LEFSCHETZ ELEMENTS OF ARTINIAN GORENSTEIN ALGEBRAS AND HESSIANS OF HOMOGENEOUS POLYNOMIALS

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ABSTRACT. We give a characterization of the Lefschetz elements in Artinian Gorenstein rings over a field of characteristic zero in terms of the higher Hessians. As an application, we give new examples of Artinian Gorenstein rings which do not have the strong Lefschetz property.

0. Introduction

The Lefschetz property is a ring-theoretic abstraction of the Hard Lefschetz Theorem for compact Kähler manifolds (see e.g., [7]). The following are fundamental problems on the study of the Lefschetz property for Artinian graded algebras:

PROBLEM 0.1. For a given graded Artinian algebra A, decide whether or not A has the strong (or weak) Lefschetz property.

PROBLEM 0.2. When a graded Artinian algebra A has the strong Lefschetz property, determine the set of Lefschetz elements in A_1.

In [16], it was shown that "most" Artinian Gorenstein algebras have the strong Lefschetz property. However, it is a difficult problem to know whether a given graded Artinian algebra has the strong (or weak) Lefschetz property. In principle, if a graded Artinian algebra A over a field k is given with a presentation

$$A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m),$$

we have an algorithm to answer the above problems since it is sufficient to compute the determinants of the matrix expression of the multiplication map

by a general element of A_1 with respect to an arbitrary homogeneous linear basis of A. In particular, the complement of the set of Lefschetz elements in A_1 has a structure of the algebraic set defined by certain determinants. However, it is hard in general to carry out the computation based on this algorithm even with the help of computer.

In the present paper, we give a simple criterion to answer these problems for Artinian Gorenstein algebras over a field k of characteristic zero. It is known that a graded Artinian Gorenstein algebra is characterized by the "Poincaré duality" which holds for the cohomology ring of the compact oriented manifolds. Hence, graded Artinian Gorenstein algebras with the strong Lefschetz property are a natural class of commutative algebras comparable to the cohomology ring of compact Kähler manifolds.

A typical example of graded Artinian Gorenstein algebras is the coinvariant algebra of finite Coxeter groups. In fact, the coinvariant algebra of the Weyl group is isomorphic to the cohomology ring of the corresponding flag variety. In [11] and [12], it has been shown that the coinvariant algebra of any finite Coxeter group has the Lefschetz property and that the set of the Lefschetz elements is the complement of the union of the reflection hyperplanes except for type H_4 case. The determination of the set of the Lefschetz elements is still open for H_4 because of the computational complexity.

Let us consider the polynomial ring $R = k[x_1, \ldots, x_n]$ and the algebra of differential operators

$$Q = k\left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right].$$

Every graded Artinian Gorenstein algebra has the presentation

$$A \cong Q/\operatorname{Ann}_Q F, \quad \operatorname{Ann}_Q F = \{\varphi(\partial_1, \ldots, \partial_n) \in Q | \varphi(\partial_1, \ldots, \partial_n) F(x) = 0\}$$

for some homogeneous polynomial $F \in k[x_1, \ldots, x_n]$. We introduce the higher Hessians $\operatorname{Hess}^{(d)} F$, $1 \leq d \leq [\deg F/2]$, of the polynomial F in order to describe the condition for an element $L \in A_1$ to be a strong Lefschetz element. The set of the strong Lefschetz elements of A is a Zariski open set in A_1, which is given as the complement of all the zero loci of the higher Hessians. We will discuss the explicit description of the set of Lefschetz elements of $A = Q/\operatorname{Ann}_Q F$ for the Fermat type polynomial $F = \sum_{i=1}^n x_i^n - n(n-1)s \prod_{i=1}^n x_i$.

When one of the higher Hessians of F is identically zero, the algebra $Q/\operatorname{Ann}_Q F$ does not have the strong Lefschetz property. In [8], [10] and [17], examples of Artinian Gorenstein algebras which do not have the strong Lefschetz property are given. The examples in [8] and [17] are based on the polynomials with the zero Hessian. In the last section, we give some polynomials F such that $\operatorname{Hess} F \neq 0$ and $\operatorname{Hess}^{(2)} F = 0$ to get new examples of Artinian Gorenstein algebras which do not have the strong Lefschetz property.

1. Lefschetz properties

DEFINITION 1.1. Let $A = \bigoplus_{d=0}^{D} A_d$, $A_D \neq 0$, be a graded Artinian algebra.

(1) We say that A has the strong Lefschetz property if there exists an element $L \in A_1$ such that the multiplication map

$$\times L^d : A_i \to A_{i+d}$$

is of full rank (i.e., injective or surjective) for all $0 \leq i \leq D$ and $0 \leq d \leq D - i$. We call $L \in A_1$ with this property a strong Lefschetz element.

(2) If we assume the existence of $L \in A_1$ such that

$$\times L : A_i \to A_{i+1}$$

is of full rank for $i = 0, \ldots, D - 1$, we say that A has the weak Lefschetz property.

If a graded Artinian algebra A over a field k is generated by A_1 as a k-algebra, we say that A has the standard grading. The weak Lefschetz property implies the unimodality of the Hilbert function, provided that the k-algebra A has the standard grading.

DEFINITION 1.2. Let $A = \bigoplus_{d=0}^{D} A_d$, $A_D \neq 0$, be a graded Artinian algebra. We say that A has the strong Lefschetz property in the narrow sense if there exists an element $L \in A_1$ such that the multiplication map

$$\times L^{D-2i} : A_i \to A_{D-i}$$

is bijective for $i = 0, \ldots, [D/2]$.

If a graded Artinian k-algebra A has the strong Lefschetz property in the narrow sense, then the Hilbert function of A is unimodal and symmetric. When a graded Artinian k-algebra A has a symmetric Hilbert function, the notion of the strong Lefschetz property on A coincides with the one in the narrow sense. Our main interest in this paper is to consider Artinian *Gorenstein* algebras, so the strong Lefschetz property will be used in the narrow sense in the subsequent sections. Throughout this paper, graded Artinian k-algebras $A = \bigoplus_{d=0}^{D} A_d$ are assumed to satisfy the conditions $A_0 \cong k$ and $\dim_k A_1, \dim_k A_D > 0$.

2. Artinian Gorenstein algebra

Throughout, k denotes a field.

DEFINITION 2.1 (See [13, Section 6.5]). A finite-dimensional graded k-algebra $A = \bigoplus_{d=0}^{D} A_d$ is called the Poincaré duality algebra if $\dim_k A_D = 1$ and the bilinear pairing

$$A_d \times A_{D-d} \to A_D \cong k$$

is non-degenerate for $d = 0, \ldots, [D/2]$.

The following is a well-known fact (see e.g., [4]).

PROPOSITION 2.1. *A graded Artinian k-algebra A is a Poincaré duality algebra if and only if A is Gorenstein.*

Proof. Assume that $A = \bigoplus_{d=0}^{D} A_d$ is a Poincaré duality algebra. For any element $f \in A \setminus \{0\}$ of degree less than D, there exisits an element $g \in A \setminus A_0$ such that $fg \neq 0$. Hence, the socle ideal $\mathrm{Soc}(A)$ of A coincides with a one-dimensional k-subspace A_D. This means that A is Gorenstein. Conversely, if A is Gorenstein, the socle ideal $\mathrm{Soc}(A)$ is the one-dimensional k-vector space. Since the maximal degree part A_D is contained in $\mathrm{Soc}(A)$, we have that $A_D = \mathrm{Soc}(A)$ and $\dim_k A_D = 1$. We will prove the following claim by induction on d:

$(*)_d$ if $f \in A_{D-d}$ satisfies $fg = 0$ for all $g \in A_d$, then $f = 0$.

If $f \in A_{D-1}$ satisfies $fg = 0$ for all $g \in A_1$, then $fg = 0$ for all $g \in A_{>0}$ for degree reasons. This implies that $f \in A_{D-1} \cap \mathrm{Soc}(A) = 0$, so $(*)_1$ follows. Let us assume that a nonzero element $f \in A_{D-d} \setminus \{0\}$, $d > 1$, satisfies $fg = 0$ for all $g \in A_d$, and that there exists an element $h \in A_i$, $1 \leq i < d$, such that $\varphi := fh \neq 0$. By the induction hypothesis $(*)_{d-i}$, we can find an element $h' \in A_{d-i}$ such that $\varphi h' \neq 0$ for the nonzero element $\varphi \in A_{D-d+i}$. Then we have $\varphi h' = f(hh') \neq 0$, which is a contradiction since $hh' \in A_d$. We have proved that if $f \in A_{D-d}$ satisfies $fg = 0$ for all $g \in A_d$, then we have $f = 0$ by contradiction. Now the claim $(*)_d$ is proved. The claims $(*)_d$ for $d = 1, \ldots, D$ imply that the pairing $A_d \times A_{D-d} \to A_D$ is non-degenerate for $d = 1, \ldots, [D/2]$. $\qquad \square$

REMARK 2.1. (1) The above proposition shows that the even part of the cohomology ring $H^{\mathrm{even}}(M, k)$ with coefficient in a field k of characteristic zero of any compact orientable manifold M of even dimension is Gorenstein.

(2) The Poincaré duality algebra is an abstraction of the property of the cohomology ring of compact orientable manifolds, whereas the strong Lefschetz property is inspired by the Hard Lefschetz Theorem for compact Kähler manifolds. Though the Kähler manifold is always oriented, the strong Lefschetz property does not imply the Poincaré duality. In other words, there exist examples of graded Artinian *non*-Gorenstein algebras with the strong Lefschetz property. For example, $A = k[x, y]/(x^2, xy, y^3)$ is a non-Gorenstein algebra with the strong Lefschetz property. At the same time, the Poincaré duality does not imply the strong Lefschetz property. See Example 2.1 and Section 5.

For simplicity, we assume that the characteristic of the field k to be zero in the rest of this paper, though our main results hold also when the characteristic of k is greater than the socle degree D of the Gorenstein algebra A. Let us regard the polynomial algebra $R := k[x_1, \ldots, x_n]$ as a module over the algebra $Q := k[X_1, \ldots, X_n]$ via the identification $X_i = \partial/\partial x_i$. For a polynomial $F \in R$,

we define the ideal $\operatorname{Ann}_Q F$ of Q by

$$\operatorname{Ann}_Q F := \{a(X_1, \ldots, X_n) \in Q | a(\partial_1, \ldots, \partial_n) F = 0\}.$$

The following theorem is a well-established fact among the experts. In fact, it is immediate if the theory of the inverse system is taken for granted ([2], [3], [6]). However, the theory of inverse system does not seem to be well-known to nonspecialists, so we give a direct proof for it.

THEOREM 2.1. *Let I be an ideal of $Q = k[X_1, \ldots, X_n]$ and $A = Q/I$ the quotient algebra. Denote by $\mathfrak{m}$ the maximal ideal $(X_1, \ldots, X_n)$ of Q. Then $\sqrt{I} = \mathfrak{m}$ and the k-algebra A is Gorenstein if and only if there exists a polynomial $F \in R = k[x_1, \ldots, x_n]$ such that $I = \operatorname{Ann}_Q F$.*

Proof. Assume that $I = \operatorname{Ann}_Q F$ for some polynomial $F \in R$. Since F is annihilated by differential operators of sufficiently high order, $\operatorname{Ann}_Q F$ contains $\mathfrak{m}^l$ for sufficiently large l. Since we are working over a field k of characteristic zero, it is clear that there exists a polynomial $G \in Q$ such that $G(X)F(x) \in k^\times$. We will show in the following that $I : \mathfrak{m} = I + k \cdot G$. Since $G(X)F(x)$ is a constant, it immediately follows that

$$\partial_i G(\partial_1, \ldots, \partial_n) F(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, n.$$

This shows that $G \in I : \mathfrak{m}$. Now let $a(X_1, \ldots, X_n) \in I : \mathfrak{m}$ be any element. By definition of the ideal $I : \mathfrak{m}$, we have $X_i a(X_1, \ldots, X_n) \in I = \operatorname{Ann}_Q F$ for all $i = 1, \ldots, n$. This means that

$$\partial_i a(\partial_1, \ldots, \partial_n) F(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, n.$$

Hence, we have that $a(X)F(x)$ is a constant. As we have already seen that $G(X)F(x)$ is a nonzero constant, we have that $a(X) - cG(X) \in \operatorname{Ann}_Q F$ for some constant $c \in k$. We have shown that $I : \mathfrak{m} = I + k \cdot G$. In other words, the k-vector space $(I : \mathfrak{m})/I$ is one-dimensional. Thus, $A = Q/\operatorname{Ann}_Q F$ is a Gorenstein algebra (see e.g., [2]).

Now let us prove the converse implication. Assume that $A = Q/I$ is an Artinian Gorenstein algebra. Then we have the isomorphism $\operatorname{Hom}_k(A, k) \cong A$ as an A-module. The Q-module $\operatorname{Hom}_k(Q, k)$ is identified with the ring of formal power series $\hat{R} = k[[x_1, \ldots, x_n]]$ regarded as a Q-module. From the exact sequence $Q \to A \to 0$, we have the exact sequence of Q-modules:

$$0 \to \operatorname{Hom}_k(A, k) \cong A \xrightarrow{\theta} \operatorname{Hom}_k(Q, k) \cong k[[x_1, \ldots, x_n]].$$

Define $F \in k[[x_1, \ldots, x_n]]$ as the image of $1 \in A$ by the homomorphism θ. From the assumption that I contains $\mathfrak{m}^l$ for $l \gg 0$, the image of θ annihilates polynomials in Q of sufficiently large degrees, so F is a polynomial in R. Finally, we have that

$$\operatorname{Ann}_Q F = \{a \in Q | ab \in I, \forall b \in Q\} = I,$$

so $A = Q/\operatorname{Ann}_Q F$. $\qquad\qquad\qquad\square$

REMARK 2.2. Let k be the field $\mathbf{C}$ of complex numbers. In this case, for a polynomial

$$F = \sum c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}, \quad c_{i_1 \cdots i_n} \in \mathbf{C},$$

we can choose the complex conjugate

$$\overline{F} = \sum \overline{c}_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$$

as a generator of the socle of $A = Q/\operatorname{Ann}_Q F$.

REMARK 2.3. When I is a homogeneous ideal (i.e., A is graded), the condition $\sqrt{I} = \mathfrak{m}$ is satisfied. In this case, we can choose F as a homogeneous polynomial.

EXAMPLE 2.1. Stanley [14] gave an example of Artinian Gorenstein algebra with a non-unimodal Hilbert function. Let us take the polynomial

$$F(u, v, w, x_1, \ldots, x_{10}) = \sum_{i=1}^{10} x_i M_i(u, v, w) \in k[u, v, w, x_1, \ldots, x_{10}],$$

where $M_1(u, v, w), \ldots, M_{10}(u, v, w)$ are monomials in u, v and w of degree 3 in an arbitrary ordering. Stanley's example is given as $A = Q/\operatorname{Ann}_Q F$ corresponding to the polynomial F defined above. The algebra A has the Hilbert function $(1, 13, 12, 13, 1)$, so it does not have the strong or weak Lefschetz property. More generally, it is shown in [1] and [9] that there exist Artinian Gorenstein algebras A with a non-unimodal Hilbert function for $\dim A_1 \geq 5$. In Section 5, we will construct Artinian Gorenstein algebras with a unimodal Hilbert function which do not have the strong Lefschetz property.

3. Characterization of Lefschetz elements

In this section, we discuss the set of the Lefschetz elements for graded Artinian Gorenstein rings $A = k[X_1, \ldots, X_n]/\operatorname{Ann}_Q F$ with the standard grading.

DEFINITION 3.1. Let G be a polynomial in $k[x_1, \ldots, x_n]$. When a family $\mathbf{B}_d = \{\alpha_i^{(d)}\}_i$ of homogeneous polynomials of degree $d > 0$ is given, we call the polynomial

$$\det\left(\left(\alpha_i^{(d)}(X)\alpha_j^{(d)}(X)G(x)\right)_{i,j=1}^{\#\mathbf{B}_d}\right) \in k[x_1, \ldots, x_n]$$

the dth Hessian of G with respect to $\mathbf{B}_d$, and denote it by $\operatorname{Hess}_{\mathbf{B}_d}^{(d)} G$. We denote the dth Hessian simply by $\operatorname{Hess}^{(d)} G$ if the choice of $\mathbf{B}_d$ is clear.

When $d = 1$ and $\alpha_j^{(1)}(X) = X_j$, $j = 1, \ldots, n$, the first Hessian $\operatorname{Hess}^{(1)} G$ coincides with the usual Hessian:

$$\operatorname{Hess}^{(1)} G = \operatorname{Hess} G := \det\left(\frac{\partial^2 G}{\partial x_i \, \partial x_j}\right)_{ij}.$$

Let us consider the case $A = Q/\operatorname{Ann}_Q F$ and the higher Hessians of F with respect to a k-linear basis $\mathbf{B}_d = \{\alpha_i^{(d)}\}$ of A_d. Note that if we change the k-linear basis of A_d, the corresponding higher Hessians $\operatorname{Hess}_{\mathbf{B}_d}^{(d)}$ are just multiplied by nonzero scalars in $k^\times$.

THEOREM 3.1 ([17, Theorem 4]). *Fix an arbitrary k-linear basis $\mathbf{B}_d$ of A_d for $d = 1, \ldots, [D/2]$. An element $L = a_1 X_1 + \cdots + a_n X_n \in A_1$ is a strong Lefschetz element of $A = Q/\operatorname{Ann}_Q F$ if and only if $F(a_1, \ldots, a_n) \neq 0$ and*

$$\left(\operatorname{Hess}_{\mathbf{B}_d}^{(d)} F\right)(a_1, \ldots, a_n) \neq 0$$

for $d = 1, \ldots, [D/2]$.

Proof. Define the identification $[\cdot] : A_D \xrightarrow{\sim} k$ by $[\omega(X)] := \omega(X)F(x)$ for any $\omega(X) \in A_D$. Note that $\omega(X)F(x) \in k$, because $\deg \omega = \deg F = D$. Since A is a Poincaré duality algebra, the necessary and sufficient condition for $L = a_1 X_1 + \cdots + a_n X_n \in A_1$ to be a strong Lefschetz element is that the bilinear pairing

$$A_d \times A_d \to A_D \cong k,$$
$$(\xi, \eta) \mapsto L^{D-2d} \xi \eta \mapsto [L^{D-2d} \xi \eta]$$

is non-degenerate for $d = 0, \ldots, [D/2]$. Therefore, L is a Lefschetz element if and only if the matrix

$$\left(L^{D-2d} \alpha_i^{(d)}(X) \alpha_j^{(d)}(X) F(x)\right)_{ij}$$

has nonzero determinant. For a homogeneous polynomial $G(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ of degree d, we have the formula

$$(a_1 X_1 + \cdots + a_n X_n)^d G(x_1, \ldots, x_n) = d! G(a_1, \ldots, a_n),$$

so

$$L^{D-2d} \alpha_i^{(d)}(X) \alpha_j^{(d)}(X) F(x)$$
$$= (D - 2d)! \alpha_i^{(d)}(X) \alpha_j^{(d)}(X) F(x)|_{(x_1, \ldots, x_n) = (a_1, \ldots, a_n)}. \quad \square$$

COROLLARY 3.1. (1) *The algebra $A = Q/\operatorname{Ann}_Q F$ has the strong Lefschetz property if and only if all the higher Hessians $\operatorname{Hess}_{\mathbf{B}_d}^{(d)} F$ with respect to a k-linear basis $\mathbf{B}_d$ of A_d, $d = 1, \ldots, [D/2]$, are nonzero polynomials.*

(2) *Assume that the socle degree of A is less than 5. An element $L = a_1 X_1 + \cdots + a_n X_n$ is a strong Lefschetz element if and only if*

$$F(a_1, \ldots, a_n) \neq 0 \quad and \quad \operatorname{Hess} F(a_1, \ldots, a_n) \neq 0.$$

Here, $\operatorname{Hess} F$ is the first Hessian of F with respect to a linear basis of A_1.

4. Set of Lefschetz elements

In this section, we discuss the set of Lefschetz elements for some simple examples of Gorenstein algebras with the strong Lefschetz property based on Corollary 3.1.

EXAMPLE 4.1. Let us consider the Gorenstein ring $A = k[X_1, \ldots, X_n]/$ $\mathrm{Ann}_Q F$ associated to the Fermat type polynomial

$$F = \sum_{i=1}^{n} x_i^n - n(n-1)s \prod_{i=1}^{n} x_i,$$

where $s \in k$ is a parameter. One can check that A has the strong Lefschetz property for any $s \in k$ as follows. For $s = 0$, it is easy to see that A has the strong Lefschetz property. When $s \neq 0$, the monomials $\alpha_1 := x_1^d, \ldots, \alpha_n := x_n^d$ and

$$\alpha_I := \prod_{i \in I} x_i, \quad I \subset \{1, \ldots, n\}, \#I = d$$

form a linear basis $\mathbf{B}_d$ of A_d for $d > 1$. The matrix $M^{(d)} = (\alpha \beta F)_{\alpha, \beta \in \mathbf{B}_d}$ is of form

$$M^{(d)} = \begin{pmatrix} M_1^{(d)} & 0 \\ 0 & M_2^{(d)} \end{pmatrix},$$

where $M_1^{(d)}$ is a diagonal matrix of size n with $\det M_1^{(d)} \neq 0$, and $M_2^{(d)}$ is a matrix of size $\binom{n}{d}$. Let us consider the monomial $G := x_1 \cdots x_n$ and the corresponding algebra $A' := Q/\mathrm{Ann}_Q G$. Then we have

$$A' \cong k[X_1, \ldots, X_n]/(X_1^2, \ldots, X_n^2),$$

so A' has the strong Lefschetz property. Thus, the dth Hessian $\mathrm{Hess}^{(d)} G$ with respect to the linear basis $\{\alpha_I | \#I = d\}$ is nonzero. Since

$$\det M_2^{(d)} = \left(-n(n-1)s\right)^{\binom{n}{d}} \cdot \mathrm{Hess}^{(d)} G \neq 0,$$

we have $\mathrm{Hess}_{\mathbf{B}_d}^{(d)} F \neq 0$. Hence, A has the strong Lefschetz property.

We give the explicit condition for the Lefschetz element for $n = 3, 4$. For $n = 3$ and $F = x^3 + y^3 + z^3 - 6s \cdot xyz$, A has the following structure:

Case $s^3 \neq 0, 1,$ $\quad A \cong k[X, Y, Z]/(sX^2 + YZ, sY^2 + XZ, sZ^2 + XY),$

Case $s = 0,$ $\quad A \cong k[X, Y, Z]/(X^3 - Y^3, X^3 - Z^3, XY, YZ, XZ),$

Case $s^3 = 1,$ $\quad A \cong k[X, Y, Z]/(X^2 + YZ, Y^2 + XZ, Z^2 + XY, XZ^2, YZ^2).$

Note that A is a complete intersection for $s^3 \neq 0, 1$. The Hilbert function of A is $\mathrm{Hilb}(A) = (1, 3, 3, 1)$ for all $s \in k$. The condition for $L = aX + bY + cZ \in A_1$ to be a strong Lefschetz element is that

$$a^3 + b^3 + c^3 - 6s \cdot abc \neq 0$$

and
$$s^2 a^3 + s^2 b^3 + s^2 c^3 - (1 - 2s^3)abc \neq 0.$$
It is remarkable that the above condition is described by a single condition
$$a^3 + b^3 + c^3 - 6s \cdot abc \neq 0$$
for $s = 1/2, (-1 \pm \sqrt{-3})/4$. This is exactly when F decomposes into the product of three linear forms.

For $n = 4$ and $F = x^4 + y^4 + z^4 + w^4 - 12s \cdot xyzw$, the Hilbert function of A is as follows:
$$\text{Hilb}(A) = (1, 4, 4, 4, 1) \quad \text{for } s = 0,$$
$$\text{Hilb}(A) = (1, 4, 10, 4, 1) \quad \text{for } s \neq 0.$$
In this case, the condition for $L = aX + bY + cZ + dW \in A_1$ to be a strong Lefschetz element is that
$$a^4 + b^4 + c^4 + d^4 - 12s \cdot abcd \neq 0$$
and
$$(1 - 2s^3 - s^4)a^2 b^2 c^2 d^2 - 2s^3 \cdot \text{symm}(a^5 bcd) - s^2 \cdot \text{symm}(a^4 b^4) \neq 0,$$
where $\text{symm}(\cdot)$ means the symmetrization of the indicated monomial.

EXAMPLE 4.2. Stanley [15] studied the strong Lefschetz property of the coinvariant algebra of finite Coxeter groups to show the Sperner property for the Bruhat ordering on finite Coxeter groups. In [11], the set of the Lefschetz elements for the coinvariant algebra of the finite Coxeter group is determined except for type H_4. Let V be the standard reflection representation of the finite irreducible Coxeter group W. Then W acts on the polynomial ring $R = \text{Sym}_{\mathbf{R}} V^*$ and the W-invariant subalgebra R^W is generated by the fundamental W-invariants $f_1, \ldots, f_r$, $r = \dim V$. The coinvariant algebra R_W is defined as the quotient algebra $R/(f_1, \ldots, f_r)$. It is known that R_W is Gorenstein (see e.g., [13, Theorem 7.5.1]). When W is crystallographic, R_W is isomorphic to the cohomology ring of the corresponding flag variety. In [11], it was shown that the set of Lefschetz elements in $V^* = (R_W)_1$ is the complement of the union of the reflection hyperplanes. For crystallographic case, their argument is based on the ampleness criterion for the $\mathbf{R}$-divisors on the flag variety, so it is applicable only when the field k of coefficients is the field $\mathbf{R}$ of real numbers.

Let us consider the case $W = S_3$ and
$$R_W = \mathbf{R}[X, Y, Z]/(X + Y + Z, XY + YZ + ZX, XYZ).$$
The algebra R_W is also given by $R_W = \mathbf{R}[X, Y, Z]/\text{Ann}\,\Delta$ with $\Delta = (x - y)(x - z)(y - z)$. The degree one part $(R_W)_1$ has a linear basis $\mathbf{B}_1 = \{X, Y\}$. Then we have
$$\text{Hess}_{\mathbf{B}_1}^{(1)} \Delta = -4(x^2 + y^2 + z^2 - xy - yz - zx),$$

which is a negative definite quadratic form. Hence, the set of the Lefschetz elements is given by

$$\{(x, y, z)|\Delta(x, y, z) \neq 0\} \subset V^*.$$

If we work in $V_{\mathbf{C}}^*$, we have to take care of the condition $x^2 + y^2 + z^2 - xy - yz - zx \neq 0$, too.

5. Gorenstein algebras which do not have the strong Lefschetz property

The result in Section 3 shows that a polynomial F gives an example of Gorenstein algebra which does not have the strong Lefschetz property if one of the higher Hessians of F is identically zero. In [8, Section 4] and [17], some examples of F with the zero Hessian are discussed. In [5], it is proved, among other things, that the Hessian of a polynomial in 4 variables does not vanish, unless a variable can be eliminated by means of a linear transformation of the variables. The polynomial $F = x_0 u^2 + x_1 uv + x_2 v^2$ is the simplest example whose Hessian vanishes, but no variables can be eliminated by a linear transformation of the variables (see [17, Example 1]).

Here, we give examples of forms F such that $\operatorname{Hess} F \neq 0$ and $\operatorname{Hess}^{(2)} F = 0$. By using these forms, we can also give examples of Gorenstein algebras $A = Q/\operatorname{Ann}_Q F$ which do not satisfy the strong Lefschetz property.

EXAMPLE 5.1. Let us consider the polynomial

$$F := \sum_{j=0}^{n} x_j^2 u^{n-j} v^j \in k[u, v, x_0, \ldots, x_n]$$

and the corresponding algebra $A = Q/\operatorname{Ann}_Q F$, where $Q = k[U, V, X_0, \ldots, X_n]$, $U = \partial/\partial u$, $V = \partial/\partial v$ and $X_i = \partial/\partial x_i$. Linear bases of A_1 and A_2 are given in the following table:

	Linear basis
$\dim A_1 = n + 3$	$U, V, X_0, \ldots, X_n$
$\dim A_2 = 3n + 4$	$\alpha_1 := U^2, \alpha_2 := UV, \alpha_3 := V^2,$
	$\alpha_4 := UX_0, \ldots, \alpha_{n+3} := UX_{n-1},$
	$\alpha_{n+4} := VX_1, \ldots, \alpha_{2n+3} := VX_n,$
	$\alpha_{2n+4} := X_0^2, \ldots, \alpha_{3n+4} := X_n^2$

It is easy to see that $\dim A_0 = \dim A_{n+2} = 1$, $\dim A_1 = \dim A_{n+1} = n + 3$ and $\dim A_d = (d + 1)n - d^2 + 2d + 4$ for $2 \leq d \leq n$. The Hessian of F with

respect to the basis above is expressed as follows:

$$\text{Hess } F = 2^{n+1}(uv)^{\frac{n(n-1)}{2}}$$

$$\times \left\{ \left(\sum_{j=0}^{n-1} (n-j)(n-j+1)x_j^2 u^{n-j-1} v^j \right) \left(\sum_{j=1}^{n} j(j+1)x_j^2 u^{n-j} v^{j-1} \right) \right.$$

$$\left. - uv \left(\sum_{j=1}^{n-1} j(n-j)x_j^2 u^{n-j-1} v^{j-1} \right)^2 \right\} \neq 0.$$

On the other hand, we have

$$X_i^2 U^2 F = 2(n-i)(n-i-1)u^{n-i-2}v^i,$$
$$X_i^2 UVF = 2(n-i)iu^{n-i-1}v^{i-1},$$
$$X_i^2 V^2 F = 2i(i-1)u^{n-i}v^{i-2}$$

and $X_i^2 \alpha_j(X)F = 0$ for $i = 0, \ldots, n$ and $j \geq 4$, so the vectors

$$\vec{\xi_i} := (X_i^2 \alpha_1(X)F, X_i^2 \alpha_2(X)F, \ldots, X_i^2 \alpha_{3n+4}(X)F), \quad i = 0, \ldots, n,$$

are linearly dependent. Hence, we can see that the second Hessian is identically zero, i.e., $\text{Hess}^{(2)} F = 0$ in $k[u, v, x_0, \ldots, x_n]$. This means that the algebra A does not have the strong Lefschetz property.

For $n = 3$, the algebra $A = Q/\text{Ann}_Q F$ has the Hilbert function $(1, 6, 13, 13, 6, 1)$. Since the multiplication map $\times L : A_2 \to A_3$ cannot be bijective for all $L \in A_1$, A does not have the weak Lefschetz property either.

EXAMPLE 5.2. There exists an example of a polynomial F of degree 5 with 5 variables such that $\text{Hess } F \neq 0$ and $\text{Hess}^{(2)} F = 0$. Let us choose

$$F = x^2 u^3 + xyu^2 v + y^2 uv^2 + z^2 v^3 \in k[u, v, x, y, z].$$

Then

$$U, V, X, Y, Z \in A = k[U, V, X, Y, Z]/\text{Ann}_Q F$$

are linearly independent. So we have

$$\text{Hess } F = 48u^3 v^3 (u^5 x^4 + 8u^4 vx^3 y + 16u^3 v^2 x^2 y^2 + 19u^2 v^3 x^2 z^2$$
$$+ 9u^2 v^3 xy^3 + 13uv^4 xyz^2 + 2uv^4 y^4 + 4v^5 y^2 z^2) \neq 0.$$

The monomials

$$\begin{array}{llll}
\alpha_1 = U^2, & \alpha_2 = V^2, & \alpha_3 = UV, & \alpha_4 = X^2, \\
\alpha_5 = Y^2, & \alpha_6 = Z^2, & \alpha_7 = XY, & \alpha_8 = UX, \\
\alpha_9 = UY, & \alpha_{10} = VX, & \alpha_{11} = VY, & \alpha_{12} = VZ
\end{array}$$

form a linear basis of A_2. We have

$$(\alpha_4 \alpha_1 F, \alpha_4 \alpha_2 F, \alpha_4 \alpha_3 F) = (12u, 0, 0),$$
$$(\alpha_5 \alpha_1 F, \alpha_5 \alpha_2 F, \alpha_5 \alpha_3 F) = (0, 4v, 4u),$$

$$(\alpha_6\alpha_1 F, \alpha_6\alpha_2 F, \alpha_6\alpha_3 F) = (0, 0, 12v),$$
$$(\alpha_7\alpha_1 F, \alpha_7\alpha_2 F, \alpha_7\alpha_3 F) = (2v, 2u, 0)$$

and $\alpha_i\alpha_j F = 0$ for $i = 4, 5, 6, 7$ and $j \geq 4$. Hence, the vectors

$$\vec{\xi_i} = (\alpha_i\alpha_1 F, \ldots, \alpha_i\alpha_{12} F), \quad j = 4, 5, 6, 7,$$

are linearly dependent and $\mathrm{Hess}^{(2)} F = 0$. The algebra $A = Q/\mathrm{Ann}_Q F$ has the Hilbert function $(1, 5, 12, 12, 5, 1)$. Since we have $\mathrm{Hess}^{(2)} F = 0$, the algebra A does not have the weak Lefschetz property.

EXAMPLE 5.3. The following example is due to Ikeda [10]. Let us choose the polynomial $F = w^3xy + wx^3z + y^3z^2$. Then the corresponding algebra $A = Q/\mathrm{Ann}_Q F$ has the Hilbert function $(1, 4, 10, 10, 4, 1)$. We choose the linear bases X, Y, Z, W of A_1 and

$$\alpha_1 = W^2, \qquad \alpha_2 = X^2, \qquad \alpha_3 = Y^2, \qquad \alpha_4 = Z^2, \qquad \alpha_5 = WX,$$
$$\alpha_6 = WY, \qquad \alpha_7 = WZ, \qquad \alpha_8 = XY, \qquad \alpha_9 = XZ, \qquad \alpha_{10} = YZ$$

of A_2. The Hessian is given as follows:

$$\mathrm{Hess}\, F = 8(3w^7xy^4 + 8w^6x^6 - 27w^5x^3y^3z + 27w^4y^6z^2$$
$$- 45w^3x^5y^2z^2 - 54w^2x^2y^5z^3 + 9wx^7yz^3 + 27x^4y^4z^4).$$

It is easy to check that the four vectors $(\alpha_6\alpha_i F)_i, (\alpha_7\alpha_i F)_i, (\alpha_8\alpha_i F)_i, (\alpha_9\alpha_i F)_i$ are linearly dependent, so the second Hessian $\mathrm{Hess}^{(2)} F$ is identically zero.

REMARK 5.1. In the above examples, we see that $\dim A_1$ takes each value greater than 3. It is known that if $\dim A_1 = 2$, the Artinian Gorenstein algebra A with the standard grading has the strong Lefschetz property [8, Proposition 4.4], [9, Theorem 2.9]. It is still open whether the Artinian Gorenstein algebra with $\dim A_1 = 3$ has the strong (or weak) Lefschetz property.

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