3) that P-a.s.

$$R^{Q_3,0}_{\gamma\vee t} = R^{Q_3}(\gamma\vee t) = \mathbf{1}_A R^{Q_1}(\gamma\vee t) + \mathbf{1}_{A^c} R^{Q_2}(\gamma\vee t) = \mathbf{1}_A R^{Q_1,0}_{\gamma\vee t} + \mathbf{1}_{A^c} R^{Q_2,0}_{\gamma\vee t}.$$

Since $R^{Q_i,0}$, $i=1,2,3$ are all RCLL processes, we have $R^{Q_3,0}_{\gamma\vee t} = \mathbf{1}_A R^{Q_1,0}_{\gamma\vee t} + \mathbf{1}_{A^c} R^{Q_2,0}_{\gamma\vee t}, \forall t\in[0,T]$ outside a null set N, and this implies that P-a.s.

$$\begin{aligned}\tau^{Q_3}(\nu) &= \inf\{t\in[\nu,T] : R^{Q_3,0}_t = Y_t\} \le \inf\{t\in[\gamma,T] : R^{Q_3,0}_t = Y_t\}\\ &= \mathbf{1}_A \inf\{t\in[\gamma,T] : R^{Q_1,0}_t = Y_t\}\\ &\quad + \mathbf{1}_{A^c} \inf\{t\in[\gamma,T] : R^{Q_2,0}_t = Y_t\}.\end{aligned}\tag{5.22}$$

Since $R^{Q_j,0}_{\tau^{Q_j}(\nu)} = Y_{\tau^{Q_j}(\nu)}$, P-a.s. for $j=1,2$, and since $\gamma = \tau^{Q_1}(\nu)\wedge\tau^{Q_2}(\nu)$, it holds P-a.s. that Y_γ is equal either to $R^{Q_1,0}_\gamma$ or to $R^{Q_2,0}_\gamma$. Then the definition of the set A shows that $R^{Q_1,0}_\gamma = Y_\gamma$ holds P-a.s. on A, and that $R^{Q_2,0}_\gamma = Y_\gamma$ holds P-a.s. on A^c, both of which further imply that P-a.s.

$$\mathbf{1}_A \inf\{t\in[\gamma,T] : R^{Q_1,0}_t = Y_t\} = \gamma\mathbf{1}_A$$

and

$$\mathbf{1}_{A^c} \inf\{t\in[\gamma,T] : R^{Q_2,0}_t = Y_t\} = \gamma\mathbf{1}_{A^c}.$$

We conclude from (5.22) that $\tau^{Q_3}(\nu)\le\gamma = \tau^{Q_1}(\nu)\wedge\tau^{Q_2}(\nu)$ holds P-a.s., hence the family $\{\tau^Q(\nu)\}_{Q\in\mathcal{Q}^k_\nu}$ is directed downwards. Thanks to [21, page 121], we can find a sequence $\{Q^{(k)}_n\}_{n\in\mathbb{N}}$ in $\mathcal{Q}^k_\nu$, such that

$$\tau_k(\nu) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^k_\nu} \tau^Q(\nu) = \lim_{n\to\infty}\downarrow \tau^{Q^{(k)}_n}(\nu), \quad P\text{-a.s.}$$

The limit $\lim_{n\to\infty}\downarrow\tau^{Q^{(k)}_n}(\nu)$ is also a stopping time in $\mathcal{S}_{\nu,T}$. □

Proof of Lemma 3.7. It is easy to see from (3.12) and (𝔣3) that

$$\theta^{Q'} = \theta^Q = 0, \quad dt\otimes dP\text{-a.e. on } [\![0,\nu]\!],\tag{5.23}$$

and that

$$f(t,\theta_t^{Q'}) = \mathbf{1}_{\{t\le\gamma\}} f(t,\theta_t^Q) + \mathbf{1}_{\{t>\gamma\}} f(t,\theta_t^{\widetilde{Q}}), \quad dt\otimes dP\text{-a.e.}\tag{5.24}$$

As a result

$$\begin{aligned}&E_{Q'}\int_\nu^T f(s,\theta_s^{Q'})\,ds\\ &\quad = E_{Q'}\int_\nu^\gamma f(s,\theta_s^{Q})\,ds + E_{Q'}\int_\gamma^T f(s,\theta_s^{\widetilde{Q}})\,ds\end{aligned}$$

$$
\begin{aligned}
&\leq E_Q \int_\nu^\gamma f(s,\theta_s^Q)\,ds + E_{Q'}\int_\gamma^T k\,ds \\
&\leq E_Q \int_\nu^T f(s,\theta_s^Q)\,ds + kT < \infty,
\end{aligned}
$$

thus $Q' \in \mathcal{Q}_\nu$. If $Q \in \mathcal{Q}_\nu^k$, we see from (3.12) and (5.24) that $dt \otimes dP$-a.e.

$$
|\theta_t^{Q'}(\omega)| \vee f(t,\omega,\theta_t^{Q'}(\omega)) = \begin{cases} |\theta_t^{Q}(\omega)| \vee f(t,\omega,\theta_t^{Q}(\omega)) \leq k, & \text{on }]\!]\nu,\gamma]\!], \\ |\theta_t^{\widetilde{Q}}(\omega)| \vee f(t,\omega,\theta_t^{\widetilde{Q}}(\omega)) \leq k, & \text{on }]\!]\gamma,T]\!], \end{cases}
$$

which, together with (5.23), shows that $Q' \in \mathcal{Q}_\nu^k$.

Now we fix $\sigma \in \mathcal{S}_{\gamma,T}$. For any $\delta \in \mathcal{S}_{\sigma,T}$, Bayes' Rule shows that P-a.s.

$$
\begin{aligned}
E_{Q'}\left[Y_\delta + \int_\sigma^\delta f(s,\theta_s^{Q'})\,ds \Big| \mathcal{F}_\sigma\right] &= E_{Q'}\left[Y_\delta + \int_\sigma^\delta f(s,\theta_s^{\widetilde{Q}})\,ds \Big| \mathcal{F}_\sigma\right] \\
&= E_{\widetilde{Q}}\left[Y_\delta + \int_\sigma^\delta f(s,\theta_s^{\widetilde{Q}})\,ds \Big| \mathcal{F}_\sigma\right],
\end{aligned}
$$

and (3.3) implies that P-a.s.

$$
\begin{aligned}
R_\sigma^{Q',0} = R^{Q'}(\sigma) &= \operatorname*{ess\,sup}_{\delta \in \mathcal{S}_{\sigma,T}} E_{Q'}\left[Y_\delta + \int_\sigma^\delta f(s,\theta_s^{Q'})\,ds \Big| \mathcal{F}_\sigma\right] \\
&= \operatorname*{ess\,sup}_{\delta \in \mathcal{S}_{\sigma,T}} E_{\widetilde{Q}}\left[Y_\delta + \int_\sigma^\delta f(s,\theta_s^{\widetilde{Q}})\,ds \Big| \mathcal{F}_\sigma\right] = R^{\widetilde{Q}}(\sigma) = R_\sigma^{\widetilde{Q},0}. \qquad \square
\end{aligned}
$$

Proof of Theorem 3.9. Fix $Q \in \mathcal{Q}_\nu$. For any $m,k \in \mathbb{N}$, we consider the probability measure $Q^{m,k} \in \mathcal{Q}_\nu^k$ as defined in (5.5). In light of Lemma 3.6, for any $l \in \mathbb{N}$ there exists a sequence $\{Q_n^{(l)}\}_{n\in\mathbb{N}}$ in $\mathcal{Q}_\nu^l$ such that

$$
\tau_l(\nu) = \lim_{n\to\infty} \downarrow \tau^{Q_n^{(l)}}(\nu), \quad P\text{-a.s.}
$$

Now let $k,l,m,n \in \mathbb{N}$ with $k \leq l$. Lemma 3.7 implies that the predictable process

$$
\theta_t^{Q_n^{m,k,l}} \triangleq \mathbf{1}_{\{t\leq\tau_l(\nu)\}}\theta_t^{Q^{m,k}} + \mathbf{1}_{\{t>\tau_l(\nu)\}}\theta_t^{Q_n^{(l)}}, \quad t \in [0,T]
$$

induces a probability measure $Q_n^{m,k,l} \in \mathcal{Q}_\nu^l$ via $dQ_n^{m,k,l} = \mathscr{E}(\theta^{Q_n^{m,k,l}} \bullet B)_T\,dP$, such that for any $t \in [0,T]$, we have $R_{\tau_l(\nu)\vee t}^{Q_n^{m,k,l},0} = R_{\tau_l(\nu)\vee t}^{Q_n^{(l)},0}$, P-a.s. Since $R^{Q_n^{m,k,l},0}$ and $R^{Q_n^{(l)},0}$ are both RCLL processes, outside a null set N we have

$$
R_{\tau_l(\nu)\vee t}^{Q_n^{m,k,l},0} = R_{\tau_l(\nu)\vee t}^{Q_n^{(l)},0} \quad \forall t \in [0,T]
$$

and this, together with the fact that $\tau_l(\nu) \leq \tau^{Q_n^{m,k,l}}(\nu) \wedge \tau^{Q_n^{(l)}}(\nu)$, P-a.s. implies that P-a.s.

$$
\begin{aligned}
\tau^{Q_n^{m,k,l}}(\nu) &= \inf\{t \in [\nu,T] : R_t^{Q_n^{m,k,l},0} = Y_t\} \\
&= \inf\{t \in [\tau_l(\nu),T] : R_t^{Q_n^{m,k,l},0} = Y_t\} \\
&= \inf\{t \in [\tau_l(\nu),T] : R_t^{Q_n^{(l)},0} = Y_t\} \\
&= \inf\{t \in [\nu,T] : R_t^{Q_n^{(l)},0} = Y_t\} = \tau^{Q_n^{(l)}}(\nu).
\end{aligned}
\tag{5.25}
$$

Similar to (5.6), we have that $dt \otimes dP$-a.e.

$$
f(t,\theta_t^{Q_n^{m,k,l}}) = \mathbf{1}_{\{t \le \tau_l(\nu)\}} f(t,\theta_t^{Q^{m,k}}) + \mathbf{1}_{\{t > \tau_l(\nu)\}} f(t,\theta_t^{Q_n^{(l)}}).
\tag{5.26}
$$

Then one can deduce from (5.25) and (5.26) that

$$
\begin{aligned}
\overline{V}(\nu) &= \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_\nu} R^Q(\nu) \le R^{Q_n^{m,k,l}}(\nu) \\
&= E_{Q_n^{m,k,l}}\left(Y_{\tau^{Q_n^{m,k,l}}(\nu)} + \int_\nu^{\tau^{Q_n^{m,k,l}}(\nu)} f(s,\theta_s^{Q_n^{m,k,l}})\,ds\Big|\mathcal{F}_\nu\right) \\
&= E_{Q_n^{m,k,l}}\left[Y_{\tau^{Q_n^{(l)}}(\nu)} + \int_{\tau_l(\nu)}^{\tau^{Q_n^{(l)}}(\nu)} f(s,\theta_s^{Q_n^{m,k,l}})\,ds\Big|\mathcal{F}_\nu\right] \\
&\quad + E_{Q^{m,k}}\left[\int_\nu^{\tau_l(\nu)} f(s,\theta_s^{Q_n^{m,k,l}})\,ds\Big|\mathcal{F}_\nu\right] \\
&= E\bigg[\big(Z^{Q_n^{m,k,l}}_{\nu,\tau^{Q_n^{(l)}}(\nu)} - Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}\big) \\
&\qquad \times \left(Y_{\tau^{Q_n^{(l)}}(\nu)} + \int_{\tau_l(\nu)}^{\tau^{Q_n^{(l)}}(\nu)} f(s,\theta_s^{Q_n^{(l)}})\,ds\right)\Big|\mathcal{F}_\nu\bigg] \\
&\quad + E\left[Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}\left(Y_{\tau^{Q_n^{(l)}}(\nu)} + \int_{\tau_l(\nu)}^{\tau^{Q_n^{(l)}}(\nu)} f(s,\theta_s^{Q_n^{(l)}})\,ds\right)\Big|\mathcal{F}_\nu\right] \\
&\quad + E_{Q^{m,k}}\left[\int_\nu^{\tau_l(\nu)} f(s,\theta_s^{Q^{m,k}})\,ds\Big|\mathcal{F}_\nu\right] \\
&\le (\|Y\|_\infty + lT)\cdot E\big[\big|Z^{Q_n^{m,k,l}}_{\nu,\tau^{Q_n^{(l)}}(\nu)} - Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}\big|\big|\mathcal{F}_\nu\big] \\
&\quad + E\big[Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}\big(Y_{\tau^{Q_n^{(l)}}(\nu)} + k\big(\tau^{Q_n^{(l)}}(\nu) - \tau_l(\nu)\big)\big)\big|\mathcal{F}_\nu\big] \\
&\quad + E_{Q^{m,k}}\left[\int_\nu^{\tau_l(\nu)} f(s,\theta_s^{Q^{m,k}})\,ds\Big|\mathcal{F}_\nu\right], \quad P\text{-a.s.}
\end{aligned}
\tag{5.27}
$$

Because $E(\int_{\tau_l(\nu)}^{\tau^{Q_n^{(l)}}(\nu)} \theta_s^{Q_n^{(l)}}\,dB_s)^2 = E\int_{\tau_l(\nu)}^{\tau^{Q_n^{(l)}}(\nu)} |\theta_s^{Q_n^{(l)}}|^2\,ds \le l^2 E[\tau^{Q_n^{(l)}}(\nu) - \tau_l(\nu)]$, which goes to zero as $n \to \infty$, using similar arguments to those that lead to

(5.8), we can find a subsequence of $\{Q_n^{(l)}\}_{n\in\mathbb{N}}$ (we still denote it by $\{Q_n^{(l)}\}_{n\in\mathbb{N}}$) such that $\lim_{n\to\infty} Z^{Q_n^{m,k,l}}_{\nu,\tau^{Q_n^{(l)}}(\nu)} = Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}$, P-a.s. Since $E[Z^{Q_n^{m,k,l}}_{\nu,\tau^{Q_n^{(l)}}(\nu)}|\mathcal{F}_\nu] = E[Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}|\mathcal{F}_\nu] = 1$, P-a.s. for any $n\in\mathbb{N}$, Scheffé's lemma implies

$$\lim_{n\to\infty} E\big(\big|Z^{Q_n^{m,k,l}}_{\nu,\tau^{Q_n^{(l)}}(\nu)} - Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}\big||\mathcal{F}_\nu\big) = 0, \quad P\text{-a.s.} \tag{5.28}$$

On the other hand, since

$$Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}\big|Y_{\tau^{Q_n^{(l)}}(\nu)} + k\big(\tau^{Q_n^{(l)}}(\nu) - \tau_l(\nu)\big)\big| \le Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}(\|Y\|_\infty + kT), \quad P\text{-a.s.},$$

and since Y is right-continuous, the Dominated Convergence theorem gives

$$\begin{aligned}\lim_{n\to\infty} E\big[Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}\big(Y_{\tau^{Q_n^{(l)}}(\nu)} + k\big(\tau^{Q_n^{(l)}}(\nu) - \tau_l(\nu)\big)\big)|\mathcal{F}_\nu\big] \\ = E\big[Z^{Q^{m,k}}_{\nu,\tau_l(\nu)}Y_{\tau_l(\nu)}|\mathcal{F}_\nu\big] = E_{Q^{m,k}}[Y_{\tau_l(\nu)}|\mathcal{F}_\nu], \quad P\text{-a.s.}\end{aligned} \tag{5.29}$$

Therefore, letting $n\to\infty$ in (5.27), we can deduce from (5.28) and (5.29) that

$$\overline{V}(\nu) \le E_{Q^{m,k}}\bigg[Y_{\tau_l(\nu)} + \int_\nu^{\tau_l(\nu)} f(s,\theta_s^{Q^{m,k}})\,ds\bigg|\mathcal{F}_\nu\bigg], \quad P\text{-a.s.}$$

As $l\to\infty$, the Bounded Convergence theorem gives

$$\overline{V}(\nu) \le E_{Q^{m,k}}\bigg[Y_{\tau(\nu)} + \int_\nu^{\tau(\nu)} f(s,\theta_s^{Q^{m,k}})\,ds\bigg|\mathcal{F}_\nu\bigg], \quad P\text{-a.s.}$$

whence, just as in (5.7), we deduce

$$\begin{aligned}\overline{V}(\nu) &\le E_{Q^{m,k}}\bigg[Y_{\tau(\nu)} + \int_\nu^{\tau(\nu)} f(s,\theta_s^{Q^{m,k}})\,ds\bigg|\mathcal{F}_\nu\bigg] \\ &\le (\|Y\|_\infty + m)\cdot E\big[\big|Z^{Q^{m,k}}_{\nu,\tau(\nu)} - Z^{Q}_{\nu,\tau(\nu)\wedge\delta_m^Q}\big||\mathcal{F}_\nu\big] \\ &\quad + \|Y\|_\infty\cdot E\big[\big|Z^{Q}_{\nu,\tau(\nu)\wedge\delta_m^Q} - Z^{Q}_{\nu,\tau(\nu)}\big||\mathcal{F}_\nu\big] \\ &\quad + E_Q\bigg[Y_{\tau(\nu)} + \int_\nu^{\tau(\nu)} f(s,\theta_s^{Q})\,ds\bigg|\mathcal{F}_\nu\bigg], \quad P\text{-a.s.}\end{aligned} \tag{5.30}$$

By analogy with (5.9) and (5.11), one can show that for any $m\in\mathbb{N}$ we have $\lim_{k\to\infty} E[|Z^{Q^{m,k}}_{\nu,\tau(\nu)} - Z^{Q}_{\nu,\tau(\nu)\wedge\delta_m^Q}||\mathcal{F}_\nu] = 0$, P-a.s. and that

$$\lim_{m\to\infty} E\big[\big|Z^{Q}_{\nu,\tau(\nu)\wedge\delta_m^Q} - Z^{Q}_{\nu,\tau(\nu)}\big||\mathcal{F}_\nu\big] = 0, \quad P\text{-a.s.}$$

Therefore, letting $k\to\infty$ and subsequently letting $m\to\infty$ in (5.30), we obtain

$$\overline{V}(\nu) \le E_Q\bigg[Y_{\tau(\nu)} + \int_\nu^{\tau(\nu)} f(s,\theta_s^{Q})\,ds\bigg|\mathcal{F}_\nu\bigg], \quad P\text{-a.s.}$$

Taking the essential infimum of the right-hand side over $Q \in \mathcal{Q}_\nu$ yields

$$\overline{V}(\nu) \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} E_Q\left[Y_{\tau(\nu)} + \int_\nu^{\tau(\nu)} f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]$$
$$\le \operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} E_Q\left[Y_\gamma + \int_\nu^{\gamma} f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right] = \underline{V}(\nu) \le \overline{V}(\nu), \quad P\text{-a.s.}$$

and the result follows. □

Proof of Proposition 3.10. For each fixed $k \in \mathbb{N}$, there exists on the strength of Lemma 3.6 a sequence $\{Q_n^{(k)}\}_{n\in\mathbb{N}}$ in $\mathcal{Q}_\nu^k$ such that

$$\tau_k(\nu) = \lim_{n\to\infty} \downarrow \tau^{Q_n^{(k)}}(\nu), \quad P\text{-a.s.}$$

For any $n \in \mathbb{N}$, the predictable process $\theta_t^{\widetilde{Q}_n^{(k)}} \triangleq \mathbf{1}_{\{t>\tau_k(\nu)\}}\theta_t^{Q_n^{(k)}}$, $t\in[0,T]$ induces a probability measure $\widetilde{Q}_n^{(k)}$ by $d\widetilde{Q}_n^{(k)} \triangleq \mathscr{E}(\theta^{\widetilde{Q}_n^{(k)}} \bullet B)_T\,dP = Z^{Q_n^{(k)}}_{\tau_k(\nu),T}\,dP$. Since $\nu \le \sigma \triangleq \tau(\nu) \le \tau_k(\nu) \le \tau^{\widetilde{Q}_n^{(k)}}(\nu)$, P-a.s., we have $\widetilde{Q}_n^{(k)} \in \mathcal{Q}^k_{\tau_k(\nu)} \subset \mathcal{Q}^k_\sigma \subset \mathcal{Q}^k_\nu$ and

$$\begin{aligned}\tau^{\widetilde{Q}_n^{(k)}}(\nu) &= \inf\{t\in[\nu,T] : R_t^{\widetilde{Q}_n^{(k)},0} = Y_t\}\\ &= \inf\{t\in[\sigma,T] : R_t^{\widetilde{Q}_n^{(k)},0} = Y_t\} = \tau^{\widetilde{Q}_n^{(k)}}(\sigma), \quad P\text{-a.s.}\end{aligned} \tag{5.31}$$

We also know from Lemma 3.7 that for any $t \in [0,T]$:

$$R^{\widetilde{Q}_n^{(k)},0}_{\tau_k(\nu)\vee t} = R^{Q_n^{(k)},0}_{\tau_k(\nu)\vee t}, \quad P\text{-a.s.}$$

Since $R^{\widetilde{Q}_n^{(k)},0}$ and $R^{Q_n^{(k)},0}$ are both RCLL processes, there exists a null set N outside which we have $R^{\widetilde{Q}_n^{(k)},0}_{\tau_k(\nu)\vee t} = R^{Q_n^{(k)},0}_{\tau_k(\nu)\vee t}, \forall t\in[0,T]$. By analogy with (5.25) and (5.6), respectively, we have

$$\tau^{\widetilde{Q}_n^{(k)}}(\nu) = \tau^{Q_n^{(k)}}(\nu), \quad P\text{-a.s.} \tag{5.32}$$

and $f(t,\theta_t^{\widetilde{Q}_n^{(k)}}) = \mathbf{1}_{\{t>\tau_k(\nu)\}} f(t,\theta_t^{Q_n^{(k)}})$, $dt\otimes dP$-a.e. Then we can deduce from (5.31), (5.32) that

$$\begin{aligned}V(\sigma) = \overline{V}(\sigma) &= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\sigma} R^Q(\sigma) \le R^{\widetilde{Q}_n^{(k)}}(\sigma)\\ &= E_{\widetilde{Q}_n^{(k)}}\left(Y_{\tau^{Q_n^{(k)}}(\nu)} + \int_\sigma^{\tau^{Q_n^{(k)}}(\nu)} \mathbf{1}_{\{s>\tau_k(\nu)\}} f(s,\theta_s^{Q_n^{(k)}})\,ds\Big|\mathcal{F}_\sigma\right)\\ &= E\Bigg[\big(Z^{\widetilde{Q}_n^{(k)}}_{\sigma,\tau^{Q_n^{(k)}}(\nu)} - 1\big)\\ &\qquad\cdot\left(Y_{\tau^{Q_n^{(k)}}(\nu)} + \int_{\tau_k(\nu)}^{\tau^{Q_n^{(k)}}(\nu)} f(s,\theta_s^{Q_n^{(k)}})\,ds\right)\Big|\mathcal{F}_\sigma\Bigg]\end{aligned} \tag{5.33}$$

$$
\begin{aligned}
&+E\biggl[Y_{\tau^{Q_n^{(k)}}(\nu)}+\int_{\tau_k(\nu)}^{\tau^{Q_n^{(k)}}(\nu)} f(s,\theta_s^{Q_n^{(k)}})\,ds\Big|\mathcal{F}_\sigma\biggr]\\
&\le (\|Y\|_\infty+kT)\cdot E\bigl[\bigl|Z^{Q_n^{(k)}}_{\tau_k(\nu),\tau^{Q_n^{(k)}}(\nu)}-1\bigr|\big|\mathcal{F}_\sigma\bigr]\\
&\quad +E\bigl[Y_{\tau^{Q_n^{(k)}}(\nu)}+k\bigl(\tau^{Q_n^{(k)}}(\nu)-\tau_k(\nu)\bigr)|\mathcal{F}_\sigma\bigr],\quad P\text{-a.s.}
\end{aligned}
$$

Just as in (5.28), it can be shown that

$$
\lim_{n\to\infty}E\bigl(\bigl|Z^{Q_n^{(k)}}_{\tau_k(\nu),\tau^{Q_n^{(k)}}(\nu)}-1\bigr|\big|\mathcal{F}_\sigma\bigr)=0,\quad P\text{-a.s.};
$$

on the other hand, the Bounded Convergence theorem implies

$$
\lim_{n\to\infty}E\bigl(Y_{\tau^{Q_n^{(k)}}(\nu)}+k\bigl(\tau^{Q_n^{(k)}}(\nu)-\tau_k(\nu)\bigr)|\mathcal{F}_\sigma\bigr)=E\bigl[Y_{\tau_k(\nu)}|\mathcal{F}_\sigma\bigr],\quad P\text{-a.s.}
$$

Letting $n\to\infty$ in (5.33) yields $V(\sigma)\le E[Y_{\tau_k(\nu)}|\mathcal{F}_\sigma]$, P-a.s., and applying the Bounded Convergence theorem we obtain that

$$
V(\sigma)\le\lim_{k\to\infty}E\bigl[Y_{\tau_k(\nu)}|\mathcal{F}_\sigma\bigr]=E[Y_\sigma|\mathcal{F}_\sigma]=Y_\sigma,\quad P\text{-a.s.}
$$

The reverse inequality is rather obvious. □

Proof of Proposition 3.11. Fix $k\in\mathbb{N}$. In light of (3.10), we can find a sequence $\{Q_n^{(k)}\}_{n\in\mathbb{N}}\subset\mathcal{Q}_\gamma^k$ such that

$$
\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k}R^Q(\gamma)=\lim_{n\to\infty}\downarrow R^{Q_n^{(k)}}(\gamma),\quad P\text{-a.s.}\tag{5.34}
$$

For any $n\in\mathbb{N}$, Lemma 3.7 implies that the predictable process

$$
\theta_t^{\widetilde{Q}_n^{(k)}}\triangleq\mathbf{1}_{\{t\le\gamma\}}\theta_t^Q+\mathbf{1}_{\{t>\gamma\}}\theta_t^{Q_n^{(k)}},\quad t\in[0,T]
$$

induces a probability measure $\widetilde{Q}_n^{(k)}\in\mathcal{Q}_\gamma$ via $d\widetilde{Q}_n^{(k)}\triangleq\mathscr{E}(\theta^{\widetilde{Q}_n^{(k)}}\bullet B)_T\,dP$, such that for any $t\in[0,T]$, $R^{\widetilde{Q}_n^{(k)}}(\gamma)=R^{Q_n^{(k)}}(\gamma)$, P-a.s. Since $\gamma\le\tau(\nu)\le\tau^{\widetilde{Q}_n^{(k)}}(\nu)$, P-a.s., applying (3.4) yields

$$
\begin{aligned}
V(\nu)\le R^{\widetilde{Q}_n^{(k)}}(\nu)&=E_{\widetilde{Q}_n^{(k)}}\biggl[R^{\widetilde{Q}_n^{(k)}}(\gamma)+\int_\nu^\gamma f(s,\theta_s^{\widetilde{Q}_n^{(k)}})\,ds\Big|\mathcal{F}_\nu\biggr]\\
&=E_{\widetilde{Q}_n^{(k)}}\biggl[R^{Q_n^{(k)}}(\gamma)+\int_\nu^\gamma f(s,\theta_s^{Q})\,ds\Big|\mathcal{F}_\nu\biggr]\\
&=E_Q\biggl[R^{Q_n^{(k)}}(\gamma)+\int_\nu^\gamma f(s,\theta_s^{Q})\,ds\Big|\mathcal{F}_\nu\biggr],\quad P\text{-a.s.}
\end{aligned}\tag{5.35}
$$

It follows from (3.2) that

$$
-\|Y\|_\infty\le Y_\gamma\le R^{Q_n^{(k)}}(\gamma)\le\|Y\|_\infty+kT,\quad P\text{-a.s.}\tag{5.36}
$$

Letting $n \to \infty$ in (5.35), we can deduce from the Bounded Convergence theorem that P-a.s.

$$V(\nu) \le E_Q\Big[\lim_{n\to\infty} \downarrow R^{Q_n^{(k)}}(\gamma)|\mathcal{F}_\nu\Big] + E_Q\bigg[\int_\nu^\gamma f(s,\theta_s^Q)\,ds\bigg|\mathcal{F}_\nu\bigg]$$
$$= E_Q\bigg[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k} R^Q(\gamma) + \int_\nu^\gamma f(s,\theta_s^Q)\,ds\bigg|\mathcal{F}_\nu\bigg].$$

Letting $n \to \infty$ in (5.36), one sees from (5.34) that

$$-\|Y\|_\infty \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k} R^Q(\gamma) \le \|Y\|_\infty + kT, \quad P\text{-a.s.},$$

which leads to that

$$-\|Y\|_\infty \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k} R^Q(\gamma) \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^1} R^Q(\gamma) \le \|Y\|_\infty + T, \quad P\text{-a.s.}$$

From the Bounded Convergence theorem and Lemma 3.4, we obtain now

$$V(\nu) \le E_Q\Big[\lim_{k\to\infty} \downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k} R^Q(\gamma)\big|\mathcal{F}_\nu\Big] + E_Q\bigg[\int_\nu^\gamma f(s,\theta_s^Q)\,ds\bigg|\mathcal{F}_\nu\bigg]$$
$$= E_Q\bigg[V(\gamma) + \int_\nu^\gamma f(s,\theta_s^Q)\,ds\bigg|\mathcal{F}_\nu\bigg], \quad P\text{-a.s.} \qquad \square$$

Proof of Lemma 3.12. Fix $k \in \mathbb{N}$. For any $Q \in \mathcal{Q}_\nu^k$, the predictable process $\theta_t^{\widetilde{Q}} \triangleq \mathbf{1}_{\{t>\nu\vee\gamma\}}\theta_t^Q$, $t \in [0,T]$ induces a probability measure $\widetilde{Q}$ by $(d\widetilde{Q}/dP) \triangleq \mathscr{E}(\theta^{\widetilde{Q}} \bullet B)_T = Z^Q_{\nu\vee\gamma,T}$. Remark 3.3 shows that $\widetilde{Q} \in \mathcal{Q}^k_{\nu\vee\gamma} \subset \mathcal{Q}_\nu^k \cap \mathcal{Q}_\gamma^k$. By analogy with (5.6), we have $f(t,\theta_t^{\widetilde{Q}}) = \mathbf{1}_{\{t>\nu\vee\gamma\}} f(t,\theta_t^Q)$, $dt\otimes dP$-a.e. Then one can deduce that

$$\begin{aligned}
(5.37)\qquad &\mathbf{1}_{\{\nu=\gamma\}} E_{\widetilde{Q}}\bigg[Y_{\sigma\vee\gamma} + \int_\gamma^{\sigma\vee\gamma} f(s,\theta_s^{\widetilde{Q}})\,ds\bigg|\mathcal{F}_\gamma\bigg]\\
&= \mathbf{1}_{\{\nu=\gamma\}} E_{\widetilde{Q}}\bigg[Y_{\sigma\vee\gamma} + \int_\gamma^{\sigma\vee\gamma} \mathbf{1}_{\{s>\nu\vee\gamma\}} f(s,\theta_s^Q)\,ds\bigg|\mathcal{F}_\nu\bigg]\\
&= E_{\widetilde{Q}}\bigg[\mathbf{1}_{\{\nu=\gamma\}}\bigg(Y_{\sigma\vee\nu} + \int_\nu^{\sigma\vee\nu} f(s,\theta_s^Q)\,ds\bigg)\bigg|\mathcal{F}_\nu\bigg]\\
&= E\bigg[E_Q\bigg[\mathbf{1}_{\{\nu=\gamma\}}\bigg(Y_{\sigma\vee\nu} + \int_\nu^{\sigma\vee\nu} f(s,\theta_s^Q)\,ds\bigg)\bigg|\mathcal{F}_{\nu\vee\gamma}\bigg]\bigg|\mathcal{F}_\nu\bigg]\\
&= E\bigg[\mathbf{1}_{\{\nu=\gamma\}} E_Q\bigg[Y_{\sigma\vee\nu} + \int_\nu^{\sigma\vee\nu} f(s,\theta_s^Q)\,ds\bigg|\mathcal{F}_\nu\bigg]\bigg|\mathcal{F}_\nu\bigg]\\
&= \mathbf{1}_{\{\nu=\gamma\}} E_Q\bigg[Y_{\sigma\vee\nu} + \int_\nu^{\sigma\vee\nu} f(s,\theta_s^Q)\,ds\bigg|\mathcal{F}_\nu\bigg], \quad P\text{-a.s.},
\end{aligned}$$

which implies that P-a.s.

$$\begin{aligned}&\mathbf{1}_{\{\nu=\gamma\}}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&\quad\geq\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k}E_Q\left[Y_{\sigma\vee\gamma}+\int_\gamma^{\sigma\vee\gamma}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\gamma\right].\end{aligned}$$

Taking the essential infimum of the left-hand side over $Q\in\mathcal{Q}_\nu^k$, one can deduce from Lemma 2.3 that P-a.s.

$$\begin{aligned}&\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&\quad=\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}\mathbf{1}_{\{\nu=\gamma\}}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&\quad\geq\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k}E_Q\left[Y_{\sigma\vee\gamma}+\int_\gamma^{\sigma\vee\gamma}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\gamma\right].\end{aligned}$$

Letting $k\to\infty$, we see from Lemma 3.4(1) that P-a.s.

$$\begin{aligned}&\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&\quad\geq\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma}E_Q\left[Y_{\sigma\vee\gamma}+\int_\gamma^{\sigma\vee\gamma}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\gamma\right].\end{aligned}$$

Reversing the roles of ν and γ, we obtain (3.16).

On the other hand, taking essential supremum over $\sigma\in\mathcal{S}_{0,T}$ on both sides of (5.37), we can deduce from Lemma 2.3 that P-a.s.

$$\begin{aligned}\mathbf{1}_{\{\nu=\gamma\}}R^{\widetilde{Q}}(\gamma)&=\operatorname*{ess\,sup}_{\sigma\in\mathcal{S}_{0,T}}\mathbf{1}_{\{\nu=\gamma\}}E_{\widetilde{Q}}\left[Y_{\sigma\vee\gamma}+\int_\gamma^{\sigma\vee\gamma}f(s,\theta_s^{\widetilde{Q}})\,ds\Big|\mathcal{F}_\gamma\right]\\&=\operatorname*{ess\,sup}_{\sigma\in\mathcal{S}_{0,T}}\mathbf{1}_{\{\nu=\gamma\}}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&=\mathbf{1}_{\{\nu=\gamma\}}R^Q(\nu),\end{aligned}$$

which implies that $\mathbf{1}_{\{\nu=\gamma\}}R^Q(\nu)\geq\mathbf{1}_{\{\nu=\gamma\}}\operatorname{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k}R^Q(\gamma)$, P-a.s. Taking the essential infimum of the left-hand side over $Q\in\mathcal{Q}_\nu^k$, one can deduce from Lemma 2.3 that P-a.s.

$$\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}R^Q(\nu)=\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}\mathbf{1}_{\{\nu=\gamma\}}R^Q(\nu)\geq\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma^k}R^Q(\gamma).$$

Letting $k\to\infty$, we see from Lemma 3.4(2) that P-a.s.

$$\mathbf{1}_{\{\nu=\gamma\}}V(\nu)=\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}R^Q(\nu)\geq\mathbf{1}_{\{\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma}R^Q(\gamma)=\mathbf{1}_{\{\nu=\gamma\}}V(\gamma).$$

Reversing the roles of ν and γ, we obtain (3.17). $\square$

Proof of Theorem 3.13.

Proof of (1).

Step 1: For any $\sigma,\nu\in\mathcal{S}_{0,T}$, we define

$$\Psi^\sigma(\nu)\triangleq\mathbf{1}_{\{\sigma\le\nu\}}Y_\sigma+\mathbf{1}_{\{\sigma>\nu\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right].$$

We see from (3.7) that

$$\begin{aligned}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}&E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&=\lim_{k\to\infty}\downarrow\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right],\quad P\text{-a.s.}\end{aligned}\tag{5.38}$$

Fix $k\in\mathbb{N}$. In light of (3.9), we can find a sequence $\{Q_n^{(k)}\}_{n\in\mathbb{N}}$ in $\mathcal{Q}_\nu^k$ such that

$$\begin{aligned}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}&E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&=\lim_{n\to\infty}\downarrow E_{Q_n^{(k)}}\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^{Q_n^{(k)}})\,ds\Big|\mathcal{F}_\nu\right],\quad P\text{-a.s.}\end{aligned}\tag{5.39}$$

By analogy with (5.36), we have

$$-\|Y\|_\infty\le E_{Q_n^{(k)}}\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^{Q_n^{(k)}})\,ds\Big|\mathcal{F}_\nu\right]\le\|Y\|_\infty+kT\tag{5.40}$$

P-a.s.; letting $n\to\infty$, we see from (5.39) that P-a.s.

$$-\|Y\|_\infty\le\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\le\|Y\|_\infty+kT.$$

Therefore, it holds P-a.s. that

$$\begin{aligned}-\|Y\|_\infty&\le\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\\&\le\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^1}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\le\|Y\|_\infty+T.\end{aligned}\tag{5.41}$$

Letting $k\to\infty$, we see from (5.38) that P-a.s.

$$-\|Y\|_\infty\le\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\le\|Y\|_\infty+T,$$

which implies that

$$-\|Y\|_\infty\le\Psi^\sigma(\nu)\le\|Y\|_\infty+T,\quad P\text{-a.s.}\tag{5.42}$$

Let $\gamma\in\mathcal{S}_{0,T}$. It follows from (3.16) that P-a.s.

$$\begin{aligned}&\mathbf{1}_{\{\nu=\gamma\}}\Psi^\sigma(\nu)\\&\quad=\mathbf{1}_{\{\sigma\le\nu=\gamma\}}Y_\sigma+\mathbf{1}_{\{\sigma>\nu=\gamma\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu}E_Q\left[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu}f(s,\theta_s^Q)\,ds\Big|\mathcal{F}_\nu\right]\end{aligned}\tag{5.43}$$

$$= \mathbf{1}_{\{\sigma\le\gamma=\nu\}} Y_\sigma + \mathbf{1}_{\{\sigma>\gamma=\nu\}} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\gamma} E_Q\left[Y_{\sigma\vee\gamma} + \int_\gamma^{\sigma\vee\gamma} f(s,\theta_s^Q)\,ds \Big| \mathcal{F}_\gamma\right]$$
$$= \mathbf{1}_{\{\nu=\gamma\}} \Psi^\sigma(\gamma).$$

Step 2: Fix $\sigma\in\mathcal{S}_{0,T}$. For any $\zeta\in\mathcal{S}_{0,T}$, $\nu\in\mathcal{S}_{\zeta,T}$ and $k\in\mathbb{N}$, we let $\{Q_n^{(k)}\}_{n\in\mathbb{N}}\subset\mathcal{Q}_\nu^k$ be the sequence described in (5.39). Then we can deduce that P-a.s.

$$\begin{aligned}
\Psi^\sigma(\zeta) &\le \mathbf{1}_{\{\sigma\le\zeta\}} Y_\sigma + \mathbf{1}_{\{\sigma>\zeta\}} E_{Q_n^{(k)}}\left[Y_{\sigma\vee\zeta} + \int_\zeta^{\sigma\vee\zeta} f(s,\theta_s^{Q_n^{(k)}})\,ds \Big| \mathcal{F}_\zeta\right] \\
&= \mathbf{1}_{\{\sigma\le\zeta\}} Y_{\sigma\wedge\zeta} \\
&\quad + \mathbf{1}_{\{\sigma>\zeta\}} E\left[E_{Q_n^{(k)}}\left[Y_{\sigma\vee\zeta} + \int_\zeta^{\sigma\vee\zeta} f(s,\theta_s^{Q_n^{(k)}})\,ds \Big| \mathcal{F}_\nu\right]\Big|\mathcal{F}_\zeta\right] \\
&= E\Bigg[\mathbf{1}_{\{\sigma\le\zeta\}} Y_{\sigma\wedge\zeta} \\
&\quad + \mathbf{1}_{\{\sigma>\zeta\}} E_{Q_n^{(k)}}\left[Y_{\sigma\vee\zeta} + \int_\zeta^{\sigma\vee\zeta} f(s,\theta_s^{Q_n^{(k)}})\,ds \Big| \mathcal{F}_\nu\right]\Bigg|\mathcal{F}_\zeta\Bigg].
\end{aligned} \tag{5.44}$$

On the other hand, it holds P-a.s. that

$$\begin{aligned}
&\mathbf{1}_{\{\sigma>\nu\}} E_{Q_n^{(k)}}\left[Y_{\sigma\vee\zeta} + \int_\zeta^{\sigma\vee\zeta} f(s,\theta_s^{Q_n^{(k)}})\,ds \Big| \mathcal{F}_\nu\right] \\
&= E_{Q_n^{(k)}}\left[\mathbf{1}_{\{\sigma>\nu\}}\left(Y_\sigma + \int_\zeta^{\sigma} f(s,\theta_s^{Q_n^{(k)}})\,ds\right) \Big| \mathcal{F}_\nu\right] \\
&= E_{Q_n^{(k)}}\left[\mathbf{1}_{\{\sigma>\nu\}}\left(Y_{\sigma\vee\nu} + \int_\nu^{\sigma\vee\nu} f(s,\theta_s^{Q_n^{(k)}})\,ds\right) \Big| \mathcal{F}_\nu\right] \\
&= \mathbf{1}_{\{\sigma>\nu\}} E_{Q_n^{(k)}}\left[Y_{\sigma\vee\nu} + \int_\nu^{\sigma\vee\nu} f(s,\theta_s^{Q_n^{(k)}})\,ds \Big| \mathcal{F}_\nu\right]
\end{aligned}$$

and that

$$\begin{aligned}
&\mathbf{1}_{\{\zeta<\sigma\le\nu\}} E_{Q_n^{(k)}}\left[Y_{\sigma\vee\zeta} + \int_\zeta^{\sigma\vee\zeta} f(s,\theta_s^{Q_n^{(k)}})\,ds \Big| \mathcal{F}_\nu\right] \\
&= E_{Q_n^{(k)}}\left[\mathbf{1}_{\{\zeta<\sigma\le\nu\}}\left(Y_\sigma + \int_\zeta^{\sigma} f(s,\theta_s^{Q_n^{(k)}})\,ds\right) \Big| \mathcal{F}_\nu\right] \\
&= E_{Q_n^{(k)}}\left[\mathbf{1}_{\{\zeta<\sigma\le\nu\}} Y_{\sigma\wedge\nu}|\mathcal{F}_\nu\right] = \mathbf{1}_{\{\zeta<\sigma\le\nu\}} Y_{\sigma\wedge\nu} = \mathbf{1}_{\{\zeta<\sigma\le\nu\}} Y_\sigma;
\end{aligned}$$

recall the definitions of the classes $\mathcal{P}_\nu$, $\mathcal{Q}_\nu$ from Section 1.1. Therefore, (5.44) reduces to that P-a.s.

$$\Psi^\sigma(\zeta) \le E\left[\mathbf{1}_{\{\sigma\le\nu\}} Y_\sigma + \mathbf{1}_{\{\sigma>\nu\}} E_{Q_n^{(k)}}\left[Y_{\sigma\vee\nu} + \int_\nu^{\sigma\vee\nu} f(s,\theta_s^{Q_n^{(k)}})\,ds \Big| \mathcal{F}_\nu\right]\Big|\mathcal{F}_\zeta\right].$$

We obtain then from (5.39), (5.40) and the Bounded Convergence theorem, that P-a.s.

$$\begin{aligned}\Psi^\sigma(\zeta) &\le \lim_{n\to\infty}\downarrow E\biggl[\mathbf{1}_{\{\sigma\le\nu\}}Y_\sigma \\ &\qquad + \mathbf{1}_{\{\sigma>\nu\}}E_{Q_n^{(k)}}\biggl[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu} f(s,\theta_s^{Q_n^{(k)}})\,ds\Big|\mathcal{F}_\nu\biggr]\Big|\mathcal{F}_\zeta\biggr] \\ &= E\biggl[\mathbf{1}_{\{\sigma\le\nu\}}Y_\sigma \\ &\qquad + \mathbf{1}_{\{\sigma>\nu\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} E_Q\biggl[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu} f(s,\theta_s^{Q_n^{(k)}})\,ds\Big|\mathcal{F}_\nu\biggr]\Big|\mathcal{F}_\zeta\biggr].\end{aligned}$$

On the other hand, we can deduce from (5.38), (5.41) and the Bounded Convergence theorem once again that P-a.s.

$$\begin{aligned}\Psi^\sigma(\zeta) &\le \lim_{k\to\infty}\downarrow E\biggl[\mathbf{1}_{\{\sigma\le\nu\}}Y_\sigma \\ &\qquad + \mathbf{1}_{\{\sigma>\nu\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu^k} E_Q\biggl[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu} f(s,\theta_s^{Q_n^{(k)}})\,ds\Big|\mathcal{F}_\nu\biggr]\Big|\mathcal{F}_\zeta\biggr] \\ &= E\biggl[\mathbf{1}_{\{\sigma\le\nu\}}Y_\sigma \\ &\qquad + \mathbf{1}_{\{\sigma>\nu\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} E_Q\biggl[Y_{\sigma\vee\nu}+\int_\nu^{\sigma\vee\nu} f(s,\theta_s^{Q})\,ds\Big|\mathcal{F}_\nu\biggr]\Big|\mathcal{F}_\zeta\biggr] \\ &= E[\Psi^\sigma(\nu)|\mathcal{F}_\zeta],\end{aligned}\tag{5.45}$$

which implies that $\{\Psi^\sigma(t)\}_{t\in[0,T]}$ is a submartingale. Therefore [13, Proposition 1.3.14] shows that

$$P\Bigl(\text{the limit } \Psi_t^{\sigma,+}\triangleq\lim_{n\to\infty}\Psi^\sigma(q_n(t)) \text{ exists for any } t\in[0,T]\Bigr)=1\tag{5.46}$$

(where $q_n(t)\triangleq\frac{\lceil 2^n t\rceil}{2^n}\wedge T$), and that $\Psi^{\sigma,+}$ is an RCLL process.

Step 3: For any $\nu\in\mathcal{S}_{0,T}$ and $n\in\mathbb{N}$, $q_n(\nu)$ takes values in a finite set $\mathcal{D}_T^n\triangleq([0,T)\cap\{k2^{-n}\}_{k\in\mathbb{Z}})\cup\{T\}$. Given an $\lambda\in\mathcal{D}_T^n$, it holds for any $m\ge n$ that $q_m(\lambda)=\lambda$ since $\mathcal{D}_T^n\subset\mathcal{D}_T^m$. It follows from (5.46) that

$$\Psi_\lambda^{\sigma,+}=\lim_{m\to\infty}\Psi^\sigma(q_m(\lambda))=\Psi^\sigma(\lambda),\quad P\text{-a.s.}$$

Then one can deduce from (5.43) that P-a.s.

$$\begin{aligned}\Psi_{q_n(\nu)}^{\sigma,+} &= \sum_{\lambda\in\mathcal{D}_T^n}\mathbf{1}_{\{q_n(\nu)=\lambda\}}\Psi_\lambda^{\sigma,+}=\sum_{\lambda\in\mathcal{D}_T^n}\mathbf{1}_{\{q_n(\nu)=\lambda\}}\Psi^\sigma(\lambda) \\ &= \sum_{\lambda\in\mathcal{D}_T^n}\mathbf{1}_{\{q_n(\nu)=\lambda\}}\Psi^\sigma(q_n(\nu))=\Psi^\sigma(q_n(\nu)).\end{aligned}$$

Thus the right-continuity of the process $\Psi^{\sigma,+}$ implies that

$$\Psi^{\sigma,+}_\nu = \lim_{n\to\infty} \Psi^{\sigma,+}_{q_n(\nu)} = \lim_{n\to\infty} \Psi^\sigma(q_n(\nu)), \quad P\text{-a.s.} \tag{5.47}$$

Hence, (5.45), (5.42) and the Bounded Convergence theorem imply

$$\Psi^\sigma(\nu) \le \lim_{n\to\infty} E[\Psi^\sigma(q_n(\nu))|\mathcal{F}_\nu] = E[\Psi^{\sigma,+}_\nu|\mathcal{F}_\nu] = \Psi^{\sigma,+}_\nu, \quad P\text{-a.s.} \tag{5.48}$$

In the last equality we used the fact that $\Psi^{\sigma,+}_\nu = \lim_{n\to\infty} \Psi^\sigma(q_n(\nu)) \in \mathcal{F}_\nu$, thanks to the right-continuity of the Brownian filtration $\mathbf{F}$.

Step 4: Set $\nu,\gamma \in \mathcal{S}_{0,T}$ and $\zeta \stackrel{\triangle}{=} \tau(\nu)\wedge\gamma, \zeta_n \stackrel{\triangle}{=} \tau(\nu)\wedge q_n(\gamma)$, $\forall n\in\mathbb{N}$. Now, let $\sigma \in \mathcal{S}_{\zeta,T}$. Since $\lim_{n\to\infty}\uparrow \mathbf{1}_{\{\tau(\nu)>q_n(\gamma)\}} = \mathbf{1}_{\{\tau(\nu)>\gamma\}}$,

$$\{\tau(\nu)>\gamma\} \subset \bigl\{q_n(\gamma) = q_n\bigl(\tau(\nu)\wedge\gamma\bigr)\bigr\} \quad \forall n\in\mathbb{N},$$

and

$$\{\tau(\nu) > q_n(\gamma)\} \subset \{q_n(\gamma) = \tau(\nu)\wedge q_n(\gamma)\} \quad \forall n \in \mathbb{N},$$

one can deduce from (5.48), (5.47), and (5.43) that P-a.s.

$$\begin{aligned} &\mathbf{1}_{\{\tau(\nu)>\gamma\}}\Psi^\sigma(\zeta) \\ &\quad\le \mathbf{1}_{\{\tau(\nu)>\gamma\}}\Psi^{\sigma,+}_\zeta = \mathbf{1}_{\{\tau(\nu)>\gamma\}} \lim_{n\to\infty}\Psi^\sigma(q_n(\zeta)) \\ &\quad= \lim_{n\to\infty}\mathbf{1}_{\{\tau(\nu)>\gamma\}}\Psi^\sigma\bigl(q_n\bigl(\tau(\nu)\wedge\gamma\bigr)\bigr) = \lim_{n\to\infty}\mathbf{1}_{\{\tau(\nu)>\gamma\}}\Psi^\sigma(q_n(\gamma)) \\ &\quad= \lim_{n\to\infty}\mathbf{1}_{\{\tau(\nu)>q_n(\gamma)\}}\Psi^\sigma(q_n(\gamma)) \\ &\quad= \lim_{n\to\infty}\mathbf{1}_{\{\tau(\nu)>q_n(\gamma)\}}\Psi^\sigma\bigl(\tau(\nu)\wedge q_n(\gamma)\bigr) \\ &\quad= \mathbf{1}_{\{\tau(\nu)>\gamma\}}\lim_{n\to\infty}\Psi^\sigma(\zeta_n). \end{aligned} \tag{5.49}$$

For any $n\in\mathbb{N}$, we see from (3.14) and Lemma 2.3 that P-a.s.

$$\begin{aligned} V(\zeta_n) &= \underline{V}(\zeta_n) = \operatorname*{ess\,sup}_{\beta\in\mathcal{S}_{\zeta_n,T}}\left(\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_{\zeta_n}} E_Q\left[Y_\beta + \int_{\zeta_n}^\beta f(s,\theta^Q_s)\,ds\Big|\mathcal{F}_{\zeta_n}\right]\right) \\ &\ge \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_{\zeta_n}} E_Q\left[Y_{\sigma\vee\zeta_n} + \int_{\zeta_n}^{\sigma\vee\zeta_n} f(s,\theta^Q_s)\,ds\Big|\mathcal{F}_{\zeta_n}\right] \\ &= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_{\zeta_n}} E_Q\left[\mathbf{1}_{\{\sigma\le\zeta_n\}}Y_{\zeta_n} + \mathbf{1}_{\{\sigma>\zeta_n\}}\left(Y_{\sigma\vee\zeta_n} + \int_{\zeta_n}^{\sigma\vee\zeta_n} f(s,\theta^Q_s)\,ds\right)\Big|\mathcal{F}_{\zeta_n}\right] \\ &= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_{\zeta_n}}\left(\mathbf{1}_{\{\sigma\le\zeta_n\}}Y_{\zeta_n} + \mathbf{1}_{\{\sigma>\zeta_n\}}E_Q\left[Y_{\sigma\vee\zeta_n} + \int_{\zeta_n}^{\sigma\vee\zeta_n} f(s,\theta^Q_s)\,ds\Big|\mathcal{F}_{\zeta_n}\right]\right) \\ &= \mathbf{1}_{\{\sigma\le\zeta_n\}}Y_{\zeta_n} + \mathbf{1}_{\{\sigma>\zeta_n\}}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_{\zeta_n}} E_Q\left[Y_{\sigma\vee\zeta_n} + \int_{\zeta_n}^{\sigma\vee\zeta_n} f(s,\theta^Q_s)\,ds\Big|\mathcal{F}_{\zeta_n}\right]. \end{aligned}$$

Since $\{\tau(\nu)\le\gamma\}\subset\{\zeta_n=\zeta=\tau(\nu)\}$ and $\{\sigma>\zeta_n\}\subset\{\sigma>\zeta\}$, it follows from

(3.16) that P-a.s.

$$
\begin{aligned}
V(\zeta_n) \geq{}& \mathbf{1}_{\{\sigma\le\zeta_n\}} Y_{\zeta_n} \\
&+ \mathbf{1}_{\{\sigma>\zeta_n,\tau(\nu)>\gamma\}} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_{\zeta_n}} E_Q\left[Y_{\sigma\vee\zeta_n} + \int_{\zeta_n}^{\sigma\vee\zeta_n} f(s,\theta_s^Q)\,ds \Big| \mathcal{F}_{\zeta_n}\right] \\
&+ \mathbf{1}_{\{\sigma>\zeta_n,\tau(\nu)\le\gamma\}} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_{\zeta}} E_Q\left[Y_{\sigma\vee\zeta} + \int_{\zeta}^{\sigma\vee\zeta} f(s,\theta_s^Q)\,ds \Big| \mathcal{F}_{\zeta}\right] \\
={}& \mathbf{1}_{\{\sigma\le\zeta_n\}} Y_{\zeta_n} + \mathbf{1}_{\{\sigma>\zeta_n,\tau(\nu)>\gamma\}} \Psi^\sigma(\zeta_n) + \mathbf{1}_{\{\sigma>\zeta_n,\tau(\nu)\le\gamma\}} \Psi^\sigma(\zeta).
\end{aligned}
$$

As $n\to\infty$, the right-continuity of processes Y, (5.49) as well as Lemma 2.3 show that

$$
\begin{aligned}
\varliminf_{n\to\infty} V(\zeta_n) &\geq \mathbf{1}_{\{\sigma=\zeta\}} Y_\zeta + \mathbf{1}_{\{\sigma>\zeta,\tau(\nu)>\gamma\}} \varliminf_{n\to\infty} \Psi^\sigma(\zeta_n) + \mathbf{1}_{\{\sigma>\zeta,\tau(\nu)\le\gamma\}} \Psi^\sigma(\zeta) \\
&\geq \mathbf{1}_{\{\sigma=\zeta\}} Y_\zeta + \mathbf{1}_{\{\sigma>\zeta\}} \Psi^\sigma(\zeta) \\
&= \mathbf{1}_{\{\sigma=\zeta\}} Y_\zeta + \mathbf{1}_{\{\sigma>\zeta\}} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\zeta} E_Q\left[Y_{\sigma\vee\zeta} + \int_\zeta^{\sigma\vee\zeta} f(s,\theta_s^Q)\,ds \Big| \mathcal{F}_\zeta\right] \\
&= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\zeta} \left(\mathbf{1}_{\{\sigma=\zeta\}} Y_\zeta + \mathbf{1}_{\{\sigma>\zeta\}} E_Q\left[Y_{\sigma\vee\zeta} + \int_\zeta^{\sigma\vee\zeta} f(s,\theta_s^Q)\,ds \Big| \mathcal{F}_\zeta\right]\right) \\
&= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\zeta} E_Q\left[\mathbf{1}_{\{\sigma=\zeta\}} Y_\zeta + \mathbf{1}_{\{\sigma>\zeta\}} \left(Y_\sigma + \int_\zeta^\sigma f(s,\theta_s^Q)\,ds\right) \Big| \mathcal{F}_\zeta\right] \\
&= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\zeta} E_Q\left[Y_\sigma + \int_\zeta^\sigma f(s,\theta_s^Q)\,ds \Big| \mathcal{F}_\zeta\right], \quad P\text{-a.s.}
\end{aligned}
$$

Taking the essential supremum of the right-hand side over $\sigma\in\mathcal{S}_{\zeta,T}$, we obtain

$$
\begin{aligned}
\varliminf_{n\to\infty} V(\zeta_n) &\geq \operatorname*{ess\,sup}_{\sigma\in\mathcal{S}_{\zeta,T}} \left(\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\zeta} E_Q\left[Y_\sigma + \int_\zeta^\sigma f(s,\theta_s^Q)\,ds \Big| \mathcal{F}_\zeta\right]\right) \\
&= \underline{V}(\zeta) = V(\zeta), \quad P\text{-a.s.}
\end{aligned}
\tag{5.50}
$$

Let us show the reverse inequality. Fix $Q\in\mathcal{Q}_\zeta$ and $n\in\mathbb{N}$. For any $k,m\in\mathbb{N}$, the predictable process

$$
\theta_t^{Q_n^{m,k}} \triangleq \mathbf{1}_{\{\zeta_n<t\le\delta_m^{Q,n}\}} \mathbf{1}_{A_{\zeta,k}^Q} \theta_t^Q, \quad t\in[0,T]
$$

induces a probability measure $Q_n^{m,k}\in\mathcal{Q}_{\zeta_n}^k$ by $dQ_n^{m,k}\triangleq\mathscr{E}(\theta^{Q_n^{m,k}}\bullet B)_T\,dP$, where $\delta_m^{Q,n}$ is defined by

$$
\delta_m^{Q,n} \triangleq \inf\left\{t\in[\zeta_n,T] : \int_{\zeta_n}^t f(s,\theta_s^Q)\,ds > m\right\} \wedge T, \quad m\in\mathbb{N}.
$$

For any $\beta \in \mathcal{S}_{\zeta_n,T}$, using arguments similar to those that lead to (5.7), we obtain that P-a.s.

$$E_{Q_n^{m,k}}\left[Y_\beta + \int_{\zeta_n}^{\beta} f(s, \theta_s^{Q_n^{m,k}})\, ds \Big| \mathcal{F}_{\zeta_n}\right]$$
$$\le (\|Y\|_\infty + m) E[|Z_{\zeta_n,T}^{Q_n^{m,k}} - Z^Q_{\zeta_n, \delta_m^{Q,n}}||\mathcal{F}_{\zeta_n}]$$
$$+ \|Y\|_\infty \cdot E[|Z^Q_{\zeta_n,\delta_m^{Q,n}} - Z^Q_{\zeta_n,T}||\mathcal{F}_{\zeta_n}] + E_Q\left[Y_\beta + \int_{\zeta_n}^{\beta} f(s,\theta_s^Q)\, ds \Big| \mathcal{F}_{\zeta_n}\right].$$

Then taking the essential supremum of both sides over $\beta \in \mathcal{S}_{\zeta_n,T}$ yields that

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}^k_{\zeta_n}} R^Q(\zeta_n) \le R^{Q_n^{m,k}}(\zeta_n)$$
$$\le (\|Y\|_\infty + m) E[|Z_{\zeta_n,T}^{Q_n^{m,k}} - Z^Q_{\zeta_n, \delta_m^{Q,n}}||\mathcal{F}_{\zeta_n}]$$
$$+ \|Y\|_\infty \cdot E[|Z^Q_{\zeta_n,\delta_m^{Q,n}} - Z^Q_{\zeta_n,T}||\mathcal{F}_{\zeta_n}]$$
$$+ R^Q(\zeta_n), \quad P\text{-a.s.} \tag{5.51}$$

Just as in (5.9), we can show that

$$\lim_{k\to\infty} E[|Z_{\zeta_n,T}^{Q_n^{m,k}} - Z^Q_{\zeta_n, \delta_m^{Q,n}}||\mathcal{F}_{\zeta_n}] = 0, \quad P\text{-a.s.}$$

Therefore, letting $k \to \infty$ in (5.51), we know from Lemma 3.4(2) that

$$V(\zeta_n) = \lim_{k\to\infty} \downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^k_{\zeta_n}} R^Q(\zeta_n)$$
$$\le \|Y\|_\infty \cdot E[|Z^Q_{\zeta_n,\delta_m^{Q,n}} - Z^Q_{\zeta_n,T}||\mathcal{F}_{\zeta_n}] + R^Q(\zeta_n), \quad P\text{-a.s.} \tag{5.52}$$

Next, by analogy with (5.11), we have

$$\lim_{m\to\infty} E(|Z^Q_{\zeta_n,\delta_m^{Q,n}} - Z^Q_{\zeta_n,T}||\mathcal{F}_{\zeta_n}) = 0, \quad P\text{-a.s.}$$

Letting $m \to \infty$ in (5.52), we obtain $V(\zeta_n) \le R^Q(\zeta_n) = R^{Q,0}_{\zeta_n}$, P-a.s. from (3.3). Then the right-continuity of the process $R^{Q,0}$, as well as (3.3), imply that

$$\varlimsup_{n\to\infty} V(\zeta_n) \le \lim_{n\to\infty} R^{Q,0}_{\zeta_n} = R^{Q,0}_{\zeta} = R^Q(\zeta), \quad P\text{-a.s.}$$

Taking the essential infimum of $R^Q(\zeta)$ over $Q \in \mathcal{Q}_\zeta$ yields

$$\varlimsup_{n\to\infty} V(\zeta_n) \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\zeta} R^Q(\zeta) = \overline{V}(\zeta) = V(\zeta), \quad P\text{-a.s.}$$

This inequality, together with (5.50), shows that

$$\lim_{n\to\infty} V\big(\tau(\nu)\wedge q_n(\gamma)\big) = V\big(\tau(\nu)\wedge\gamma\big), \quad P\text{-a.s.} \tag{5.53}$$

Step 5: Now fix $\nu \in \mathcal{S}_{0,T}$. It is clear that $P \in \mathcal{Q}_\nu$ and that $\theta^P_\cdot \equiv 0$. For any $t \in [0,T]$, (3.17) implies that

$$\mathbf{1}_{\{t\ge\nu\}} V\big(\tau(\nu)\wedge t\big) = \mathbf{1}_{\{t\ge\nu\}} V\big(\tau(\nu)\wedge(t\vee\nu)\big), \quad P\text{-a.s.},$$

since $\{t \ge \nu\} \subset \{\tau(\nu)\wedge t = \tau(\nu)\wedge(t\vee\nu)\}$. Then we can deduce from (3.15), ($\mathfrak{f}$3), and (3.14) that for any $s \in [0,t)$

$$\begin{aligned}
&\mathbf{1}_{\{s\ge\nu\}} V\big(\tau(\nu)\wedge s\big)\\
&\quad = \mathbf{1}_{\{s\ge\nu\}} V\big(\tau(\nu)\wedge(s\vee\nu)\big)\\
&\quad \le \mathbf{1}_{\{s\ge\nu\}} E\bigg[V\big(\tau(\nu)\wedge(t\vee\nu)\big) + \int_{\tau(\nu)\wedge(s\vee\nu)}^{\tau(\nu)\wedge(t\vee\nu)} f(r,\theta^P_r)\,dr\bigg|\mathcal{F}_{\tau(\nu)\wedge(s\vee\nu)}\bigg]\\
&\quad = \mathbf{1}_{\{s\ge\nu\}} E\bigg[V\big(\tau(\nu)\wedge(t\vee\nu)\big)\bigg|\mathcal{F}_{\tau(\nu)\wedge s}\bigg]\\
&\quad = E\big[\mathbf{1}_{\{s\ge\nu\}} V\big(\tau(\nu)\wedge(t\vee\nu)\big)|\mathcal{F}_{\tau(\nu)\wedge s}\big]\\
&\quad \le E\big[\mathbf{1}_{\{t\ge\nu\}} V\big(\tau(\nu)\wedge(t\vee\nu)\big) + \mathbf{1}_{\{t\ge\nu>s\}}\|Y\|_\infty|\mathcal{F}_{\tau(\nu)\wedge s}\big]\\
&\quad = E\big[E\big[\mathbf{1}_{\{t\ge\nu\}}\big(V\big(\tau(\nu)\wedge t\big)+\|Y\|_\infty\big)|\mathcal{F}_{\tau(\nu)}\big]|\mathcal{F}_s\big] - \mathbf{1}_{\{s\ge\nu\}}\|Y\|_\infty\\
&\quad = E\big[\mathbf{1}_{\{t\ge\nu\}}\big(V\big(\tau(\nu)\wedge t\big)+\|Y\|_\infty\big)|\mathcal{F}_s\big] - \mathbf{1}_{\{s\ge\nu\}}\|Y\|_\infty, \quad P\text{-a.s.},
\end{aligned}$$

which shows that $\{\mathbf{1}_{\{t\ge\nu\}}(V(\tau(\nu)\wedge t)+\|Y\|_\infty)\}_{t\in[0,T]}$ is a submartingale. Hence, it follows from [13, Proposition 1.3.14] that

$$P\Big(\text{the limit } V^{0,\nu}_t \stackrel{\triangle}{=} \lim_{n\to\infty} \mathbf{1}_{\{q_n(t)\ge\nu\}} V\big(\tau(\nu)\wedge q_n(t)\big) \text{ exists}, \forall t\in[0,T]\Big) = 1,$$

and that $V^{0,\nu}$ is an RCLL process.

Let $\zeta \in \mathcal{S}^\star_{0,T}$ take values in a finite set $\{t_1 < \cdots < t_m\}$. For any $\lambda \in \{1\cdots m\}$ and $n \in \mathbb{N}$, since $\{\zeta = t_\lambda\} \subset \{\tau(\nu)\wedge q_n(\zeta) = \tau(\nu)\wedge q_n(t_\lambda)\}$, one can deduce from (3.17) that

$$\mathbf{1}_{\{\zeta=t_\lambda\}} V\big(\tau(\nu)\wedge q_n(\zeta)\big) = \mathbf{1}_{\{\zeta=t_\lambda\}} V\big(\tau(\nu)\wedge q_n(t_\lambda)\big), \quad P\text{-a.s.}$$

As $n\to\infty$, (5.53) shows

$$\begin{aligned}
\mathbf{1}_{\{\zeta=t_\lambda\}} V^{0,\nu}_\zeta = \mathbf{1}_{\{\zeta=t_\lambda\}} V^{0,\nu}_{t_\lambda} &= \mathbf{1}_{\{t_\lambda\ge\nu\}} \lim_{n\to\infty} \mathbf{1}_{\{\zeta=t_\lambda\}} V\big(\tau(\nu)\wedge q_n(t_\lambda)\big)\\
&= \mathbf{1}_{\{t_\lambda\ge\nu\}} \lim_{n\to\infty} \mathbf{1}_{\{\zeta=t_\lambda\}} V\big(\tau(\nu)\wedge q_n(\zeta)\big)\\
&= \mathbf{1}_{\{\zeta\ge\nu\}} \mathbf{1}_{\{\zeta=t_\lambda\}} V\big(\tau(\nu)\wedge\zeta\big), \quad P\text{-a.s.}
\end{aligned}$$

Summing the above expression over λ, we obtain $V^{0,\nu}_\zeta = \mathbf{1}_{\{\zeta\ge\nu\}} V(\tau(\nu)\wedge\zeta)$, P-a.s. Then for any $\gamma \in \mathcal{S}_{0,T}$, the right-continuity of the process $V^{0,\nu}$ and (5.53) imply that P-a.s.

$$V^{0,\nu}_\gamma = \lim_{n\to\infty} V^{0,\nu}_{q_n(\gamma)} = \lim_{n\to\infty} \mathbf{1}_{\{q_n(\gamma)\ge\nu\}} V\big(\tau(\nu)\wedge q_n(\gamma)\big) = \mathbf{1}_{\{\gamma\ge\nu\}} V\big(\tau(\nu)\wedge\gamma\big),$$

proving (3.18). In particular, $V^{0,\nu}$ is an RCLL modification of the process $\{\mathbf{1}_{\{t\ge\nu\}} V(\tau(\nu)\wedge t)\}_{t\in[0,T]}$.

Proof of (2).

Proposition 3.10 and (3.18) imply that $V^{0,\nu}_{\tau(\nu)} = V(\tau(\nu)) = Y_{\tau(\nu)}$, P-a.s. Hence, we can deduce from the right-continuity of processes $V^{0,\nu}$ and Y that $\tau_V(\nu)$ in (3.19) is a stopping time belonging to $\mathcal{S}_{\nu,\tau(\nu)}$ and that

$$Y_{\tau_V(\nu)} = V^{0,\nu}_{\tau_V(\nu)} = V(\tau_V(\nu)), \quad P\text{-a.s.},$$

where the second equality is due to (3.18). Then it follows from (3.15) that for any $Q \in \mathcal{Q}_\nu$

$$\begin{aligned} V(\nu) &\le E_Q\left[V(\tau_V(\nu)) + \int_\nu^{\tau_V(\nu)} f(s,\theta_s^Q)\,ds \middle| \mathcal{F}_\nu\right] \\ &= E_Q\left[Y_{\tau_V(\nu)} + \int_\nu^{\tau_V(\nu)} f(s,\theta_s^Q)\,ds \middle| \mathcal{F}_\nu\right], \quad P\text{-a.s.} \end{aligned}$$

Taking the essential infimum of the right-hand side over $Q \in \mathcal{Q}_\nu$ yields that

$$\begin{aligned} V(\nu) &\le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} E_Q\left[Y_{\tau_V(\nu)} + \int_\nu^{\tau_V(\nu)} f(s,\theta_s^Q)\,ds \middle| \mathcal{F}_\nu\right] \\ &\le \operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu,T}} \left(\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\nu} E_Q\left[Y_\gamma + \int_\nu^{\gamma} f(s,\theta_s^Q)\,ds \middle| \mathcal{F}_\nu\right]\right) \\ &= \underline{V}(\nu) = V(\nu), \quad P\text{-a.s.}, \end{aligned}$$

from which the claim follows. □

5.2. Proofs of results in Section 4.

Proof of Theorem 4.2. It is easy to see from (i) that

$$Y_{\sigma_*} = V(\sigma_*) = R^{Q^*}(\sigma_*), \quad P\text{-a.s.} \tag{5.54}$$

which together with (ii) and (iii) shows that for any $Q \in \mathcal{Q}_0$

$$E_{Q^*}[Y^{Q^*}_{\sigma_*}] = E_{Q^*}[V^{Q^*}(\sigma_*)] = V^{Q^*}(0) = V(0) \le E_Q[V^Q(\sigma_*)] = E_Q[Y^Q_{\sigma_*}].$$

Thus the second inequality in (4.1) holds for (Q^*, σ_*). Now we show that (Q^*, σ_*) satisfies the first inequality in (4.1) in three steps:

- When $\nu \in \mathcal{S}_{0,\sigma_*}$, property (iii) and (5.54) imply that P-a.s.

$$Y^{Q^*}_\nu \le V^{Q^*}(\nu) = E_{Q^*}[V^{Q^*}(\sigma_*)|\mathcal{F}_\nu] = E_{Q^*}[Y^{Q^*}_{\sigma_*}|\mathcal{F}_\nu]. \tag{5.55}$$

Taking the expectation E_{Q^*} on both sides yields that $E_{Q^*}[Y^{Q^*}_\nu] \le E_{Q^*}[Y^{Q^*}_{\sigma_*}]$.

- When $\nu \in \mathcal{S}_{\sigma_*,T}$, it follows from (5.54) that

$$\begin{aligned} E_{Q^*}[Y^{Q^*}_\nu] &= E_{Q^*}\left[E_{Q^*}\left[Y_\nu + \int_{\sigma_*}^{\nu} f(s,\theta_s^{Q^*})\,ds \middle| \mathcal{F}_{\sigma_*}\right] + \int_0^{\sigma_*} f(s,\theta_s^{Q^*})\,ds\right] \\ &\le E_{Q^*}\left[R^{Q^*}(\sigma_*) + \int_0^{\sigma_*} f(s,\theta_s^{Q^*})\,ds\right] = E_{Q^*}[Y^{Q^*}_{\sigma_*}]. \end{aligned}$$

• For a general stopping time $\nu \in \mathcal{S}_{0,T}$, let us define $\nu_1 = \nu \wedge \sigma_* \in \mathcal{S}_{0,\sigma_*}$ and $\nu_2 = \nu \vee \sigma_* \in \mathcal{S}_{\sigma_*,T}$. Since $\{\nu \le \sigma_*\} \in \mathcal{F}_{\nu\wedge\sigma_*} = \mathcal{F}_{\nu_1}$, one can deduce from (5.55) that

$$\begin{aligned}
E_{Q^*}[Y_\nu^{Q^*}] &= E_{Q^*}\big[E_{Q^*}\big[\mathbf{1}_{\{\nu\le\sigma_*\}}Y_{\nu_1}^{Q^*} + \mathbf{1}_{\{\nu>\sigma_*\}}Y_{\nu_2}^{Q^*}|\mathcal{F}_{\sigma_*}\big]\big] \\
&= E_{Q^*}\big[\mathbf{1}_{\{\nu\le\sigma_*\}}Y_{\nu_1}^{Q^*} + \mathbf{1}_{\{\nu>\sigma_*\}}E_{Q^*}\big[Y_{\nu_2}^{Q^*}|\mathcal{F}_{\sigma_*}\big]\big] \\
&\le E_{Q^*}\left[\mathbf{1}_{\{\nu\le\sigma_*\}}Y_{\nu_1}^{Q^*} + \mathbf{1}_{\{\nu>\sigma_*\}}\left(R^{Q^*}(\sigma_*) + \int_0^{\sigma_*} f(s,\theta_s^{Q^*})\,ds\right)\right] \\
&= E_{Q^*}\big[\mathbf{1}_{\{\nu\le\sigma_*\}}Y_{\nu_1}^{Q^*} + \mathbf{1}_{\{\nu>\sigma_*\}}Y_{\sigma_*}^{Q^*}\big] \\
&= E_{Q^*}\big[\mathbf{1}_{\{\nu\le\sigma_*\}}Y_{\nu_1}^{Q^*} + \mathbf{1}_{\{\nu>\sigma_*\}}E_{Q^*}[Y_{\sigma_*}^{Q^*}|\mathcal{F}_{\nu_1}]\big] \\
&\le E_{Q^*}\big[\mathbf{1}_{\{\nu\le\sigma_*\}}E_{Q^*}[Y_{\sigma_*}^{Q^*}|\mathcal{F}_{\nu_1}] + \mathbf{1}_{\{\nu>\sigma_*\}}E_{Q^*}[Y_{\sigma_*}^{Q^*}|\mathcal{F}_{\nu_1}]\big] \\
&= E_{Q^*}[Y_{\sigma_*}^{Q^*}]. \qquad \square
\end{aligned}$$

Proof of Lemma 4.8. Fix $t \in [0,T]$. For any $\gamma \in \mathcal{S}_{\nu\vee t,T}$, we see from (4.5) that P-a.s.

$$\widetilde{\Gamma}_{\nu\vee t} = \widetilde{\Gamma}_\gamma + \int_{\nu\vee t}^{\gamma} f(s,\theta_s^{*,\nu})\,ds + \widetilde{K}_\gamma - \widetilde{K}_{\nu\vee t} - \int_{\nu\vee t}^{\gamma} \widetilde{\mathcal{Z}}_s\,dB_s^{Q^{*,\nu}}.$$

Applying $E_{Q^{*,\nu}}[\cdot|\mathcal{F}_{\nu\vee t}]$ to both sides, we obtain

$$\widetilde{\Gamma}_{\nu\vee t} = E_{Q^{*,\nu}}\left[\widetilde{\Gamma}_\gamma + \int_{\nu\vee t}^{\gamma} f(s,\theta_s^{*,\nu})\,ds + \widetilde{K}_\gamma - \widetilde{K}_{\nu\vee t}\Big|\mathcal{F}_{\nu\vee t}\right] \tag{5.56}$$

$$\ge E_{Q^{*,\nu}}\left[Y_\gamma + \int_{\nu\vee t}^{\gamma} f(s,\theta_s^{*,\nu})\,ds\Big|\mathcal{F}_{\nu\vee t}\right], \quad P\text{-a.s.} \tag{5.57}$$

Let $\sigma^*_{\nu\vee t} \triangleq \inf\{s \in [\nu\vee t, T] : \widetilde{\Gamma}_s = Y_s\} \in \mathcal{S}_{\nu\vee t,T}$. The flat-off condition satisfied by $(\widetilde{\Gamma}, \widetilde{\mathcal{Z}}, \widetilde{K})$ and the continuity of $\widetilde{K}$ imply that

$$\begin{aligned}
0 &= \int_{[\nu\vee t,\sigma^*_{\nu\vee t})} \mathbf{1}_{\{\widetilde{\Gamma}_s > Y_s\}}\,d\widetilde{K}_s = \int_{[\nu\vee t,\sigma^*_{\nu\vee t})} d\widetilde{K}_s \\
&= \lim_{s\nearrow\sigma^*_{\nu\vee t}} \widetilde{K}_s - \widetilde{K}_{\nu\vee t} = \widetilde{K}_{\sigma^*_{\nu\vee t}} - \widetilde{K}_{\nu\vee t}, \quad P\text{-a.s.}
\end{aligned}$$

Hence, taking $\gamma = \sigma^*_{\nu\vee t}$ in (5.56), we obtain that

$$\widetilde{\Gamma}_{\nu\vee t} = E_{Q^{*,\nu}}\left[Y_{\sigma^*_{\nu\vee t}} + \int_{\nu\vee t}^{\sigma^*_{\nu\vee t}} f(s,\theta_s^{*,\nu})\,ds\Big|\mathcal{F}_{\nu\vee t}\right], \quad P\text{-a.s.},$$

which, together with (5.57) and (3.3), shows that

$$\begin{aligned}
\widetilde{\Gamma}_{\nu\vee t} &= \operatorname*{ess\,sup}_{\gamma\in\mathcal{S}_{\nu\vee t,T}} E_{Q^{*,\nu}}\left[Y_\gamma + \int_{\nu\vee t}^{\gamma} f(s,\theta_s^{*,\nu})\,ds\Big|\mathcal{F}_{\nu\vee t}\right] \\
&= R^{Q^{*,\nu}}(\nu\vee t) = R^{Q^{*,\nu},0}_{\nu\vee t}, \quad P\text{-a.s.}
\end{aligned}$$

Then the right-continuity of the processes $\widetilde{\Gamma}$ and $R^{Q^{*,\nu},0}$ implies (4.6). $\square$

Proof of Theorem 4.10. We shall show that $(Q^*, \tau^{Q^*}(0))$ satisfies conditions (i)–(iii) of Theorem 4.2:

(1) It follows easily from Proposition 3.2 that $Y_{\tau^{Q^*}(0)} = R^{Q^*,0}_{\tau^{Q^*}(0)} = R^{Q^*}(\tau^{Q^*}(0))$, P-a.s.

(2) For any $k \in \mathbb{N}$ and $Q \in \mathcal{Q}_0^k$, we can deduce from (4.9), the right-continuity of processes $R^{Q^*,0}$ and $\widetilde{\Gamma}$, as well as (4.8) that P-a.s.

$$R^{Q^*,0}_t = \widetilde{\Gamma}_t \le R^{Q,0}_t \quad \forall t \in [0,T].$$

In particular, we have $Y_{\tau^Q(0)} \le R^{Q^*,0}_{\tau^Q(0)} \le R^{Q,0}_{\tau^Q(0)} = Y_{\tau^Q(0)}$, P-a.s. Hence $Y_{\tau^Q(0)} = R^{Q^*,0}_{\tau^Q(0)}$, P-a.s., which implies further that $\tau^{Q^*}(0) \le \tau^Q(0)$, P-a.s. Taking the essential infimum of right-hand side over $Q \in \mathcal{Q}_0^k$ and letting $k \to \infty$, we deduce that, in the notation of (3.11), we have

$$\tau^{Q^*}(0) \le \lim_{k\to\infty} \downarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_0^k} \tau^Q(0) = \tau(0), \quad P\text{-a.s.}$$

Then (3.15) shows $V(0) \le E_Q[V^Q(\tau^{Q^*}(0))]$ for any $Q \in \mathcal{Q}_0$.

(3) For any $\nu \in \mathcal{S}_{0,\tau^{Q^*}(0)}$, and since $\nu \le \tau^{Q^*}(0) \le \tau^{Q^*}(\nu)$ holds P-a.s., one can deduce from (4.9) and (3.4) that

$$\begin{aligned}
V^{Q^*}(\nu) &= R^{Q^*}(\nu) + \int_0^\nu f(s,\theta_s^*)\,ds \\
&= E_{Q^*}\left[R^{Q^*}(\tau^{Q^*}(0)) + \int_\nu^{\tau^{Q^*}(0)} f(s,\theta^*_s)\,ds \middle| \mathcal{F}_\nu\right] + \int_0^\nu f(s,\theta^*_s)\,ds \\
&= E_{Q^*}\left[R^{Q^*}(\tau^{Q^*}(0)) + \int_0^{\tau^{Q^*}(0)} f(s,\theta^*_s)\,ds \middle| \mathcal{F}_\nu\right] \\
&= E_{Q^*}[V^{Q^*}(\tau^{Q^*}(0))|\mathcal{F}_\nu], \quad P\text{-a.s.}
\end{aligned}$$

□

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