ON ZEROS OF THE DERIVATIVE OF THE THREE-DIMENSIONAL SELBERG ZETA FUNCTION

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ABSTRACT. In this article, we study the distribution of zeros of
the derivative of the Selberg zeta function for compact three-
dimensional hyperbolic spaces. We obtain an asymptotic formula
for the counting function of its zeros. This is a three-dimensional
version of the celebrated work of Wenzhi Luo. We also deduce
other asymptotic formulas relating to its zeros from the above
formula.

1. Introduction

Let $\mathbb{H}^2$ be an upper half plane and Γ a cocompact discrete subgroup of
$PSL(2, \mathbb{R})$ without elliptic elements. Regarding $\Gamma \backslash \mathbb{H}^2$ as a compact Riemann
surface of genus $g \geq 2$, we define the Selberg zeta function $Z_\Gamma(s)$ associated
with $\Gamma \backslash \mathbb{H}^2$ by

$$Z_\Gamma(s) := \prod_{\{P_0\}} \prod_{l=0}^{\infty} \left(1 - N(P_0)^{-s-l}\right),$$

for $\mathrm{Re}(s) > 1$, where $\{P_0\}$ is taken over all primitive hyperbolic conjugacy
classes of Γ and $N(P_0)$ is the norm of P_0 defined by $N(P_0) := |a(P_0)|^2$, where
$a(P_0)$ is the eigenvalue of P_0 with $|a(P_0)| > 1$. This function is extended as
an analytic function to the whole plane $\mathbb{C}$ by the Selberg trace formula. As
is well known, infinitely many nontrivial zeros of $Z_\Gamma(s)$ lie on the critical line
$\mathrm{Re}(s) = 1/2$, except finitely many real zeros, that is to say, essentially, the
Riemann hypothesis holds for the Selberg zeta function. (See Selberg [Sel] or
Hejhal [Hej].)

Since more than two-third of nontrivial zeros of the Riemann zeta function
$\zeta(s)$ are simple under the Riemann hypothesis (see Montgomery [Mon]), a
study on multiplicity of zeros of $Z_\Gamma(s)$ is a natural problem. Also, as zeros of

$Z_\Gamma(s)$ are corresponding to eigenvalues of the Laplacian on $L^2(\Gamma\backslash\mathbb{H}^2)$, such a study is important from the aspect of the multiplicity problem for eigenvalues of the Laplacian.

One of the approaches to multiplicity of zeros of a function is to search for zeros of derivatives of the function. As being connected with the Riemann hypothesis, the zeros of the derivative of the Riemann zeta function $\zeta'(s)$ and more generally, the zeros of k-th derivatives $\zeta^{(k)}(s)$ were studied by Speiser [Spe], Spira [Spi1], [Spi2], [Spi3]. Furthermore, Berndt [Ber] and Levinson and Montgomery [LM] studied counting functions of nonreal zeros of $\zeta^{(k)}(s)$. Soundararajan [Sou] and others also studied zeros of $\zeta'(s)$.

In the case of Selberg zeta functions, Luo studied the distribution of nonreal zeros of the derivative of the Selberg zeta function $Z'_\Gamma(s)$ for compact Riemann surfaces [Luo2]. Let $\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)}$ be zeros of $Z_\Gamma^{(k)}(s)$. We define the counting function $N_k(T)$ for the above zeros with ordinate less than T as follows:

$$N_k(T) := \sharp\{\beta^{(k)} + i\gamma^{(k)} \mid Z_\Gamma^{(k)}(\beta^{(k)} + i\gamma^{(k)}) = 0, 0 < \gamma^{(k)} \le T\}.$$

Luo obtained an asymptotic formula for $N_1(T)$.

THEOREM 1.1 (Luo [Luo2, p. 1142, Theorem 1]).

$$N_1(T) = \frac{\text{area}(\Gamma\backslash\mathbb{H}^2)}{4\pi}T^2 + O(T) \quad (as\ T \to \infty),$$

where $\text{area}(\Gamma\backslash\mathbb{H}^2)$ is the area of $\Gamma\backslash\mathbb{H}^2$ with respect to the hyperbolic measure.

This theorem is an analogue of the Weyl law (see Hejhal [Hej, p. 115, Theorem 7.1; p. 118, Theorem 7.4; p. 119, Theorem 8.1])

$$N_0(T) = \frac{\text{area}(\Gamma\backslash\mathbb{H}^2)}{4\pi}T^2 + O\left(\frac{T}{\log T}\right) \quad (as\ T \to \infty).$$

The study of Luo is motivated by Luo [Luo1] and Phillips and Sarnak [PS1], namely, the aim of his study is to obtain a good bound for the multiplicity of zeros of $Z_\Gamma(s)$ or the multiplicity of eigenvalues of the hyperbolic Laplacian Δ.

His work moved the author to study the distribution of zeros of the derivative of the Selberg zeta function associated with the real three-dimensional compact upper half space. Let $\mathbb{H}^3$ be the three-dimensional upper half space and Γ a cocompact discrete subgroup of $PSL(2, \mathbb{C})$. For the Selberg zeta function $Z_\Gamma(s)$ associated with the compact quotient $\Gamma\backslash\mathbb{H}^3$ (see the next section), we define the counting function $N_k(T)$ of nonreal zeros of $Z_\Gamma^{(k)}(s)$ in the same way as above. The aim of this paper is to prove the following theorem which is an analogue of the above theorem of Luo.

THEOREM 1.2.

$$N_1(T) = \frac{\text{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}T^3 + O(T^2) \quad (as\ T \to \infty),$$

where $\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)$ *is the volume of* $\Gamma\backslash\mathbb{H}^3$ *with respect to the hyperbolic measure.*

In the three-dimensional case, the Weyl law is given by the following theorem.

THEOREM 1.3 (Elstrodt, Grunewald and Mennicke [EGM, p. 215, Theorem 5.6]).

$$N_0(T) = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}T^3 + O(T^2) \quad (as\ T \to \infty).$$

From the above two theorems, we have

$$N_0(T) = N_1(T) + O(T^2).$$

REMARKS. (i) We mentioned that the work of Luo is motivatied by [PS1], but the theory of Phillips and Sarnak does not hold for hyperbolic spaces of dimensions ≥ 3. This is because their theory depends on the deformation theory of cofinite discrete subgroups of $PSL(2,\mathbb{R})$, and the deformation theory works efficiently for $PSL(2,\mathbb{R})$ only. (See Phillips and Sarnak [PS2], Sarnak [Sar1], [Sar2] and Mostow [Mos].) In spite of the shortage of an three-dimensional analogue of [PS1], our main theorem (Theorem 1.2) holds as an analogue of the above theorem of Luo (Theorem 1.1).

(ii) In Section 5, we will obtain other asymptotic formulas related to zeros of the derivative of the Selberg zeta function by Theorem 1.2 and Lemma 4.3 in Section 4. We shall deduce plain zero density estimates by following the method of Levinson and Montgomery [LM].

2. Preliminaries

In this section, we refer the definition of the Selberg zeta function and the resolvent trace formula for the three-dimensional hyperbolic space (Elstrodt, Grunewald and Mennicke [EGM]).

Let $\mathbb{H}^3$ be the three-dimensional upper half space

$$\mathbb{H}^3 := \{(z,r) \mid z = x + iy \in \mathbb{C}, r > 0\}$$

with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}.$$

The volume measure is given by

$$\frac{dx\,dy\,dr}{r^3},$$

and the Lapalace–Beltrami operator is given by

$$\Delta := -r^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\right) + r\frac{\partial}{\partial r}.$$

Let Γ be a discontinuous subgroup of $PSL(2,\mathbb{C})$. The group Γ acts on $\mathbb{H}^3$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z,r) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}r^2}{|cz+d|^2 + |c|^2 r^2}, \frac{r}{|cz+d|^2 + |c|^2 r^2} \right),$$

transitively.

In order to define the Selberg zeta funcion $Z_\Gamma(s)$ associated with $\Gamma\backslash\mathbb{H}^3$, we classify any element $P \in \Gamma - \{I\}$ into 4-classes as follows:

If $|\operatorname{tr}(P)| = 2$ and $\operatorname{tr}(P) \in \mathbb{R}$, then P is called parabolic.
If $|\operatorname{tr}(P)| < 2$ and $\operatorname{tr}(P) \in \mathbb{R}$, then P is called elliptic.
If $|\operatorname{tr}(P)| > 2$ and $\operatorname{tr}(P) \in \mathbb{R}$, then P is called hyperbolic.
In all other cases, P is called loxodromic.

The Selberg zeta function $Z_\Gamma(s)$ is defined by all primitive hyperbolic or loxodromic conjugacy classes $\{P_0\}$. A primitive element means that it is not an essential power of any other element.

DEFINITION 2.1 (*Elstrodt, Grunewald, and Mennicke* [EGM, *p.* 206, *Definition* 4.1]). For $\operatorname{Re}(s) > 2$, $Z_\Gamma(s)$ is defined as

$$(2.1) \qquad Z_\Gamma(s) := \prod_{\{P_0\}} \prod_{(l,m)} \left(1 - a(P_0)^{-2l} \overline{a(P_0)}^{-2m} N(P_0)^{-s} \right),$$

where $\{P_0\}$ is taken over all primitive hyperbolic or loxodromic conjugacy classes of Γ, the pair (l,m) runs over all pairs of nonnegative integers satisfying the congruence relation $l \equiv m (\operatorname{mod} m(P_0))$. Here $m(P)$ is the order of the torsion of the centralizer of the hyperbolic or loxodromic element $P \in \Gamma$.

In this article, we only consider a cocompact discrete subgroup Γ of $PSL(2,\mathbb{C})$. Then the quotient $\Gamma\backslash\mathbb{H}^3$ is a compact Riemannian space. Let $\lambda_n = 1 + r_n^2$, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be eigenvalues of Δ on the compact space $\Gamma\backslash\mathbb{H}^3$.

In the compact case, the completed Selberg zeta function $\Xi_\Gamma(s)$ is defined by the following definition.

DEFINITION 2.2 (*Elstrodt, Grunewald, and Mennicke* [EGM, *p.* 206, (4.13)]).

$$\Xi_\Gamma(s) := Z_I(s) Z_E(s) Z_\Gamma(s),$$

where

$$Z_I(s) := \exp\left(-\frac{\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi}(s-1)^3 \right),$$

$$Z_E(s) := \exp\left(E(s-1) \right),$$

which are "gamma factors". The first gamma factor $Z_I(s)$ is a contribution from the identity of Γ, $\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)$ is the volume of $\Gamma\backslash\mathbb{H}^3$ by the above hyperbolic measure. The second gamma factor $Z_E(s)$ is a contribution from the

elliptic elements of Γ, the positive constant E is defined by

$$E := \sum_{\{R\}:\text{elliptic}} \frac{\log N(T_0)}{m(R)|\operatorname{tr}(R)^2 - 4|},$$

where $\{R\}$ is taken over all elliptic conjugacy classes of Γ, T_0 is a hyperbolic or loxodromic element of the centralizer of R such that $N(T_0)$ is minimal and $m(R)$ denotes the order of maximal finite group in the centralizer of R.

Elstrodt, Grunewald, and Mennicke proved the resolvent trace formula.

THEOREM 2.3 (Elstrodt, Grunewald, and Mennicke [EGM, p. 208, Theorem 4.3]). *For all s, $a \in \mathbb{C} \setminus \{s_n = 1 \pm ir_n | n \geq 0\}$,*

$$(2.2) \qquad \frac{1}{2s-2}\frac{\Xi'_\Gamma(s)}{\Xi_\Gamma(s)} - \frac{1}{2a-2}\frac{\Xi'_\Gamma(a)}{\Xi_\Gamma(a)} = \sum_{n=0}^{\infty}\left(\frac{1}{(s-1)^2 + r_n^2} - \frac{1}{(a-1)^2 + r_n^2}\right).$$

This formula implies that zeros of $\Xi_\Gamma(s)$, $Z_\Gamma(s)$ are of the form $1 \pm ir_n$. The analogue of the Riemann hypothesis holds except those λ_n satisfying $\lambda_n < 1$.

From the trace formula, the functional equation of $Z_\Gamma(s)$ is deduced.

THEOREM 2.4 (Elstrodt, Grunewald, and Mennicke [EGM, p. 209, Corollary 4.4]).

$$(2.3) \qquad Z_\Gamma(2-s) = \exp\left(-\frac{\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)}{3\pi}(s-1)^3 + 2E(s-1)\right)Z_\Gamma(s),$$

Therefore, the critical line of the zeta functions is $\operatorname{Re}(s) = 1$. For more details on the Sleberg zeta function, see [EGM].

3. Zero-free regions for $Z'(s)$

Hereafter, we denote the Selberg zeta function $Z_\Gamma(s)$ briefly by $Z(s)$. Let $s = \sigma + it$ $(\sigma, t \in \mathbb{R})$. In order to prove our main theorem, we need two types of zero-free regions for $Z'(s)$.

First, one is the zero-free region on the right.

PROPOSITION 3.1. *There exists a sufficiently large $\sigma_0 \geq 4$ such that*

$$(3.1) \qquad Z'(\sigma + it) \neq 0 \quad \text{for } \sigma \geq \sigma_0.$$

Proof. For $\operatorname{Re}(s) > 2$, we have [EGM, p. 208]

$$\frac{Z'(s)}{Z(s)} = \sum_{\{P\}} \frac{\Lambda(P)}{N(P)^s},$$

where, $\{P\}$ are hyperbolic or loxodromic conjugacy classes of Γ, and

$$\Lambda(P) := \frac{N(P)\log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2} > 0.$$

Since

$$\Lambda(P) \le \frac{\log N(P_0)}{|1 - a(P)^{-1}|^2} \ll \log N(P_0) < N(P)^\varepsilon \quad (\varepsilon > 0),$$

we have

$$\text{(3.2)} \qquad \sum'_{\{P\}} \frac{\Lambda(P)}{N(P)^s} \ll \int_{C_S N(P_{00})}^{\infty} \frac{d\pi_0(x)}{x^{\sigma - \varepsilon}},$$

where

$$N(P_{00}) := \min_{\{P_0\}}\{N(P_0)\} > 1,$$

$$C_S := \frac{\min\{N(P) | N(P_{00}) < N(P)\}}{N(P_{00})} > 1,$$

$$\pi_0(x) := \sharp\{\{P\} | N(P) \le x\},$$

and the primed summation symbol means the sum over all hyperbolic or loxodromic conjugacy classes $\{P\}$ except for $\{P\}$ with $N(P) = N(P_{00})$. It is known that $\pi_0(x) = O(x^2)$ as $x \to \infty$ ([EGM, p. 70, Lemma 6.3]).

By (3.2), the following estimate is obtained:

$$\text{(3.3)} \qquad \sum'_{\{P\}} \frac{\Lambda(P)}{N(P)^s} = O\left(\frac{1}{\{C_S N(P_{00})\}^\sigma}\right) \quad (\sigma \ge 4).$$

We choose a sufficiently large $\sigma_0 \ge 4$ such that

$$\sum'_{\{P\}} \frac{\Lambda(P)}{N(P)^\sigma} \le \frac{1}{2} \frac{L}{N(P_{00})^\sigma}$$

for $\sigma \ge \sigma_0$, where

$$L := \sum_{N(P) = N(P_{00})} \Lambda(P).$$

Then

$$\left| \frac{Z'(s)}{Z(s)} \right| \ge \frac{L}{N(P_{00})^\sigma} - \sum'_{\{P\}} \frac{\Lambda(P)}{N(P)^\sigma} \ge \frac{1}{2} \frac{L}{N(P_{00})^\sigma}.$$

This is the three-dimensional analogue of (4) in Luo [Luo2, p. 1143]. That is

$$|Z'(s)| \ge \frac{1}{2} \frac{L}{N(P_{00})^\sigma} |Z(s)| > 0.$$

This implies the assertion of Proposition 3.1. $\qquad\square$

On the other hand, the zero-free region of $Z'(s)$ on the left was determined by the author's former paper [Min].

THEOREM 3.2 ([Min, Theorem 3]).

$$Z'(s) \ne 0 \quad \text{for } \operatorname{Re}(s) < 1, \ \operatorname{Im}(s) \ne 0.$$

It is proved by the estimate of $\mathrm{Im}\{Z'(\sigma+it)/(2s-2)Z(\sigma+it)\}$ with the Hadamard product expression of $\Xi_\Gamma(s)$.

REMARK. We obtain the following equality from the proof of [Min, Theorem 3].

$$\mathrm{Im}\left\{\frac{1}{2ti}\frac{Z'(1+it)}{Z(1+it)}\right\} = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{4\pi}t + \frac{E}{2t}.$$

That is, the function $Z'(1+it)$ only vanishes at the zeros of $Z(s)$ when $E\neq 0$, and except for $t=0$ when $E=0$. From the viewpoint of the multiplicity problem for the eigenvalues of the Laplacian, we are interested in the number of zeros of $Z'(s)$ on $\mathrm{Re}(s)=1$, or in $\mathrm{Re}(s)>1$.

4. The number of zeros of $Z'(s)$

In this section, we shall prove our main theorem (Theorem 1.2) by the method of Luo [Luo2]. In order to count the number of nonreal zeros of $Z'(s)$, we define the function $X(s)$ as follows:

DEFINITION 4.1 (*cf. Luo* [Luo2, *p. 1143*]).

$$(4.1) \qquad\qquad X(s) := \frac{N(P_{00})^s}{L}Z'(s).$$

The function $X(s)$ has the following property.

PROPOSITION 4.2. *For any $t\in\mathbb{R}$ and any $\sigma\geq 4$, there exists a constant c_Γ $(0<c_\Gamma<1)$ such that*

$$(4.2) \qquad\qquad X(s) = 1 + O(c_\Gamma^\sigma) \quad (\sigma\geq 4),$$

where c_Γ is independent of σ and t.

Moreover, there exist a sufficiently large $\sigma_0\geq 4$ and a constant $C=C(\sigma_0)$ such that

$$(4.3) \qquad\qquad |\mathrm{Re}\,X(\sigma_0+it)| > C > 0,$$

where C depends on σ_0, but C and σ_0 are independent of t.

Proof. First, we shall prove (4.2). By Definition 4.1, and the proof of Proposition 3.1 in Section 3,

$$
\begin{aligned}
X(s) &= \frac{N(P_{00})^s}{L}\frac{Z'(s)}{Z(s)}Z(s)\\
&= \frac{N(P_{00})^s}{L}\left(\frac{L}{N(P_{00})^s} + {\sum_{\{P\}}}'\frac{\Lambda(P)}{N(P)^s}\right)Z(s)\\
&= \left(1 + O\left(\frac{1}{C_S^\sigma}\right)\right)Z(s).
\end{aligned}
$$

By [EGM, p. 214, (5.20)], there exists some positive constant M such that

$$(4.4) \qquad |\log Z(s)| \leq M \sum_{\{P_0\}} \frac{1}{N(P_0)^\sigma}, \quad \sigma > 2.$$

Then by the same argument as that of Hejhal [Hej, p. 74, Proposition 4.13], $Z(s)$ is estimated by

$$(4.5) \qquad Z(s) = 1 + O\left(\frac{1}{N(P_{00})^\sigma}\right) \quad (\sigma \geq 3).$$

Therefore, there exists a constant c_Γ $(0 < c_\Gamma < 1)$ which satisfies (4.2). It follows that there exists a sufficiently large $\sigma_0 \geq 4$ such that

$$(4.6) \qquad |X(\sigma_0 + it) - 1| < \frac{1}{2}.$$

This proves the existence of a positive constant $C > 0$ which is independent of t such that (4.3) holds. $\qquad\qquad\square$

To prove Theorem 1.2, we need the following lemma.

LEMMA 4.3. *Let* $\beta' + i\gamma'$ *be zeros of* $Z'(s)$ *and* $0 < \alpha < 1$. *Then we have*

$$(4.7) \qquad \sum_{0 < \gamma' \leq T} (\beta' - \alpha) = (1 - \alpha) \frac{\mathrm{vol}(\Gamma \backslash \mathbb{H}^3)}{6\pi^2} T^3 + O(T^2).$$

Proof. We choose σ_0 which satisfies the assertions of Proposition 3.1 in Section 3 and Proposition 4.2, and assume that a constant $t_0 \geq 2$ and T is sufficiently large. Let R_1 be the rectangle with the vertices $\alpha + it_0$, $\sigma_0 + it_0$, $\sigma_0 + iT'$ and $\alpha + iT'$ where α is a constant such that $0 < \alpha < 1$ and $T - 1 \leq T' \leq T$. Without loss of generality, we can assume that $Z'(s)$ has no zeros on the boundary of the rectangle R_1.

By Littlewood's theorem [Tit2, p. 132], we have

$$(4.8) \qquad 2\pi \sum_{\rho' \in R_1} (\beta' - \alpha)$$
$$= \int_{t_0}^{T'} \log|X(\alpha + it)|\, dt - \int_{t_0}^{T'} \log|X(\sigma_0 + it)|\, dt$$
$$- \int_\alpha^{\sigma_0} \arg X(\sigma + it_0)\, d\sigma + \int_\alpha^{\sigma_0} \arg X(\sigma + iT')\, d\sigma$$
$$= I_1 + I_2 + I_3 + I_4,$$

say, where $\rho' = \beta' + i\gamma'$ denotes the zeros of $Z'(s)$.

By estimating I_i $(i = 1, 2, 3, 4)$, we shall prove Theorem 1.2.

[**The estimate of** I_3]: Since the function $\arg X(\sigma + it_0)$ is independent of the parameter T,

$$(4.9) \qquad I_3 = -\int_\alpha^{\sigma_0} \arg X(\sigma + it_0)\, d\sigma = O(1).$$

[**The estimate of** I_4]: Using Titchmarsh [Tit1, p. 213, Section 9.4], we shall estimate I_4.

By Proposition 4.2, we have $|\operatorname{Re} X(\sigma_0 + iT')| > 0$, and

$$(4.10) \qquad |\arg X(\sigma_0 + iT')| < \frac{\pi}{2}.$$

Now we assume that $\operatorname{Re} X(s)$ vanishes q times on the segment between $\alpha + iT'$ and $\sigma_0 + iT'$. Then

$$(4.11) \qquad |\arg X(\sigma + iT')| \leq (q+1)\pi \quad (\alpha \leq \sigma < \sigma_0).$$

In order to estimate the number q, we define a function $Y(s)$ as follows:

$$Y(s) := \frac{1}{2}\{X(s + iT') + X(s - iT')\} \quad (\alpha \leq \sigma \leq \sigma_0).$$

The number q is expressed by the number of zeros of $Y(\sigma)$ $(\sigma \in \mathbb{R})$.

Let $n(r)$ be the number of zeros of $Y(s)$ for $|s - \sigma_0| \leq r$. Then we have

$$(4.12) \qquad q \leq n(\sigma_0 - \alpha).$$

Clearly, $n(\sigma_0 - \alpha)$ satisfies

$$(4.13) \qquad n(\sigma_0 - \alpha)\log\frac{\sigma_0 - \alpha'}{\sigma_0 - \alpha} \leq \int_0^{\sigma_0 - \alpha'} \frac{n(r)}{r}\,dr \quad (0 < \alpha' < \alpha < 1).$$

By Jensen's theorem

$$(4.14) \qquad \int_0^{\sigma_0 - \alpha'} \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi} \log\big|Y\big(\sigma_0 + (\sigma_0 - \alpha')e^{i\theta}\big)\big|\,d\theta - \log|Y(\sigma_0)|.$$

Let

$$M(\sigma, t) = \max_{\substack{\sigma \leq \sigma' \leq 2\sigma_0, \\ 2 \leq |t'| \leq t}} |X(\sigma' + it')|.$$

Then we have

$$(4.15) \qquad \big|Y\big(\sigma_0 + (\sigma_0 - \alpha')e^{i\theta}\big)\big| \leq M(\alpha', T' + \sigma_0)$$

and

$$(4.16) \qquad |Y(\sigma_0)| > C > 0,$$

hence, by (4.13), (4.14), (4.16), and (4.15), the number q is estimated by

$$(4.17) \qquad q \leq \frac{1}{\log\frac{\sigma_0 - \alpha'}{\sigma_0 - \alpha}}\left(\log M(\alpha', T' + \sigma_0) + \log\frac{1}{C}\right).$$

In order to estimate $M(\alpha', T' + \sigma_0)$, we need the following lemma.

LEMMA 4.4. *If $t \neq 0$ and $\sigma \in \mathbb{R}$ is situated in a finite interval, then there exists some constant $C' > 0$ such that*

$$(4.18) \qquad |Z'(\sigma + it)| \ll e^{C't^2}.$$

Proof. By the Cauchy integral formula, we have

$$(4.19) \qquad Z'(s) = \frac{1}{2\pi i} \int_{\tilde{C}} \frac{Z(\xi)}{(\xi - s)^2} \, d\xi,$$

where $\tilde{C}$ is a circle centered at s of suitable radius.

Here, we recall [EGM, p. 213, Corollary 5.4]: There exists some constant $C'' > 0$ such that

$$(4.20) \qquad Z(s) \ll \exp(C'' t^2),$$

where $\mathrm{Re}(s)$ is located in a finite interval.

Therefore, the estimate (4.18) is deduced from the estimate (4.19), and (4.20). □

By (4.11), (4.17), and (4.18), we have

$$|\arg X(\sigma + iT')| \ll T'^2, \quad (\alpha \le \sigma < \sigma_0)$$

and

$$(4.21) \qquad I_4 = \int_{\alpha}^{\sigma_0} \arg X(\sigma + iT') \, d\sigma = O(T'^2) = O(T^2).$$

[**The estimate of** I_2]: In order to estimate the integral I_2, let R_2 be the rectangle with the vertices $\sigma_0 + it_0$, $R + it_0$, $R + iT'$, and $\sigma_0 + iT'$, where $R > \sigma_0$. Since $X(s) \ne 0$ inside R_2, we may apply Cauchy's theorem to $X(s)$. Then we have

$$(4.22) \qquad -i \int_{t_0}^{T'} \log X(\sigma_0 + it) \, dt = \int_{R}^{\sigma_0} \log X(\sigma + iT') \, d\sigma$$

$$- i \int_{t_0}^{T'} \log X(R + it) \, dt$$

$$+ \int_{\sigma_0}^{R} \log X(\sigma + it_0) \, d\sigma.$$

On the right-hand side of (4.22), we have

$$\int_{\sigma_0}^{R} \log X(\sigma + it_0) \, d\sigma = O\left(\int_{\sigma_0}^{R} c_{\Gamma}^{\sigma} \, d\sigma \right) = O(1)$$

by the estimation (4.2) for $X(s)$. Similarly,

$$\int_{R}^{\sigma_0} \log X(\sigma + iT') \, d\sigma = O(1),$$

and

$$\int_{t_0}^{T'} \log X(R + it) \, dt = O\left(\int_{t_0}^{T'} c_{\Gamma}^{R} \, dt \right) \to 0 \quad (\text{as } R \to \infty).$$

Then

$$(4.23) \qquad |I_2| \leq \left| \int_{t_0}^{T'} \log X(\sigma_0 + it) \, dt \right| = O(1) \quad (\text{as } R \to \infty).$$

[**The estimate of** I_1]: This part is the longest and the most important in the estimates of the integrals in (4.8).

In the functional equation (2.3), we put

$$f(s) := \exp\left(\frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{3\pi} (s-1)^3 - 2E(s-1) \right).$$

Differentiating the functional equation, we have

$$Z'(s) = Z(2-s)\left(f'(s) - \frac{Z'(2-s)}{Z(2-s)} f(s) \right),$$

where

$$f'(s) = \left(\frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{\pi} (s-1)^2 - 2E \right) f(s).$$

Then $X(s)$ is expressed by

$$(4.24) \qquad X(s) = \frac{N(P_{00})^s}{L} Z(2-s) f(s) \left(\frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{\pi} (s-1)^2 - 2E \right)$$

$$\times \left(1 - \frac{1}{\frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{\pi}(s-1)^2 - 2E} \frac{Z'(2-s)}{Z(2-s)} \right).$$

We recall the zero-free region theorem (Theorem 3.2 in Section 3), i.e.,

$$(4.25) \qquad 1 - \frac{1}{\frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{\pi}(s-1)^2 - 2E} \frac{Z'(2-s)}{Z(2-s)} \neq 0,$$

for $\text{Re}(s) < 1$ and $\text{Im}(s) > 0$. We shall use it later.

We examine the integral I_1. In (4.24), we put $s = \alpha + it$ and take the absolute values and the logarithms, then

$$(4.26) \int_{t_0}^{T'} \log |X(\alpha + it)| \, dt$$

$$= \int_{t_0}^{T'} \log \left| \frac{N(P_{00})^\alpha}{L} \right| dt + \int_{t_0}^{T'} \log |Z(2 - \alpha - it)| \, dt$$

$$+ \int_{t_0}^{T'} \log |f(\alpha + it)| \, dt + \int_{t_0}^{T'} \log \left| \frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{\pi} (\alpha - 1 + it)^2 - 2E \right| dt$$

$$+ \int_{t_0}^{T'} \log \left| 1 - \frac{1}{\frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{\pi}(\alpha - 1 + it)^2 - 2E} \frac{Z'(2 - \alpha - it)}{Z(2 - \alpha - it)} \right| dt,$$

where we call the integrals of the right-hand side of (4.26) I_{1i} $(i = 1, 2, \ldots, 5)$ by turns.

[**The estimate of** I_{11}]: Obviously,

$$(4.27) \qquad I_{11} = \int_{t_0}^{T'} \log\left|\frac{N(P_{00})^{\alpha}}{L}\right| dt = O(T).$$

[**The estimate of** I_{12}]: In order to estimate I_{12}, let R_3 be the rectangle with vertices $2 - \alpha + it_0$, $3 + it_0$, $3 + iT'$ and $2 - \alpha + iT'$. Since $Z(s) \neq 0$ inside R_3, the function $\log Z(s)$ is holomorphic on this domain. By the Cauchy integral theorem,

$$0 = \int_{2-\alpha}^{3} \log Z(\sigma + it_0)\, d\sigma + i \int_{t_0}^{T'} \log Z(3 + it)\, dt$$
$$+ \int_{3}^{2-\alpha} \log Z(\sigma + iT')\, d\sigma + i \int_{T'}^{t_0} \log Z(2 - \alpha + it)\, dt.$$

By taking the imaginary parts, we have

$$\int_{t_0}^{T'} \log|Z(2 - \alpha + it)|\, dt = \int_{2-\alpha}^{3} \arg Z(\sigma + it_0)\, d\sigma + \int_{t_0}^{T'} \log|Z(3 + it)|\, dt$$
$$+ \int_{3}^{2-\alpha} \arg Z(\sigma + iT')\, d\sigma,$$

where we name the integrals of the right-hand side of the above equality I_{12i} $(i = 1, 2, 3)$ by turns.

Obviously, $I_{121} = O(1)$. By [EGM, p. 214, (5.20)] or (4.4), $\log Z(s) = O(1)$ for $\sigma > 2$, therefore, we have $\log|Z(3 + it)| = O(1)$. Hence, $I_{122} = O(T)$. In order to estimate I_{123}, we remark

$$(4.28) \quad \log Z(\sigma + iT') = \log Z(3 + iT') - \int_{\sigma}^{3} \frac{Z'(u + iT')}{Z(u + iT')}\, du \quad (2 - \alpha \le \sigma \le 3).$$

To estimate the second term in the above equality, we apply [Nak, p. 324, Lemma 3.4, (3.7)] (see [Hej, p. 102, Proposition 6.6]):

$$(4.29) \qquad \frac{Z'(1 + \varepsilon + it)}{Z(1 + \varepsilon + it)} \ll \frac{t^2}{\varepsilon} \quad (\varepsilon > 0, |t| \ge 2)$$

to (4.28). Then

$$|\log Z(\sigma + iT')| \le |\log Z(3 + iT')| + \int_{\sigma}^{3} \left|\frac{Z'(u + iT')}{Z(u + iT')}\right| du$$
$$= O(1) + O(T'^2).$$

We remark that the number α is a constant, not a parameter.

This implies

$$\arg Z(\sigma + iT') = O(T^2).$$

Then $I_{123} = O(T^2)$. By the above three estimates, we get

$$(4.30) \qquad I_{12} = \int_{t_0}^{T'} \log |Z(2 - \alpha - it)| \, dt = O(T^2).$$

[**The estimate of I_{13}**]: The estimate of I_{13} leads to the main term in our asymptotic formula for $N_1(T)$. We recall that

$$f(s) := \exp\left(\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{3\pi} (s - 1)^3 - 2E(s - 1) \right),$$

and put $s = \alpha + it$. Then

$$|f(\alpha + it)| = \exp\left(\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{3\pi} (\alpha - 1)^3 - \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi} (\alpha - 1)t^2 - 2E(\alpha - 1) \right).$$

Therefore,

$$(4.31) \qquad I_{13} = \int_{t_0}^{T'} \log |f(\alpha + it)| \, dt = (1 - \alpha) \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{3\pi} T^3 + O(T^2).$$

[**The estimate of I_{14}**]: Since

$$\left| \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi} (\alpha - 1 + it)^2 - 2E \right|$$

$$= \left| \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi} t^2 \times \left\{ \left(\frac{\alpha - 1}{it} + 1 \right)^2 + \frac{2E}{\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi} t^2} \right\} \right|,$$

we have

$$(4.32) \quad I_{14} = \int_{t_0}^{T'} \log \left| \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi} (\alpha - 1 + it)^2 - 2E \right| \, dt = 2T \log T + O(T).$$

[**The estimate of I_{15}**]: A rough estimate $I_{15} = O(T \log T)$ is deduced from [Nak, p. 324, Lemma 3.4, (3.10)] easily. For future possibilities of this sort of studies, we shall search for a better estimate than the aforesaid estimate. Let D be the trapezoid with vertices $\alpha + it_0$, $\alpha + iT'$, $-T' + iT'$ and $-t_0 + it_0$. Here, we recall (4.25). By this, the function

$$\log\left(1 - \frac{1}{\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi} (s - 1)^2 - 2E} \frac{Z'(2 - s)}{Z(2 - s)} \right)$$

is holomorphic inside D. Then we apply the Cauchy theorem to get the estimate of I_{15}, that is

$$i \int_{t_0}^{T'} \log\left(1 - \frac{1}{\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi} (\alpha - 1 + it)^2 - 2E} \frac{Z'(2 - \alpha - it)}{Z(2 - \alpha - it)} \right) dt$$

$$= \int_{-T'}^{\alpha} \log\left(1 - \frac{1}{\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi} (\sigma - 1 + iT')^2 - 2E} \frac{Z'(2 - \sigma - iT')}{Z(2 - \sigma - iT')} \right) d\sigma$$

$$+ (i-1) \int_{t_0}^{T'} \log\left(1 - \frac{1}{\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi}(\sigma + 1 - i\sigma)^2 - 2E} \frac{Z'(2 + \sigma - i\sigma)}{Z(2 + \sigma - i\sigma)}\right) d\sigma$$

$$- \int_{-t_0}^{\alpha} \log\left(1 - \frac{1}{\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi}(\sigma - 1 + it_0)^2 - 2E} \frac{Z'(2 - \sigma - it_0)}{Z(2 - \sigma - it_0)}\right) d\sigma$$

$$= I_{151} + I_{152} + I_{153},$$

say. By estimating I_{15i} $(i = 1, 2, 3)$, we shall prove $I_{15} = O(T)$.

We apply [Nak, p. 324, Lemma 3.4, (3.10)]:

$$\frac{Z'(\sigma + it)}{Z(\sigma + it)} \ll |t|^{2\max(0, 2-\sigma)} \log|t| \quad \left(\sigma > 1 + \frac{1}{\log|t|}, |t| \geq 2\right)$$

to the integrand of I_{151}, namely we see that

$$\frac{1}{\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi}(\sigma - 1 + iT')^2 - 2E} \frac{Z'(2 - \sigma - iT')}{Z(2 - \sigma - iT')} \ll \frac{T^{2\alpha}\log T}{T^2},$$

where $-T' \leq \sigma \leq \alpha$. Then the integral is estimated by $I_{151} = O(T)$. Obviously, we have $I_{153} = O(1)$. By the estimate (3.3) in Section 3, we have

$$\frac{Z'(s)}{Z(s)} = O\left(\frac{1}{N(P_{00})^\sigma}\right) \quad (\sigma \geq 4).$$

Hence, we get $I_{152} = O(1)$. These estimates imply that

$$(4.33) \quad I_{15} = \int_{t_0}^{T'} \log\left|1 - \frac{1}{\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{\pi}(\alpha - 1 + it)^2 - 2E} \frac{Z'(2 - \alpha - it)}{Z(2 - \alpha - it)}\right| dt$$
$$= O(T).$$

Substituting (4.27), (4.30), (4.31), (4.32), and (4.33) into (4.26), we have the estimate of I_1,

$$(4.34) \qquad I_1 = \int_{t_0}^{T'} \log|X(\alpha + it)| \, dt = (1 - \alpha)\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{3\pi}T^3 + O(T^2).$$

By (4.8), (4.9), (4.21), (4.23), (4.34), and Theorem 3.2, the proof of Lemma 4.3 is completed. $\qquad\square$

Proof of Theorem 1.2. In (4.7), we change α to $\alpha/2$;

$$(4.35) \qquad \sum_{0 < \gamma' \leq T}\left(\beta' - \frac{\alpha}{2}\right) = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}\left(1 - \frac{\alpha}{2}\right)T^3 + O(T^2).$$

Subtract (4.7) from (4.35) and divide it by $\alpha/2$. Then we have the asymptotic formula

$$(4.36) \qquad N_1(T) = \sum_{0 < \gamma' \leq T} 1 = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}T^3 + O(T^2).$$

This completes the proof of Theorem 1.2. $\qquad\square$

REMARKS. (i) The asymptotic formula for the number of zeros of $\zeta^{(k)}(s)$ was given by Berndt [Ber].

(ii) Error terms in (4.7) and (4.35) may depend on the constant α, but it is clear that the error term in (4.36) is independent of the constant α.

5. Concluding remarks

A good bound of multiplicity for zeros of the Selberg zeta function $Z(s)$ cannot be obtained by Theorem 1.2 (Lemma 4.3). However, the following formulas are deduced from Theorem 1.2 and Lemma 4.3.

THEOREM 5.1. *Let $\beta' + i\gamma'$ be zeros of $Z'(s)$. Then*

$$(5.1) \qquad \sum_{0<\gamma'\leq T} \beta' = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}T^3 + O(T^2),$$

$$(5.2) \qquad \sum_{0<\gamma'\leq T} (\beta' - 1) = O(T^2),$$

$$(5.3) \qquad \sum_{0<\gamma'\leq T} (\beta' - \lambda) = (1-\lambda)\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}T^3 + O(T^2),$$

where λ is any constant.

Proof. Multiplying the both sides of (4.35) by 2, we get

$$(5.4) \qquad \sum_{0<\gamma\leq T} (2\beta' - \alpha) = (2-\alpha)\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}T^3 + O(T^2).$$

Subtract (4.7) from (5.4), then formula (5.1) is deduced.

Next, subtract λ-times the formula $N_1(T)$ from (5.1), then formula (5.3) is proved.

The formula (5.2) is the special case $\lambda = 1$ in (5.3). $\square$

Formula (5.3) implies that formula (4.7) in Lemma 4.3 holds for any constant α. Formula (5.2), which is a three-dimensional analogue of [Luo2, p. 1147, Theorem 2] (see also Theorem 5.4 below), suggests that the most of the zeros of $Z'(s)$ are located near the critical line $\mathrm{Re}(s) = 1$. In fact, we can prove the following two corollaries.

COROLLARY 5.2. *Let $N_1^+(c,T)$ denote the number of nonreal zeros of $Z'(s)$ such that $\beta' \geq c$ and $0 < \gamma' \leq T$. Then, for any positive constant $\varepsilon > 0$,*

$$(5.5) \qquad N_1^+(1+\varepsilon, T) \ll \frac{T^2}{\varepsilon} \ll \frac{N_1(T)}{\varepsilon T}.$$

Proof. By Theorem 3.2 in Section 3, and formula (5.2), we have

$$\varepsilon N_1^+(1+\varepsilon, T) \leq \sum_{0<\gamma'\leq T} (\beta' - 1) \ll T^2.$$

This proves the corollary. □

This is an analogue of the corresponding result for $\zeta'(s)$, due to Levinson and Montgomery [LM, p. 50, Theorem 2].

Another corollary is the plain zero density estimate.

COROLLARY 5.3. *For any positive constant $\varepsilon > 0$, we have*

$$\text{(5.6)} \qquad \lim_{T \to \infty} \frac{\sharp\{\rho' | 1 \leq \beta' < 1 + \varepsilon, 0 < \gamma' \leq T\}}{N_1(T)} = 1.$$

Proof. Denote the number of nonreal zeros of $Z'(s)$ such that $\beta' < c$, and $0 < \gamma' \leq T$ by $\widetilde{N}_1^-(c, T)$. Since

$$\text{(5.7)} \qquad N_1(T) = \widetilde{N}_1^-(1 + \varepsilon, T) + N_1^+(1 + \varepsilon, T),$$

the following is obtained:

$$\text{(5.8)} \qquad 1 = \frac{\widetilde{N}_1^-(1 + \varepsilon, T)}{N_1(T)} + \frac{N_1^+(1 + \varepsilon, T)}{N_1(T)}.$$

By Corollary 5.2,

$$\text{(5.9)} \qquad \frac{\widetilde{N}_1^-(1 + \varepsilon, T)}{N_1(T)} = 1 + O\left(\frac{1}{\varepsilon T}\right),$$

that is

$$\text{(5.10)} \qquad \lim_{T \to \infty} \frac{\widetilde{N}_1^-(1 + \varepsilon, T)}{N_1(T)} = 1.$$

By the above and Theorem 3.2, the statement is proved. □

Finally, we will apply the above argument to the case of compact Riemann surfaces of $g \geq 2$. We use the same symbols yet written. However, notice that the critical line is $\text{Re}(s) = 1/2$.

By Luo [Luo2], the following formula was given.

THEOREM 5.4 ([Luo2, p. 1147, Theorem 2]).

$$\text{(5.11)} \qquad \sum_{0 < \gamma' \leq T} \left(\beta' - \frac{1}{2}\right) = \frac{T \log T}{2\pi} + O(T).$$

By using [Luo2, p. 1147, (8)] and the same argument in the proof of Theorem 5.1, we have the following:

THEOREM 5.5.

$$\text{(5.12)} \qquad \sum_{0 < \gamma' \leq T} \beta' = \frac{1}{2} \frac{\text{area}(\Gamma \backslash \mathbb{H}^2)}{4\pi} T^2 + \frac{T \log T}{2\pi} + O(T),$$

$$\text{(5.13)} \qquad \sum_{0 < \gamma' \leq T} (\beta' - \lambda) = \left(\frac{1}{2} - \lambda\right) \frac{\text{area}(\Gamma \backslash \mathbb{H}^2)}{4\pi} T^2 + \frac{T \log T}{2\pi} + O(T),$$

where λ is any constant.

By the same method as in the proof of Corollay 5.2, the following result is deduced from Theorem 5.4.

COROLLARY 5.6. *For any positive constant $\varepsilon > 0$, we have*

$$(5.14) \qquad N_1^+(1/2 + \varepsilon, T) \ll \frac{T \log T}{\varepsilon}.$$

In view of Theorem 1.1, we have

$$(5.15) \qquad N_1^+(1/2 + \varepsilon, T) \ll \frac{N_1(T) \log T}{\varepsilon T}.$$

Moreover, the following result is obtained by [Min, Theorem 2] and Corollary 5.6.

COROLLARY 5.7. *For any positive constant $\varepsilon > 0$,*

$$(5.16) \qquad \lim_{T \to \infty} \frac{\sharp\{\rho' | \frac{1}{2} \le \beta' < \frac{1}{2} + \varepsilon, 0 < \gamma' \le T\}}{N_1(T)} = 1.$$

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