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UNCERTAINTY PRINCIPLES FOR COMPACT GROUPS

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ABSTRACT. We establish an uncertainty principle over arbitrary compact groups, generalizing several previous results. Specifically, we show that if P and R are operators on $L^2(G)$ such that P commutes with projection onto every measurable subset of G and R commutes with left-multiplication by elements of G, then $\| \mathrm{PR} \| \leq \|\mathrm{P} \cdot \chi_G\|_2 \|\mathrm{R}\|_2$, where $\chi_G : g \mapsto 1$ is the characteristic function of G. As a consequence, we show that every nonzero function f in $L^2(G)$ satisfies $\mu(\mathbf{supp}\, f) \cdot \sum_{\rho \in \hat{G}} d_\rho \, \mathbf{rank}\, \hat{f}(\rho) \geq 1$.

1. Introduction

Uncertainty principles assert, roughly, that *a function and its Fourier transform cannot simultaneously be highly concentrated.* One example of this is the well-known Heisenberg uncertainty principle concerning position and momentum wavefunctions in quantum physics. Several uncertainty principles have been formulated for complex-valued functions on groups. For finite Abelian groups, perhaps the most basic of these is an inequality which relates the sizes of the supports of f and its transform $\hat{f}$ to the size of the group. It states that

$$|\,\mathbf{supp}\, f||\,\mathbf{supp}\, \hat{f}| \geq |G|, \tag{1.1}$$

unless f is identically zero [13]. One can generalize this inequality by establishing an analogous statement concerning the associated projection operators on the group algebra $\mathbb{C}G$. Specifically, let P be the operator which, when expressed in the group basis, projects to the support of f; let R be the operator which, when expressed in the Fourier basis, projects to the support of $\hat{f}$. Since the operator norm of PR is equal to 1, we can rewrite (1.1) as

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$\|\mathrm{PR}\|^2 \leq |\mathbf{supp}\, f||\mathbf{supp}\, \hat{f}|/|G|$. It is natural to ask if a similar fact holds for *any* pair P_S and R_T, where P_S projects (in the group basis) to $S \subset G$ and R_T projects (in the Fourier basis) to $T \subset \widehat{G}$. Indeed, a generalized uncertainty principle for finite Abelian groups states that

$$\|\mathrm{P}_S \mathrm{R}_T\|^2 \leq \frac{|S||T|}{|G|}. \tag{1.2}$$

The above principles were proved for the real and finite cyclic case by Donoho and Stark [3] and extended to locally compact Abelian groups by Smith [12]. See also Terras's presentation [13] of these results in the finite Abelian case, and the excellent survey by Folland and Sitaram [6].

In this article, we study generalizations of the above bounds to general compact groups. An immediate difficulty in generalizing estimates such as (1.1) and (1.2) to nonAbelian groups is to settle upon an appropriate interpretation of $|\mathbf{supp}\, \hat{f}|$, as the Fourier transform is now a collection of linear operators. We show that a natural analogue of (1.2) still holds when $\widehat{G}$ is the collection of irreducible representations of a compact group G. We also refine this bound by establishing an uncertainty principle for a wider class of operators, that may operate inside the spaces of the various irreducibles representations of G. On finite groups, this principle subsumes the results of Donoho and Stark [3], Meshulam [11] and the present authors [1]. A corollary of our principle appropriately generalizes the statement (1.1) to compact groups, improving upon the result of Kok Seng and Wee Seng [9]. This corollary is also an improvement, in the setting of compact groups, over the principle for which Kaniuth [8] recently characterized all minimizing functions. Our primary contribution is the following theorem.

THEOREM 1. *Let G be a compact group with Haar measure μ, and let* P *and* R *be operators on $L^2(G)$. If* P *commutes with projection onto every measurable subset of G and* R *commutes with left multiplication by elements of G, then $\|\mathrm{PR}\| \leq \|\mathrm{P} \cdot \chi_G\|_2 \|\mathrm{R}\|_2$.*

Here, $\|\cdot\|$ denotes the operator norm, χ_G denotes the characteristic function of G, and $\|\cdot\|_2$ denotes the L^2-norm of a function as well as the Hilbert–Schmidt norm of an operator. We remark that Theorem 1 still holds when "left" is replaced by "right," corresponding to a different choice of Fourier transform. Theorem 1 allows us to prove the following uncertainty principle concerning functions on a compact group.

THEOREM 2. *Let G be a compact group with Haar measure μ, and f a nonzero element of $L^2(G)$. Then $\mu(\mathbf{supp}\, f) \cdot \sum_{\rho \in \hat{G}} d_\rho\, \mathbf{rk}\, \hat{f}(\rho) \geq 1$.*

This corollary improves upon the previously known result in [9], where the size of the support of $\hat{f}$ was given by $\sum_{\{\rho : \hat{f}(\rho) \neq 0\}} d_\rho^2$. Theorem 1 also implies the following result on finite groups.

Corollary 1. *Let* P *and* R *be projection operators on the group algebra of a finite group* G. *If* P *commutes with projection onto elements of* G, *and* R *commutes with left-multiplication by elements of* G, *then* $\|\,\mathrm{PR}\,\|^2 \leq \mathbf{rk}\,\mathrm{P}\cdot\mathbf{rk}\,\mathrm{R}/|G|$.

This corollary, in turn, easily implies a result of the form (1.2) for all finite groups.

Corollary 2. *Let* G *be a finite group, and let* $S\subset G$ *and* $T\subset\hat{G}$. *If* P_S *denotes projection onto* S (*in the group basis*), *and* R_T *denotes projection onto* T (*in the Fourier basis*), *then*

$$\|\,\mathrm{PR}\,\|^2 \leq \frac{|S|\cdot\sum_{\rho\in T} d_\rho^2}{|G|}.$$

In the next section, we discuss the so-called time-limiting and band-limiting operators appearing in the statements above. We then prove Theorems 1 and 2, and end with a discussion of the implications of our theorems to finite groups.

2. Preliminaries

2.1. Time-limiting operators. The definitions of *time-limiting* and *band-limiting* operators, which act on the space of square-integrable functions on a group, are inspired by the signal processing applications of the uncertainty principle considered by Donoho and Stark [3]. In what follows, G will denote a compact group with Haar measure μ, normalized so that $\mu G=1$. A time-limiting projection operator P_S acts on $L^2(G)$ simply by projecting to a subset S of the group. More generally, we may consider the linear operator P_f given by pointwise multiplication by a bounded function f. While the operator norm of P_f is equal to the L^∞-norm of f, the Hilbert–Schmidt norm of P_f is unbounded unless $f=0$ almost everywhere. The following proposition shows that these operators are a natural generalization of time-limiting projections.

Proposition 1. *Let* G *be a compact group. A bounded linear operator* P *on* $L^2(G)$ *commutes with projection onto every measurable subset of* G *if and only if* $\mathrm{P}=\mathrm{P}_f$ *for some bounded* $f\in L^2(G)$.

Proof. The reverse direction is clear. For the forward direction, we first define $f=\mathrm{P}\cdot\chi_G$ and note that P acts by pointwise multiplication by f on all simple functions. This is enough to show that f must also be bounded. Supposing otherwise, let $S_k=\{x : f(x)\in[k,k+1]\}$ and define a simple function g by

$$g(S_k)=\begin{cases}\frac{1}{k\sqrt{\mu(S_k)}}, & \text{if } \mu(S_k)>0,\\ 0, & \text{otherwise.}\end{cases}$$

Evidently, $g \in L^2(G)$, but

$$\|P \cdot g\|_2^2 = \int_G |f(x)g(x)|^2 \, d\mu(x) \geq \sum_k \mu(S_k) \cdot k|g(S_k)|^2,$$

and an infinite number of terms in this sum are equal to 1. Hence, $\mathrm{P} \cdot g \notin L^2(G)$, and thus, f must be bounded. Finally, an arbitrary function $g \in L^2(G)$ is an L^2-limit of simple functions g_n. Since P is bounded (and hence continuous), and f is a bounded function, we have

$$\mathrm{P} \cdot g = \mathrm{P} \cdot \lim_{n \to \infty} g_n = \lim_{n \to \infty} \mathrm{P} \cdot g_n = \lim_{n \to \infty} f g_n = fg. \qquad \square$$

2.2. Band-limiting operators. Recall that the Fourier transform of a function $f \in L^2(G)$ at the irreducible representation ρ is given by

$$\hat{f}(\rho) = \int f(x)\rho(x)^\dagger \, d\mu(x). \tag{2.1}$$

Using the Fourier inversion formula, we may then define band-limiting projection onto a subset T of $\hat{G}$ by setting

$$\mathrm{R}_T \cdot f(x) = \sum_{\rho \in T} d_\rho \, \mathbf{tr}[\hat{f}(\rho)\rho(x)].$$

We remark that these operators commute with both the left and the right action of G on $L^2(G)$. In terms of the irreducible decomposition of $L^2(G)$ given by the Peter–Weyl theorem, R_T operates on $L^2(G)$ by projecting onto the subspace $\bigoplus_{\rho \in T} \mathcal{E}_\rho$, where $\mathcal{E}_\rho$ denotes the ρ-isotypic subspace. While this is satisfactory for studying Abelian groups, where $\dim \mathcal{E}_\rho = 1$, such operators become increasingly coarse as the dimensions of the various irreducible representations of G increase. For this reason, we wish to consider more general band-limiting operators in the nonAbelian setting. A general band-limiting operator will be described by a collection $\mathrm{R} = \{\mathrm{R}_\rho\}_{\rho \in \hat{G}}$ of linear operators, where R_ρ operates on the space of ρ. The action of such an operator on a function $f \in L^2(G)$ is then given by

$$\mathrm{R} \cdot f(x) = \sum_{\rho \in \hat{G}} d_\rho \, \mathbf{tr}[\mathrm{R}_\rho \hat{f}(\rho)\rho(x)]. \tag{2.2}$$

The previous notion of a band-limiting operator R_T is the special case where $\mathrm{R}_\rho = \mathbb{1}_\rho$ for $\rho \in T$ and $\mathrm{R}_\rho = 0$ otherwise.

Given a group element $x \in G$, let L_x denote the operator corresponding to the left action of x on $L^2(G)$, i.e., $[\mathrm{L}_x \cdot f](y) = f(x^{-1}y)$. If an operator R satisfies $\mathrm{R}\mathrm{L}_x = \mathrm{L}_x\mathrm{R}$ for every $x \in G$, then we say that R *commutes with left multiplication.*

PROPOSITION 2. *Let G be a compact group, and* R *a linear operator on $L^2(G)$. Then* R *commutes with left multiplication if and only if* $\mathrm{R} \cdot f(x) =$

$\sum_{\rho\in\hat{G}} d_\rho\, \mathbf{tr}[\mathrm{R}_\rho \hat{f}(\rho)\rho(x)]$ *for some collection* $\{\mathrm{R}_\rho\}_{\rho\in\hat{G}}$, *where each* R_ρ *is a linear operator on the space of* ρ.

Proof. We first observe that

$$\widehat{\mathrm{L}_y \cdot f}(\rho) = \int f(y^{-1}x)\rho(x)^\dagger\, d\mu(x) = \int f(x)\rho(yx)^\dagger\, d\mu(x) = \hat{f}(\rho)\rho(y^{-1}).$$

If a linear operator R satisfies (2.2), then

$$[\mathrm{RL}_y \cdot f](x) = \sum_{\rho\in\hat{G}} d_\rho\, \mathbf{tr}[\mathrm{R}_\rho \hat{f}(\rho)\rho(yx)] = [\mathrm{L}_y \mathrm{R}\cdot f](x),$$

i.e. R commutes with left multiplication.

Now, let R be any linear operator on $L^2(G)$ which commutes with left multiplication. By Schur's Lemma, such an operator decomposes into a direct sum of linear operators A_ρ, one for each irreducible representation $\rho\in\hat{G}$. Each A_ρ acts on the ρ-isotypic subspace $\mathcal{E}_\rho$ from the decomposition of $L^2(G)$ given by the Peter–Weyl theorem. Schur's Lemma also asserts that each A_ρ is determined by a linear operator R_ρ acting on a single space of the representation ρ. The action of A_ρ on the Fourier transform of f is precisely by matrix multiplication by R_ρ on the left. Taken together with the Fourier inversion formula, this means precisely that R has the form (2.2). □

3. Results

3.1. Operator uncertainty principles. Our primary technical contribution is the following.

THEOREM 3. *Let G be a compact group, and f a bounded measurable function on G. Let* R *be an operator on $L^2(G)$ which commutes with right multiplication by elements of G. Then*

$$\|\mathrm{P}_f\mathrm{R}\|_2^2 = \|f\|_2^2 \sum_{\rho\in\hat{G}} d_\rho \|\mathrm{R}_\rho\|_2^2,$$

where R_ρ is the operator on the representation space of ρ from the decomposition of R *implied by Proposition 2.*

Proof. For every irreducible representation ρ of G, let Π_ρ denote projection onto the ρ-isotypic subspace $\mathcal{E}_\rho \subset L^2(G)$. By the Peter–Weyl theorem, the Π_ρ form a resolution $\sum_\rho \Pi_\rho = \mathbb{1}$ of the identity operator on $L^2(G)$. We can thus express the Hilbert–Schmidt norm of $\mathrm{P}_f\mathrm{R}$ as follows:

$$\begin{aligned}\|\mathrm{P}_f\mathrm{R}\|_2^2 &= \mathbf{tr}[\mathrm{R}^\dagger \mathrm{P}_f^\dagger \mathrm{P}_f \mathrm{R}] = \mathbf{tr}\left[\sum_\rho \Pi_\rho \mathrm{R}^\dagger \mathrm{P}_{|f|^2}\mathrm{R}\sum_\sigma \Pi_\sigma\right]\\ &= \sum_\rho\sum_\sigma \mathbf{tr}\left[\Pi_\rho \mathrm{R}^\dagger \mathrm{P}_{|f|^2}\mathrm{R}\Pi_\sigma\right] = \sum_\rho \mathbf{tr}\left[\Pi_\rho \mathrm{R}^\dagger \mathrm{P}_{|f|^2}\mathrm{R}\Pi_\rho\right],\end{aligned}$$

where we have made use of the fact that $(\mathrm{P}_f)^\dagger = \mathrm{P}_{\bar{f}}$. We remark that the operators Π_ρ also commute with left multiplication by elements of G. Indeed, their images are invariant subspaces of $L^2(G)$ viewed as a representation of G under the left-multiplication action. We thus have, for every $x \in G$,

$$\begin{aligned}\|\mathrm{P}_f\mathrm{R}\|_2^2 &= \sum_\rho \mathbf{tr}\big[\mathrm{L}_x\mathrm{L}_{x^{-1}}\Pi_\rho\mathrm{R}^\dagger\mathrm{P}_{|f|^2}\mathrm{R}\Pi_\rho\big]\\ &= \sum_\rho \mathbf{tr}\big[\mathrm{L}_{x^{-1}}\Pi_\rho\mathrm{R}^\dagger\mathrm{P}_{|f|^2}\mathrm{R}\Pi_\rho\mathrm{L}_x\big]\\ &= \sum_\rho \mathbf{tr}\big[\Pi_\rho\mathrm{R}^\dagger\mathrm{L}_{x^{-1}}\mathrm{P}_{|f|^2}\mathrm{L}_x\mathrm{R}\Pi_\rho\big].\end{aligned}$$

Integrating both sides over G, we conclude that

$$\begin{aligned}\|\mathrm{P}_f\mathrm{R}\|_2^2 = \int_x \|\mathrm{P}_f\mathrm{R}\|_2^2\,d\mu(x) &= \int_x\sum_\rho \mathbf{tr}\big[\Pi_\rho\mathrm{R}^\dagger\mathrm{L}_{x^{-1}}\mathrm{P}_{|f|^2}\mathrm{L}_x\mathrm{R}\Pi_\rho\big]\,d\mu(x)\\ &= \sum_\rho \mathbf{tr}\left[\Pi_\rho\mathrm{R}^\dagger\int_x \mathrm{L}_{x^{-1}}\mathrm{P}_{|f|^2}\mathrm{L}_x\,d\mu(x)\,\mathrm{R}\Pi_\rho\right].\end{aligned}$$

The last integral above denotes the so-called *weak operator integral.* One may arrive at the proper definition of such an integral, for instance, by requiring that the integral $\int\langle\Phi(x)u, v\rangle$ of each matrix element is equal to the corresponding matrix element of the integral $\int\Phi(x)$. This is well defined for any bounded operator-valued function Φ, i.e., if there exists M such that $\|\Phi(x)\| \le M$ for every x. Moreover, the integral thus defined commutes with taking the trace, and commutes with composition with bounded operators on the left and the right; in fact, this is precisely the integral appearing in the operator-valued Fourier transform (2.1) for compact groups. For further details on issues of integrability and measurability of operator-valued functions, consult Fell and Doran [5] or Conway [2]. Returning to our calculation, we now wish to show that the operator $\int_G \mathrm{L}_{x^{-1}}\mathrm{P}_{|f|^2}\mathrm{L}_x\,d\mu(x)$ is in fact just scaling by $\|f\|_2^2$, i.e., the operator $\mathrm{P}_{\|f\|_2^2}$. Indeed, we see that for any $g, h \in L^2(G)$,

$$\begin{aligned}\left\langle\int_x \mathrm{L}_{x^{-1}}\mathrm{P}_{|f|^2}\mathrm{L}_x\,d\mu(x)\cdot g, h\right\rangle &= \int_x\big\langle\mathrm{L}_{x^{-1}}\mathrm{P}_{|f|^2}\mathrm{L}_x\cdot g, h\big\rangle\,d\mu(x)\\ &= \int_x\int_y \mathrm{L}_{x^{-1}}\mathrm{P}_{|f|^2}\mathrm{L}_x\cdot g(y)\overline{h(y)}\,d\mu(y)\,d\mu(x)\\ &= \int_x\int_y |f(xy)|^2 g(y)\overline{h(y)}\,d\mu(y)\,d\mu(x)\\ &= \int_y\int_x |f(xy)|^2\,d\mu(x)\,g(y)\overline{h(y)}\,d\mu(y)\\ &= \langle\|f\|_2^2\cdot g, h\rangle,\end{aligned}$$

where the second-to-last step follows from Fubini's Theorem. Finally, we have

$$\begin{aligned}\|\mathrm{P}_f \mathrm{R}\|_2^2 &= \sum_\rho \mathbf{tr}\big[\Pi_\rho \mathrm{R}^\dagger \mathrm{P}_{\|f\|_2^2} \mathrm{R}\Pi_\rho\big] = \|f\|_2^2 \sum_\rho \mathbf{tr}[\Pi_\rho \mathrm{R}^\dagger \mathrm{R}\Pi_\rho] \\ &= \|f\|_2^2 \sum_\rho \|\mathrm{R}\Pi_\rho\|_2^2 = \|f\|_2^2 \sum_\rho \|\mathrm{A}_\rho\|_2^2 = \|f\|_2^2 \sum_\rho d_\rho \|\mathrm{R}_\rho\|_2^2,\end{aligned}$$

where the last two steps follow from the proof of Proposition 2. □

The following operator uncertainty principle follows easily from the above theorem.

THEOREM 1. *Let G be a compact group, and let* P *and* R *be operators on $L^2(G)$. If* P *commutes with projection onto every measurable subset of G and* R *commutes with left multiplication by elements of G, then*

$$\|\,\mathrm{PR}\,\| \le \|\mathrm{P}\cdot\chi_G\|_2 \|\mathrm{R}\|_2.$$

Proof. By Proposition 1, $\mathrm{P} = \mathrm{P}_f$, where $f = \mathrm{P}\cdot\chi_G \in L^\infty(G)$. Theorem 3 then asserts that

$$\|\,\mathrm{PR}\,\|_2^2 = \|\mathrm{P}\cdot\chi_G\|_2^2 \sum_\rho d_\rho \|\mathrm{R}_\rho\|_2^2 = \|\mathrm{P}\cdot\chi_G\|_2^2 \|\mathrm{R}\|_2^2.$$

The result now follows from the fact that the operator norm is bounded above by the Hilbert–Schmidt norm. □

3.2. Function uncertainty principles. In this section, we apply Theorem 1 to prove a "classical" uncertainty principle along the lines of (1.1), relating the support of a nonzero function on a compact group to the support of its Fourier transform. For simplicity of notation, for $f \in L^2(G)$ we define

$$\Sigma f = \{x \in G : f(x) \neq 0\} \quad \text{and} \quad \Sigma \hat{f} = \{\rho \in \hat{G} : \hat{f}(\rho) \neq 0_\rho\}.$$

Since we are given a normalized measure μ on G, there is a canonical and natural way to quantify the size of the support of f, i.e., $|\Sigma f| = \mu(\Sigma f)$. The size of the support of $\hat{f}$, on the other hand, involves a nontrivial choice of dual measure on $\hat{G}$. For finite groups, a natural choice is *Plancherel measure*, which assigns mass $d_\rho^2/|G|$ to each representation ρ. As this is the dimension-wise fraction of the group algebra consisting of irreps isomorphic to ρ, the Plancherel measure of $\hat{G}$ is equal to the (normalized) Haar measure of G. For general compact groups, we will assign measure d_ρ^2 to each representation ρ. This is still a natural choice, since d_ρ^2 is equal to the dimension of the ρ-isotypic subspace of $L^2(G)$. We thus set, for $T \subset \hat{G}$, $|T| = \sum_{\rho\in T} d_\rho^2$; meanwhile, for $S \subset G$, we set $|S| = \mu S$. Now, consider the following statement:

$$\text{If } |\Sigma f| < |G| \text{ and } |\Sigma \hat{f}| < |\hat{G}|, \text{ then } f = 0. \tag{3.1}$$

We emphasize that for general compact groups, and indeed in all cases of interest for (3.1), the quantity $|\hat{G}|$ is infinite. Hogan [7] showed that if G is infinite and compact, then (3.1) is valid if and only if G is connected. Echterhoff, Kaniuth, and Kumar [4] showed that if G has a noncompact, nondiscrete, cocompact normal subgroup H that satisfies (3.1), then G satisfies it as well. Another principle, given in Kutyniok [10], states that

$$\text{(3.2)} \qquad \textit{if } |\Sigma f||\Sigma \hat{f}|_1 < 1, \textit{ then } f = 0,$$

where $|\Sigma \hat{f}|_1 = \sum_{\rho \in \Sigma \hat{f}} d_\rho$. Kutyniok proves that a compact group G satisfies (3.2) if and only if G modulo the connected component of the identity is an Abelian group. While the choice between $|\cdot|$ and $|\cdot|_1$ has no impact on Abelian groups, or on the principle (3.1), it is quite significant for the principle (3.2) on arbitrary compact groups. For instance, if G has a finite quotient G/H, then $|\Sigma \chi_H| = |H|$ while

$$|\Sigma \hat{\chi}_H| = \sum_{\rho \in \Sigma \hat{\chi}_H} d_\rho^2 = \sum_{\rho \in \widehat{G/H}} d_\rho^2 = [G : H] = 1/|H|.$$

Hence, $|\Sigma \chi_H||\Sigma \hat{\chi}_H| = 1$, while any smaller choice of measure for $\Sigma \hat{\chi}_H$ would result in inequality. In fact, the entire proof of the main theorem of [10] still holds if we assign measure 1 to one-dimensional irreps, and measure 0 to any ρ satisfying $d_\rho > 1$.

For arbitrary compact groups, our Theorem 3 implies the following result.

THEOREM 2. *Let G be a compact group, and f a nonzero element of $L^2(G)$. Then*

$$|\Sigma f| \sum_{\rho \in \hat{G}} d_\rho \, \mathbf{rk} \, \hat{f}(\rho) \geq 1.$$

Proof. Let f be a nonzero element of $L^2(G)$. For each $\rho \in \hat{G}$, let R_ρ be the operator on the space of ρ which projects to $\mathbf{Im} \, \hat{f}(\rho)$, as a subspace of the representation space of ρ. Define the operator R on $L^2(G)$ by

$$\mathrm{R} \cdot g(x) = \sum_\rho d_\rho \, \mathbf{tr}[\mathrm{R}_\rho \hat{g}(\rho)\rho(x)]$$

and recall that R commutes with left multiplication by Proposition 2. Set $\mathrm{P} = \mathrm{P}_{\Sigma f}$, i.e., L^2 projection onto the support of f. By Theorem 3,

$$\| \mathrm{PR} \|_2^2 = |\Sigma f| \sum_{\rho \in \hat{G}} d_\rho \|\mathrm{R}_\rho\|_2^2 = |\Sigma f| \sum_{\rho \in \hat{G}} d_\rho \, \mathbf{rk} \, \hat{f}(\rho).$$

As both P and R are orthogonal projections, and $\mathrm{PR} \cdot f = f$, we have that $\| \mathrm{PR} \| = 1$. Since the operator norm is bounded above by the Hilbert–Schmidt norm, we conclude that

$$1 \leq |\Sigma f| \sum_{\rho \in \hat{G}} d_\rho \, \mathbf{rk} \, \hat{f}(\rho). \qquad \square$$

An obvious consequence is that, for G compact and any nonzero $f \in L^2(G)$, $|\Sigma f||\Sigma \hat{f}| \geq 1$. Indeed, this is precisely the uncertainty principle appearing in [9].

3.3. Consequences in finite groups. In this section, we collect two interesting corollaries of the main theorem, in the setting of finite groups.

COROLLARY 1. *Let* P *and* R *be projection operators on the group algebra of a finite group* G. *If* P *commutes with projection onto elements of* G, *and* R *commutes with left-multiplication by elements of* G, *then* $\|\mathrm{PR}\|^2 \leq \mathbf{rk}\,\mathrm{P}\,\mathbf{rk}\,\mathrm{R}/|G|$.

Proof. A projection operator on $\mathbb{C}G$ commutes with projection onto elements of G if and only if it projects onto some set $S \subset G$, in which case its rank is $|S|$. On the other hand, a projection operator commutes with left-multiplication exactly when it respects the decomposition of $\mathbb{C}G = L^2(G)$ into irreducible spaces according to the left action. Since the restrictions R_ρ of R to these irreducible spaces are themselves projection operators, we have

$$\mathbf{rk}\,\mathrm{R} = \sum_{\rho \in \hat{G}} d_\rho \cdot \mathbf{rk}\,\mathrm{R}_\rho = \sum_{\rho \in \hat{G}} d_\rho \cdot \|\mathrm{R}_\rho\|_2^2.$$

The result now follows from Theorem 3. □

COROLLARY 2. *Let* G *be a finite group, and let* $S \subset G$ *and* $T \subset \hat{G}$. *If* P_S *denotes projection onto* S (*in the group basis*), *and* R_T *denotes projection onto* T (*in the Fourier basis*), *then*

$$\|\mathrm{PR}\|^2 \leq \frac{|S| \cdot \sum_{\rho \in T} d_\rho^2}{|G|}.$$

Proof. By Proposition 2, R_T commutes with left multiplication; its rank is clearly $\sum_{\rho \in T} d_\rho^2$. Also, P_S obviously commutes with projection onto elements of G, and its rank is $|S|$. The result now follows from Corollary 1. □

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