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SL_k-TILINGS OF THE PLANE

FRANÇOIS BERGERON AND CHRISTOPHE REUTENAUER

ABSTRACT. We study properties of (bi-infinite) arrays having all adjacent $k \times k$ adjacent minors equal to one. If we further add the condition that all adjacent $(k-1) \times (k-1)$ minors be nonzero, then these arrays are necessarily of rank k. It follows that we can explicit construct all of them. Several nice properties are made apparent. In particular, we revisit, with this perspective, the notion of frieze patterns of Coxeter. This shed new light on their properties. A connexion is also established with the notion of T-systems of Statistical Physics.

1. Introduction

As discussed in [1], the study of cluster algebras naturally leads to the special case $k = 2$ of the notion of SL_k-tiling introduced in the present paper. Our SL_k-tilings are simply $\mathbb{Z} \times \mathbb{Z}$ arrays of numbers (or elements of a commutative ring) having all adjacent $k \times k$ minors equal to one. Not only are they a natural extension of notions already considered, but one can recast in their guise such notions as *T-systems* of Theoretical Physics (see [8]), or *frieze patterns* of Coxeter (see [6]). An instance of a positive integer SL_2-tiling is given in Figure 1.

Clearly, any SL_k-tiling $\mathcal{A}$ is of rank at least k (when considered as a bi-infinite matrix). As we will see, the SL_k-tilings that are of minimal rank are of particular interest, not only by themselves, but as well as for the cases when they correspond to frieze patterns or T-systems. We call *tame* such minimal rank SL_k-tilings, and we give several general results regarding them. Among these interesting results, we show that to any tame SL_k-tiling there corresponds another interesting tame SL_k-tiling, that we call its *dual*. The

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$$
\begin{array}{cccccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 887 & 567 & 247 & 174 & 101 & \boxed{28} & 11 & 5 & 4 & 3 & 2 & 1 & 1 & \cdots \\
\cdots & 158 & 101 & 44 & 31 & 18 & \boxed{5} & 2 & 1 & 1 & 1 & 1 & 1 & 2 & \cdots \\
\cdots & 61 & 39 & 17 & 12 & 7 & \boxed{2} & 1 & 1 & 2 & 3 & 4 & 5 & 11 & \cdots \\
\cdots & 25 & 16 & 7 & 5 & 3 & \boxed{1} & 1 & 2 & 5 & 8 & 11 & 14 & 31 & \cdots \\
 & \boxed{14} & \boxed{9} & \boxed{4} & \boxed{3} & \boxed{2} & \boxed{1} & \boxed{2} & \boxed{5} & \boxed{13} & \boxed{21} & \boxed{29} & \boxed{37} & \boxed{82} & \\
\cdots & 3 & 2 & 1 & 1 & 1 & \boxed{1} & 3 & 8 & 21 & 34 & 47 & 60 & 133 & \cdots \\
\cdots & 1 & 1 & 1 & 2 & 3 & \boxed{4} & 13 & 35 & 92 & 149 & 206 & 263 & 583 & \cdots \\
\cdots & 1 & 2 & 3 & 7 & 11 & \boxed{15} & 49 & 132 & 347 & 562 & 777 & 992 & 2199 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

FIGURE 1. A SL_2-tiling with values in $\mathbb{N}^*$.

entries of these dual tilings are obtained by computing adjacent $(k-1)\times(k-1)$-minors. It is striking that this duality is actually an involution. We also rederive, in a new an elegant manner, the main results of Conway–Coxeter [4, 6] concerning frieze-patterns: their periodicity, their glided symmetry, and their construction using quiddities (which are the sequence of multiplicities at the vertices of the diagonals in a triangulation of an n-gon). Indeed, our approach allows new tools to bear on this subject, especially because we can now make use of linear algebra and particular presentations for $SL_2(\mathbb{Z})$.

Our approach also opens the door for the study of generalized frieze patterns, including those that have already been considered in [5]. This corresponds to the study of SL_k-tilings that afford two periods (in two linearly independent directions). We call *toric* such SL_k-tilings, since they are evidently characterized by their value on a torus. Once again the tame situation is of particular interest. More on this will be the subject of a planed sequel for this paper.

It was observed by the referee that several results of the present paper may be obtained as consequences of results of the theory of T-systems. This theory was initiated by Bazhanov and Reshetikhin [2] in the context of the representation theory of quantum affine algebras. It was first written in published form by Kuniba, Nakanishi and Suzuki [12]. T-systems were identified as cluster algebras by Di Francesco and Kedem [9] and solved in [8], or more generally by Di Francesco in [7]. In particular, after suitable translation of results on T-systems, our equation (5) is Theorem 3.2 in [8]; the involution deduced from our duality result (Proposition 6) may be deduced from the symmetry of T-systems (cf., e.g., Lemma 2.2 in [9]); and the Laurent phenomenon observed in Proposition 9 and Corollary 10 may be deduced from the Laurent phenomenon of a particular cluster algebra [7].

2. Definitions

We consider arrays $\mathcal{A} = (a_{ij})_{i,j\in\mathbb{Z}}$, with values a_{ij} lying in a field $\mathbb{K}$. For two equal cardinality finite subsets I and J of $\mathbb{Z}$, we denote by $\mathcal{A}_{IJ}$ the submatrix of $\mathcal{A}$ obtained by selecting the rows indexed by the elements of I and columns indexed by the elements of J. The corresponding minor is denoted[1] by M_{IJ}, that is: $M_{IJ} := \det \mathcal{A}_{IJ}$. Since we often need to write down *adjacent* $k\times k$ minors, we introduce the short hand notations:

$$\mathcal{A}_{ij}^{(k)} := \mathcal{A}_{\{i,\dots,i+k-1\},\{j,\dots,j+k-1\}} \quad \text{and} \quad M_{ij}^{(k)} := \det \mathcal{A}_{ij}^{(k)}. \tag{1}$$

We say that $\mathcal{A}$ is a *SL_k-tiling* of the plane if all its adjacent $k\times k$ minors of $\mathcal{A}$ are equal to one. This is to say that it satisfies the *SL_k-property*:

$$M_{ij}^{(k)} = 1 \quad \text{for all } i \text{ and } j \text{ in } \mathbb{Z}, \tag{2}$$

We sometimes consider *partial SL_k-tilings*, only defined on some subset S (called *shape*) of $\mathbb{Z}\times\mathbb{Z}$, with condition (2) applying only if all the entries considered belong to the underlying subset. As usual, a *rectangle* in $\mathbb{Z}\times\mathbb{Z}$ is a (possibly infinite) shape S such that $(u, v+s)$ and $(u+r, v)$ lie in S, whenever (u,v) and $(u+r, v+s)$ both lie in S. A partial SL_k-tiling is said to be a *SL_k-array* if its shape is a rectangle. In particular, $\mathbb{N}\times\mathbb{N}$ and $\mathbb{Z}\times\mathbb{Z}$ shaped SL_k-tilings are SL_k-arrays. Clearly, linear combinations of rows (or columns) make sense for SL_k-arrays, so that we may consider the notion of rank of a such SL_k-tilings. In particular, any SL_k-array is at least of rank k, since any k consecutive rows have to be linearly independent in view of the SL_k-property. We say that a (partial) SL_k-tiling is *tame* if it has rank k. Otherwise, we call it *wild.*

A word of warning is in order concerning our convention for the underlying coordinate system. Indeed, as in the example of Figure 1, we use the usual matrix convention for coordinates, so that the x-axis points downwards, and the y-axis points to the right.

A family of examples. The *positive integer frieze patterns* of Coxeter (see [4, 6, 14]) give rise to an interesting family of nonzero *partial* SL_2-tilings. Up to a 45° degree tilting, the original description of Coxeter may be formulated as follows. One considers partial SL_2-tilings[2] such as the one illustrated in Figure 2, assuming that all a_{ij} are positive integers. Note that the number of "diagonals" is n.

As shown in [6], one of the striking property of frieze patterns is that they are necessarily periodic along the direction $y = x$. This is to say that there exists some p in $\mathbb{Z}$ such that $a_{i+p,j+p} = a_{ij}$ for all i and j, with $p = n+1$ (n

[1] Without explicit reference to the underlying bi-infinite matrix $\mathcal{A}$.
[2] The SL_2 condition applies only when it makes sense.

$$
\begin{array}{cccccccccccccccc}
\ddots & \ddots & \ddots & \cdots & \ddots & \ddots & & & & & & & & & & \\
 & & 1 & a_{10} & a_{20} & \cdots & a_{n-2,0} & 1 & & & & & & & & \\
 & & & 1 & a_{11} & a_{21} & \cdots & a_{n-2,1} & 1 & & & & & & & \\
 & & & & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & & & & & & \\
 & & & & & 1 & a_{1j} & a_{2j} & \cdots & a_{n-2,j} & 1 & & & & & \\
 & & & & & & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & & & &
\end{array}
$$

FIGURE 2. Conway–Coxeter frieze patterns.

being the number of diagonals as above).

We may turn frieze patterns into full SL_2-tilings, by the simple device of extending them (skew) periodically both along the x and y directions, that is, setting:

$$
\begin{aligned}
a_{i+p,j} &= -a_{ij} \quad \text{and} \\
a_{i,j+p} &= -a_{ij}.
\end{aligned}
$$

One needs only check that this is consistent with the SL_2-condition at the "boundary." Such a tiling is illustrated in Figure 3 in the case of a generic[3] frieze pattern having 4 diagonals. In this SL_2-tiling, a and b may assume any value as long as we have

$$
c = \frac{1+b}{a}, \qquad d = \frac{1+a+b}{ab}, \qquad e = \frac{1+a}{b}.
$$

Observe the further symmetry corresponding to a transposition followed by a diagonal translation. The number of frieze patterns having n diagonals, and for which all entries are positive integers, has been shown in [4] to be another

$$
\begin{array}{cccccccccccc}
a & b & 1 & 0 & -1 & -a & -b & -1 & 0 & 1 & a & b \\
1 & c & d & 1 & 0 & -1 & -c & -d & -1 & 0 & 1 & c \\
0 & 1 & e & a & 1 & 0 & -1 & -e & -a & -1 & 0 & 1 \\
-1 & 0 & 1 & b & c & 1 & 0 & -1 & -b & -c & -1 & 0 \\
-e & -1 & 0 & 1 & d & e & 1 & 0 & -1 & -d & -e & -1 \\
-a & -b & -1 & 0 & 1 & a & b & 1 & 0 & -1 & -a & -b \\
-1 & -c & -d & -1 & 0 & 1 & c & d & 1 & 0 & -1 & -c \\
0 & -1 & -e & -a & -1 & 0 & 1 & e & a & 1 & 0 & -1 \\
1 & 0 & -1 & -b & -c & -1 & 0 & 1 & b & c & 1 & 0 \\
e & 1 & 0 & -1 & -d & -e & -1 & 0 & 1 & d & e & 1 \\
a & b & 1 & 0 & -1 & -a & -b & -1 & 0 & 1 & a & b \\
1 & c & d & 1 & 0 & -1 & -c & -d & -1 & 0 & 1 & c
\end{array}
$$

FIGURE 3. Skew-periodic extension of a Conway–Coxeter frieze pattern.

[3] All positive frieze patterns of width 2 may be obtained from it by specialization.

incarnation of the ubiquitous Catalan numbers

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

A nice exposition of classical results regarding frieze patterns, as well as many new results tying their study to the type-A cluster algebras of Fomin and Zelevinsky, is given by Propp in [14].

It may readily be checked that the example of Figure 3 is a rank 2 bi-infinite matrix. We will show in Section 8 how we may construct all frieze patterns using our theory of SL_k-tilings (for $k=2$), by extending them to complete SL_k-tilings. Moreover, using our theory, we give new proofs of all the results obtained by Coxeter and Conway. Although our exploration of this point of view will mainly be for the case $k=2$, many of our results actually hold (with the necessary adaptations) in the general context of a suitable notion of SL_k-frieze patterns (see Section 9).

3. Tame SL_k-tilings

Not all SL_k-tiling are tame, as seen in example (3) for $k=2$.

$$\begin{array}{ccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 1 & x_{11} & -1 & x_{12} & 1 & x_{13} & -1 & x_{14} & 1 & x_{15} & \cdots \\
\cdots & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
\cdots & -1 & x_{21} & 1 & x_{22} & -1 & x_{23} & 1 & x_{24} & -1 & x_{25} & \cdots \\
\cdots & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\
\cdots & 1 & x_{31} & -1 & x_{32} & 1 & x_{33} & -1 & x_{34} & 1 & x_{35} & \cdots \\
\cdots & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
\cdots & -1 & x_{41} & 1 & x_{42} & -1 & x_{43} & 1 & x_{44} & -1 & x_{45} & \cdots \\
\cdots & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\
\cdots & 1 & x_{51} & -1 & x_{52} & 1 & x_{53} & -1 & x_{54} & 1 & x_{55} & \cdots \\
\cdots & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{3}$$

Here the x_{ij} may be chosen at will (or as independent variables). In particular, this example shows that there are SL_2-tilings of any rank ≥ 2.

In part, the interest of considering tame tilings comes from the fact that they are easily characterized by their value on relatively small subsets of $\mathbb{Z}\times\mathbb{Z}$. But we will also make evident that tame tilings have very nice properties. We first illustrate tameness with the following special case.

0-*free tilings.* We say that a SL_k-tiling is 0-*free* if all its $(k-1)\times(k-1)$ adjacent subminors are nonzero. Note that in the case $k=2$: a SL_2-tiling is 0-free if its values are nonzero; in particular if they are positive integers, as in Figure 1 or the SL_2-tilings constructed in [1].

The proof of the following proposition uses Dodgson[4] “Condensation Law of Determinants” [10], that can be stated in the format:

$$M_{ij}^{(r+1)}M_{i+1,j+1}^{(r-1)} = \det\begin{pmatrix} M_{ij}^{(r)} & M_{i,j+1}^{(r)} \\ M_{i+1,j}^{(r)} & M_{i+1,j+1}^{(r)} \end{pmatrix} \tag{4}$$

for all r. In fact, this is a direct consequence of a result of Desnanot and Jacobi (see [3, Theorem 3.12, page 111]). For instance, with $r=2$, we get the identity

$$\det\begin{pmatrix} a_{ij} & a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j} & a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j} & a_{i+2,j+1} & a_{i+2,j+2} \end{pmatrix} \det\left(a_{i+1,j+1}\right)$$
$$= \det\begin{pmatrix} \begin{vmatrix} a_{ij} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} & \begin{vmatrix} a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j+1} & a_{i+1,j+2} \end{vmatrix} \\ \begin{vmatrix} a_{i+1,j} & a_{i+1,j+1} \\ a_{i+2,j} & a_{i+2,j+1} \end{vmatrix} & \begin{vmatrix} a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j+1} & a_{i+2,j+2} \end{vmatrix} \end{pmatrix}.$$

PROPOSITION 1. *Any* 0*-free* SL_k*-array is tame.*

Proof. Consider any adjacent $(k-1)\times(k-1)$ subarray of a SL_k-array, we observe that the determinant of the corresponding submatrix does not vanish, since this is precisely the 0-free condition. On the other hand, for $r=k$, the right-hand side of (4) is zero in all instances, since the four $k\times k$ minors considered are all equal to 1. We thus conclude that any adjacent $(k+1)\times(k+1)$ subarray of a 0-free SL_k-array must necessarily have vanishing determinant. The proof then follows from Lemma 14 (see Section 7). $\square$

We may construct all tame SL_k-tilings as follows. Given a SL_k-tiling $\mathcal{A}$, let us denote by R_i and C_j its rows and columns. Then each C_j is a linear combination of the k preceding columns, that is, of $C_{j-1},\dots,C_{j-k}$. The linear combination may be written as

$$(-1)^k C_0 - (-1)^k a_1 C_1 + \cdots - a_{k-1}C_{k-1} + C_k = 0. \tag{5}$$

Indeed this follows from the SL_k-property and from the next lemma, which is an exercise in linear algebra (expansion of the determinant with respect to the rows), left to the reader.

LEMMA 2. *Let A be a rank k matrix with $k+1$ columns $C_0,\dots,C_k$ and rows indexed by $\mathbb{Z}$. Then*

$$\begin{aligned}(-1)^k M_{01}^{(k)} C_0 &- (-1)^k \det A_{\{0,\dots,k-1\},\{0,2,\dots,k-1\}} C_1 \\ &+ \cdots - \det A_{\{0,\dots,k-1\},\{0,\dots,k-2,k\}} C_{k-1} + M_{00}^{(k)} C_k = 0.\end{aligned} \tag{6}$$

[4] a.k.a. Lewis Carrol.

Hence, to each j in $\mathbb{Z}$ we associate a row vector $\lambda_j = (a_1, \dots, a_{k-1})$ of dimension $m = k-1$ over the field $\mathbb{K}$, and we simply denote by λ the resulting element of $(\mathbb{K}^{1\times m})^{\mathbb{Z}}$. Similarly, we associate to each i a column vector γ_i of dimension m over $\mathbb{K}$, which expresses the linear dependence of R_i on the m preceding rows, and denote by γ the resulting element of $(\gamma_i) \in (\mathbb{K}^{m\times 1})^{\mathbb{Z}}$. These row and columns vectors are called the *linearization coefficients* of the SL_k-tiling $\mathcal{A}$. We call *linearization data* the triple

$$(7) \qquad \big(\mathcal{A}_{00}^{(k)}, \lambda, \gamma\big) \quad \text{in } SL_k(\mathbb{K}) \times (\mathbb{K}^{1\times m})^{\mathbb{Z}} \times (\mathbb{K}^{m\times 1})^{\mathbb{Z}}, \text{ where } m = k-1.$$

Proposition 3. *The mapping*

$$\mathcal{A} \mapsto \big(\mathcal{A}_{00}^{(k)}, \lambda, \gamma\big),$$

which associates to a tame SL_k-tiling its linearization data, is a bijection between the set of tame SL_k-tilings and the set $SL_k(\mathbb{K}) \times (\mathbb{K}^{1\times m})^{\mathbb{Z}} \times (\mathbb{K}^{m\times 1})^{\mathbb{Z}}$, with $m = k-1$.

Proof. The fact that this mapping is well defined and injective follows from the remarks preceding the proof. For surjectivity, let the data in $SL_k(\mathbb{K}) \times (\mathbb{K}^{1\times m})^{\mathbb{Z}} \times (\mathbb{K}^{m\times 1})^{\mathbb{Z}}$ be given. Then clearly there exists a $\mathbb{Z}\times\mathbb{Z}$ array $\mathcal{A}$ of rank k which maps onto this data. We have only to verify that the SL_k-property holds. This is a consequence of the following easy fact (and its dual): let M be a $k\times(k+1)$-matrix with columns $C_0, \dots, C_k$ such that (5) holds. Then the matrix of its first k columns is of determinant 1 if and only if the matrix of its k last columns is of determinant 1. Indeed,

$$\begin{aligned}
&\det(C_1, \dots, C_k)\\
&\quad = \det\big(C_1, \dots, C_{k-1}, -(-1)^k C_0 + (-1)^k a_1 C_1 - \dots + a_{k-1} C_{k-1}\big)\\
&\quad = \det\big(C_1, \dots, C_{k-1}, -(-1)^k C_0\big)\\
&\quad = \det(C_0, \dots, C_{k-1}).
\end{aligned}$$

□

Proposition 4. *Let $\mathcal{A}$ be an SL_k-tiling. Then $\mathcal{A}$ is tame if and only if the infinite matrix $\mathcal{M} := (M_{IJ})_{I,J}$, with I and J varying in k-subsets of $\mathbb{Z}$, is of rank 1. In particular, if I_0, J_0 are intervals in $\mathbb{Z}$ and I, J are any k-subsets, then we have*

$$(8) \qquad M_{IJ} = M_{IJ_0} M_{I_0 J}.$$

Proof. If $\mathcal{A}$ is tame, then the fact that $\mathcal{M}$ has rank 1 is a consequence of the study of $\mathbb{N}\times\mathbb{N}$ tilings in Section 8.2. The converse follows from Lemma 15.

Now take I, J, I_0, and J_0 as in the statement. Then

$$(9) \qquad \begin{pmatrix} M_{I_0 J_0} & M_{I_0 J} \\ M_{I J_0} & M_{IJ} \end{pmatrix}$$

is a submatrix of $\mathcal{M}$. But $M_{I_0J_0}=1$ (since $\mathcal{A}$ is a SL_k-tiling), and the determinant of (9) mush vanish, that is,

$$M_{IJ}-M_{I_0J}M_{IJ_0}=0,$$

since $\mathcal{M}$ is of rank 1. Thus, we get the desired equality. □

The easy direct proof of the next result is left to the reader. It will be useful in the sequel.

LEMMA 5. *Let C_1, C_2, and C_3 be three consecutive columns* (*resp. rows*), *of a tame SL_2-tiling, that are such that $C_2=C_1+C_3$. Then a new tame SL_2-tiling may be constructed by suppressing the column* (*resp. row*) *C_2.*

Group actions on tilings. There is a natural translation action of $\mathbb{Z}^2$ on SL_2-tilings. Formally, the action of the vector (p,q) replaces the tiling $\mathcal{A}=(a_{ij})$ by

$$(p,q)\cdot\mathcal{A}=(a_{i+p,j+q}).$$

We denote by $\mathcal{A}_x$ the translate of $\mathcal{A}$ by $(1,0)$, and by $\mathcal{A}_y$ the translate by $(0,1)$. We may describe these last translates intrinsically in terms of the data given by the bijection of Proposition 3. More specifically, let $(\mathcal{S},\lambda,\gamma)$ be the linearization data corresponding to $\mathcal{A}$ via this bijection. Then the linearization data corresponding to $\mathcal{A}_x$ is $(\mathcal{S}_x,\lambda,\gamma')$, with

$$\mathcal{S}_x=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & 1\\ (-1)^{k-1} & (-1)^k(\gamma_k)_1 & \dots & & (\gamma_k)_{k-1}\end{pmatrix}\mathcal{S},$$

and $\gamma'_i=\gamma_{i+1}$. Transposition of matrices (which amounts to exchanging rows and columns) also preserves SL_k-tilings, as is easily observed. In terms of linearization data, it amounts to transposing the initial matrix and exchanging λ and γ. We can thus easily describe $\mathcal{A}_y$ using these remarks.

Observe that if $k\equiv 0$ or $1 \bmod 4$, then a vertical or an horizontal symmetry also preserves SL_k-tilings, since in this case, such a symmetry preserves the determinant of $k\times k$ matrices (for $k\equiv 2$ or $3 \bmod 4$, the determinant is clearly multiplied by -1).

4. Dual tilings

To any array $\mathcal{A}$, we associate the *m-derived array*:

$$\partial_m\mathcal{A}:=\big(M_{ij}^{(m)}\big)_{i,j},\tag{10}$$

consisting of the adjacent $m\times m$ minors of $\mathcal{A}$. For SL_k-arrays, we are specially interested in the case $m=k-1$, in which case the resulting array is called the *dual* array of $\mathcal{A}$. We also write $\mathcal{A}^*$ for $\partial_{k-1}\mathcal{A}$. Clearly, ∂_1 is the identity operator, and it is natural to set $\partial_0\mathcal{A}$ equal to the tiling whose value is 1 in all positions. As an illustration of the above definition, the dual of the SL_3-tiling

$$\begin{array}{ccccccccccc} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 8997 & 1782 & 353 & 70 & 14 & 3 & 1 & 1 & \cdots \\ \cdots & 1782 & 353 & 70 & 14 & 3 & 1 & 1 & 2 & \cdots \\ \cdots & 353 & 70 & 14 & 3 & 1 & 1 & 2 & 5 & \cdots \\ \cdots & 70 & 14 & 3 & 1 & 1 & 2 & 5 & 14 & \cdots \\ \cdots & 14 & 3 & 1 & 1 & 2 & 5 & 14 & 42 & \cdots \\ \cdots & 3 & 1 & 1 & 2 & 5 & 14 & 42 & 131 & \cdots \\ \cdots & 1 & 1 & 2 & 5 & 14 & 42 & 131 & 417 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{11}$$

is the tiling

$$\begin{array}{ccccccccc} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 417 & 131 & 42 & 14 & 5 & 2 & 1 & \cdots \\ \cdots & 131 & 42 & 14 & 5 & 2 & 1 & 1 & \cdots \\ \cdots & 42 & 14 & 5 & 2 & 1 & 1 & 3 & \cdots \\ \cdots & 14 & 5 & 2 & 1 & 1 & 3 & 14 & \cdots \\ \cdots & 5 & 2 & 1 & 1 & 3 & 14 & 70 & \cdots \\ \cdots & 2 & 1 & 1 & 3 & 14 & 70 & 353 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{12}$$

We have the following property of derivation of arrays, that will be proved in Section 7.

PROPOSITION 6. *The dual of a tame SL_k-tiling is a tame SL_k-tiling. Moreover, for any natural integers r, s such that $r + s = k$, we have*

$$\partial_r\mathcal{A}^* = (r-1, r-1)\cdot(\partial_s\mathcal{A}). \tag{13}$$

In particular, $(\mathcal{A}^*)^*$ and $\mathcal{A}$ coincide up to translation.

Observe also that, with $r = k-1$, identity (4) gives $(\mathcal{A}^*)^* = a_{11}\det(\mathcal{A})$ for any 3×3 matrix

$$\mathcal{A} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}.$$

Hence, for any SL_3-tiling $\mathcal{A}$, we have

$$(\mathcal{A}^*)^* = \left(a_{i+1,j+1}M_{ij}^{(3)}\right)_{i,j\in\mathbb{Z}}. \tag{14}$$

It follows that $(\mathcal{A}^*)^* = \mathcal{A}$ (up to the necessary translation) for any SL_3-tiling, wether they be tame or wild. However, for $k \geq 4$, it may be checked that the tameness condition is necessary for (13) to hold.

5. Tilings associated to paths

We consider *paths* π as lists of *points*, that is, elements of $\mathbb{Z} \times \mathbb{Z}$,

$$\pi = (i_0, j_0), (i_1, j_1), \ldots, (i_N, j_N),$$

starting at $\mathrm{s}(\pi) := (i_0, j_0)$ and ending at $\mathrm{e}(\pi) := (i_N, j_N)$, and such that

$$(i_{s+1}, j_{s+1}) = \begin{cases} (i_s, j_s - (1,0) & \text{or} \\ (i_s, j_s) + (0,1) \end{cases}$$

for points along the path. To understand why signs appear here, it may be good to recall our convention for the orientation of the x and y axis (see Section 2). If we fix the start and end points (i_0, j_0) and (i_N, j_N), it is well known that these paths number $\binom{m+n}{m}$ (with $(-m, n) = (i_N, j_N) - (i_0, j_0)$), and that they are in bijection with words

$$w = w_1 w_2 \cdots w_{n+m} \tag{15}$$

on the alphabet $\mathcal{A} = \{x, y\}$, having m occurrences of the letter x, and n occurrences of the letter y. Recall that the corresponding classical bijection, between paths and words, is realized by choosing $(i_{s+1}, j_{s+1}) = (i_s, j_s) - (1, 0)$ if $w_s = x$, and $(i_{s+1}, j_{s+1}) = (i_s, j_s) + (0, 1)$ if $w_s = y$. We denote by π_w the resulting path.

We also consider words, and associated paths, that are infinite in both directions,

$$w = \cdots w_{-3} w_{-2} w_{-1} w_0 w_1 w_2 w_3 \cdots \tag{16}$$

and say that they are *bi-infinite* words (or paths). Such a word (and the associated path) is said to be *admissible* if there are infinitely many x' s and y's in both directions.

Let us now associate to a given word w (finite or bi-infinite) a tiling $\mathcal{A} = \mathcal{A}_{w;k}$, whose entries are obtained by the (weighted) enumeration of paths starting and ending at some points of π_w. We restrict these paths to stay within some "distance" k of π_w. This is made more precisely below after the introduction of more notation and terminology (some of which will only be used later).

Given a path π_w as above, for each point $p = (i, j)$ in $\mathbb{Z} \times \mathbb{Z}$ we denote by $\gamma_w(p) = (i, \beta_\pi(p))$ (resp. $\chi_w(p) = (\alpha_\pi(p), j)$) the leftmost (resp. topmost) point that lies on the path π_w, which has the same first (resp. second) coordinate as p. We say that these are respectively, the *horizontal projection* and *vertical projection* of p on π_w. A point $p = (i, j)$ is said to lie *below* the path π if we have the inequalities $\beta_\pi(p) \leq j$, or equivalently, $\alpha_\pi(p) \leq i$. Otherwise, we say that p lies *above* the path.

We now consider the word $w(p)$: associated to the portion of the path π_w going from the horizontal projection $\gamma_w(p)$ of p on π_w, to its vertical projection $\chi_w(p)$. This word is used to define the notion of *projection word* of a point p, denoted by w_p, as follows. We simply set $w_p := w(p)$ whenever p lies below the path. Otherwise, when p lies above the path, we set $w_p := \overline{w(p)}$. Here, $\overline{w}$ is the operation corresponding to reading the letters of a word w in reverse order, replacing each x by $\overline{x}$ and each y by $\overline{y}$. For example, with $w = yyxyxyyyx$, we get $\overline{w} = \overline{x}\overline{y}\overline{y}\overline{x}\overline{y}\overline{x}\overline{x}\overline{y}\overline{y}$.

Let p be a point lying below the path π_w, and suppose that w_p factors as $x^i u y^j$ (with i and j maximal). Then we say that $u_p := u$ is the *short projection word* of p on w. Illustrating with the tiling of Figure 5, one may check that for the points p corresponding to the entries with value equal to 6 (lying below the path), one has $w_p = xxyyxxyy$ and $u_p = yyxx$ (for all instances of 6); whereas for the points p corresponding to the entries 30 (lying above the path), one has $w_p = \bar{y}\bar{y}\bar{x}\bar{x}$ (likewise, for all instances of 30).

For a point p lying below a path π, the *distance* between p and π is the unique integer $k \in \mathbb{N}$ such that $(i-k+1, j-k+1)$ lies on π. Observe that there is but one point of π lying on any given diagonal $x = y + c$. Our definition makes it so that points lying on the path are considered to be at distance 1 of it (this will make our life easier later). We further consider the notion of *k-fringe*, $\Phi_k(\pi)$ of a path π, that is, the points lying below the path that are within distance k of it. Thus, we have

$$\Phi_k(\pi) := \{(i+m, j+m) \mid (i,j) \in \pi, \text{ and } 0 \le m < k\} \tag{17}$$

Some of these notions are illustrated in Figure 4.

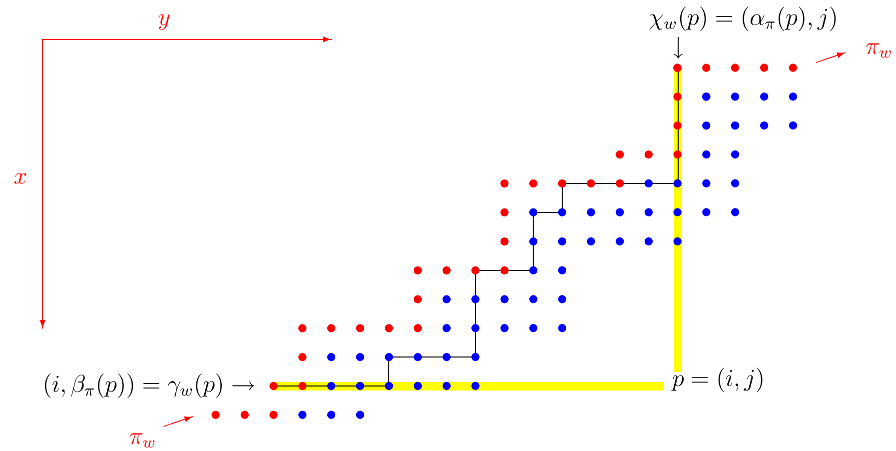

FIGURE 4. Path from $\gamma_w(p)$ to $\chi_w(p)$ in the 3-fringe of π_w.

Given two points p and q on the path π_w of a word w, we consider the sets of paths

$$\mathcal{P}_{w;k}(p,q) := \{\theta \text{ a path} \mid \mathrm{s}(\theta) = p, \mathrm{e}(\theta) = q, \text{ and } \theta \subset \Phi_k(\pi_w)\}, \tag{18}$$

that start at p, end at q, while staying inside the k-fringe of π_w. We also denote by $\mathcal{P}_{w;k}$ the set of all such paths, disregarding start and end points. The tiling $\mathcal{A}_{w;k} = (a_{ij})_{i,j}$ is then defined, for points $p = (i,j)$ lying below the path π_w, by setting

$$a_{ij} := \#\mathcal{P}_{w;k}(\chi_w(p), \gamma_w(p)). \tag{19}$$

For instance, for the word $w = \cdots yyxxyxyyyx \cdots$ and $k = 2$, the resulting (partial) tiling is as follows:

$$A_{w;2} = \begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & 1 & 1 & 1 & 1 \\
 & & 1 & 1 & 2 & 3 & 4 \\
 & & 1 & 2 & 5 & 8 & 11 \\
1 & 1 & 1 & 3 & 8 & 13 & 18
\end{array}$$

Observe that, for fixed d, any path θ in the k-fringe of π goes through at most one of the the k points of the set $\Phi_k(\pi) \cap \Delta_d$, where Δ_d denotes the *diagonal*

$$\Delta_d := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i - j = d\}. \tag{20}$$

It is useful to have the following terminology: given a bi-infinite path and a tiling $\mathcal{A}$, let us call *principal minors of order m* (relative to π_w), the minors of $\mathcal{A}$ that are of the form $M_{ij}^{(m)}$, with (i,j) lying on the path π_w. In other words, a principal minor is an adjacent minor of $\mathcal{A}$ whose upper left corner lies on the path (in particular, it is contained in the $k-1$ fringe of the path when $m \le k$).

For $h \in \mathbb{Z}$, we also say that a minor $M_{ij}^{(m)}$ is *located on the h-th diagonal* if we have $h = j - i$. To tie all this to our study of SL_k-tilings, we now give entirely combinatorial arguments for the following statements.

Proposition 7. *The partial tiling $\mathcal{A}_{w;k}$ is a 0-free SL_k-tiling with principal minors of order $< k$ all equal to 1. Moreover, for any k-subsets I and J such that $I \times J$ is contained in the shape of $\mathcal{A}$ (all the points lying below π_w), we have*

$$M_{IJ} = M_{I,\{j,\dots,j+k-1\}} M_{\{i,\dots,i+k-1\},J}. \tag{21}$$

Finally, if the path π_w is admissible, then $\mathcal{A}_{w;k}$ extends uniquely to a complete tame SL_k-tiling.

There is some redundance here, since the 0-free property implies that $\mathcal{A}$ is tame (Proposition 1), and thus (21) holds by Proposition 4. Notwithstanding, we want to make evident that nice combinatorial methods may be used to understand all this.

Proof of Proposition 7. It follows from a theorem of Gessel–Viennot (see [13]) that we may interpret combinatorially any minor of $\mathcal{A}_{w;k}$ as follows. Recall that a family of paths is said to be noncrossing if no pair of paths in the family has a common point. Given equal cardinality subsets I of rows and J of columns, such that $I\times J$ lies below π_w, we denote by NoX_{IJ} the set of all noncrossing families of paths in $\mathcal{P}_{w;k}$ linking the horizontal projection of I on π_w to the vertical projection of J on π_w. More precisely, let

$$I=\{i_1<i_2<\cdots<i_m\} \quad \text{and} \quad J=\{j_1<j_2<\cdots<j_m\},$$

and denote by $p_1,\dots,p_m$ and $q_1,\dots,q_m$ the respective horizontal and vertical projections on π_w. Then the elements of NoX_{IJ} are “configurations” $\{\pi_1,\pi_2,\dots,\pi_m\}$ of paths π_s in $\mathcal{P}_{w;k}$, with

$$\mathrm{s}(\pi_s)=p_s \quad \text{and} \quad \mathrm{e}(\pi_s)=q_s, \tag{22}$$

no two of which cross. In our context, the aforementioned theorem of [13] states that we have

$$M_{IJ}=\#\mathrm{NoX}_{IJ}. \tag{23}$$

Recall that this is shown by constructing a sign changing involution on the set of crossing configurations, thus showing that they can be eliminated from a global signed counting that clearly corresponds to the evaluation of the determinant considered.

Observe that noncrossing path configurations $\{\pi_1,\pi_2,\dots,\pi_m\}$ intersect any given diagonal Δ_d in at most m points. In fact, this intersection number is exactly equal to m for the diagonals that pass through points of π_w lying between p_1 and q_1. This forces all the sets NoX_{IJ} to be empty whenever $\#I=\#J>k$. Hence, the corresponding minors all vanish, so that the tiling is of rank k.

To continue with our combinatorial argument, let us write $\mathrm{NoX}_{ij}^{(m)}$ when

$$I=\{i,\dots,i+m-1\} \quad \text{and} \quad J=\{j,\dots,j+m-1\},$$

so that we have

$$M_{ij}^{(m)}=\#\mathrm{NoX}_{ij}^{(m)}. \tag{24}$$

In the k-fringe of π_w, there is room for one and exactly one configuration of k noncrossing paths having adjacent starting points and adjacent end points, so that we must necessarily have $M_{ij}^{(k)}=1$ for all point (i,j) in the tiling, hence the SL_k-condition is verified.

The tiling is 0-free, and in fact we have $M_{ij}^{(m)}\geq 1$ for all m between 1 and k, since there exists corresponding noncrossing path configurations for all these m. Finally, the multiplicative property (21) of $k\times k$-minors can easily be explained as follows, in terms of configurations of k noncrossing paths. With I and J satisfying the hypothesis of the proposition, the required identity follows from a simple bijection

$$\mathrm{NoX}_{IJ} \to \mathrm{NoX}_{I,\{j,\dots,j+k-1\}} \times \mathrm{NoX}_{\{i,\dots,i+k-1\},J}, \tag{25}$$

obtained by breaking up the paths considered into three portions as follows. For a path π_s (starting at p_s and ending at q_s) in $\Phi_k(\pi_w)$, denote respectively, by p and q the points of π_s that lie on the diagonals respectively containing p_1 and q_1. These exist since $p_s \leq p_1 \leq q_1 \leq q_s$. We decompose π_s as the concatenation

$$\pi_s = \pi_s^{(1)}\pi_s^{(2)}\pi_s^{(3)},$$

with

- $\pi_s^{(1)}$ being the portion of π_s going from p_s to p,
- $\pi_s^{(2)}$ being the portion of π_s going from p to q, and
- $\pi_s^{(3)}$ being the portion of π_s going from q to q_s.

In particular, all the paths $\pi_s^{(2)}$ start on the same diagonal (the one that contains p_1) and end on the same diagonal (the one that contains q_1). Since these k paths are noncrossing and all lie in the k-fringe, there is but one possibility for the resulting configuration $\{\pi_1^{(2)},\dots,\pi_k^{(2)}\}$. We easily identify the configurations $\{\pi_1^{(1)},\dots,\pi_k^{(1)}\}$ with elements of $\mathrm{NoX}_{I,\{j,\dots,j+k-1\}}$ (by application of the same decomposition as above to these last elements, observing that in this cases third components are trivial). Likewise, we identify the configurations $\{\pi_1^{(3)},\dots,\pi_k^{(3)}\}$ with elements of $\mathrm{NoX}_{\{i,\dots,i+k-1\},J}$. This establishes the bijection.

The SL_k-tiling $\mathcal{A}_{w;k}$ may be uniquely completed into a tame SL_k-array by Lemma 8 below. ☐

For example, with $k=4$ and the word $w=\cdots xxyyxxyyxxyy\cdots$, we first get the partial array (lying below the path) of Figure 5 by path enumeration, and then complete it to get a SL_4-tiling of $\mathbb{Z}\times\mathbb{Z}$.

Lemma 8. *Consider a partial SL_k-tiling which is defined on every point below a given path, and such that all of its adjacent $(k+1)\times(k+1)$-subminor (lying entirely in its shape) vanishes. If the path is admissible, then the partial SL_k-tiling extends uniquely to a complete tame SL_k-tiling.*

Proof. Let i, j and k be integers such that $\{i,i+1,i+2,\dots\}\times\{j-k,\dots,j-1,j\}$ is contained in the shape of $\mathcal{A}$. denote by $C_{j-k},\dots,C_{j-1},C_j$ the columns of the corresponding submatrix of $\mathcal{A}$. Then, it follows from the vanishing $(k+1)\times(k+1)$-subminor condition, that we have a relation of the form

$$(-1)^kC_{j-k}-(-1)^ka_1C_{j-k+1}+\cdots-a_{k-1}C_{j-1}+C_j=0$$

for some $a_1,\dots,a_{k-1}$ in $\mathbb{K}$. Note that the coefficients a_h are independent of the i chosen.

$$
\begin{array}{ccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 1437 & 457 & 30 & 10 & 1 & 1 & 1 & \cdots \\
\cdots & 457 & 146 & 10 & 4 & 1 & 2 & 3 & \cdots \\
\cdots & 30 & 10 & 1 & 1 & 1 & 3 & 6 & \cdots \\
\cdots & 10 & 4 & 1 & 2 & 3 & 10 & 22 & \cdots \\
\cdots & 1 & 1 & 1 & 3 & 6 & 22 & 53 & \cdots \\
\cdots & 1 & 2 & 3 & 10 & 22 & 84 & 211 & \cdots \\
\cdots & 1 & 3 & 6 & 22 & 53 & 211 & 553 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

FIGURE 5. The SL_4-tiling associated to $\cdots xxyyxxyyxxyy\cdots$.

Considering the analogous argument for rows, and assuming that the origin of the plane is in the shape of $\mathcal{A}$, we obtain a linearization data (see Section 3). Using this linearization data, we may apply Proposition 3 to get a complete tame tiling of the plane. Call it $\mathcal{A}'$. It follows from the construction that $\mathcal{A}$ and $\mathcal{A}'$ coincide on the shape of $\mathcal{A}$, which proves the lemma. $\square$

Weighted word tilings. We now extend the previous construction to the situation where paths are given Laurent monomial weights.

At a point $p=(i,j)$, along a path θ, we say that we have a *right-turn* (resp. *left-turn*) if both $(i+1,j)$ and $(i,j+1)$ (resp. $(i,j-1)$ and $(i-1,j)$) belong to the path θ. This is illustrated in Figure 6. For a given (bi-infinite) word w, we start by giving a weight $\nu(p)$ to each point $p=(i,j)$ in the k-fringe of π_w, setting

$$\nu(p) := \frac{t_{j-i,r}}{t_{j-i,r-1}},$$

where r is the distance between p and π_w, We assume here that the $t_{m,r}$ are independent commuting variables, setting $t_{m,r}=1$ whenever $r\leq 0$ or $r\geq k$. With all this at hand, define the *weight* $\omega(\theta)$ (with respect to the word w) of a nonempty path θ to be the product

$$\omega(\theta) := \alpha\beta \prod_{p \text{ left-turn of } \theta} \nu(p) \cdot \prod_{p \text{ right-turn of } \theta} \nu(p)^{-1}, \tag{26}$$

where we set $\alpha := \nu(p_s)$, if θ starts at p_s by a vertical step. Otherwise we set $\alpha=1$. Likewise, we set $\beta:=\nu(p_e)$, if θ ends at p_e after an horizontal step. Otherwise, we set $\beta=1$. Finally, when θ is the empty path, both starting and ending at p, we simply set $\omega(\theta):=\nu(p)$.

We then consider the partial tiling $\mathcal{B}_{w;k} := (b_{ij})_{i,j}$, for point (i,j) lying below the path, obtained by setting

$$b_{ij} := \sum_{\theta} \omega(\theta) \tag{27}$$

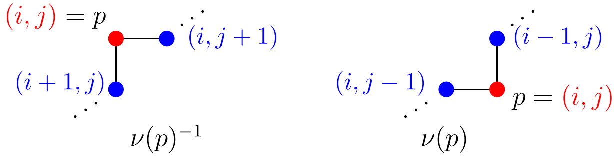

FIGURE 6. Right and left turns at p, and corresponding weight.

for θ varying in the set $\mathcal{P}_{w;k}(\chi_w(i,j), \gamma_w(i,j))$, of paths starting at $\chi_w(i,j)$ and ending at $\gamma_w(i,j)$.

PROPOSITION 9. *There is a unique tame SL_k-tiling of $\mathbb{Z} \times \mathbb{Z}$ extending $\mathcal{B}_{w;k}$, with entries Laurent polynomials in the variables t_{hr}. More precisely, the values are in the subsemiring generated by these variables and their inverses. Moreover, each principal minor of order r, $r = 1, \ldots, k-1$, located on the hth diagonal, is equal to t_{hr}.*

Proof. Again, we simply apply the Gessel–Viennot technique, verifying that the involution (as in their original proof), required to show that crossing path configurations may be eliminated, is weight preserving. There are several cases, left to the reader. It follows, as in the proof of Proposition 7, that b_{ij} is a tame SL_k-tiling. It is clear that (27) is in the described semiring. Moreover, by the noncrossing path description, each principal minor of order $r < k$ is equal to t_{hr}, if the minor is located on the hth diagonal. □

This proposition may be used to construct SL_k-tilings having arbitrary values (variables) as entries in the $(k-1)$-fringe of the path π_w. It turns out that the entries of the resulting tiling actually lie in the polynomial ring generated by these variables as well as the inverses of all principal minors (relative to π_w) of order at most $k-1$. This is a Laurent-like phenomenon (see [11]). Moreover, we may in fact replace "ring" by "semiring," so that we actually get a positivity result, just as is the case in the theory of cluster algebras.

COROLLARY 10. *An admissible path π_w being given, associate to each point in its $(k-1)$-fringe, a distinct commutative variable. Then this assignment extends uniquely into a complete tame SL_k-tiling of the plane whose values are in the semiring generated by the principal minors of order $< k$ and their inverses.*

Proof. Consider the tiling of Proposition 9. Let a_{ij} denote its value at the point (i,j). Then, by the same proposition, each a_{ij} is in the semiring generated by the variables t_{hr} and their inverses, where $h \in \mathbb{Z}$ and $r = 1, \ldots, k-1$.

Let $s_{hr} = t_{hr}/t_{h,r-1}$ for $h \in \mathbb{Z}$ and $r = 1, \ldots, k-1$. Recall that the t_{hr} are distinct commuting variables, and that $t_{h,0} = 1$. The field of fractions $\mathbb{K}$ in the variables t_{hr} is also generated by the s_{hr}, and the mapping $t_{hr} \mapsto s_{hr}$ is an automorphism of this field. For $h \in \mathbb{Z}$ and r going from 1 to $k-1$, denote by α_{hr} the entry a_{ij}, if the point (i,j) lies at distance r below the path π_w, on the hth diagonal. By the path description, we see that, α_{hr} is the sum of s_{hr} and of a fraction in the $t_{h'r'}$ with $(h',r') < (h,r)$ for the natural order on $\mathbb{Z}^2$. The latter fraction, when expressed in the $s_{h'r'}$, involves only variables $s_{h'r'}$ with the same condition. Hence, the function $t_{hr} \mapsto \alpha_{hr}$ defines an automorphism of $\mathbb{K}$.

Now, let x_{hr} be a family of distinct commuting variables, for $h \in \mathbb{Z}$ and $r = 1, \ldots, k-1$. Let $\mathbb{L}$ be its field of fractions. The fields $\mathbb{K}$ and $\mathbb{L}$ are isomorphic (e.g., by the mapping $t_{ij} \mapsto x_{ij}$). By what we have just seen, the mapping $\alpha_{hr} \mapsto x_{hr}$, $h \in \mathbb{Z}$ and $r = 1, \ldots, k-1$ defines an isomorphism from $\mathbb{K}$ onto $\mathbb{L}$. If we map each a_{ij} under this isomorphism, we obtain a tame SL_k-tiling $X = (b_{ij})$ such that $b_{hr} = x_{hr}$ for $h \in \mathbb{Z}$ and $r = 1, \ldots, k-1$. This also implies that we may find elements τ_{hr} in the field $\mathbb{L}$ such that b_{ij} is in the semiring generated by the τ_{hr} and their inverses, $h \in \mathbb{Z}$ and $r = 1, \ldots, k-1$. Furthermore, by Proposition 9, the principal $r \times r$-minor of X $(r < k)$, located on the hth diagonal, is equal to τ_{hr}.

Unicity follows from the following lemma, which of independent interest. □

LEMMA 11. *An admissible path π_w being given, associate to each point in its $(k-1)$-fringe, an element of some field. Suppose that the $(k-1)$-principal minors relative to π_w are nonzero. Then this partial tiling extends uniquely to a tame SL_k-tiling of the plane.*

Proof. Indeed, under the nonzero $(k-1)$-principal minor hypothesis, the SL_k property imposes that we have a unique extension of the partial tiling to its k-fringe. This furnishes enough $k \times (k+1)$ and $(k+1) \times k$ submatrices so that we may compute the linearization data for any tame SL_k-tiling that would extend the k-fringe shaped partial tiling (see the remark following the proof of Proposition 3). This proves unicity, in view of the same proposition. Existence, which will not be used here, is left to the reader. □

In [1], one may find many SL_2-tilings associated to paths, both over the integers, and with arbitrary variables on the path. The case SL_3 has an extra interesting feature. Indeed, a consequence of (14) is that we can very elegantly characterize any SL_3-tiling in tandem with its dual tiling. Indeed, under the assumption that $\mathcal{A}$ is SL_3 and writing $\mathcal{A}^* = (a^*_{ij})_{i,j}$, the tiling identity considered is equivalent to the family of equalities

$$\begin{aligned} a_{ij} &= \frac{1}{a_{i-1,j-1}}(a^*_{i-1,j-1} + a_{i-1,j}a_{i,j-1}), \\ a^*_{ij} &= \frac{1}{a^*_{i-1,j-1}}(a_{i-1,j-1} + a^*_{i-1,j}a^*_{i,j-1}). \end{aligned} \tag{28}$$

$$
\begin{array}{cccccccccccccc}
 & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & & & & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & 1 & & 1 & & 2 & & 3 & & 4 & & \cdots & & \\
 & & & 1 & & 1 & & 3 & & 6 & & \cdots & & & \\
 & & 1 & & 2 & & 5 & & 9 & & \cdots & & & & \\
 & & & 1 & & 3 & & 14 & & \cdots & & & & & \\
1 & & 1 & & 3 & & 9 & & \cdots & & & & & & \\
 & 1 & & 1 & & 6 & & \cdots & & & & & & & \\
1 & & 2 & & 7 & & \cdots & & & & & & & &
\end{array}
$$

FIGURE 7. Joint calculation of a SL_3-tiling and its dual.

This makes it evident (in another fashion) that the tiling constructed from a path is positive (and nonzero) for points lying below the path, since entries of $\mathcal{A}$ and $\mathcal{A}^*$ may be calculated recursively in parallel using the positive expression on the right-hand side of (28). In the case of integer tilings, this is illustrated in Figure 7. Large entries correspond to the a_{ij}'s, and smaller ones correspond to the a^*_{ij}'s. The entry a^*_{ij} sits immediately to the south-east of a_{ij}. Clearly, the recursion process may be continued where it is left off. It corresponds to the statement that each number is obtained as the determinant of the 4 numbers that immediately surround it.

6. Matrix description

Consider the morphism μ, from the free group $F_{x,y}$ (on the letters x and y) to the group SL_k, which is obtained by setting

$$\mu(x) := \mathrm{Id} + N \quad \text{and} \quad \mu(y) := \mathrm{Id} + N^{\mathrm{tr}}, \tag{29}$$

where we denote by N the matrix nilpotent $k \times k$ matrix

$$N := \begin{pmatrix} 0 & 1 & 0 & & \dots & 0 \\ & 0 & 1 & 0 & & \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & 1 & 0 \\ 0 & & \dots & & 0 & 1 \end{pmatrix}.$$

(Recall that N is nilpotent of order k, so that $N^k = 0$, and $N^i \neq 0$ when $i < k$.) We denote $\overline{x}$ the inverse of x in the free group $F_{x,y}$, and likewise for y. We then define the function $\mathcal{T}_w : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ as

$$\mathcal{T}_w(p) := \mathbf{e}_k \mu(w_p) \mathbf{e}_k^{\mathrm{tr}}, \tag{30}$$

where $\mathbf{e}_k$ denotes the unit k-vector $(0,\ldots,0,1)$. Recall that the projection word w_p has been defined in Section 5.

PROPOSITION 12. *For any admissible bi-infinite word w, the function $\mathcal{T}_w$ is a tame SL_k-tiling, whose principal minors of order $< k$ are all equal to 1. It coincides with the tiling of Proposition 7.*

For the proof of Proposition 12, see Section 7. Observe that this result easily implies the following.

COROLLARY 13. *With the same hypothesis as in Proposition 12, we have*

(i) *If p lies below π_w, then*

$$\mathcal{T}_w(p) = \mathbf{e}_k \mu(u_p) \mathbf{e}_k^{\mathrm{tr}}. \tag{31}$$

In other words, we can replace the projection word w_p by the short projection word u_p in our calculations.

(ii) *If p lies above π_w,with $w_p = \overline{x_1}\cdots\overline{x_n}$, $x_1,\ldots,x_n \in \{x,y\}$, then*

$$\mathcal{T}_w(p) = \mathbf{e}_k \mu'(x_1\cdots x_n)\mathbf{e}_k^{\mathrm{tr}}, \tag{32}$$

where μ' is the morphism such that

$$\mu'(x) := (\mathrm{Id} - N)^{-1} \quad \textit{and} \quad \mu'(y) := \mu'(x)^{\mathrm{tr}}.$$

In particular, we conclude that $\mathcal{T}_w(p)$ is positive for all p.

Proof. Assume that $w = w_p = x^i u y^j$ with $u = u_p$. To show (i), we first observe that

$$\mu(x^i) = (\mathrm{Id} + N)^i = \mathrm{Id} + \sum_{j=0}^{i} \binom{i}{j} N^i$$

is upper unitriangular, hence $\mathbf{e}_k\mu(x^i) = \mathbf{e}_k$. Likewise, $\mu(y^i)$ is lower unitriangular, so that $\mu(y^j)\mathbf{e}_k^{\mathrm{tr}} = \mathbf{e}_k^{\mathrm{tr}}$. Thus, we directly calculate that

$$\mathcal{T}_w(p) = \mathbf{e}_k\mu(x^i u y^j)\mathbf{e}_k^{\mathrm{tr}} = \mathbf{e}_k\mu(x^i)\mu(u)\mu(y^j)\mathbf{e}_k^{\mathrm{tr}} = \mathbf{e}_k\mu(u)\mathbf{e}_k^{\mathrm{tr}}$$

as announced.

For (ii), we make use of the matrix isomorphism

$$\alpha(A) := D_k A D_k^{-1},$$

with D_k standing for the diagonal matrix with entry equal to $(-1)^{i+1}$ on the diagonal. Clearly, $\alpha(a_{ij}) = ((-1)^{i+j}a_{ij})$. Thus, $\alpha(\mu'(x)) = \mu(\overline{x})$ and $\alpha(\mu'(y)) = \mu(\overline{y})$ as is easily verified. Hence, for any $x_1,\ldots,x_n \in \{x,y\}$, we have $\alpha(\mu(x_1\cdots x_n)) = \mu(\overline{x_1}\cdots\overline{x_n})$. We conclude since $\alpha(A_{kk}) = \mathcal{A}_{kk}$. For the final assertion, note that μ' has nonnegative coefficients. $\square$

7. Proofs

To prove some of our previous assertions, we first need a few linear algebra lemmas.

LEMMA 14. *If a matrix has all its adjacent* $(k+1)\times(k+1)$ *minors vanishing, whereas no adjacent* $k\times k$ *vanishes, then it is of rank* k.

Proof. It is enough to show that for any choice of $k+1$ successive columns $C_0,\dots,C_k$ of this matrix, C_0 (resp. C_k) is a linear combination of $C_1,\dots,C_k$ (resp. of $C_0,\dots,C_{k-1}$). By symmetry, the property with C_0 will suffice. Let v_i denote the rows of the matrix $(C_0,\dots,C_k)$. Note that v_i is of length $k+1$.

To show our assertion, let us construct a nonvanishing linear form φ that annihilates all v_i. The existence of such a linear form implies the existence of scalars $a_0,\dots,a_k$ such that $\sum_{j=0,\dots,k} a_jC_j=0$. Moreover a_0 has to be nonzero, since otherwise it would contradict the assumption on the nonvanishing k-minors.

Such a linear form exists for $k+1$ successive rows of M, since $\det(M)=0$ by assumption. Consider $k+2$ successive rows of M, and two nonvanishing linear forms φ and ψ such that the first $k+1$ rows are in $\mathrm{Ker}(\varphi)$ and the $k+1$ last are in $\mathrm{Ker}(\psi)$. Then we argue as follows to show that φ and ψ must necessarily be proportional. If we restrict the two linear forms to the k intermediate rows, $v_1,\dots,v_k$ say, we see that φ and ψ, considered as column vectors of length $k+1$, are both annihilated by the $(k\times(k+1))$-matrix

$$\begin{pmatrix} v_1\\ v_2\\ \vdots\\ v_k\end{pmatrix}, \tag{33}$$

whose rows are the vectors v_i. By assumption, this matrix is of rank k, hence it has a kernel of dimension 1. It follows that its columns vectors are proportional, and thus so are φ and ψ. □

LEMMA 15. *Let* A *be a square matrix of order* $k+1$ *such the matrix of its* $k\times k$*-minors* $(\det(A_{IJ})_{I,J}$, *with* I *and* J *running through all* k*-subsets of* $\{1,\dots,k+1\}$, *is of rank* 1. *Then* $\det(A)=0$.

Proof. If the central $(k-1)\times(k-1)$-minor $\det A_{\{2,\dots,k\},\{2,\dots,k\}}$ of A is nonzero, then (4), with $r=k$, implies that $\det(A)=0$. If some $(k-1)\times(k-1)$-minor of M is nonzero, we may bring it into central position by row and column permutations; these operations amount to row and column permutations of the matrix of $k\times k$-minors of A; hence, by the previous argument, $\det(A)=0$. Finally, if all the $(k-1)\times(k-1)$-minors of A vanish, then so does $\det(A)$. □

Lemma 16. *Let V be a vector space, and consider a finite ordered set of indices K for which we have selected vectors v_k, v'_k in V, as well as u_k, u'_k in the dual space V^*. Assuming that for all k in K we have the relations*[5]

$$v'_k = v_k + \sum_{\ell<k} \star v_\ell,$$
$$u'_k = u_k + \sum_{\ell<k} \star u_\ell.$$

Then we have the equality

$$\det(u_k(v_\ell))_{k,\ell\in K} = \det(u'_k(v'_\ell))_{k,\ell\in K}. \tag{34}$$

Proof. We simply pass from one matrix to the other by multiplication on the left and on the right by uni-triangular matrices. □

Consider now intervals of cardinality $k-1$ of the set $\{2,\dots,2k-1\}$, of the form

$$I_q := \{q+1, q+2, \dots, q+k-1\}, \quad q = 1,\dots,k.$$

For convenience sake, we write $K_q := [k] \setminus \{q\}$ (with $[k]$ standing as usual for $\{1,\dots,k\}$). Let $e_1,\dots,e_{2k-1}$ be elements of some vector space V. For $J = \{j_1,\dots,j_s\}$ such that

$$1 \le j_1 \le \cdots \le j_s \le 2k-1,$$

we denote by e_J the wedge product

$$e_J := e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_s}.$$

Then the following holds.

Lemma 17. *For vectors $e_1,\dots,e_{2k-1}$ in V which are such that*

$$e_j = (-1)^{k-1} e_{j-k} + \star e_{j-k+1} + \cdots + \star e_{j-1} \quad \textit{for } j = k+1,\dots,2k-1,$$

the following identity holds for all $q = 1,\dots,k$:

$$e_{I_q} = (-1)^{q+1} e_{K_q} + \star e_{K_{q-1}} + \cdots + \star e_{K_1}. \tag{35}$$

Proof. Writing $E := e_{q+1} \wedge e_{q+2} \wedge \cdots \wedge e_k$ and $\varepsilon := (-1)^{k-1}$, we calculate that

$$\begin{aligned}
e_{I_q} &= E \wedge e_{k+1} \wedge \cdots \wedge e_{k+q-1} \\
&= E \wedge e_{k+1} \wedge \cdots \wedge e_{k+q-2} \wedge (\varepsilon e_{q-1} + \star e_q + \cdots + \star e_{k+q-2}) \\
&= E \wedge e_{k+1} \wedge \cdots \wedge e_{k+q-2} \wedge (\varepsilon e_{q-1} + \star e_q) \\
&\qquad \text{(since } e_{q+1},\dots,e_{k+q-2} \text{ appear as factors} \\
&\qquad\ \text{in the product } E \wedge e_{k+1} \wedge \cdots \wedge e_{k+q-2}.) \\
&= E \wedge e_{k+1} \wedge \cdots \wedge e_{k+q-3} \wedge (\varepsilon e_{q-2} + \star e_{q-1} + \cdots + \star e_{k+q-3})
\end{aligned}$$

[5] Here, as in the sequel of this section, the stars ($\star$) stand for some coefficients that we do not actually need to specify.

$$
\begin{aligned}
&\quad \wedge (\varepsilon e_{q-1} + \star e_q) \\
&= E \wedge e_{k+1} \wedge \cdots \wedge e_{k+q-3} \wedge (\varepsilon e_{q-2} + \star e_{q-1} + \star e_q) \wedge (\varepsilon e_{q-1} + \star e_q) \\
&= \cdots \\
&= E \wedge (\varepsilon e_1 + \star e_2 + \cdots + \star e_q) \wedge \cdots \wedge (\varepsilon e_{q-2} + \star e_{q-1} + \star e_q) \\
&\quad \wedge (\varepsilon e_{q-1} + \star e_q) \\
&= (-1)^{(q-1)(k-q)} (\varepsilon e_1 + \star e_2 + \cdots + \star e_q) \wedge \cdots \wedge (\varepsilon e_{q-2} + \star e_{q-1} + \star e_q) \\
&\quad \wedge (\varepsilon e_{q-1} + \star e_q) \wedge E.
\end{aligned}
$$

The product that precedes E is evidently in the $(q-1)$th-exterior power of the span of $e_1, \dots, e_q$. It is thus a linear combination of the $e_{[q]\setminus\{i\}}$, for $i = 1, \dots, q$. It follows (as we are multiplying these $e_{[q]\setminus\{i\}}$ by E on the right) that we have expressed e_{I_q} as a linear combination of the E_{K_i}, for $i = 1, \dots, q$. Moreover, e_{K_q} appears only once in the resulting expression. Its coefficient is thus $(-1)^{(q-1)(k-q)}(-1)^{(q-1)(k-1)}$. We conclude that (35) holds, since $(q-1)(k-q) + (q-1)(k-1) \equiv (q-1)(-q-1) \equiv (q+1)^2 \equiv (q+1)$ modulo 2. $\square$

In the next result, the first row and first column of matrices are indexed by 1.

Proposition 18. *Consider a $(2k-1) \times (2k-1)$ matrix A of rank k having all of its adjacent $k \times k$ minors equal to 1, and write $B = A_{11}^{(k)}$, $C = \partial_{k-1} A$, and $D = C_{22}^{(k)}$. Then, for all $h \le k$, we have*

$$\det D_{11}^{(h)} = \det B_{h+1,h+1}^{(k-h)}.$$

Observe that the square matrices A, B, C, D are respectively of order $2k-1$, k, $k+1$ and k (as illustrated in Figure 8). Recall also that $\partial_{k-1} A$ is the matrix of adjacent $(k-1)$-minors of A.

Proof. Let $e_1, \dots, e_{2k-1}$ be the column vectors of A, and consider the vector space V that they span. Dually, let $\varphi_1, \dots, \varphi_{2k-1}$ be the restriction to V of the $2k-1$ projections of column vectors on the underlying field of scalars.

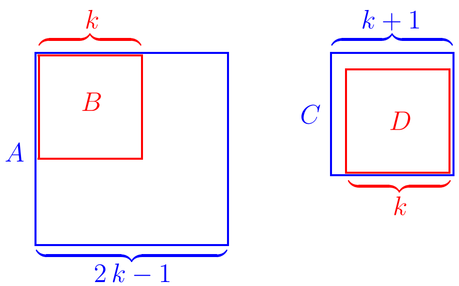

Figure 8. The square matrices A, B, C, and D of Proposition 18.

We clearly have $\varphi_i(e_j) = a_{ij}$. Using the usual duality[6] between $(V^*)^{\wedge k}$ and $V^{\wedge k}$, we see that $d_{ij} = c_{i+1,j+1} = \langle \varphi_{I_i}, e_{I_j} \rangle$, with the notations introduced before Lemma 17. Thus, the determinant of $D_{11}^{(h)}$ is equal to $\det(\varphi_{I_i}(e_{I_j}))_{1\le i,j\le h}$.

In view of the hypotheses on A, we have

$$e_j = \varepsilon e_{j-k} + \star e_{j-k+1} + \cdots + \star e_{j-1} \quad \bigl(\text{where as before } \varepsilon := (-1)^{k-1}\bigr)$$

for all $j = k+1, \ldots, 2k-1$ (as in the hypothesis of Lemma 17). Dually, we have

$$\varphi_j = \varepsilon \varphi_{j-k} + \star \varphi_{j-k+1} + \cdots + \star \varphi_{j-1}$$

for all $j = k+1, \ldots, 2k-1$. Applying Lemma 17, we get for $1 \le i, j \le h$ that

$$\varphi_{I_i} = (-1)^{i+1} \varphi_{K_i} + \star \varphi_{K_{i-1}} + \cdots + \star \varphi_{K_1}$$

and

$$e_{I_j} = (-1)^{j+1} e_{K_j} + \star e_{K_{j-1}} + \cdots + \star e_{K_1}.$$

Using Lemma 16, we conclude that the above determinant is equal to

$$\det((-1)^{i+j} \varphi_{K_i}(e_{K_j}))_{1\le i,j\le h},$$

which is exactly the $h \times h$-minor of the adjoint matrix of B, corresponding to rows and columns going from 1 to h.

To finish the argument, we apply a result Jacobi stating that (in the case of matrices of determinant 1) a minor is equal to the complementary minor of the adjoint matrix. □

Proof of Proposition 6. Proposition 18 implies (13). This equation, for $r = k$ and $s = 0$ implies that the dual is a SL_k-tiling. For $r = k-1$ and $s = 1$, it implies that the tiling coincides with its bidual, up to the necessary translation.

In order to show that the dual is tame, we proceed as follows. Observe that for any matrix (finite or infinite) (a_{ij}) of rank at most k, there exist a vector space E of dimension at most k, vectors $u_j \in E$, and linear forms φ_i on E, all such that $a_{ij} = \varphi_i(u_j)$ (take the space spanned by the columns and the linear function obtained by projections of the columns). Conversely, such a data gives a matrix (a_{ij}) of rank at most k.

Now we form the matrix $(\langle \varphi_I, u_J \rangle)_{IJ}$, over some family of $(k-1)$-subsets of the row and column indices. Then u_j is in the $(k-1)$th exterior power of E, which is of dimension at most k. Hence, this new matrix is of rank at most k. This implies that the dual is at rank at most k, since it is obtained from the original tiling by such a construction. □

Our proof of Proposition 12 relies on the following two lemmas.

[6] $\langle \psi_1 \wedge \cdots \wedge \psi_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\psi_i(v_j))_{1\le i,j\le k}$.

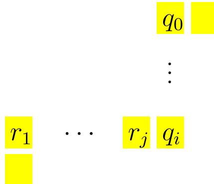

FIGURE 9. Points of the path π_w that lie on the same row and column as q.

LEMMA 19. *Let p and q be two points that are adjacent horizontally, i.e.: $p=(a,b)$ and $q=(a,b+1)$. Then $w_q = w_p x y^i$, where $i+1$ is the number of points lying on the path π_w in the same vertical as q.*

Proof. Denote by $q_0,\ldots,q_i$ these $i+1$ points (starting from the top), and by $r_1,\ldots,r_j$ all the points of π_w lying to the left of q_i (labeled from left to right). This corresponds to the portion of the path π_w illustrated in Figure 9. Clearly we have $w_{q_k} = y^k$, for $k<i$, and $w_{q_i} = x^j y^i$. Each of the following cases is clear (it helps to consider Figure 9), using the definition of w_p in Section 5:

(1) If q lies strictly above q_0, we have
$$w_p = w_q \overline{y}^i \overline{x},$$
implying that $w_q = w_p x y^i$ as required.

(2) When $q = q_j$, for $0 \le j \le i-1$, then
$$w_p = \overline{y}^{i-j}\overline{x} = y^j \overline{y}^i \overline{x}$$
so that again we have $w_q = y^j = w_p x y^i$.

(3) Finally, when q lies below q_i, we evidently have $w_q = w_p x y^i$, thus the assertion is verified for all possible cases. □

LEMMA 20. *For $m_1,\ldots,m_k$ in $\mathbb{N}$, let*
$$u_i := x^{m_{i-1}} y \cdots x^{m_1} y,$$
when $0 \le i \le k$. (In particular $u_0 = 1$.) Then we have
$$\begin{pmatrix} \mathbf{e}_k \mu(u_0) \\ \mathbf{e}_k \mu(u_1) \\ \vdots \\ \mathbf{e}_k \mu(u_{k-1}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \star & \ldots & \star & \star \end{pmatrix}.$$

Proof. We recursively show that $\mathbf{e}_k \mu(u_i) = (0,\ldots,0,1,\star,\ldots,\star)$ with 1 sitting in position $(k-i)$. If $i=0$, we have $\mathbf{e}_k \mu(u_0) = \mathbf{e}_k = (0,\ldots,0,1)$, so that 1 indeed sits in position k. By induction we may assume that the first nonzero value of the vector

$$\mathbf{v} := \mathbf{e}_k \mu(x^{m_{i-1}} y \cdots x^{m_2} y)$$

is a 1 sitting in position $(k-i)$. Then the the first nonzero value of the vector

$$\mathbf{e}_k \mu(x^{m_{i-1}} y \cdots x^{m_1}) = \mathbf{v}\mu(x^{m_1}) = (0, \dots, 0, 1, \star, \dots, \star),$$

also sits in position $(k-i)$, since $\mu(x^{m_1})$ is upper unitriangular. We can thus easily conclude since $\mathbf{e}_k \mu(u_i)$ is obtained by multiplying (on the right) this last vector by $\mu(y) = \mathrm{Id} + N^{\mathrm{tr}}$, hence its first nonzero value lies in position $(k-i-1)$. □

Proof of Proposition 12. Let us first check that $\mathcal{T}_w$ is indeed a SL_k-tiling. Consider any set of points p_{ij}, $0 \le i, j \le k-1$, forming an adjacent $k \times k$ subarray of $\mathbb{Z} \times \mathbb{Z}$, and let us write w_{ij} for the projection word $w_{p_{ij}}$ associated to these points p_{ij}. From Lemma 19 and its symmetric statement, there exists integers $m_0, \dots, m_{k-1}$ and $n_0, \dots, n_{k-1}$ such that

$$w_{ij} = u_i w_{00} v_j,$$

with

$$u_i = x^{m_{i-1}} y \cdots x^{m_1} y \quad \text{and} \quad v_j = x y^{n_1} \cdots x y^{m_{j-1}}.$$

We have the matrix identity

$$\begin{aligned}(\mathbf{e}_k \mu(u_i w_{00} v_j) \mathbf{e}_k^{\mathrm{tr}})_{0 \le i,j \le k-1} &= (\mathbf{e}_k \mu(u_i) \mu(w_{00}) \mu(v_j) \mathbf{e}_k^{\mathrm{tr}})_{0 \le i,j \le k-1} \\ &= \begin{pmatrix} \mathbf{e}_k \mu(u_0) \\ \vdots \\ \mathbf{e}_k \mu(u_{k-1}) \end{pmatrix} \mu(w_{00}) (\mathbf{e}_k \mu(v_0), \dots, \mathbf{e}_k \mu(v_{k-1}).\end{aligned}$$

The fact that $\det(w_{00}) = 1$, together with Lemma 20, implies that this matrix has determinant 1 as announced.

To show that $\mathcal{T}_w$ is tame, we argue as follows. First, observe that any given row of $\mathcal{T}_w$ is of the form

$$(\mathbf{e}_k \mu(m) \mathbf{e}_k^{\mathrm{tr}})_{m \in \mathcal{M}},$$

where $\mathcal{M}$ is the (ordered) set of projection words of the points on this row. Choose k other rows, lying below the given row. These k rows are (successively) of the form

$$(\mathbf{e}_k \mu(m_i \cdots m_1 m) \mathbf{e}_k^{\mathrm{tr}})_{m \in \mathcal{M}}$$

for i running from 1 to k, and suitable words $m_1, \dots, m_k$. Now, the $k+1$ row vectors $\mathbf{e}_k \mu(m_i \cdots m_1)$, $i = 0, \dots, k$, are perforce linearly dependent, since they are all of length k. Multiplying, this linear combination by $\mu(m)\mathbf{e}_k^{\mathrm{tr}}$ on the right, we find that the $k+1$ chosen rows of $\mathcal{T}_w$ are linearly dependent, and hence $\mathcal{T}_w$ is of rank $\le k$.

The proof that $\mathcal{T}_w$ is the same SL_k-tiling as the one described in Proposition 7, using Lemma 8, is left to the reader. □

8. Applications

8.1. SL_2-Frieze patterns revisited. The aim of this section is to show that the frieze patterns of Coxeter may be realized in terms of SL_2-tilings. This gives a new slant on their study, with emphasis on their link with representations of $SL_2(\mathbb{Z})$.

PROPOSITION 21. *Let a_i, $i\in\mathbb{Z}$ be nonzero elements in the field K. There exists a unique tame SL_2-tiling that extends the partial tiling of* (36).

$$\begin{array}{ccccccccc} \ddots & & & & & & & & \\ \ddots & 1 & & & & & & & \\ & a_{-1} & 1 & & & & & & \\ & & a_0 & 1 & & & & & \\ & & & a_1 & 1 & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & a_i & 1 & & \\ & & & & & & \ddots & \ddots & \end{array} \tag{36}$$

To better study such tilings, let us consider the notion of *signed continuant polynomials* $q_n(x_1,\dots,x_n)$ defined by the recurrence

$$q_n(x_1,\dots,x_n) := x_n q_{n-1}(x_1,\dots,x_{n-1}) - q_{n-2}(x_1,\dots,x_{n-2}), \tag{37}$$

whenever $n>0$, setting $q_{-1}:=0$ and $q_0:=1$. We omit indices when possible, writing simply $q(x_1,\dots,x_n)$ for $q_n(x_1,\dots,x_n)$. Let us now consider the particular SL_2 matrices

$$Y(t) := \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}$$

for which one may easily show by induction that

$$Y(x_1)Y(x_2)\cdots Y(x_n) = \begin{pmatrix} -q(x_2,\dots,x_{n-1}), & -q(x_2,\dots,x_n) \\ q(x_1,\dots,x_{n-1}), & q(x_1,\dots,x_n) \end{pmatrix}. \tag{38}$$

Proof of Proposition 21. To prove unicity, we exploit the fact that the SL_2-tiling contains subarrays of the form

$$\begin{array}{ccc} 1 & 0 & * \\ a_i & 1 & 0. \end{array} \tag{39}$$

Indeed, this follows directly from the SL_2-property. Let C_1, C_2, and C_3 be the three corresponding columns, from left to right. Then, since the tiling is of rank 2, we have $C_1-\alpha C_2+C_3=0$, which forces $\alpha=a_i$. Thus, the coefficients of linearization are completely determined. Moreover, we are given at least one adjacent (2×2)-subarray, namely lower left (2×2)-submatrix of (39). Thus, unicity of the tiling follows by Proposition 3.

For the existence of the tiling, we check that we may define its entries as follows:

$$
\begin{array}{ccccccc}
0 & -1 & -a_i & & \cdots & & \beta \\
1 & 0 & -1 & \ddots & & & \vdots \\
a_i & 1 & 0 & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 & -1 & -a_j \\
\vdots & & & \ddots & 1 & 0 & -1 \\
\alpha & \cdots & \cdots & \cdots & a_j & 1 & 0
\end{array}
$$

with $\beta = -\alpha$, and

$$\alpha = q(a_i, \dots, a_j). \tag{40}$$

We then need only verify that the resulting tiling has the right the linearization coefficients (as in the first part of the proof). To this aim, let us denote by α' and α'' the two entries of the tiling that sit immediately to the right of α, so that we have

$$\alpha \qquad \alpha' \qquad \alpha''$$

and therefore $\alpha' = q(a_{i+1}, \dots, a_j)$, and $\alpha'' = q(a_{i+2}, \dots, a_j)$. But the recurrence (37) implies that

$$\alpha - a_i\alpha' + \alpha'' = 0.$$

Hence, since the proof for β is analoguous, the tiling defined above has the desired linearization coefficients. $\square$

Consider now any frieze pattern, as below, with the a_i positive integers, having n diagonals (see Figure 2).

$$
\begin{array}{ccccccccccc}
 & & & \ddots & & & & & & & \\
 & & & \ddots & 1 & & & & & & \\
 & & & & a_{-2} & 1 & & & & & \\
 & & & \iddots & & a_{-1} & 1 & & & & \\
\ddots & & \iddots & & & & a_0 & 1 & & & \\
 & 1 & & & & & & a_1 & 1 & & \\
 & & 1 & & & & & & a_2 & 1 & \\
 & & & 1 & & & & \iddots & & \ddots & \ddots \\
 & & & & 1 & & \iddots & & & & \\
 & & & & & 1 & & & & & \\
 & & & & & & \ddots & & & &
\end{array}
\tag{41}
$$

Observe that, since the coefficients of the frieze pattern are positive, they are completely characterized by the a_i, in view of the SL_2-property. Hence, this frieze pattern extends uniquely to the same complete SL_2-tiling $\mathcal{A}$ as the one given by Proposition 21. Note that this extension has a few values more, immediately deduced from the positivity of the entries of the frieze pattern, and the SL_2-property, without resulting to tameness. These are the 0's and -1's given below. We may therefore extract from the tiling $\mathcal{A}$ the following subarray, where n is the number of diagonals of the frieze pattern and $i \in \mathbb{Z}$:

$$\begin{array}{cccccccc}
a_{i+1} & & & & & & & \\
* & a_{i+2} & & & & & & \\
\vdots & \ddots & \ddots & & & & & \\
\vdots & & & \ddots & & & & \\
1 & \dots & \dots & * & a_{i+n-1} & & & \\
0 & 1 & \dots\dots & & * & a_{i+n} & & \\
-1 & 0 & \dots\dots & & \dots & & * & a_{i+n+1}
\end{array}$$

By formula (40) for the entries of the tiling $\mathcal{A}$, we obtain

$$q(a_{i+1},\dots,a_{i+n+1}) = -1, \qquad q(a_{i+1},\dots,a_{i+n}) = 0,$$
$$q(a_{i+2},\dots,a_{i+n}) = 1, \qquad q(a_{i+2},\dots,a_{i+n+1}) = 0,$$

and, using (38), we conclude that

$$Y(a_{i+1})Y(a_{i+2})\cdots Y(a_{i+n+1}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{42}$$

Using this, we may prove the following corollary.

Corollary 22. *Let $\mathcal{A} = (a_{ij})$ be the unique tame SL_2-tiling $\mathcal{A} = (a_{ij})$ extending a given frieze pattern $\mathcal{F}$ with n diagonals.*

(i) *$\mathcal{A}$ has diagonal period $n+1$, that is to say $a_{i+n+1,j+n+1} = a_{ij}$.*
(ii) ([4], [6]) *In particular, the frieze pattern has diagonal period $n+1$.*
(iii) *Moreover, $\mathcal{A}$ has horizontal and vertical skew-period $n+1$, this is to say that*
$$a_{i+n+1,j} = -a_{ij} = a_{i,j+n+1}.$$
(iv) ([4], [6]) *Finally, $\mathcal{A}$ and $\mathcal{F}$ are invariant under a glided symmetry, which is the symmetry with respect to the middle diagonal of $\mathcal{F}$ followed by the diagonal translation of length $\frac{n+1}{2}$.*

Proof. Referring to (37), let C_i denote the column containing the coefficient a_i. Then we have $C_i - a_iC_{i+1} + C_{i+2} = 0$, as is shown at the beginning of the proof of Proposition 21. Thus, we have the following recurrence between the $\mathbb{Z}\times 2$ matrices (C_i, C_{i+1}):

$$(C_i, C_{i+1}) = (C_{i+1}, C_{i+2})Y(a_i).$$

Thus, (42) implies that $\mathcal{A}$ has horizontal skew-period $n+1$. Vertical periodicity follows by symmetry, and the diagonal periodicity follows at once. In order to prove (iv), note that (42) implies

$$Y(a_{i+2})\cdots Y(a_{i+n+1}) = -Y(a_{i+1}^{-1}) = \begin{pmatrix} -a_{i+1} & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, $a_{i+1} = q(a_{i+3}, \dots, a_{i+n})$ by (38). This shows, by taking $i = n, 0, 1, \dots$ and recalling that we have the diagonal period $n+1$ (hence $a_i = a_{i+n+1}$) that: $a_{n+1} = q(a_{n+3}, \dots, a_{2n}) = q(a_2, \dots, a_{n-1})$, $a_1 = q(a_3, \dots, a_n)$, $\dots$. Hence, using (40), we see that R has the following form, extending (36):

$$\begin{array}{cccccccccc}
\ddots & & & & & & & & & \\
\ddots & 1 & & & & & & & & \\
 & a_1 & 1 & & & & & & & \\
 & & a_2 & 1 & & & & & & \\
 & & & a_3 & 1 & & & & & \\
\ddots & \ddots & & & \ddots & \ddots & & & & \\
 & 1 & a_{n+1} & & & a_{n-1} & 1 & & & \\
 & & 1 & a_1 & & & a_n & 1 & & \\
 & & & 1 & a_2 & & & a_{n+1} & 1 & \\
 & & & & \ddots & \ddots & & & \ddots & \ddots
\end{array}$$

We conclude by using a symmetric version of Proposition 21. $\square$

Following Conway–Coxeter (in [4]), we call *quiddity* a sequence $a_1, \dots, a_{n+1}$, where a_i gives the number of triangle incident to the vertex i in a triangulation of a convex $(n+1)$-gone, whose vertex are successively labeled 1 to $n+1$ turning around the n-gone. They show [4, p. 180] that any quiddity may be obtained from the particular quiddity 111 by successive applications of the local rewriting rule

$$\cdots ab \cdots \quad \to \quad \cdots a+11b+1 \cdots.$$

We prove below their result that quiddities and frieze patterns are in one-to-one correpondence. For this, we make a detour through presentations of the group $SL_2(\mathbb{Z})$.

Proposition 23. *Consider the rewriting rule in the free monoid $\mathbb{P}^*$ generated by $\mathbb{P}$*

$$(a+1)1(b+1) \quad \to \quad ab, \tag{43}$$

where $a, b \in \mathbb{P}$. Then

$$Y(w) := Y(n_1)\cdots Y(n_k) = \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\varepsilon = \pm 1),$$

if and only if $w \to^* 1^k$, *with* $k \equiv 0 \pmod 6$ *when* $\varepsilon = 1$, *and* $k \equiv 3 \pmod 6$ *when* $\varepsilon = -1$. *In this case, if* w *is not a power of* 1, *then it contains a factor* $(a+1)1(b+1)$.

One direction of the proposition easily follows from the identities

$$Y(1)^3 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y(a+1)Y(1)Y(b+1) = Y(a)Y(b), \tag{44}$$

both of which can be easily checked by direct computation.

Moreover, for further use, it is easily checked that

$$Y(1)Y(2)Y(1)Y(2) = -1 \tag{45}$$

and also, recursively, that

$$Y(n) = (-1)^n (Y(2)Y(1)^2)^{n-2} Y(2). \tag{46}$$

We now give a proof of Proposition 23 after recalling some facts regarding presentations of $SL_2(\mathbb{Z})$. To simplify our discussion, let us informally[7] write "-1" for a central element of SL_2 whose square is the identity (denoted by 1).

LEMMA 24. *Denoting* $Y(1)$ *by* y_1, *and* $Y(2)$ *by* y_2,

(i) $SL_2(\mathbb{Z})$ *affords the presentation*

$$\langle y_1, y_2 \mid y_1^3 = -1, (y_1 y_2)^2 = -1 \rangle. \tag{47}$$

(ii) $SL_2(\mathbb{Z})$ *affords the confluent presentation*

$$\langle y_1, y_2 \mid y_1^3 \to -1, y_2 y_1 y_2 \to y_1^2 \rangle. \tag{48}$$

Proof. To show (i), let

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is well known that $SL_2(\mathbb{Z})$ has the presentation

$$\langle a, b \mid \bar{a} b \bar{a} = b \bar{a} b, (\bar{a} b \bar{a})^4 = 1 \rangle. \tag{49}$$

Direct calculations show that

$$a = \bar{y}_1 y_2, \qquad b = y_1 \bar{y}_2. \tag{50}$$

Thus, y_1, y_2 generate $SL_2(\mathbb{Z})$. The relations in (i) hold by (44) and (45). Therefore, it is enough to show that these relations imply the relations in (49), once a, b have been replaced using (50). By direct substitution, we get

$$\bar{a} b \bar{a} = \bar{y}_2 y_1 y_1 \bar{y}_2 \bar{y}_2 y_1 \quad \text{and} \quad b \bar{a} b = y_1 \bar{y}_2 \bar{y}_2 y_1 y_1 \bar{y}_2.$$

[7] This can easily be made formally correct by adding a generator, with straightforward relations, to our presentations.

Now, since $y_1^3 = -1$, we have $-\bar{y}_1 = y_1^2$, and hence $y_1^2 \bar{y}_2^3(-\bar{y}_1) = (-\bar{y}_1)\bar{y}_2^3 y_1^2$. Thus, $y_1^2 \bar{y}_2^2(-\bar{y}_2\bar{y}_1) = (-\bar{y}_1\bar{y}_2)\bar{y}_2^2 y_1^2$. But, since $y_1y_2y_1y_2 = -1$, we also have $y_2y_1y_2y_1 = -1$, and therefore $-\bar{y}_2\bar{y}_1 = y_1y_2$ and $-\bar{y}_1\bar{y}_2 = y_2y_1$. Hence,

$$y_1^2\bar{y}_2^2 y_1 y_2 = y_2 y_1 \bar{y}_2^2 y_1^2.$$

Multiplying this both on the left and on the right by $\bar{y}_2$, we obtain

$$\bar{y}_2 y_1^2 \bar{y}_2^2 y_1 = y_1 \bar{y}_2^2 y_1^2 \bar{y}_2,$$

so that $\bar{a}b\bar{a} = b\bar{a}b$.

On the other hand, we have $\bar{a}b = \bar{y}_2 y_1^2 \bar{y}_2$, and we have seen that $y_1y_2 = -\bar{y}_2\bar{y}_1$. Thus,

$$y_1 = -\bar{y}_2\bar{y}_1\bar{y}_2 = -\bar{y}_2(-y_1^2)\bar{y}_2 = \bar{y}_2 y_1^2 \bar{y}_2,$$

since $\bar{y}_1 = -y_1^2$. Thus, $(\bar{y}_2 y_1^2 \bar{y}_2)^6 = 1$. It follows, using $\bar{a}b\bar{a} = b\bar{a}b$, that

$$(\bar{a}b\bar{a})^4 = \bar{a}b\bar{a}b\bar{a}b\bar{a}b\bar{a}b\bar{a}b = (\bar{a}b)^6 = (\bar{y}_2 y_1^2 \bar{y}_2)^6 = 1.$$

(ii) We conclude from the first part that

$$\langle y_1, y_2 \mid y_1^3 = -1, y_2y_1y_2 = y_1^2\rangle \tag{51}$$

is a presentation of $SL_2(\mathbb{Z})$. Orienting the equalities, we obtain a rewriting system, whose confluence we must prove. This follows since the only non-trivial critical pair that needs to be examined is

$$\begin{tikzcd}[column sep=small] & y_2y_1y_2y_1y_2 \arrow[dl] \arrow[dr] & \\ (y_1^2)\, y_1y_2 \arrow[dr] & & y_2y_1\,(y_1^2) \arrow[dl] \\ & -y_2 & \end{tikzcd}$$

This ends our proof. □

Proof of Proposition 23. We need only show that if $w \neq 1^n$ is such that $Y(w) = \pm\,\mathrm{Id}$, then w must contain a factor of the form $(a+1)1(b+1)$. To see this, formally replace each letter n in $\mathbb{P}$ by y_n in words w in $\mathbb{P}^*$. Then, assuming that w is different from y_1^n, we may consider the canonical expansion

$$w = y_1^{n_0} y_{m_1} y_1^{n_1} y_{m_2} \cdots y_{m_k} y_1^{n_k},$$

where each $m_i \geq 2$ and $k \geq 1$. Using (46), we replace in this expansion each y_m by $(-1)^m (y_2 y_1^2)^{m-2} y_2$, we obtain a word in $\{y_1, y_2\}^*$ containing at least one instance of y_2. Since the system (48) is confluent, this word must contain $y_2y_1y_2$, hence one of the n_i must be equal to 1, thus proving our assertion. □

Corollary 25 (Conway–Coxeter [4]). *For each frieze pattern of the form* (41)*, the bi-infinite sequence of positive integers* $\cdots a_{-2}a_{-1}a_0a_1a_2a_3\cdots$ *is equal to* $\cdots wwwwwww \cdots$ *for some quiddity* w*.*

Proof. Denote by $\mathcal{F}$ this frieze pattern, let n be the number of diagonals of $\mathcal{F}$ and denote by $\mathcal{A}$ the SL_2-tiling obtained through Proposition 21. Then, by the discussion before Corollary 22, the coefficients a_i satisfy

$$Y(a_1)Y(a_2)\cdots Y(a_{n+1}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, by Corollary 22, $\cdots a_{-2}a_{-1}a_0a_1a_2a_3\cdots = \cdots wwwwwww\cdots$, with $w = a_1\cdots a_{n+1}$. Moreover, by Proposition 23, we have

$$a_{i-1} > 1, \qquad a_i = 1 \quad \text{and} \quad a_{i+1} > 1$$

for some i, $1 < i < n+1$. Thus, we find in $\mathcal{A}$ the subarray

$$\begin{array}{ccccc}
a_{i-2} & 1 & 0 & & \\
 & a_{i-1} & 1 & 0 & \\
 & (a_{i-1}-1) & 1 & 1 & 0 \\
 & & (a_{i+1}-1) & a_{i+1} & 1 \\
 & & & & a_{i+2}
\end{array}$$

Let C_j and R_j denote the columnn and row containing a_j. Then, as in the discussion at the beginning of the proof of Proposition 23, we have $C_i - C_{i+1} + C_{i+2} = 0$ and $R_i - R_{i-1} + R_{i-2} = 0$. Thus, by Lemma 5, we may suppress both the column C_{i+1} and the row R_{i-1}, to get the tame SL_2-tiling

$$\begin{array}{cccc}
a_{i-2} & 1 & & \\
 & (a_{i-1}-1) & 1 & \\
 & & (a_{i+1}-1) & 1 \\
 & & & a_{i+2}
\end{array}$$

We may clearly do this periodically for each column $C_{i+1+p(n+1)}$ and each row $R_{i-1+p(n+1)}$, for $p \in \mathbb{Z}$. Since both $\mathcal{A}$ and $\mathcal{F}$ have the diagonal period $n+1$, by Corollary 22, we obtain a tame SL_2-tiling $\mathcal{A}'$ and a frieze pattern $\mathcal{F}'$ with $n-1$ diagonals, such that $\mathcal{A}'$ is the unique extension of $\mathcal{F}'$ according to Proposition 21. This proves the corollary, once it is noted that the initial case corresponds to the frieze patterns reduced to $n = 2$ diagonals containing only 1's (here considered as having a diagonal period equal to 3). $\square$

8.2. $\mathbb{N}\times\mathbb{N}$ SL_k**-tilings.** When we restrict ourselves to $\mathbb{N}\times\mathbb{N}$ arrays, we may apply tools from matrix algebra and generating series. Assume that, for an invertible $k \times k$ matrix, we have a $(k+h)\times(k+h)$ matrix of rank k that decomposes into blocks in the following manner

$$\begin{pmatrix} \mathcal{S} & \Lambda \\ \Gamma & \mathcal{X} \end{pmatrix}, \tag{52}$$

with h possibly infinite. Then we must necessarily have $\mathcal{X} = \Gamma\mathcal{S}^{-1}\Lambda$. Indeed, it is clear in the following simple matrix identity

$$\begin{pmatrix} \mathcal{S} & \Lambda \\ \Gamma & \mathcal{X} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_k & -\mathcal{S}^{-1}\Lambda \\ 0 & \mathrm{Id}_h \end{pmatrix} = \begin{pmatrix} \mathcal{S} & 0 \\ \Gamma & \mathcal{X} - \Gamma\mathcal{S}^{-1}\Lambda \end{pmatrix}$$

that the right-hand side is also a matrix of rank k, since we are multiplying our original rank k matrix by an invertible one. However, we already know that $\mathcal{S}$ is of rank k. This forces $\mathcal{X} - \Gamma\mathcal{S}^{-1}\Lambda$ to vanish, and we have the required identity.

Let us assume that $\mathcal{A}$ is a tame *quarter-plane* (of shape $\mathbb{N}\times\mathbb{N}$) SL_k-array. Choose $\mathcal{S} := \mathcal{A}_{00}^{(k)}$, and let Γ (resp. Λ) stand for the subarray consisting of the first k columns (resp. rows) of $\mathcal{A}$. Then, we deduce from the above identity that we have

$$\mathcal{A} = \Gamma\mathcal{S}^{-1}\Lambda, \tag{53}$$

whenever $\mathcal{A}$ and $\mathcal{S}$ are both of rank k. It follows that for any subset I (resp. J) of rows (resp. columns, with $\#I = \#J$), we have

$$\mathcal{A}_{IJ} = \Gamma_I\mathcal{S}^{-1}\Lambda_J. \tag{54}$$

The simplest possible case of this identity allows the calculation of entries of $\mathcal{A}$ in the form

$$a_{ij} = \Gamma_i\mathcal{S}^{-1}\Lambda_j. \tag{55}$$

Now, if we choose both I and J to be of cardinality k, and take the determinant of both sides, we deduce from the fact that $\det(\mathcal{S}) = 1$, the identity

$$M_{IJ} = M_{I,\{1,\ldots,k\}}M_{\{1,\ldots,k\},J}. \tag{56}$$

A straightforward encoding of the $\mathbb{N}\times\mathbb{N}$-array $\mathcal{A}$ is through its bivariate *generating function*:

$$\mathcal{A}(x,y) := \sum_{(i,j)} a_{ij}x^iy^j, \tag{57}$$

with the sum running over all pairs (i,j) belonging to the shape of $\mathcal{A}$. An equivalent description may be given in terms of matrices, considering $X = (x^i)_{0\le i}$ as an infinite one-line matrix, and likewise $Y = (y^j)_{0\le j}$ as an infinite one-column matrix. We then have $\mathcal{A}(x,y) = X\mathcal{A}Y$. Now, when $\mathcal{A}$ is tame, it follows from (53) that we have

$$\begin{aligned} \mathcal{A}(x,y) &= X\mathcal{A}Y \\ &= X\Gamma\mathcal{S}^{-1}\Lambda Y \\ &= \begin{pmatrix} C_1(x) & \cdots & C_k(x) \end{pmatrix} \mathcal{S}^{-1} \begin{pmatrix} L_1(y) \\ \vdots \\ L_k(y) \end{pmatrix}, \end{aligned} \tag{58}$$

where the $C_i(x)$ are respectively the generating functions of the first k columns of $\mathcal{A}$. Likewise the $L_j(x)$ are the respective generating functions of the first k rows of $\mathcal{A}$. Often, as below, we have $\mathcal{S} = \mathrm{Id}$.

To illustrate the situation considered above, one may show that the SL_k-property holds for the matrix of binomial coefficients

$$\Gamma := \left(\binom{j}{i}\right)_{0\le i, 0\le j<k}.$$

From this, we get a SL_k-array $\mathcal{A} := \Gamma\Lambda$, with Λ equal to the transpose of Γ. Observe that the generating function of the jth-column (resp. ith) of Γ (resp. Λ) is evidently

$$\frac{1}{(1-x)^j} = \sum_{i\ge 0} \binom{i+j}{i} x^i \quad \text{for } j = 0, 1, \ldots, k-1$$

(resp. $(1-y)^i$). After calculation, using (58), we get that the generating function of $\mathcal{A}$ is

$$\mathcal{A}(x,y) = \sum_{\ell=1}^{k} \frac{x^{\ell-1}y^{\ell-1}}{(1-x)^\ell(1-y)^\ell}. \tag{59}$$

The following result follows from the constructions in Section 5.

Proposition 26. *The tiling given by* (59) *has all minors of the from* $M_{i0}^{(m)}$ *and* $M_{0j}^{(m)}$ *equal to* 1, *whenever* $m < k$, *and it is a* SL_k*-tiling.*

Using (55), or directly from (59), one may calculate that the individual entries of $\mathcal{A}$ are given by the formula

$$a_{ij} = \sum_{\ell=0}^{k-1} \binom{i}{\ell}\binom{j}{\ell}. \tag{60}$$

It follows also from Section 5, that for (i,j) in the k-fringe, $a_{ij} = \binom{i+j}{i}$. For example, with $k = 3$, we get the array of Figure 10.

$$\mathcal{A} = \begin{array}{|cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\
1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & \cdots \\
1 & 4 & 10 & 19 & 31 & 46 & 64 & 85 & 109 & \cdots \\
1 & 5 & 15 & 31 & 53 & 81 & 115 & 155 & 201 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Figure 10. A SL_3-tiling of $\mathbb{N} \times \mathbb{N}$.

$$
\begin{array}{ccccccc}
 & & & & & & \iddots \\
 & & & & & 1 & 1 \\
 & & & & 1 & 1 & 2 \\
 & & & 1 & 1 & 2 & 5 \\
 & & 1 & 1 & 2 & 5 & 13 \\
 & 1 & 1 & 2 & 5 & 13 & 34 \\
1 & 1 & 2 & 5 & 13 & 34 & 89 \\
\iddots & & & & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & & & & \iddots \\
 & & & & & 1 & 1 \\
 & & & & 1 & 1 & 2 \\
 & & & 1 & 1 & 2 & 5 \\
 & & 1 & 1 & 2 & 5 & 14 \\
 & 1 & 1 & 2 & 5 & 14 & 42 \\
1 & 1 & 2 & 5 & 14 & 42 & 132 \\
\iddots & & & & & &
\end{array}
$$

$k = 2$ $\qquad\qquad$ $k \to \infty$

FIGURE 11. Zigzag path tiling.

It may readily be shown that the dual of $\mathcal{A}$ affords the generating function

$$
\mathcal{A}^*(x,y) = \frac{1}{(1-x)(1-y)} + \sum_{\ell=2}^{k} \frac{xy}{(1-x)^\ell(1-y)^\ell}. \tag{61}
$$

8.3. Zigzag path. It is shown in [1] that the SL_2-tiling associated to the bi-infinite word $\cdots xyxyxyxyx\cdots$ has entries equal to the Fibonacci numbers of even rank (if we set $F_{n+2} = F_{n+1} + F_n$, $F_0 = F_1 = 1$), see the left part of Figure 11. If we let k go to infinity, then by Section 5, the entries of the resulting tiling are the Catalan numbers. In particular, it is noteworthy that the value of the $(k \times k)$-principal minors given by Proposition 7 corresponds in this situation to the classical result stating that for any natural integer k, the Hankel matrix $(C_{h+i+j})_{i,j=0,\ldots,k}$ (with either $h=0$, or $h=1$) has determinant equal to 1 (here, as usual, we have $C_n = \frac{1}{n+1}\binom{2n}{n}$), see the right part of Figure 11.

9. Closing remarks

A converse. Experiments suggest that a "converse" of Proposition 6 holds, namely that for any tame tiling, if (13) holds for some pair (r,s) for which $r+s=k$, then the tiling is necessarily a SL_k-tiling. Special cases, for small values of k, are easy to prove using generic value tilings and Gröbner basis computations.

Generalized frieze patterns. A notion of generalized frieze patterns, for $k>2$, has been considered in [5]. These are best understood in terms of certain tame "toric" SL_k-tilings $\mathcal{A}$. More precisely, we say that a tiling has a *skew-period* (p,q) in $\mathbb{Z}\times\mathbb{Z}$ $(\neq (0,0))$, if and only if

$$
\mathcal{A}(i+p, j+q) = (-1)^k \mathcal{A}(i,j) \quad \text{for all } (i,j) \in \mathbb{Z}\times\mathbb{Z}, \tag{62}
$$

$$
\begin{array}{ccccccccccc}
\ddots & \ddots & \ddots & \ddots & \ddots & & & & & & \\
 & 0 & 0 & 1 & a_{-1} & 1 & & & & & \\
 & & 0 & 0 & 1 & a_0 & 1 & & & & \\
 & & & 0 & 0 & 1 & a_1 & 1 & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & 0 & 0 & 1 & a_i & 1 & \\
 & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}
$$

FIGURE 12. Positive integer SL_k-frize patterns of "width" 1.

and we then say that the tiling is *skew-periodic*. A *toric* tiling is one that has two linearly independent skew-periods. A SL_k*-frieze patterns* $\mathcal{A}$ is a tame SL_k-tiling such that

$$
\mathcal{A}(i,j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i - j < k, \\ (-1)^{k-1}, & \text{if } i = j + k, \end{cases} \tag{63}
$$

which is periodic (not skew) with a period of the form $(p, -p)$, for $p > k$. In other words, on top of being periodic, the tiling is prescribed to have a diagonal of 1's, another diagonal filled with $(-1)^{k-1}$ below, with these two diagonals separated by $(k-1)$ diagonals of 0's.

Condition (63) and periodicity (together with tameness) ensure that the whole tiling is determined by its values along a band $\{(i,j)\ j \le i < j+p, j \in \mathbb{Z}\}$, with p as above. Moreover, one may show that any such tiling is toric, with skew-periods $(p,0)$ and $(0,p)$. This implies that it exhibits a frieze-like behaviour, since the tiling must necessarily have period (p,p). The generalized frieze patterns of [5] appear as special cases of this notion.

It is interesting to observe (and easy to prove) that the SL_k-frieze patterns, with $p = k+2$, that are of the form illustrated in Figure 12, include those for which the sequence $\ldots, a_{-1}, a_0, a_1, \ldots, a_i \ldots$ is a quiddity. Indeed, applying the SL_k-condition to the submatrix with main diagonal corresponding to a_i's, one easily check that we have

$$
q_k(a_i, a_{i+1}, \ldots, a_{i+k-1}) = 1,
$$

with q_k denoting the signed continuant polynomials considered in Section 8.1.

Other interesting toric SL_k-tilings seem to abound. For example, with $k=3$ and $p=4$, we have the following positive integer valued toric SL_k-tilings:

$$\mathcal{A} = \begin{array}{ccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & \cdots \\
\cdots & 1 & 3 & 6 & 4 & 1 & 3 & 6 & 4 & 1 & 3 & 6 & \cdots \\
\cdots & 1 & 2 & 4 & 3 & 1 & 2 & 4 & 3 & 1 & 2 & 4 & \cdots \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 2 & 3 & \cdots \\
\cdots & 1 & 3 & 6 & 4 & 1 & 3 & 6 & 4 & 1 & 3 & 6 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

It may be checked that this is a tame tiling, however the entries of the corresponding dual tiling are not all positive, since:

$$\partial\mathcal{A} = \begin{array}{cccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & \cdots \\
\cdots & 1 & 3 & 0 & -2 & 1 & 3 & 0 & -2 & 1 & 3 & \cdots \\
\cdots & -1 & 0 & 2 & 1 & -1 & 0 & 2 & 1 & -1 & 0 & \cdots \\
\cdots & -1 & -2 & 1 & 2 & -1 & -2 & 1 & 2 & -1 & -2 & \cdots \\
\cdots & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & \cdots \\
\cdots & 1 & 3 & 0 & -2 & 1 & 3 & 0 & -2 & 1 & 3 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

General properties of tame toric tilings, as well as results concerning SL_k-frieze patterns similar to those of Section 8.1, will be the subject of a planed sequel to this paper.

T-systems. On a closing note, it is interesting to observe that there is a close tie between tame SL_k-tilings and the notion of T-systems, which appear as solutions of the discrete Hirota equation (see [8]) of mathematical physics. Indeed, up to a simple relabelling, one can characterize the entries of T-systems in terms of derivatives of suitably chosen tame SL_k-tilings. Recall that a T-system $T:\{0,\dots,r+1\}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{N}$ must satisfy the equation

$$T_{\{\alpha,j,k+1\}}T_{\{\alpha,j,k-1\}} = T_{\{\alpha,j+1,k\}}T_{\{\alpha,j-1,k\}} + T_{\{\alpha+1,j,k\}}T_{\{\alpha-1,j,k\}}, \tag{64}$$

with boundary conditions

$$T_{\{0,j,k\}} = T_{\{r+1,j,k\}} = 1 \tag{65}$$

for all j, and k in $\mathbb{Z}$. It is shown in [8] that

$$T_{\{\alpha,j,k\}} = \det\big(T_{\{1,j-a+b,k+a+b-\alpha-1\}}\big)_{1\le a,b\le\alpha}. \tag{66}$$

From this, one can readily see that

$$T_{\alpha,j,k} = (\partial^\alpha \mathcal{A})_{s,t}$$

for s, t simple linear functions of j and k, and $\mathcal{A}$ a SL_{r+1}-tiling directly obtained from $(T_{1,j,k})_{j,k}$.

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François Bergeron, Christophe Reutenauer, Département de Mathématiques, Université du Québec à Montréal, C.P. 8888, Succ. Centre-Ville, Montréal, Québec, H3C 3P8, Canada

E-mail address: `bergeron.francois@uqam.ca`; `reutenauer.christophe@uqam.ca`