Illinois Journal of Mathematics
Volume 53, Number 3, Fall 2009, Pages 757–767
S 0019-2082

MAPPING AND CONTINUITY PROPERTIES OF THE BOUNDARY SPECTRUM IN BANACH ALGEBRAS

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ABSTRACT. We present further properties of the boundary spectrum $S_\partial(a) = \{\lambda : \lambda - a \in \partial S\}$ of a, where ∂S denotes the topological boundary of the set S of all noninvertible elements of a Banach algebra A, and where a is an element of A. In particular, we investigate the conditions under which it is true that $S_\partial(f(a)) = f(S_\partial(a))$, where f is a complex valued function which is analytic on a neighbourhood of the spectrum of a. We also consider continuity properties of the boundary spectrum.

1. Introduction and preliminaries

Let A be a complex Banach algebra with unit 1. If $\lambda \in \mathbb{C}$, then we shall write λ for the element $\lambda 1$ in A. Let $K(\mathbb{C})$ denote the set of nonempty compact subsets of the complex plane $\mathbb{C}$.

If E is a subset of A, then the topological boundary and the topological interior of E relative to A will be denoted by ∂E and $\operatorname{int} E$, respectively (or by $\partial_A E$ and $\operatorname{int}_A E$, respectively, if the particular Banach algebra needs to be emphasized). For an $\varepsilon > 0$ and an element x in A, the notation $B(x, \varepsilon)$ will be used to denote the open ball in A with centre x and radius ε. Let A^{-1} denote the group of all invertible elements of A, and let $\exp(A) = \{e^a : a \in A\}$ and $\operatorname{Exp}(A) = \{e^{c_1} \cdots e^{c_k} : k \in \mathbb{N},\ c_1, \ldots, c_k \in A\}$, i.e., $\operatorname{Exp}(A)$ is the component of 1 in A^{-1}. We denote the set of quasinilpotent elements in A by $\operatorname{QN}(A)$ and the radical of A by $\operatorname{Rad}(A)$. Recall that $\operatorname{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\} = \{a \in A : Aa \subseteq \operatorname{QN}(A)\}$. For a compact set K, let $C(K)$ indicate the Banach algebra of all continuous complex valued functions on K.

Given an element a in a Banach algebra A, recall that the spectrum of a is

$$\sigma(a) \equiv \sigma(a, A) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\} = \{\lambda \in \mathbb{C} : \lambda - a \in S\},$$

Received January 9, 2009; received in final form February 12, 2009.
2000 *Mathematics Subject Classification.* 46H05, 46H30.

where S (or S_A, if necessary) denotes the set of all noninvertible elements of A. The *boundary spectrum* of a replaces S by its boundary:

$$S_\partial(a) \equiv S_\partial(a, A) = \{\lambda \in \mathbb{C} : \lambda - a \in \partial S\}.$$

This concept was introduced in [10], and in [11] it was shown that the boundary spectrum plays an interesting role in spectral continuity in ordered Banach algebras.

In this paper, we present mapping properties of the boundary spectrum, similar in spirit to some of the work in [4] and [5]. More specifically, we investigate the conditions under which it is true that $S_\partial(f(a)) = f(S_\partial(a))$, where f is a complex valued function which is analytic on a neighbourhood of $\sigma(a)$. We also compare our results with corresponding mapping properties of the boundary $\partial\sigma$ of the spectrum. In addition, we investigate "regularity-type" properties (see [6], [8], [12], [13]) of the set $R_\partial := A \backslash \partial S = A^{-1} \cup \operatorname{int} S$, as well as the continuity properties of the boundary spectrum.

2. The boundary spectrum

Let A be a complex Banach algebra with unit 1 and let S be the (closed) set of all noninvertible elements of A. For $a \in A$, the boundary spectrum $S_\partial(a)$ of a is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda - a$ is an element of the boundary of S, that is

$$S_\partial(a) = \{\lambda \in \mathbb{C} : \lambda - a \in \partial S\}.$$

From [10], we know that $S_\partial(a)$ is a nonempty, compact subset of the complex plane such that $\partial\sigma(a) \subseteq S_\partial(a) \subseteq \sigma(a)$. In [10], several other properties of the boundary spectrum were established. In particular, we recall the following results.

Lemma 2.1 ([10, Lemma 2.6]). *Let A be a Banach algebra, $a \in \partial S$ and d an invertible element. Then $ad \in \partial S$ and $da \in \partial S$.*

Theorem 2.2 ([10, Corollary 2.12]). *Let B be a closed subalgebra of a Banach algebra A such that B contains the unit element 1 of A. If $a \in B$, then $S_\partial(a, B) \subseteq S_\partial(a, A)$.*

We also showed that, in general, $\partial\sigma(a) \neq S_\partial(a)$ [10, Example 2.3]. See also Example 3.7. In the following example, we illustrate that, in general, $S_\partial(a) \neq \sigma(a)$ as well. For this purpose, let Γ denote the circle with centre 0 and radius 1 and D the closed disk with centre 0 and radius 1 in $\mathbb{C}$.

Example 2.3. *Let $A = C(\Gamma)$ and B the subalgebra of A consisting of all elements of A which can be extended to a function on D which is analytic on its interior. If $f(z) = z$ for all $z \in D$, then $f \in B$ and $S_\partial(f, B) \neq \sigma(f, B)$.*

Proof. Since $\sigma(f,A)=\Gamma$ and $\sigma(f,B)=D$ [14, Problem 9, p. 399], we have that $D\backslash\Gamma\subseteq\sigma(f,B)\backslash S_\partial(f,A)\subseteq\sigma(f,B)\backslash S_\partial(f,B)$, by Theorem 2.2. Therefore, $S_\partial(f,B)\neq\sigma(f,B)$. $\square$

In [10], an elementary mapping property of the boundary spectrum was established.

Proposition 2.4 ([10, Proposition 2.7]). *Let a be an invertible element of a Banach algebra A. Then $S_\partial(a^{-1})=(S_\partial(a))^{-1}$.*

Some mapping properties of $\partial\sigma$ and S_∂ are fairly obvious. In the interest of completeness, we provide these properties in the remainder of this section.

Proposition 2.5. *Let a be an element of a Banach algebra A and let $\lambda\in\mathbb{C}$. Then*

1. $\partial\sigma(\lambda a)=\lambda\partial\sigma(a)$ *and* $\partial\sigma(a+\lambda)=\partial\sigma(a)+\lambda$.
2. $S_\partial(\lambda a)=\lambda S_\partial(a)$ *and* $S_\partial(a+\lambda)=S_\partial(a)+\lambda$.

Proof. For any set $K\subseteq\mathbb{C}$, we have $\partial(\lambda K)=\lambda\partial K$ and $\partial(K+\lambda)=\partial K+\lambda$. This, together with the spectral mapping theorem, yields (1). For (2), since the case $\lambda=0$ is trivial, we suppose that $\lambda\neq 0$ and that $\mu\in S_\partial(\lambda a)$, so that $\mu-\lambda a\in\partial S$. By Lemma 2.1, we have $\gamma-a\in\partial S$, where $\gamma=\frac{\mu}{\lambda}$. Since $\gamma\in S_\partial(a)$, it follows that $\mu=\lambda\gamma\in\lambda S_\partial(a)$. This yields the inclusion $S_\partial(\lambda a)\subseteq\lambda S_\partial(a)$. The inclusion $\lambda S_\partial(a)\subseteq S_\partial(\lambda a)$ and the other equation are proved similarly. $\square$

The open mapping property of analytic functions shows that if f is analytic on a neighbourhood of the spectrum of a, then it is always the case [5, Proposition 2.2], that the boundary of the spectrum of the element $f(a)$ is contained in the image under f of the boundary of the spectrum of the element a.

Proposition 2.6. *Let a be an element of a Banach algebra A and let f be a complex valued function which is analytic on a neighbourhood of $\sigma(a)$. Then $\partial\sigma(f(a))\subseteq f(\partial\sigma(a))$.*

Proof. Let $\lambda_0\in\partial\sigma(f(a))$. Then by the spectral mapping theorem $\lambda_0=f(\mu_0)$ for some $\mu_0\in\sigma(a)$. If f is constant on the component K of $\sigma(a)$ to which μ_0 belongs, then $\lambda_0=f(\omega_0)$ for any $\omega_0\in\partial K$, so that $\lambda_0\in f(\partial\sigma(a))$. So suppose that f is not constant on the component K of $\sigma(a)$ containing μ_0 and suppose that $\mu_0\in\operatorname{int}\sigma(a)$, say $B(\mu_0,\varepsilon)\subseteq K$. By the open mapping theorem [2, p. 99] $f(B(\mu_0,\varepsilon))$ is open, say $B(\lambda_0,\varepsilon')\subseteq f(B(\mu_0,\varepsilon))\subseteq f(\sigma(a))$. This yields the contradiction $\lambda_0\in\operatorname{int}f(\sigma(a))$, so that $\mu_0\in\partial\sigma(a)$, and the result follows. $\square$

Under certain circumstances, it is the case that

$$\partial\sigma(f(a))=f(\partial\sigma(a))=f(S_\partial(a))=S_\partial(f(a)). \tag{2.1}$$

For instance, if the spectrum under consideration has empty interior, then some results are immediate.

Proposition 2.7. *Let a be an element of a Banach algebra A and let f be a complex valued function which is analytic on a neighbourhood of $\sigma(a)$.*

1. *If $\sigma(a)$ has no interior points, then $S_\partial(f(a)) \subseteq f(S_\partial(a))$.*
2. *If $\sigma(f(a))$ has no interior points, then $f(\partial\sigma(a)) \subseteq \partial\sigma(f(a))$ and $f(S_\partial(a)) \subseteq S_\partial(f(a))$.*
3. *If $\sigma(a)$ has no interior points and f is one-to-one, then (2.1) holds.*

Proof. If the spectrum of an element has empty interior, then this spectrum is equal to its boundary and, hence, to the corresponding boundary spectrum. This, together with the spectral mapping theorem, yields (1) and (2). If $\sigma(a)$ has no interior points and f is one-to-one, then $\sigma(f(a))$ has no interior points, since if λ_0 were an interior point of $\sigma(f(a))$, say $B(\lambda_0, \varepsilon) \subseteq \sigma(f(a))$, then $\sigma(a)$ would contain the open set $f^{-1}(B(\lambda_0, \varepsilon))$. Hence (3), follows from (1), (2) and Proposition 2.6. □

We will show later that parts of Proposition 2.7(3) can be strengthened considerably.

The proof of the following proposition is a standard argument and will be omitted.

Proposition 2.8. *Let a be an element of a Banach algebra A.*

1. *If f is a constant function, then (2.1) holds.*
2. *Let U be an open set such that $\sigma(a) \cap \partial U = \emptyset$. If f is the characteristic function of U, then (2.1) holds.*

3. Mapping properties

In this paragraph, we shall write K^2 for the set $\{\lambda^2 : \lambda \in K\}$ (i.e. $K^2 = f(K)$ where $f(\lambda) = \lambda^2$).

If the spectrum of a is symmetric with respect to the origin, then it is easy to verify that the set of all squares of elements of the boundary of the spectrum of a is contained in the boundary of the spectrum of the element a^2. Together with Proposition 2.6, we have the following.

Proposition 3.1. *Let a be an element of a Banach algebra A such that $\sigma(a)$ is symmetric with respect to the origin. Then $(\partial\sigma(a))^2 = \partial\sigma(a^2)$.*

To obtain a corresponding result for the boundary spectrum S_∂, we need some preliminary results, starting with the following lemma.

Lemma 3.2. *Let a and b be elements of a Banach algebra A with $b \neq 0$, and let $\varepsilon_0 > 0$. Then there exist an $\varepsilon_1 > 0$ and an $\varepsilon_2 > 0$ such that $B(a, \varepsilon_1)B(b, \varepsilon_2) \subseteq B(ab, \varepsilon_0)$.*

Proof. Let $\varepsilon_1 = \frac{\varepsilon_0}{2\|b\|}$ and $\varepsilon_2 = \frac{\|b\|\varepsilon_0}{2\|a\|\|b\|+\varepsilon_0}$, and let $x \in B(a,\varepsilon_1)$ and $y \in B(b,\varepsilon_2)$. Then

$$\begin{aligned}\|xy-ab\| &\le \|x\|\|y-b\| + \|x-a\|\|b\| \\ &< \left(\frac{\varepsilon_0}{2\|b\|} + \|a\|\right)\left(\frac{\|b\|\varepsilon_0}{2\|a\|\|b\|+\varepsilon_0}\right) + \left(\frac{\varepsilon_0}{2\|b\|}\right)\|b\| \\ &= \varepsilon_0,\end{aligned}$$

so that $xy \in B(ab,\varepsilon_0)$. This yields the desired inclusion. ☐

COROLLARY 3.3. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $a_1,\dots,a_n$ be elements of a Banach algebra A with $a_j \neq 0$ for all $j = 2,\dots,n$. If $\varepsilon_0 > 0$, then there exist $\varepsilon_j > 0$ $(j = 1,\dots,n)$ such that*

$$B(a_1,\varepsilon_1)B(a_2,\varepsilon_2)\cdots B(a_n,\varepsilon_n) \subseteq B(a_1a_2\cdots a_n,\varepsilon_0).$$

THEOREM 3.4. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $a_1,\dots,a_n$ be elements of a Banach algebra A. If $a_1a_2\cdots a_n \in \operatorname{int} S$, then $a_j \notin \partial S$ for at least one $j \in \{1,\dots,n\}$.*

Proof. If $a_j = 0$ for some $j \in \{2,\dots,n\}$ the result is obvious. So suppose that $a_j \neq 0$ for all $j = 2,\dots,n$ and suppose that $a_1a_2\cdots a_n \in \operatorname{int} S$, say $B(a_1a_2\cdots a_n,\varepsilon_0) \subseteq S$. Together with Corollary 3.3, it follows that there exist $\varepsilon_j > 0$ $(j = 1,\dots,n)$ such that

$$B(a_1,\varepsilon_1)B(a_2,\varepsilon_2)\cdots B(a_n,\varepsilon_n) \subseteq S. \tag{3.1}$$

If $a_j \in \partial S$ for all $j = 1,\dots,n$, then there exists a $c_j \in B(a_j,\varepsilon_j) \cap A\backslash S$ for each $j = 1,\dots,n$. By definition of S the product $c_1c_2\cdots c_n \notin S$. On the other hand, (3.1) implies that $c_1c_2\cdots c_n \in S$. This contradiction yields the result. ☐

We are now in a position to present two corollaries containing useful results.

COROLLARY 3.5. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $a_1,\dots,a_n$ be mutually commuting elements of a Banach algebra A. If $a_j \in \partial S$ for all $j = 1,\dots,n$, then $a_1a_2\cdots a_n \in \partial S$.*

Proof. The assumptions imply that $a_1a_2\cdots a_n \in S$, so that the result follows from Theorem 3.4. ☐

COROLLARY 3.6. *Let a be an element of a Banach algebra A.*

1. *If $a \in \partial S$, then $a^n \in \partial S$ for all $n \in \mathbb{N}$.*
2. *If $0 \in S_\partial(a)$, then $0 \in S_\partial(a^n)$ for all $n \in \mathbb{N}$.*

Using Corollary 3.6, we can give another example to illustrate that the boundary of the spectrum of an element is in general properly contained in the boundary spectrum of the element.

EXAMPLE 3.7. *Let $K = \{\lambda = re^{i\theta} \in \mathbb{C} : r \in [0,1]$ and $\theta \in [0,\frac{\pi}{2}]\}$ and let $A = C(K)$. If $g(z) = z^2$ for all $z \in K$ and $a = g^2$, then $a \in A$ and $\partial\sigma(a) \neq S_\partial(a)$.*

Proof. Since $\sigma(a) = (\sigma(g))^2 = (g(K))^2 = D$ (where D is the closed unit disk with centre 0 and radius 1), clearly $0 \notin \partial\sigma(a)$. However, since $\sigma(g) = \{\lambda = re^{i\theta} \in \mathbb{C} : r \in [0,1] \text{ and } \theta \in [0,\pi]\}$, it follows that $0 \in \partial\sigma(g)$, so that $0 \in S_\partial(g)$. Corollary 3.6(2) now implies that $0 \in S_\partial(g^2) = S_\partial(a)$, so that $0 \in S_\partial(a)\backslash\partial\sigma(a)$. □

The following theorem states that the set of squares of elements of the boundary spectrum of a is contained in the boundary spectrum of the element a^2, provided that one of two conditions is satisfied, one of which is that the boundary spectrum of a is symmetric with respect to the origin.

THEOREM 3.8. *Let a be an element of a Banach algebra A. If either $S_\partial(a) \cap -\sigma(a) = \emptyset$ or $S_\partial(a) = -S_\partial(a)$ is symmetric with respect to the origin, then $(S_\partial(a))^2 \subseteq S_\partial(a^2)$.*

Proof. If $S_\partial(a) \cap -\sigma(a) = \emptyset$ and $\lambda \in S_\partial(a)$, then $\lambda - a \in \partial S$ and $\lambda + a$ is invertible, so that $\lambda^2 - a^2 = (\lambda + a)(\lambda - a) \in \partial S$, by Lemma 2.1. Hence, $\lambda^2 \in S_\partial(a^2)$.

If $S_\partial(a)$ is symmetric with respect to the origin and $\lambda \in S_\partial(a)$, then both λ and $-\lambda$ are in $S_\partial(a)$, so that both $\lambda - a$ and $\lambda + a$ are in ∂S. It follows from Corollary 3.5 that $\lambda^2 - a^2 = (\lambda + a)(\lambda - a) \in \partial S$, so that $\lambda^2 \in S_\partial(a^2)$. □

Using induction, the second part of Theorem 3.8 can be generalized slightly.

COROLLARY 3.9. *Let a be an element of a Banach algebra A and let $n \in \mathbb{N}$. If $S_\partial(a^{2^k})$ is symmetric with respect to the origin for all $k = 0, 1, \ldots, n-1$, then $(S_\partial(a))^{2^n} \subseteq S_\partial(a^{2^n})$.*

For $a \in A$ in the following lemma, the set of all complex valued functions which are analytic and one-to-one on a neighbourhood of $\sigma(a)$ is indicated by $H_1(a)$.

LEMMA 3.10. *Let A be a Banach algebra and let $w : A \to K(\mathbb{C})$ be any mapping such that $w(a) \subseteq \sigma(a)$ for all $a \in A$. Then the following statements are equivalent:*

1. *$w(f(a)) \subseteq f(w(a))$ for all $a \in A$ and all $f \in H_1(a)$.*
2. *$f(w(a)) \subseteq w(f(a))$ for all $a \in A$ and all $f \in H_1(a)$.*

Proof. If $a \in A$ and $f \in H_1(a)$, say f is analytic and one-to-one on an open set G containing $\sigma(a)$, then it follows from [2, Corollary 7.6, p. 99], that $g : H \to \mathbb{C}$ is analytic (and one-to-one) where $g = f^{-1}$ and $H = f(G)$. Consider the element $b = f(a)$ in A. By the spectral mapping theorem $\sigma(b) = f(\sigma(a)) \subseteq H$, so that $g \in H_1(b)$. Furthermore, $g(b) = (g \circ f)(a) = a$ [3, Problem 4, p. 209].

Now, if (1) holds, $a \in A$ and $f \in H_1(a)$, then $w(g(b)) \subseteq g(w(b))$, so that $w(a) \subseteq g(w(f(a)))$, by the preceding paragraph. Applying f, we obtain (2).

Similarly, if (2) holds, $a \in A$ and $f \in H_1(a)$, then $g(w(b)) \subseteq w(g(b))$, so that $g(w(f(a))) \subseteq w(a)$, and (1) follows by applying f. $\square$

Proposition 2.6 and Lemma 3.10 yield the following result, which is a stronger version of the first part of Proposition 2.2 in [5].

PROPOSITION 3.11. *Let a be an element of a Banach algebra A and let f be a complex valued function which is analytic and one-to-one on a neighbourhood of $\sigma(a)$. Then $\partial\sigma(f(a)) = f(\partial\sigma(a))$.*

Clearly, Proposition 3.11 implies Proposition 2.5(1). Proposition 3.11 also strengthens the first part of Proposition 2.7(3).

Even for f a polynomial, the property $\partial\sigma(f(a)) = f(\partial\sigma(a))$ does not hold in general if f is not one-to-one on a neighbourhood of the spectrum of a. To see this, consider the Banach algebra $A = C(K)$ with K as in Example 3.7 and let $a = g$, where $g(z) = z^2$ for all $z \in K$. Then $a \in A$ and if $f(z) = z^2$, then $\partial\sigma(f(a)) \neq f(\partial\sigma(a))$. Recalling Proposition 3.1, we also note that $\sigma(a)$ is not symmetric with respect to the origin.

Theorem 3.13 shows that Proposition 3.11 remains true if $\partial\sigma$ is replaced with S_∂. We need the following result.

THEOREM 3.12. *Let a be an element of a Banach algebra A and let f be a complex valued function which is analytic and one-to-one on a neighbourhood G of $\sigma(a)$. If $\lambda_0 \in G$ and $\beta = f(\lambda_0)$, then there exists an invertible element $y \in A$ such that $\beta - f(a) = (\lambda_0 - a)y$.*

Proof. Let $g(\lambda) = \beta - f(\lambda)$. Then λ_0 is a zero of g in G. Since f is one-to-one on G, this is the only zero of g in G. Therefore, there exists an analytic function h on G such that $g(\lambda) = (\lambda_0 - \lambda)h(\lambda)$ for all $\lambda \in G$, with $h(\lambda) \neq 0$ for all $\lambda \in G\backslash\{\lambda_0\}$. Since f is one-to-one on G, it follows from [2, Problem 4, p. 100] that $g'(\lambda) = -f'(\lambda) \neq 0$ for all $\lambda \in G$. Therefore, $h(\lambda_0) = -g'(\lambda_0) \neq 0$, so that h has no zeros on G. It follows that $0 \notin h(\sigma(a)) = \sigma(h(a))$, so that $y = h(a)$ is invertible. By the holomorphic functional calculus $\beta - f(a) = g(a) = (\lambda_0 - a)y$, and the result follows. $\square$

THEOREM 3.13. *Let a be an element of a Banach algebra A and let f be a complex valued function which is analytic and one-to-one on a neighbourhood of $\sigma(a)$. Then $S_\partial(f(a)) = f(S_\partial(a))$.*

Proof. Let $\beta = f(\lambda_0)$, where $\lambda_0 \in S_\partial(a)$. By Theorem 3.12, we have $\beta - f(a) = (\lambda_0 - a)y$, for some invertible element $y \in A$. Since $\lambda_0 - a \in \partial S$, it follows from Lemma 2.1 that $\beta - f(a) \in \partial S$, so that $\beta \in S_\partial(f(a))$. This proves the inclusion $f(S_\partial(a)) \subseteq S_\partial(f(a))$. Together with Lemma 3.10, the result follows. $\square$

In view of Theorem 3.13 and Proposition 2.8(1), the boundary spectrum is a Mobius spectrum in the sense of Harte and Wickstead [5, Definition 2.1].

Clearly, Theorem 3.13 implies Propositions 2.4 and 2.5(2), and also strengthens the last part of Proposition 2.7(3).

It follows from Theorem 3.13 that if a is an element of a Banach algebra A such that the diameter of the spectrum of a is less than 2π, then $S_\partial(e^a) = e^{S_\partial(a)}$.

Finally, we remark that, despite a number of instances where the boundary $\partial\sigma$ of the spectrum and the boundary spectrum S_∂ behave similarly (see for instance Proposition 3.11 and Theorem 3.13), it does not seem obvious whether $\partial\sigma$ can in general be replaced by S_∂ in Proposition 2.6. (By Proposition 2.7, this is the case if $\sigma(a)$ has no interior points.) Hence, we have the following open question.

PROBLEM. *Let a be an element of a Banach algebra A and let f be a complex valued function which is analytic on a neighbourhood of $\sigma(a)$. Is*

$$S_\partial(f(a)) \subseteq f(S_\partial(a))?$$

4. Regularity-type properties

The *spectrum* $\sigma_R(a)$ of an element a in a Banach algebra A *relative to* any subset R of A is defined by $\sigma_R(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin R\}$. If we define $R_\partial = A^{-1} \cup \operatorname{int} S$, then $\sigma_{R_\partial}(a) = S_\partial(a)$.

In [6], [8], [12] and [13] the concepts of regularities and semiregularities were introduced. In particular, in [12] Müller defined a nonempty subset R of a Banach algebra A to be a *lower semiregularity* if

(i) $a \in A$, $n \in \mathbb{N}$, $a^n \in R \Rightarrow a \in R$,

(ii) if a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1$ and $ab \in R$, then $a, b \in R$.

If R is a lower semiregularity, then $A^{-1} \subseteq R$ [12, Lemma 2]. Also, by [12, Remark 3], a nonempty subset R of a Banach algebra A satisfying

$$a, b \in A, \qquad ab = ba, \qquad ab \in R \quad \Rightarrow \quad a, b \in R$$

is a lower semiregularity. Some of the results in Section 3 show that we have the following (weaker) result for the set R_∂.

THEOREM 4.1. *Let A be a Banach algebra and let $a, b \in A$.*

1. *If $a^n \in R_\partial$, then $a \in R_\partial$.*
2. *If $ab = ba$ and $ab \in R_\partial$, then $a \in R_\partial$ or $b \in R_\partial$.*

Proof. Let $a^n \in R_\partial$. If $a^n \in A^{-1}$, then $a \in A^{-1} \subseteq R_\partial$. If $a^n \in \operatorname{int} S$, then it follows from Corollary 3.6(1) that $a \notin \partial S$, and so $a \in R_\partial$, yielding (1). Towards (2), let $ab = ba$ and $ab \in R_\partial$. If $ab \in A^{-1}$, then $a, b \in A^{-1} \subseteq R_\partial$, so suppose that $ab \in \operatorname{int} S$. Then $ab \notin \partial S$, so that $a \notin \partial S$ or $b \notin \partial S$, by Corollary 3.5. Hence $a \in R_\partial$ or $b \in R_\partial$. □

By (ii) above and [12, Lemma 2(iii)] (see also [6, Proposition 1.3]) a lower semiregularity R satisfies the following property:

(P1) If $a,b\in A$, $ab=ba$ and $b\in A^{-1}$, then

$$ab\in R \quad\Leftrightarrow\quad a\in R \quad\text{and}\quad b\in R.$$

We show that R_∂ satisfies property (P1):

PROPOSITION 4.2. *Let A be a Banach algebra and let $a,b\in A$ be such that $ab=ba$ and $b\in A^{-1}$. Then $ab\in R_\partial$ if and only if $a\in R_\partial$ and $b\in R_\partial$.*

Proof. Suppose that $ab=ba$ and $b\in A^{-1}$. Let $a\in R_\partial$ (and $b\in R_\partial$). If $a\in A^{-1}$, then $ab\in A^{-1}\subseteq R_\partial$, so suppose that $a\in \operatorname{int} S$. If $ab\notin R_\partial$, then $ab\in\partial S$, so that $a\in\partial S$ by Lemma 2.1. This contradiction yields $ab\in R_\partial$. Conversely, let $ab\in R_\partial$. Since $b\in A^{-1}$, $b\in R_\partial$. If $a\notin R_\partial$, then $a\in\partial S$, so that $ab\in\partial S$, by Lemma 2.1. This contradiction yields $a\in R_\partial$. $\square$

Let R be a nonempty subset of a Banach algebra A. The concept of a *regularity* was defined in [6, Definition 1.2]. By [6, Remark 3], if R satisfies the property

(P2) $ab\in R\Leftrightarrow a\in R$ and $b\in R$ for all commuting elements $a,b\in A$,

then R is a regularity.

The following example will show that R_∂ does not have property (P2). For this purpose, let D denote the closed disk with center 0 and radius 1 in $\mathbb{C}$ and let $B=A(D)$ denote the Banach algebra of all continuous complex valued functions on D which are analytic in the interior of D.

EXAMPLE 4.3. *Let $A=B\times B$ and consider $R_\partial=A^{-1}\cup \operatorname{int}_A S_A$. If $a=(f,0)\in A$ and $b=(0,f)\in A$ where $f(z)=z$ for all $z\in D$, then $ab=ba$, $a\in R_\partial$ and $b\in R_\partial$, but $ab\notin R_\partial$.*

Proof. By Example 2.3, $0\notin S_\partial(f,B)$. Since $S_\partial(a,A)=\{\lambda\in\mathbb{C}:(\lambda-f,\lambda)\in\partial_A S_A\}$ and $S_\partial(b,A)=\{\lambda\in\mathbb{C}:(\lambda,\lambda-f)\in\partial_A S_A\}$, it is easily checked that $0\notin S_\partial(a,A)$ and $0\notin S_\partial(b,A)$. Therefore, $a\in R_\partial$ and $b\in R_\partial$. However, $ab=(0,0)\in\partial_A S_A$, so that $ab\notin R_\partial$. $\square$

We conclude by giving a characterization of the radical (Corollary 4.6) in terms of R_∂. Recall that, since $\lambda-a\in\exp(A)$ whenever $|\lambda|>\|a\|$ [1, Theorem 3.3.6], we have that $A=\operatorname{Exp}(A)+\operatorname{Exp}(A)$.

THEOREM 4.4. *Let A be a Banach algebra and let B be a subset of A containing* $\operatorname{Exp}(A)$. *Then* $\operatorname{Rad}(A)=\{a\in A: Ba\subseteq\operatorname{QN}(A)\}$.

Proof. For the nontrivial inclusion, suppose that $Ba\subseteq\operatorname{QN}(A)$ and let $b\in A$ with $b=c+d$ $(c,d\in\operatorname{Exp}(A))$. Since $ca\in Ba$, it follows from the assumption and [1, Theorem 3.3.6], that $1-ca\in\exp(A)$, say $1-ca=e^x$ where $x\in A$. Again, since $e^{-x}da\in Ba$, it follows that $1-e^{-x}da\in\exp(A)$, say $1-e^{-x}da=$

e^y, where $y \in A$. Therefore, $1 - ba = (1 - ca)(1 - e^{-x}da) = e^x e^y \in \mathrm{Exp}(A) \subseteq A^{-1}$. We have proved that $1 - Aa \subseteq A^{-1}$, so that $a \in \mathrm{Rad}(A)$. □

Theorem 4.4 includes [7, Proposition 4.1], and since $A^{-1} \subseteq R$ if R is a lower semiregularity, we have the following.

Corollary 4.5. *Let A be a Banach algebra and let R be a lower semiregularity. Then* $\mathrm{Rad}(A) = \{a \in A : Ra \subseteq \mathrm{QN}(A)\}$.

Since $A^{-1} \subseteq R_\partial$, we also have the following.

Corollary 4.6. *Let A be a Banach algebra. Then* $\mathrm{Rad}(A) = \{a \in A : R_\partial a \subseteq \mathrm{QN}(A)\}$.

5. Continuity properties

We recall from [1, pp. 48, 50], that the *Hausdorff distance* between two nonempty compact sets K_1 and K_2 in the complex plane is defined by

$$\Delta(K_1, K_2) = \max\Big\{ \sup_{\lambda \in K_1} d(\lambda, K_2), \sup_{\lambda \in K_2} d(\lambda, K_1) \Big\},$$

where $d(z, K)$ is the distance from a point z to a compact set K in $\mathbb{C}$. If also $r > 0$, then $K + r$ denotes the set $\{z \in \mathbb{C} : d(z, K) \leq r\}$. A function $F : X \to K(\mathbb{C})$ (with X a normed space) is *continuous at* $x \in X$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|y - x\| < \delta$, then $\Delta(F(y), F(x)) < \varepsilon$ and F is *upper semicontinuous* on X if for every $x \in X$ and every open set U containing $F(x)$ there exists a $\delta > 0$ such that if $\|y - x\| < \delta$ then U contains $F(y)$.

Let A be a Banach algebra. We shall write $r(a)$ for the spectral radius of an element $a \in A$. In general, the function $a \mapsto S_\partial(a)$ from A to $K(\mathbb{C})$ is not continuous, as illustrated by a well-known example by Kakutani [1, p. 49]. In the case of commuting elements, we have the following property.

Proposition 5.1. *Let a and b be commuting elements of a Banach algebra A. Then $\mathbb{C} \backslash \sigma(a) \cap S_\partial(b) \subseteq S_\partial(a) + r(a - b)$.*

Proof. If $\alpha \in \mathbb{C} \backslash \sigma(a) \cap S_\partial(b)$ is such that $d(\alpha, S_\partial(a)) > r(a - b)$, then $d(\alpha, \sigma(a)) > r(a - b)$, by [10, Proposition 2.1]. From the proof of [1, Theorem 3.4.1], it follows that $\alpha \notin \sigma(b)$, and since $S_\partial(b) \subseteq \sigma(b)$, we have a contradiction. □

A somewhat neater result holds in the context of an ordered Banach algebra [10, Theorem 3.8]. (See also [9, Theorems 4.2 and 4.12].)

Corollary 5.2. *Let a and b be commuting elements of a Banach algebra A such that $S_\partial(a) \cap \sigma(b) = \emptyset = S_\partial(b) \cap \sigma(a)$. Then*

$$\Delta(S_\partial(a), S_\partial(b)) \leq r(a - b) \leq \|a - b\|.$$

Since $S_\partial(a) \subseteq \sigma(a)$ for all $a \in A$ and $A \backslash \partial S$ is open, a standard argument yields the upper semicontinuity of the map $a \mapsto S_\partial(a)$:

THEOREM 5.3. *Let a be an element of a Banach algebra A and let U be an open set containing $S_\partial(a)$. Then there exists a $\delta > 0$ such that if $\|x - a\| < \delta$, then U contains $S_\partial(x)$.*

COROLLARY 5.4. *Let a be an element of a Banach algebra A and let U be an open set such that $\sigma(a) \cap \partial U = \emptyset$ and $S_\partial(a) \cap U \neq \emptyset$. Then there exists a $\delta > 0$ such that if $\|x - a\| < \delta$, then $S_\partial(x) \cap U \neq \emptyset$.*

The above corollary follows directly from [1, Theorems 3.4.2 and 3.4.4].

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