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THE p-BANACH SAKS PROPERTY IN SYMMETRIC OPERATOR SPACES

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Abstract. Let E be a separable r.i. space which is an interpolation space between $L^r(\mathbb{R}^+)$ and $L^q(\mathbb{R}^+), 1 < r < q < \infty$. We give sufficient conditions on E implying that the symmetric operator space $E(\mathcal{M}, \tau)$ has the p-Banach-Saks property for a suitable p, for an arbitrary semifinite von Neumann algebra $\mathcal{M}$.

1. Introduction

We recall that a Banach space X has the p-Banach-Saks property (pBS for short), where $1 < p < \infty$, if every weakly null sequence in X has a p-Banach-Saks subsequence $(y_j)_{j=1}^{\infty}$, i.e., there exists a positive constant K such that

$$\overline{\lim_{n\to\infty}}\, n^{-1/p} \left\| \sum_{j=1}^{n} x_j \right\| \leq K$$

for all subsequences $(x_j)_{j=1}^{\infty} \subseteq (y_j)_{j=1}^{\infty}$.

For example, if $1 < p < \infty$, $L^p([0,1])$ has $\inf\{p, 2\}BS$ and this is sharp [B]; the same holds for non-commutative $L^p(\mathcal{M}, \tau)$ spaces; see [HRS], [S1] if the trace is finite, and [DDS, Proposition 3.2]. More generally, for a Banach space, type p implies pBS [R] (and we shall in passing give a proof of this fact in the preliminaries). Here we show that there are symmetric operator spaces $E(\mathcal{M}, \tau)$ with no type which have pBS, e.g., among Lorentz spaces (Proposition 15). More generally, we complement in the non-commutative setting results obtained for r.i. spaces $E(0, 1)$ in [SS], [ASS]: they relate the pBS property in E and the Boyd indices of E when these are non-trivial (if at least one of the indices is trivial, then E does not have pBS for any $1 < p < \infty$ [ASS, Theorem 4.2]). The Boyd indices of E are non-trivial if and only if E is an interpolation space between some L^r and L^q, $1 < r < q < \infty$ (see the definitions and references below). Assuming that E is such a space,

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we relate the pBS property in $E(\mathcal{M},\tau)$ and the disjoint pBS-property in E. Here, $(\mathcal{M},\tau)$ is an arbitrary semifinite von Neumann algebra with a faithful normal semifinite trace, $E(\mathcal{M},\tau)$ is the symmetric operator space associated with E and $(\mathcal{M},\tau)$, and the disjoint pBS property in E means that every weakly null sequence of disjointly supported elements in E has a p-Banach-Saks subsequence.

Our main results are Theorem 8 ($q \leq 2$) and Theorem 12. The pBS property in the setting of symmetric operator spaces was briefly considered in [DDS], however our results here (and some techniques) are different.

A classical application of p-Banach-Saks properties is that a Banach space with exact pBS cannot embed isomorphically in a Banach space possessing qBS, if $p < q$. We give an example in Proposition 15, which is parallel to (but does not follow from) some results in [A], [AL], [DDS].

The paper is organized as follows: in Part 1 we collect notation and preliminaries; Part 2 is devoted to the proof of Theorem 8, Part 3 to the proof of Theorem 12, Part 4 to the example of non-commutative Lorentz $L^{p,q}(\mathcal{M})$ spaces. In the Appendix we collect some facts on D*-convexity and its companion property D-convexity.

Comment: Let us compare our methods with those of previous papers: in contrast to [SS], [ASS], we do not use square functions. Besides interpolation, we use the Brunel-Sucheston Theorem as in [DDS], D*-convexity as in [DSeS, Part 5] and the characterization of relatively weakly compact sets in $E(\mathcal{M})$ [DSS]; in Part 3 we also need a splitting principle (Proposition 11) as in most papers on this topic, and a substitute of the Schmidt decomposition in non-atomic von Neumann algebras as in [DDS].

2. Definitions, notation, preliminaries

We define rearrangement invariant (r.i. for short) function spaces E on the interval $[0,a), 0 < a \leq \infty$, equipped with Lebesgue measure m, as in [LT, 2.a]. They form a subclass of fully symmetric function spaces [DDP],[LT, Definition 2.a.6, Proposition 2.a.8].

We shall deal mostly with separable r.i. spaces E: indeed, the property pBS, inherited by closed subspaces, does not hold in l^∞, and r.i. spaces which do not contain l^∞ as a closed subspace are separable.

Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra on the Hilbert space H, equipped with a faithful normal semifinite trace. We denote by $\widetilde{\mathcal{M}}$ the *-algebra of closed densely defined operators x on H which commute with the unitaries of $\mathcal{M}'$ and are $\tau-$measurable (i.e., for every $\varepsilon > 0$ there exists an orthogonal projection $P \in \mathcal{M}$ such that $\tau(\mathrm{Id} - P) < \varepsilon$ and $P(H) \subset \operatorname{dom} x, xP \in \mathcal{M}$). The sets

$$N(\epsilon,\delta)=\Big\{x\in\widetilde{\mathcal{M}}\mid \exists\, P\in\mathcal{M}, P=P^*=P^2,\\ \tau(\mathrm{Id}-P)<\varepsilon, \|xP\|_{\mathcal{M}}<\delta\Big\}$$

form a base at 0 for the (metrizable) *measure* topology on $\widetilde{\mathcal{M}}$. Let $x\in\widetilde{\mathcal{M}}$, and let $E_{|x|}$ be the spectral measure of $|x|$; the generalized singular value function of x is $\mu(x): t\to\mu_t(x)$, where, for $0\le t<\tau(\mathrm{Id}):=a$,

$$\mu_t(x)=\inf\{s\ge 0\mid \tau(E_{|x|}(s,\infty)\le t\}.$$

For $f\in\widetilde{L^\infty[0,a)}$, $\mu(f)$ is the decreasing rearrangement of $|f|$ [LT, p. 116].

Symmetric operator spaces. Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra as above and let $E=E[0,a)$ be a r.i. space. Then the symmetric operator space $E(\mathcal{M},\tau)$ is the Banach space of those $x\in\widetilde{\mathcal{M}}$ whose generalized singular value function $\mu(x)$ belongs to E, and

$$\|x\|_{E(\mathcal{M},\tau)}=\|\mu(x)\|_E\,.$$

Since there will be no ambiguity, we shall write $E(\mathcal{M})$ instead of $E(\mathcal{M},\tau)$.

A sequence $(x_n)_{n\ge1}\subseteq E(\mathcal{M})$ is two-sided disjointly supported if $r(x_n)r(x_m)=l(x_n)l(x_m)=0$, $n\ne m$, where $r(x)$ and $l(x)$ denote right and left support projections of a τ-measurable operator x.

If $E=E[0,\infty)$ is separable and $x\in E(\mathcal{M})$, then $\mu_t(x)\to_{t\to\infty}0$.

If $E=E[0,a)$ is separable, then its Köthe dual $E^\times$, coincide with its dual space E^*, and so E embeds isometrically in $E^{\times\times}$. Then the dual space of $E(\mathcal{M})$ is $E^*(\mathcal{M})=E^\times(\mathcal{M})$ and $E(\mathcal{M})$ embeds isometrically in $E^{\times\times}(\mathcal{M})$ (see [DDP], [DSS] and the references there).

In some statements of this paper we shall assume that $(\mathcal{M},\tau)$ is non-atomic, i.e., has no minimal projection. This causes no loss in generality for the main results: indeed, if $(\mathcal{M},\tau)$ has atoms, $(L^\infty[0,1]\overline{\otimes}\mathcal{M}, m\otimes\tau)$ is non-atomic and $E(\mathcal{M})$ is isometric to a closed subspace of $E(L^\infty[0,1]\overline{\otimes}\mathcal{M})$.

We set $K_n=B(l^2_n)$, identified with a closed subalgebra of $K=B(l^2)$. The canonical basis of l^2 is denoted by $(e_j)_{j\ge1}$.

We denote by $\operatorname{diag} E(K\overline{\otimes}\mathcal{M})$ the closed linear span in $E(K\overline{\otimes}\mathcal{M})$ of

$$\{\widetilde{x_j}=e_j\otimes e_j\otimes x_j\mid x_j\in E(\mathcal{M}),\quad j\ge1\}.$$

Interpolation spaces. We say that a r.i. space $E=E[0,a)$ is an interpolation space between $L^r([0,a))$ and $L^q([0,a)), 1\le r<q\le\infty$, denoted by $E\in I_{r,q}$, if every linear operator T which is bounded, $L^r([0,a))\to L^r([0,a))$ and $L^q([0,a))\to L^q([0,a))$, is also bounded on E, with $\|T\|_{E\to E}\le\max\{\|T\|_{r\to r},\|T\|_{q\to q}\}$. In particular, every r.i. space is an interpolation space between $L^1([0,a))$ and $L^\infty([0,a))$ [DDP, Theorem 2.4, Corollaries 2.6, 2.7].

If $E=E[0,\tau(\mathrm{Id}))$ belongs to $I_{r,q}$, then for every semifinite von Neumann algebra $(\mathcal{M},\tau)$, $E(\mathcal{M})$ is an interpolation space between $L^r(\mathcal{M},\tau)$ and

$L^q(\mathcal{M},\tau)$ [DDP, Theorem 3.2]. This does not imply in general that, e.g., $l^p(E(\mathcal{M}))$ is an interpolation space between $l^p(L^r(\mathcal{M},\tau))$ and $l^p(L^q(\mathcal{M},\tau))$.

D^*-convexity. An important tool when dealing with interpolation spaces is the notion of D^*-convexity:

DEFINITION 1. A r.i. space $E = E([0,a))$ is D^*-convex if there exists a constant $D > 0$ such that, for $n \geq 1$ and all $(y_i)_{i=1}^{2^n} \subseteq E$,

$$2^{-n}\sum_{i=1}^{2^n}\|y_i\|_E = \left\|\sum_{i=1}^{2^n}\chi_i^{(n)}\otimes y_i\right\|_{L^1([0,1],E)} \leq D\left\|\sum_{i=1}^{2^n}\chi_i^{(n)}\otimes y_i\right\|_{E([0,1]\times[0,a))}.$$

Here, $\chi_i^{(n)}$, $1 \leq i \leq 2^n$, denotes the characteristic function of the interval $[\frac{i-1}{2^n},\frac{i}{2^n}]$.

By [S2] a similar inequality then holds for $E(\mathcal{M})$, where $(\mathcal{M},\tau)$ is any semifinite von Neumann algebra. Considering a sequence of Rademacher functions on $[0,1]$, then replacing it by an i.i.d sequence $(\varepsilon_j)_{j\geq 1}$ of centered Bernoulli variables on a probability space $(\Omega,\mathcal{A},P)$, it follows that

$$\left\|\sum_{j=1}^{n}\varepsilon_j\otimes x_j\right\|_{L^1(\Omega,E(\mathcal{M}))} \leq D\left\|\sum_{j=1}^{n}\varepsilon_j\otimes x_j\right\|_{E(L^\infty(\Omega)\overline{\otimes}\mathcal{M}))}.$$

For more on D^*-convexity, see the appendix to this paper.

The p-Banach-Saks property.

DEFINITION 2. Let X be a Banach space and $1 < p < \infty$. A weakly null sequence $(x_j)_{j=1}^\infty \subseteq X$ is a *p-Banach-Saks sequence* if there exists a constant $K > 0$ such that, for all further subsequences $(x_{j_k})_{k=1}^\infty$,

$$\overline{\lim_{n\to\infty}}\, n^{-1/p}\left\|\sum_{k=1}^{n}x_{j_k}\right\| \leq K.$$

X has the p-Banach-Saks property (pBS) if every weakly null sequence in X has a p-Banach-Saks subsequence.

The constant K depends a priori on the sequence; by [SS, Lemma 4.2], if $X = E([0,1])$ has pBS, and $\|x_j\| \leq 1$, then the constant depends only on the space.

For technical reasons we shall also consider the following property:

DEFINITION 3. Let $E = E([0,a))$ be a r.i. space. E has the disjoint p-Banach-Saks property (disjoint pBS) if every weakly null disjointly supported sequence in E has a p-Banach-Saks subsequence.

Obviously, if E satisfies an upper p-estimate (see the definition below), then E has disjoint pBS.

p-Banach-Saks property and Boyd indices. When $X = E([0,a))$ is a separable r.i. space there are some relations between the disjoint p-Banach-Saks property and the Boyd indices $1 \leq p_E \leq q_E \leq \infty$ of E, as defined in [LT, 2.b.1 and p. 132].

Indeed, p_E is the supremum of those p such that, for some constant K,

$$\left\| \sum_{j=1}^{n} y_j \right\|_E \leq K n^{1/p}, \quad n \geq 1,$$

for every sequence $(y_j)_{j=1}^{\infty} \subseteq E$ of disjointly supported equimeasurable norm one functions, while q_E is the infimum of those q such that, for some constant K', and the same sequences,

$$\left\| \sum_{j=1}^{n} y_j \right\|_N \geq K' n^{1/q}, n \geq 1.$$

Hence, if every bounded sequence of disjointly supported functions in E is weakly convergent to 0 (see Remark 2 after Theorem 8) and if E has the p-Banach-Saks disjoint property, one must have $1 < p \leq p_E$.

Considering a sequence of Rademacher functions in $E([0,1])$, the property $q_E < \infty$ implies that the p-Banach-Saks property may hold in $E([0,a))$ only for $1 < p \leq 2$ [LT, p. 134]. Conversely, if a separable $E([0,a))$ has the p-Banach-Saks property for $1 < p \leq 2$, then $q_E < \infty$ [ASS, Theorem 4.2].

Boyd indices and interpolation. For a r.i. space $E = E([0,a))$, the following properties are equivalent:

(a) The Boyd indices are nontrivial, i.e., $1 < p_E \leq q_E < \infty$.
(b) $E \in I_{r,q}$ for some $1 < r < q < \infty$.

Moreover, in this case, $E \in I_{r,q}$ for any $r < p_E \leq q_E < q$.

Indeed, (a) implies (b) and the last assertion, by the Boyd Interpolation Theorem [LT, Theorem 2.b.11].

Conversely, let us recall that (at least if $a = \infty$, and with a suitable modification if $a < \infty$)

$$q_E = \lim_{s \to 0^+} \frac{\operatorname{Log}(s)}{\operatorname{Log} \|\sigma_s\|_{E \to E}}, \; p_E = \lim_{s \to \infty} \frac{\operatorname{Log}(s)}{\operatorname{Log} \|\sigma_s\|_{E \to E}},$$

where $\sigma_s, s > 0$, is the dilation operator:

$$\sigma_s(f)(t) = f\left(\frac{t}{s}\right), \quad t \in (0,\infty)$$

[LT, 2.b.1]. Since $p_{L^p} = q_{L^p} = p$, (b) implies, by interpolation of σ_s, that $1 < r \le p_E \le q_E \le q < \infty$, whence (a).

We recall that a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is C-*unconditional* if there exists a constant $C > 0$ such that, for all $n \ge 1$ and all scalars c_i and α_i with $|\alpha_i| = 1$,

$$\left\| \sum_{i=1}^{n} \alpha_i c_i x_i \right\| \le C \left\| \sum_{i=1}^{n} c_i x_i \right\|.$$

Hyperfinite von Neumann algebras and unconditionality constant of $E(R)$. In the following, the advantage afforded by the assumption that $(\mathcal{M},\tau) = (R,\tau)$ is hyperfinite lies in the fact that the spaces $L^r(R,\tau)$, $1 < r < \infty$, admit an unconditional finite dimensional decomposition (UFDD) $\{\mathcal{U}_n\}_{n\ge 1}, \mathcal{U}_n \subset R \cap L^1(R,\tau)$, [SF], [SF2], whose unconditionality constant C_r depends only on r. We denote by P_n the (orthogonal) projection: $L^r(R,\tau) \to \mathcal{U}_n \subset L^r(R,\tau)$. When an increasing sequence of conditional expectations $\mathcal{E}_n, n \ge 1$, onto finite dimensional Von Neumann subalgebras R_n is available (assuming $\bigcup_{n\ge 1} R_n$ is dense in R for the strong operator topology and the restriction of τ to R_n is a semifinite trace), defining $\mathcal{U}_n$ as the range of $\mathcal{E}_{n+1} - \mathcal{E}_n, n \ge 2$, $\mathcal{U}_1$ as the range of $\mathcal{E}_1$, the UFDD also comes from [PX, Remark 2.4], applied to the martingales $(\mathcal{E}_n x)_{n\ge 1}, x \in L^r(R,\tau)$.

Interpolating for every fixed choice of signs $(\varepsilon_n)_{n\ge 1}$ the operators

$$x \to \sum_{n=1}^{N} \varepsilon_n P_n(x),$$

one gets a similar UFDD in every symmetric operator space $E(R,\tau)$ associated with a separable r.i. space $E \in I_{r,q}$, $1 < r < q < \infty$, and the unconditionality constant is $C_E \le \max\{C_r, C_q\}$ [SF], [SF2].

We shall consider sequences of disjoint block projections $(\mathcal{P}_j)_{j\ge 1}$, meaning that $\mathcal{P}_j = \sum_{n=n_{j-1}}^{n_j - 1} P_n$, where $(n_j)_{j\ge 0}$ is an increasing sequence of integers and $n_0 = 1$. Then $\|\mathcal{P}_j\|_{L^r \to L^r} \le C_r$.

Note that all $L^r(R,\tau)$ for (R,τ) semifinite and hyperfinite are described in [HRS, Theorem 5.1].

Type, upper and lower estimate. Modifying the usual definition [LT, Definition 1.e.12] and using the Kahane inequalities if $p > 2$ [LT, Theorem 1.e.13], we say that a Banach space X has type $\inf\{p, 2\}$ for some $p > 1$ if there exists a constant $T_{\inf\{p,2\}}$ such that, for every finite sequence $(x_j)_{j=1}^{n} \subseteq X$, and a sequence $(\varepsilon_j)_{j\geq 1}$ of i.i.d centered Bernoulli variables on a probability space $(\Omega, \mathcal{A}, P)$,

$$\left\| \sum_{j=1}^{n} \varepsilon_j \otimes x_j \right\|_{L^p(\Omega,X)} \leq T_{\inf\{p,2\}} \left(\sum_{j=1}^{n} \|x_j\|_X^{\inf\{p,2\}} \right)^{1/\inf\{p,2\}} .$$

Let $1 \leq p, q \leq \infty$. A Banach lattice X is *p-convex*, respectively *q-concave*, if there exists a constant $C > 0$ such that for every finite sequence $(x_j)_{j=1}^n \subseteq X$,

$$\left\| \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \right\|_X \leq C \left(\sum_{j=1}^{n} \|x_j\|_X^p \right)^{1/p} ,$$

respectively,

$$\left(\sum_{j=1}^{n} \|x_j\|_X^q \right)^{1/q} \leq C \left\| \left(\sum_{j=1}^{n} |x_j|^q \right)^{1/q} \right\|_X .$$

If the inequalities above hold only for an arbitrary choice of pairwise disjoint elements $(x_i)_{i=1}^n$ in X, then X is said to satisfy an *upper p-estimate* (respectively, *lower q-estimate*).

The Brunel-Sucheston trick. The following is a classical tool when considering Banach-Saks properties; see, e.g., [HRS, p. 47].

Let $(y_j)_{j\geq 1}$ be a weakly null sequence in a Banach space X. By the Brunel-Sucheston Theorem [BS], there exists a subsequence $(x_j)_{j\geq 1}$ such that, for every $k \geq 1$, the finite sequence $(x_j)_{j=k}^{2^k}$ is $4-$unconditional.

Assume moreover that every $4-$unconditional finite subsequence of $(x_j)_{j\geq 1}$ satisfies for some $p \geq 1$, some constant K and every admissible n

$$\left\| \sum_{k=1}^{n} x_{j_k} \right\|_X \leq K n^{1/p} \sup_j \|x_j\|_X . \tag{1}$$

Then

$$\overline{\lim_{n\to\infty}}\, n^{-1/p} \left\| \sum_{j=1}^{n} x_j \right\|_X \leq K \sup_j \|x_j\|_X .$$

Indeed, for every n, choose k such that $2^{k-1} \leq n < 2^k$. Then

$$\begin{aligned} \left\| \sum_{j=1}^{n} x_j \right\|_X &\leq \sum_{j=1}^{k-1} \|x_j\|_X + \left\| \sum_{j=k}^{n} x_j \right\|_X \\ &\leq (\log_2 n + K(n - \log_2 n)^{1/p}) \sup_j \|x_j\|_X . \end{aligned}$$

The trick shows, in particular, that for a Banach space, type p implies pBS [R]. Indeed, (1) is satisfied if X has type p (with constant T_p) since, for every $4-$unconditional sequence $(z_k)_{k\geq 1}$ in X,

$$\begin{aligned}\left\|\sum_{k=1}^{n} z_k\right\|_X &\leq 4\left\|\sum_{k=1}^{n}\varepsilon_k\otimes z_k\right\|_{L^1(\Omega,X)}\\ &\leq 4T_p\left(\sum_{k=1}^{n}\|z_k\|_X^p\right)^{1/p}\\ &\leq 4T_p n^{1/p}\sup_k\|z_k\|_X .\end{aligned}$$

3. Results in the $\mathbf{I}_{r,2}$ case

In this part $(\mathcal{M},\tau)$ may have atoms.

Though type inequalities cannot be interpolated in general, one has the following result, which can be compared to [ASS, Lemma 3.5] in the commutative setting:

PROPOSITION 4. *Let $E = E(0,\infty) \in I_{r,2}, 1 < r < 2$, be a separable r.i. space. Let $(x_j)_{j=1}^n$ be a sequence in $E(\mathcal{M})$. Then:*

(a) *If $\mathcal{M} = R$ is hyperfinite, for any sequence of disjoint block projections $(\mathcal{P}_j)_{j\geq 1}$ (associated with an UFDD $\{\mathcal{U}_n\}_{n\geq 1}$)*

$$\left\|\sum_{j=1}^{n}\mathcal{P}_j x_j\right\|_{E(R)} \leq C_r^2\left\|\sum_{j=1}^{n} e_j\otimes e_j\otimes x_j\right\|_{E(K_n\overline{\otimes}R)} \qquad n\geq 1,$$

where C_r is the unconditionality constant of $\{\mathcal{U}_n\}_{n\geq 1}$ in $L^r(R,\tau)$.

(b) *If E is D^*-convex with constant D,*

$$\left\|\sum_{j=1}^{n}\varepsilon_j\otimes x_j\right\|_{L^1(\Omega,E(\mathcal{M}))} \leq D\left\|\sum_{j=1}^{n} e_j\otimes e_j\otimes x_j\right\|_{E(K_n\overline{\otimes}\mathcal{M})}.$$

Proof. By complex interpolation between 1 and 2, $L^p(\mathcal{M})$ has type p with constant $T_p = 1$ for $1\leq p\leq 2$; in particular,

$$\begin{aligned}\left\|\sum_{j=1}^{n}\varepsilon_j\otimes x_j\right\|_{L^p(\Omega,L^p(\mathcal{M}))} &\leq \left(\sum_{j=1}^{n}\|x_j\|_{L^p(\mathcal{M})}^p\right)^{1/p}\\ &= \left\|\sum_{j=1}^{n} e_j\otimes e_j\otimes x_j\right\|_{L^p(K_n\overline{\otimes}\mathcal{M})}.\end{aligned} \tag{2}$$

Since $\operatorname{diag} L^p(K\overline{\otimes}\mathcal{M})$ is $1-$complemented in $L^p(K\overline{\otimes}\mathcal{M}), 1 \leq p \leq \infty$, the functor which interpolates $E(K\overline{\otimes}\mathcal{M})$ between $L^r(K\overline{\otimes}\mathcal{M})$ and $L^2(K\overline{\otimes}\mathcal{M})$ also interpolates $\operatorname{diag} E(K\overline{\otimes}\mathcal{M})$ between $\operatorname{diag} L^r(K\overline{\otimes}\mathcal{M})$ and $\operatorname{diag} L^2(K\overline{\otimes}\mathcal{M})$.

(a) We have to prove that the linear mappings

$$A_n : \sum_{j=1}^{n} e_j \otimes e_j \otimes x_j \to \sum_{j=1}^{n} \mathcal{P}_j x_j$$

are uniformly bounded: $\operatorname{diag} E(K_n\overline{\otimes}R) \to E(R)$. By interpolation it is enough to verify that the norm of A_n: $\operatorname{diag} L^p(K_n\overline{\otimes}R) \to L^p(R)$ is uniformly bounded for $p = r > 1$ and $p = 2$.

Owing to the UFDD of $L^p(R)$, the sequence $(\mathcal{P}_j x_j)_{j=1}^n$ is C_p-unconditional in $L^p(R), 1 < p < \infty$, hence (2) implies for $1 < p \leq 2$

$$\begin{aligned}\left\|\sum_{j=1}^{n} \mathcal{P}_j x_j\right\|_{L^p(R)} &\leq C_p \left\|\sum_{j=1}^{n} \varepsilon_j \otimes \mathcal{P}_j x_j\right\|_{L^p(\Omega, L^p(R))} \\ &\leq C_p \left(\sum_{j=1}^{n} \|\mathcal{P}_j x_j\|_{L^p(R)}^p\right)^{1/p} \\ &\leq C_p^2 \left(\sum_{j=1}^{n} \|x_j\|_{L^p(R)}^p\right)^{1/p} \\ &= C_p^2 \left\|\sum_{j=1}^{n} e_j \otimes e_j \otimes x_j\right\|_{L^p(K_n\overline{\otimes}R)},\end{aligned}$$

which proves the claim since $C_2 = 1$.

(b) We have to prove that the linear mappings

$$B_n : \sum_{j=1}^{n} e_j \otimes e_j \otimes x_j \to \sum_{j=1}^{n} \varepsilon_j \otimes x_j$$

are uniformly bounded: $\operatorname{diag} E(K_n\overline{\otimes}\mathcal{M}) \to L^1(\Omega, E(\mathcal{M}))$. By D^*-convexity of E,

$$\left\|\sum_{j=1}^{n} \varepsilon_j \otimes x_j\right\|_{L^1(\Omega, E(\mathcal{M}))} \leq D \left\|\sum_{j=1}^{n} \varepsilon_j \otimes x_j\right\|_{E(L^\infty(\Omega)\overline{\otimes}\mathcal{M}))}.$$

Hence it is enough to prove that B_n has norm 1: $\operatorname{diag} E(K_n\overline{\otimes}\mathcal{M}) \to E(L^\infty(\Omega)\overline{\otimes}\mathcal{M})$. By interpolation, it is enough to know that B_n has norm 1: $\operatorname{diag} L^p(K_n\overline{\otimes}\mathcal{M}) \to L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M})) = L^p(\Omega, L^p(\mathcal{M}))$, $1 \leq p \leq 2$, which is (2). $\square$

We define $L_0(0,\infty)$ to be the norm closure of $L^1(0,\infty) \cap L^\infty(0,\infty)$ in $L^1(0,\infty) + L^\infty(0,\infty)$.

LEMMA 5. *Let $E = E(0,\infty)$ be a separable r.i. space such that $E^\times \subset L_0(0,\infty)$. Then a sequence $(x_j)_{j\geq 1}$ is relatively weakly compact in $E(\mathcal{M})$ if and only if the sequence $(\mu(x_j))_{j\geq 1}$ is relatively weakly compact in E.*

Proof. Since $E(\mathcal{M})$ isometrically embeds in $E^{\times\times}(\mathcal{M})$, the condition on $(x_j)_{j\geq 1}$ also reads: $(x_j)_{j\geq 1}$ is relatively $\sigma(E^{\times\times}(\mathcal{M}), E^\times(\mathcal{M}))$ compact, and similarly for the condition on $(\mu(x_j))_{j\geq 1}$. Hence the lemma comes from [DSS, Theorem 5.4]. □

The following result comes from [CDS, Lemma 2.6] (see also [Ra, Proposition 2.4]).

LEMMA 6. *Let $E = E(0,\infty)$ be a separable r.i. space such that $E^\times \subset L_0(0,\infty)$. Let $(\widetilde{x_j})_{j\geq 1}$ be a two-sided disjointly supported sequence in $E(\mathcal{M})$. Then there exists a sequence $(f_j)_{j\geq 1}$ of disjointly supported elements in E, satisfying $\mu(\widetilde{x_j}) = \mu(f_j), j \geq 1$, and*

$$\left\| \sum_{j=1}^{n} \widetilde{x_j} \right\|_{E(\mathcal{M})} = \left\| \sum_{j=1}^{n} f_j \right\|_{E}, \quad n \geq 1.$$

COROLLARY 7. *Let $E = E(0,\infty)$ be a separable r.i. space such that $E^\times \subset L_0(0,\infty)$ and assume that E has disjoint pBS. Let $(y_j)_{j\geq 1}$ be a weakly null sequence in $E(\mathcal{M})$. Then every two-sided disjointly supported sequence $(\widetilde{y_j})_{j\geq 1}$ in $E(\mathcal{M})$ such that $\mu(y_j) = \mu(\widetilde{y_j}), j \geq 1$, has a subsequence $(\widetilde{x_j})_{j\geq 1} \subset (\widetilde{y_j})_{j\geq 1}$ satisfying, for some constant K and all further subsequences*

$$\overline{\lim_n}\, n^{-1/p} \left\| \sum_{k=1}^{n} \widetilde{x_{j_k}} \right\|_{E(\mathcal{M})} \leq K.$$

Proof. (a) Since $(y_j)_{j\geq 1}$ is weakly null, $(\mu(y_j))_{j\geq 1}$ is relatively weakly compact in E by Lemma 5; by Lemma 5 again, so is the sequence $(f_j)_{j\geq 1}$ associated to $(\widetilde{y_j})_{j\geq 1}$ as in Lemma 6, since $\mu(y_j) = \mu(\widetilde{y_j}) = \mu(f_j)$. We claim that $(f_j)_{j\geq 1}$ is weakly null, i.e., $\int g f_j dm \to_{j\to\infty} 0$ for every $g \in E^* = E^\times$. Indeed, if this does not hold, there exist a subsequence $(f_{j_k})_{k\geq 1}$ and $\varepsilon > 0$ such that

$$\|g f_{j_k}\|_{L^1(0,\infty)} \geq \left| \int g f_{j_k} dm \right| \geq \varepsilon, \quad k \geq 1.$$

Since $(g f_{j_k})_{k\geq 1}$ is disjointly supported, it is equivalent in $L^1(0,\infty)$ to the canonical basis $(e_k)_{k\geq 1}$ of l^1; on the other hand, $(g f_j)_{j\geq 1}$ is relatively $\sigma(L^1, L^\infty)$ compact, and hence $(e_k)_{k\geq 1}$ is $\sigma(l^1, l^\infty)$ compact. This contradiction proves the claim.

(b) The sequence $(f_j)_{j\geq 1}$, being a weakly null disjointly supported sequence in E, has by assumption a p-Banach-Saks subsequence. By Lemma 6, the corresponding subsequence $(\widetilde{x_j})_{j\geq 1} \subset (\widetilde{y_j})_{j\geq 1}$ satisfies the conclusion of the corollary. □

THEOREM 8. *Let $E = E(0,\infty) \in I_{r,2}, 1 < r < 2$, be a separable r.i. space and assume that $E(0,\infty)$ has disjoint pBS, $1 < p \leq 2$. Then $E(\mathcal{M})$ has the pBS property if either $\mathcal{M} = R$ is hyperfinite, or E is D^*-convex.*

Proof. First note that $E^\times \subset L^{r'} + L^2 \subset L_0(0,\infty)$, $\frac{1}{r} + \frac{1}{r'} = 1$.

For a sequence $(x_j)_{j\geq 1}$ in $E(\mathcal{M})$, the sequence $(\widetilde{x_j})_{j\geq 1} = (e_j \otimes e_j \otimes x_j)_{j\geq 1}$ is a two-sided disjointly supported sequence in $E(K\overline{\otimes}\mathcal{M})$.

(a) Let $(y_j)_{j\geq 1}$ be a weakly null sequence in $E(R)$. We can extract a subsequence $(x_j)_{j\geq 1}$ and find a sequence of disjoint block projections $(\mathcal{P}_j)_{j\geq 1}$ such that $\|x_j - \mathcal{P}_j x_j\|_{E(R)} \leq \frac{1}{2^j}$, $j \geq 1$. By Proposition 4(a)

$$\left\|\sum_{j=1}^n \mathcal{P}_j x_j\right\|_{E(R)} \leq C_r^2 \left\|\sum_{j=1}^n \widetilde{x_j}\right\|_{E(K\overline{\otimes}R)}.$$

From the two-sided disjointly supported sequence $(\widetilde{x_j})_{j\geq 1}$ we extract a subsequence associated to a constant K as in Corollary 7. The corresponding subsequence of $(x_j)_{j\geq 1}$ is p-Banach-Saks in $E(R)$ since for further subsequences $(x_{j_k})_{k\geq 1}$

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, n^{-1/p} \left\|\sum_{k=1}^n x_{j_k}\right\|_{E(R)} &= \overline{\lim_{n\to\infty}}\, n^{-1/p} \left\|\sum_{k=1}^n \mathcal{P}_{j_k} x_{j_k}\right\|_{E(R)} \\ &\leq C_r^2 \overline{\lim_{n\to\infty}}\, n^{-1/p} \left\|\sum_{k=1}^n \widetilde{x_{j_k}}\right\|_{E(K\overline{\otimes}R)} \\ &\leq C_r^2 K.\end{aligned}$$

(b) Let $(y_j)_{j\geq 1}$ be a weakly null sequence in $E(\mathcal{M})$. By Corollary 7, there exists $K > 0$ and a subsequence $(\widetilde{z_j})_{j\geq 1} \subset (\widetilde{y_j})_{j\geq 1} \subset E(K\overline{\otimes}\mathcal{M})$, such that, for every further subsequence $(\widetilde{x_j})_{j\geq 1} \subset (\widetilde{z_j})_{j\geq 1}$, there exists N satisfying

$$\forall n \geq N \quad \left\|\sum_{j=1}^n \widetilde{x_j}\right\|_{E(K\overline{\otimes}\mathcal{M})} \leq K n^{1/p}. \tag{3}$$

Let us extract from $(z_j)_{j\geq 1} \subset (y_j)_{j\geq 1}$ (corresponding to $(\widetilde{z_j})_{j\geq 1} \subset (\widetilde{y_j})_{j\geq 1}$) a subsequence $(x_j)_{j\geq 1}$ as in the Brunel-Sucheston Theorem; let $(\widetilde{x_j})_{j\geq 1} \subset (\widetilde{z_j})_{j\geq 1}$ be the corresponding sequence and let N be defined as in (3) for this $(\widetilde{x_j})_{j\geq 1}$.

Every 4-unconditional finite subsequence $(x_{j_k})_{1\le k\le m} \subset (x_j)_{j\ge 1}$ satisfies, by Proposition 4(b) and (3), for $N \le n \le m$,

$$\begin{aligned}\left\|\sum_{k=1}^{n} x_{j_k}\right\|_{E(\mathcal{M})} &\le 4\left\|\sum_{k=1}^{n} \varepsilon_j \otimes x_{j_k}\right\|_{L^1(\Omega,E(\mathcal{M}))}\\ &\le 4D\left\|\sum_{k=1}^{n} \widetilde{x_{j_k}}\right\|_{E(\mathcal{M}\overline{\otimes}K)}\\ &\le 4DKn^{1/p}.\end{aligned}$$

Hence we may apply the Brunel-Sucheston trick to $(x_j)_{j\ge 1}$ in $E(\mathcal{M})$, so that $(x_j)_{j\ge 1}$ is a p-Banach-Saks sequence. □

Remark 9. The proof of Theorem 8 is easier if E satisfies an upper p-estimate with constant K_p instead of having only disjoint pBS, and Corollary 7 is not needed in this case.

Indeed, for the first step in the proof of (b), we define $(f_j)_{j\ge 1}$ associated to $(\widetilde{y_j})_{j\ge 1}$ as in Lemma 6, hence, for all further subsequences,

$$\left\|\sum_{k=1}^{n} \widetilde{y_j}_k\right\|_{E(K\overline{\otimes}\mathcal{M})} = \left\|\sum_{k=1}^{n} f_{j_k}\right\|_{E} \le K_p n^{1/p} \sup_j \|y_j\|_{E(\mathcal{M})}.$$

A similar argument works for (a).

Motivated by the proof of Corollary 7, one may wonder when every bounded disjointly supported sequence in a separable r.i. space is weakly null. We are indebted to Yves Raynaud for the following remark:

Remark 10. For a Banach lattice X, the following properties are equivalent:

(i) Every bounded disjointly supported sequence in X is weakly null.
(ii) X does not contain a sublattice order isomorphic to l^1.

If moreover X is a separable Köthe function space, these conditions are also equivalent to:

(iii) X does not contain a closed subspace isomorphic to l^1.

Proof.
(i)⟹(ii) is obvious since the canonical basis of l^1 does not satisfy (i).
(ii)⟹(i): Assume (i) does not hold. Then there exists a bounded disjointly supported sequence $(f_j)_{j\ge 1}$ in X, with $\|f_j\|_X \le C$, there exists $g \in X^*$, $\|g\|_{X^*} = 1$, and $\varepsilon > 0$, such that $g(f_j) \ge \varepsilon, j \ge 1$. Hence for $\lambda_j \ge 0, n \ge 1$,

$$C\sum_{j=1}^{n} \lambda_j \ge \left\|\sum_{j=1}^{n} \lambda_j f_j\right\|_{E} \ge \sum_{j=1}^{n} \lambda_j g(f_j) \ge \varepsilon \sum_{j=1}^{n} \lambda_j.$$

Since

$$\left|\sum_{j=1}^{n} \lambda_j f_j\right| = \left|\sum_{j=1}^{n} \lambda'_j f_j\right|$$

as soon as $\lambda_j = \left|\lambda'_j\right|, j \geq 1$, the closed span of the f_j's is order isomorphic to l^1, and it is obviously a sublattice of X.

(iii)$\Longrightarrow$(ii) is obvious.

(ii)$\Longrightarrow$(iii): By [MN, Proposition 2.3.12], (ii) implies that the lattice X^* does not contain a sublattice order isomorphic to l^∞. Since X^* is a lattice and a dual, it is $\sigma-$order complete; by [LT, Proposition 1.a.7] applied to X^*, X^* is $\sigma-$order continuous.

If moreover X is a separable Köthe function space on the measure space (Ω, Σ, μ), X is $\sigma-$order complete and $\sigma-$order continuous, hence $X^* = X^\times$ by [LT, Theorem 1.b.14]. By this same theorem applied to $X^\times$, $L^1 \cap L^\infty(\Omega, \Sigma, \mu)$ is norm dense in $X^\times$. Every $f \geq 0$ in $L^1 \cap L^\infty$, being a.s. a pointwise limit of an increasing sequence $(f_n)_{n\geq 1}$ of positive integrable step functions, is also a limit of $(f_n)_{n\geq 1}$ for the norm of X^* since X^* is $\sigma-$order continuous. Since (Ω, Σ, μ) is separable, it follows that $X^\times$ is separable, which together with the equality $X^* = X^\times$ implies (iii). □

4. Results in the $\mathbf{I}_{r,q}$ case

4.1. Subsequence splitting principle. If E has the Fatou property (i.e. $E = E^{\times\times}$) the following result is [DDS, Proposition 2.7]. In the general case the proof follows along the same lines (see also [ASS, Lemma 3.6]) and is therefore omitted.

Proposition 11. *Let $E = E(0,\infty)$ be a separable r.i. space and let $\mathcal{M}$ be non-atomic. Let $(y_n)_{n\geq 1}$ be a sequence in $E(\mathcal{M})$, such that $\|y_n\|_{E(\mathcal{M})} = 1, n \geq 1$. Then there exist a subsequence $(x_n)_{n\geq 1} \subseteq (y_n)_{n\geq 1}$, sequences $(u_n)_{n\geq 1}, (v_n)_{n\geq 1}$, $(w_n)_{n\geq 1}$ in $E(\mathcal{M})$, an element $u \in E(\mathcal{M})^{\times\times}$, with $\|u\|_{E(\mathcal{M})^{\times\times}} \leq 1$, such that*

$$x_n = u_n + v_n + w_n, \quad n \geq 1,$$
$$\mu(u_n) \leq \mu(u), \quad n \geq 1,$$
$$\|w_n\|_{E(\mathcal{M})} \to_{n\to\infty} 0,$$

and $(v_n)_{n\geq 1}$ is a bounded two-sided disjointly supported sequence which tends to 0 for the measure topology.

If, in addition, the sequence $(y_n)_{n\geq 1}$ is weakly null and $E^\times \subseteq L_0[0,\infty)$, then the sequences $(u_n)_{n\geq 1}, (v_n)_{n\geq 1}$ may be chosen to be weakly null as well.

If E has the Fatou property, the sequence $(u_n)_{n\geq 1}$ may be chosen equimeasurable.

The next result looks similar to Theorem 8, but is actually deeper and uses the splitting lemma. Indeed there is no trick analogous to Proposition 4.

THEOREM 12. *Let $E = E(0,\infty) \in I_{r,q}, 1 < r < q < \infty$, be a separable r.i. space. Let $\alpha = \inf\{r, 2\}$. Assume that $E(0,\infty)$ has the disjoint αBS property and that $\mathcal{M}$ is non-atomic. Then $E(\mathcal{M})$ has the αBS property as soon as either $\mathcal{M} = R$ is hyperfinite or E is D^*-convex.*

We first need two lemmas. The next one is [DDP, Theorem 3.5], a generalization of the Schmidt decomposition for compact operators on a Hilbert space.

LEMMA 13. *Let $(\mathcal{M}, \tau)$ be a semifinite non-atomic von Neumann algebra. If $x \in \widetilde{\mathcal{M}}$ and $\mu_t(x) \to_{t\to\infty} 0$, then there exists a positive rearrangement-preserving algebra $*$-isomorphism $J_{|x|}$ of $\widetilde{L^\infty(0,\infty)}$ into $\widetilde{\mathcal{M}}$ such that*

$$J_{|x|}(\mu(x)) = |x|.$$

In particular, for $f \in \widetilde{L^\infty(0,\infty)}$,

$$\mu(f) = \mu(J_{|x|}(f)).$$

LEMMA 14. *Let $(\mathcal{M}, \tau)$ be a semi-finite non-atomic von Neumann algebra. Let $U_1, \ldots, U_n \in \mathcal{M}$ be fixed partial isometries, let $h_1, \ldots, h_n$ be fixed functions in $L^\infty(0,\infty)$, with $\|h_k\|_\infty \le 1, 1 \le k \le n$, and let $J_1, \ldots, J_n$ be fixed positive *-isomorphisms: $\widetilde{L^\infty(0,\infty)}) \to \widetilde{\mathcal{M}}$ which preserve decreasing rearrangements, as in Lemma 13.*

(a) *Let $\mathcal{M} = R$ be moreover hyperfinite and let $(\mathcal{P}_k)_{k\ge1}$ be a sequence of disjoint block projections associated with the UFDD $(\mathcal{U}_n)_{n\ge1}$. Let $\mathcal{S}_n$ be the linear mapping:*

$$f \to \sum_{k=1}^{n} \mathcal{P}_k U_k J_k(h_k f).$$

Then, for $1 < p < \infty$,

$$\|\mathcal{S}_n\|_{L^p(0,\infty)\to L^p(R)} \le C_p^2 T_{\inf\{p,2\}} n^{1/\inf\{p,2\}},$$

where C_p is the unconditionality constant of $\{\mathcal{U}_n\}_{n\ge1}$ in $L^p(R,\tau)$.

(b) *Let $\mathcal{R}_n$ be the linear mapping:*

$$f \to \sum_{k=1}^{n} \varepsilon_k \otimes U_k J_k(h_k f).$$

Then, for $1 \le p < \infty$,

$$\|\mathcal{R}_n\|_{L^p(0,\infty)\to L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M})} \le T_{\inf\{p,2\}} n^{1/\inf\{p,2\}}.$$

(c) *Let* $E = E(0,\infty) \in I_{r,q}, 1 < r < q < \infty$, *be a separable r.i. space, let* $\alpha = \inf\{r, 2\}$. *Then there is a constant* C *such that*

$$\|\mathcal{S}_n\|_{E\to E(R)} \le Cn^{1/\alpha}$$

and

$$\|\mathcal{R}_n\|_{E\to E(L^\infty(\Omega)\overline{\otimes}\mathcal{M})} \le Cn^{1/\alpha}.$$

Proof. Each J_k is by definition an isometry: $L^p(0,\infty) \to L^p(\mathcal{M})$. Since $L^p(\mathcal{M})$ has type $\alpha = \inf\{p, 2\}$ and $L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M}) = L^p(\Omega, L^p(\mathcal{M}))$,

$$\left\|\sum_{k=1}^n \varepsilon_k \otimes x_k\right\|_{L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M})} \le T_\alpha \left(\sum_{k=1}^n \|x_k\|^\alpha_{L^p(\mathcal{M})}\right)^{1/\alpha}. \tag{4}$$

(a) Since every sequence $(\mathcal{P}_k x_k)_{k=1}^n$ is C_p-unconditional in $L^p(R), 1 < p < \infty$, (4) implies

$$\begin{aligned}\left\|\sum_{k=1}^n \mathcal{P}_k U_k J_k(h_k f)\right\|_{L^p(R)} &\le C_p \left\|\sum_{k=1}^n \varepsilon_k \otimes \mathcal{P}_k U_k J_k(h_k f)\right\|_{L^p(\Omega, L^p(R))} \\ &\le C_p T_\alpha \left(\sum_{k=1}^n \|\mathcal{P}_k U_k J_k(h_k f)\|^\alpha_{L^p(\mathcal{M})}\right)^{1/\alpha} \\ &\le C_p^2 T_\alpha n^{1/\alpha} \|f\|_{L^p(0,\infty)}.\end{aligned}$$

(b) Similarly, by (4),

$$\begin{aligned}\left\|\sum_{k=1}^n \varepsilon_k \otimes U_k J_k(h_k f)\right\|_{L^p(L^\infty(\Omega)\overline{\otimes}\mathcal{M})} &\le T_\alpha \left(\sum_{k=1}^n \|U_k J_k(h_k f)\|^\alpha_{L^p(\mathcal{M})}\right)^{1/\alpha} \\ &\le T_\alpha n^{1/\alpha} \|f\|_{L^p(0,\infty)}.\end{aligned}$$

(c) This follows by interpolation from (a), (b) applied to $p = r$ and $p = q$, since $\inf\{r, 2\} < \inf\{q, 2\}$. □

Proof of Theorem 12. First note that $E^\times \subset L^{r'} + L^{q'} \subset L_0(0,\infty)$, $\frac{1}{r} + \frac{1}{r'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$. By the splitting principle, applying Corollary 7 to the two-sided disjointly supported part, it suffices to find a constant K such that

$$\overline{\lim_n}\, n^{-1/\alpha} \left\|\sum_{k=1}^n x_{j_k}\right\|_{E(\mathcal{M})} \le K \sup_j \|x_j\|_{E(\mathcal{M})}$$

for all subsequences of some $(x_j)_{j\ge1} \subset (y_j)_{j\ge1}$, where $(y_j)_{j\ge1} \subset E(\mathcal{M})$ is weakly null and satisfies $\mu(y_j) \le \mu(u), j \ge 1,\ u \in E(\mathcal{M})^{\times\times}$. Let $y_j = U_j\,|y_j|$ be the polar decomposition of y_j. By Proposition 11 and Lemma 13, since E is separable, there exists J_j such that $|y_j| = J_j(\mu(y_j)), j \ge 1$.

(a) Since $(y_j)_{j\geq 1}$ is weakly null, one may extract a subsequence $(x_j)_{j\geq 1} \subset (y_j)_{j\geq 1}$, and find a sequence of disjoint block projections $(\mathcal{P}_j)_{j\geq 1}$ such that

$$\|x_j - \mathcal{P}_j x_j\|_{E(R)} \leq \frac{1}{2^j}, \quad j \geq 1.$$

For any fixed n, let $f_n(t) = \sup_{1\leq j\leq n} \mu_t(x_j) \in E$ and define $h_j^{(n)}$ by

$$\mu_t(x_j) = h_j^{(n)}(t) f_n(t), 1 \leq j \leq n.$$

In particular,

$$\|f_n\|_E \leq \|\mu(u)\|_{E^{\times\times}} \leq \sup_j \|y_j\|_{E(R)}.$$

By Lemma 14(c) applied to the operator $\mathcal{S}_n$ defined by these $\mathcal{P}_j, U_j, J_j, h_j^{(n)}$,

$$\begin{aligned}\left\|\sum_{j=1}^n \mathcal{P}_j x_j\right\|_{E(R)} &= \left\|\sum_{j=1}^n \mathcal{P}_j U_j J_j (h_j^{(n)} f_n)\right\|_{E(R)} \\ &\leq C n^{1/\alpha} \|f_n\|_E \\ &\leq C n^{1/\alpha} \sup_j \|y_j\|_{E(R)}.\end{aligned}$$

Since a similar inequality holds for all subsequences of $(x_j)_{j\geq 1}$, the claim is proved.

(b) Since $(y_j)_{j\geq 1}$ is weakly null, one may extract a subsequence $(x_j)_{j\geq 1} \subset (y_j)_{j\geq 1}$ as in the Brunel-Sucheston Theorem. By the Brunel-Sucheston trick, it suffices to get the estimate

$$\left\|\sum_{k=1}^n x_{j_k}\right\|_{E(\mathcal{M})} \leq 4 D n^{1/\alpha} \sup_j \|y_j\|_{E(\mathcal{M})}$$

when the finite subsequence $(x_{j_k})_{k=1}^n$ is $4-$unconditional in $E(\mathcal{M})$. Defining f_n and $h_{j_k}^{(n)}, 1 \leq k \leq n$, as in (a), let us consider the mapping $\mathcal{R}_n$ defined as in Lemma 14(c) by the corresponding $U_{j_k}, J_{j_k}, h_{j_k}^{(n)}$. Since E is D^*-convex, Lemma 14(c) implies

$$\begin{aligned}\left\|\sum_{k=1}^{n} x_{j_k}\right\|_{E(\mathcal{M})} &\leq 4\left\|\sum_{k=1}^{n} \varepsilon_k \otimes x_{j_k}\right\|_{L^1(\Omega,E(\mathcal{M}))} \\ &\leq 4D\left\|\sum_{k=1}^{n} \varepsilon_k \otimes x_{j_k}\right\|_{E(L^\infty(\Omega)\overline{\otimes}\mathcal{M}))} \\ &= 4D\left\|\mathcal{R}_n(f_n)\right\|_{E(L^\infty(\Omega)\overline{\otimes}\mathcal{M}))} \\ &\leq 4DCn^{1/\alpha}\left\|f_n\right\|_E \\ &\leq 4DCn^{1/\alpha}\sup_j \left\|y_j\right\|_{E(\mathcal{M})}. \qquad \square\end{aligned}$$

5. Examples and application: the Lorentz $L^{p,q}(\mathcal{M})$ spaces

We recall that $x \in L^{p,q}(\mathcal{M},\tau)$, $1 < p < \infty$, $1 \leq q \leq \infty$, if

$$\|x\|_{L^{p,q}} = \begin{cases} \left(\frac{q}{p}\int_0^\infty (\mu_t(x)t^{1/p})^q \frac{dt}{t}\right)^{1/q}, & q < \infty, \\ \sup_{0<t<\infty} \mu_t(x)t^{1/p}, & q = \infty, \end{cases}$$

is finite. The expression $\|\cdot\|_{L^{p,q}}$ is a norm for $q \leq p$ and is equivalent to a norm for $q > p$ [LT, p. 142]. If $\mathcal{M}$ coincides with the algebra l^∞ of all bounded complex sequences, $L^{p,q}(\mathcal{M})$ coincides with the classical sequence space $l^{p,q}$.

The spaces $L^{p,q} = L^{p,q}(0,\infty)$ are separable for $q < \infty$ and reflexive for $1 < q < \infty$. The spaces $L^{p,\infty}$ are not separable. However, the spaces $L_0^{p,\infty} = \overline{L_1 \cap L_\infty}^{\|\cdot\|_{L^{p,\infty}}}$ are separable r.i. spaces.

The spaces $L^{p,q}$ belong to $I_{p-\epsilon,p+\epsilon}$ for every $\varepsilon > 0$, $1 < p < \infty, 1 \leq q \leq \infty$, so that their Boyd indices are $p_E = q_E = p$.

The spaces $L^{p,q}$ are D^*-convex if $1 \leq q \leq p$ [S2].

The spaces $L^{p,q}$ have type $\inf\{p,q,2\}$ if $1 < p, q < \infty$ and $p \neq 2$ [N], [C].

The spaces $L^{2,q}$ have type q if $1 < q \leq 2$, and $(2-\varepsilon)$ for every $\varepsilon > 0$ if $2 < q < \infty$ [N], [C].

The spaces $L_0^{p,\infty}$ are 2-convex if $2 \leq p < \infty$, and satisfy an upper p-estimate for $1 < p < 2$ (see [C] or [SS, Lemma 5.6]).

The spaces $L_0^{p,\infty}$ and $L^{p,1}, 1 < p < \infty$ have no type.

The following proposition extends to the non-commutative setting the results of [SS, Part 5].

Proposition 15. *Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra. Then:*

(a) *$L^{p,q}(\mathcal{M},\tau)$ has $\inf\{p,q,2\}BS$ if $1 < p, q < \infty$ and $p \neq 2$.*
(b) *$L^{2,q}(\mathcal{M},\tau)$ has qBS if $1 < q \leq 2$ and $(2-\varepsilon)BS$ for every $\varepsilon > 0$ (but not $2BS$) if $2 < q < \infty$.*
(c) *$L^{p,1}(\mathcal{M},\tau)$ has $\inf\{p,2\}BS$ if $1 < p < \infty$ and $p \neq 2$, $\mathcal{M}$ being non-atomic.*

(d) $L^{2,1}((\mathcal{M},\tau)$ *has* $(2-\varepsilon)BS$ *for every* $\varepsilon > 0$ *(but not* $2BS$*) if* $\mathcal{M}$ *is non-atomic.*

If $(\mathcal{M},\tau) = (R,\tau)$ *is hyperfinite non-atomic, then:*

(e) $L_0^{p,\infty}(R,\tau)$ *has* $\inf\{p,2\}BS$ *if* $1 < p < \infty$ *and* $p \neq 2$.

(f) $L_0^{2,\infty}(R,\tau)$ *has* $(2-\varepsilon)BS$ *for every* $\varepsilon > 0$ *(but not* $2BS$*).*

These results are sharp.

Proof. The results are sharp because they are already sharp in the commutative setting of $E(0,1)$ r.i. spaces [SS, Part 5].

(a) and (b) are known since they come from the type property of these spaces.

(e) For $p < 2$ the claim comes from Theorem 8(a) since $L_0^{p,\infty}$ satisfies an upper p-estimate and hence the disjoint pBS property.

For $p > 2$ the claim comes from Theorem 12(a) applied with $r = p - \varepsilon > 2$ and $q = p + \varepsilon$, since $L_0^{p,\infty}$, being 2-convex, has the disjoint $2BS$ property. Note that $L_0^{p,\infty}$ is not D^*-convex (see [S2, Proposition 4.2]) and does not have the Fatou property.

(f) This comes from Theorem 12(a) applied with $r = 2 - \varepsilon$ and $q = 2 + \varepsilon$, since $L_0^{2,\infty}$, being 2-convex, has the disjoint $2 - BS$ property, and hence the disjoint $(2-\varepsilon)BS$ property.

(c) In order to apply Theorem 8(b) if $p < 2$, or Theorem 12(b) between $r = p - \varepsilon > 2$ and $q = p + \varepsilon$ if $p > 2$, it suffices to know that $L^{p,1}$ has disjoint $\inf\{p,2\}BS$, $1 < p < \infty$, which is proved in the next Lemma 16. Note that $L^{p,1}$ is D^*-convex and has the Fatou property.

(d) Similarly, in order to apply Theorem 12(b) between $r = 2 - \varepsilon$ and $q = 2 + \varepsilon$, it suffices to know that $L^{2,1}$ has the disjoint $2BS$ property, which again comes from Lemma 16. □

LEMMA 16.

(a) $L^{p,1}(0,\infty)$ *has the disjoint* pBS *property,* $1 < p < \infty$.

(b) $L^{2,1}(0,\infty)$ *and* $L^{2,q}(0,\infty)$, $2 < q < \infty$, *and* $L_0^{2,\infty}(0,\infty)$ *do not have the* $2BS$ *property.*

(c) $l^{2,1}$ *and* $l^{2,q}$, $2 < q < \infty$, *and* $l_0^{2,\infty}$ *have the* $2BS$ *property.*

Proof. (a) By [DDS, Lemma 3.13], a bounded sequence of disjointly supported functions in $L^{p,1}$ which tends to 0 in measure either converges in norm to 0 or is equivalent to the canonical basis of l^1. So, if moreover the sequence is weakly null, it converges in norm to 0.

Let $(f_j)_{j\geq 1} \subset L^{p,1}$ be a weakly null sequence of disjointly supported functions. By the splitting principle, some subsequence $(g_j)_{j\geq 1}$ admits a decomposition $g_j = u_j + v_j + w_j$, $j \geq 1$, as in Proposition 11. We just saw that $\|v_j\|_{L^{p,1}} \to_{j\to\infty} 0$, and $\|w_j\|_{L^{p,1}} \to_{j\to\infty} 0$. Since $L^{p,1}$ has the Fatou property, we may assume that $(u_j)_{j\geq 1} \subset L^{p,1}$ is a weakly null sequence of disjointly supported equimeasurable functions. Hence, by definition,

$$\left\| \sum_{j=1}^{n} u_j \right\|_{L^{p,1}} = \frac{1}{p} \int_0^\infty \mu_{t/n}(u_1) t^{\frac{1}{p}-1} dt = n^{1/p} \left\| u_1 \right\|_{L^{p,1}} .$$

(Equivalently, the Boyd indices satisfy $p_{L^{p,1}} = q_{L^{p,1}} = p$.) It follows that $(g_j)_{j\geq 1}$ is a pBS sequence.

(b) This is proved in [SS, Lemmas 5.2, 5.5 and 5.8] for $L^{2,q}(0,1)$ (resp. $L_0^{2,\infty}(0,1)$), which can be viewed as a closed subspace of $L^{2,q}(0,\infty)$ (resp. $L_0^{2,\infty}(0,\infty)$).

(c) $l^{2,1}$ can be viewed as a closed subspace of $L^{2,1}(0,\infty)$, hence has the disjoint $2BS$ property by (a). Then it has the $2BS$ property since from a weakly null sequence in $l^{2,1}$ one can extract a subsequence $(f_j)_{j\geq 1}$ such that $\|f_j - g_j\|_{l^{2,1}} \leq \frac{1}{2^j}$ for some disjointly supported sequence $(g_j)_{j\geq 1}$. In the same way, $l^{2,q}$ for $2 < q < \infty$ and $l_0^{2,\infty}$, being 2-convex, have the disjoint $2BS$ property, and hence the $2BS$ property. □

Similarly as in [DDS, Proposition 2.14], a theorem of Arazy [A] now implies the following result, which is similar in spirit to [AL, Theorem 6].

Proposition 17. *Let $C^{2,q} = l^{2,q}(K)$ be the Schatten ideal associated to $l^{2,q}$. Then if $q = 1$ or $2 < q < \infty$, $L^{2,q}(0,\infty)$ is not isomorphic to a closed subspace of $C^{2,q}$. $L_0^{2,\infty}(0,\infty)$ is not isomorphic to a closed subspace of $C_0^{2,\infty}$.*

Proof. By Lemma 16, $L^{2,q}(0,\infty)$ does not have the $2BS$ property, so it suffices to prove that $C^{2,q}$ has the $2BS$ property. Since every weakly null normalized sequence in $C^{2,q}$ has a shell-block subsequence, [A, Theorem 2.4] implies that there exists a further subsequence which is equivalent to a block basis of a permutation of the natural basis of $l^2 \oplus l^{2,q}$. Since $l^{2,q}$ has $2BS$ by Lemma 16, so does $l^2 \oplus l^{2,q}$, hence $C^{2,q}$ also has the $2BS$ property. The same argument also works for $C_0^{2,\infty}$, considering $l^2 \oplus l_0^{2,\infty}$. □

Appendix A. D-and D^*-convex r.i. spaces.

D- and D^*-convexity were defined in [S2] with different names; see Definition 1 for the second notion. Keeping the same notation we have:

Definition 18. A r.i. space $E = E(0,a)$ is D-convex if there exists a constant $D' > 0$ such that, for $n \geq 1$ and all $(y_i)_{i=1}^{2^n} \subseteq E$,

$$\left\| \sum_{i=1}^{2^n} \chi_i^{(n)} \otimes y_i \right\|_{E([0,1]\times(0,\infty))} \leq D' \max_{1\leq i\leq 2^n} \|y_i\|_E.$$

The following holds:

(a) If E is D-convex, then $E^\times$ is D^*-convex; if E is D^*-convex, then $E^\times$ is D-convex [S2, Proposition 2.3].

(b) If E is D^*-convex (respectively D-convex), one may in the above inequalities replace E by $E(\mathcal{M})$ and $E([0,1]\times(0,\infty))$ by $E(L^\infty[0,1]\overline{\otimes}\mathcal{M})$ [S2, Proposition 2.2].

(c) If E is D-convex and $(\varepsilon_j)_{j\geq 1}$ is a sequence of Rademacher functions on $[0,1]$,

$$\left\|\sum_{j=1}^n \varepsilon_j \otimes x_j\right\|_{E(L^\infty[0,1]\overline{\otimes}\mathcal{M})} \leq D' \left\|\sum_{j=1}^n \varepsilon_j \otimes x_j\right\|_{L^\infty([0,1],E(\mathcal{M}))};$$

we already mentioned the similar inequality when E is D^*-convex.

(d) The Orlicz and Lorentz function spaces $\Lambda_{\psi,q}$ on $(0,\infty)$ are D^*-convex, in particular, $L^{p,q}(0,\infty), 1 \leq q \leq p < \infty$; the Orlicz spaces on $(0,\infty)$ are D-convex [S2, Proposition 2.4, 2.5].

Independently, in the setting of r.i. spaces on a finite interval, the same properties were introduced by N. Kalton [K] in a different way and further studied in [MSS, Part 6], where it is mentioned that the Marcinkiewicz function spaces on $(0,1)$ are D-convex. We now give these second definitions and explain why they are equivalent to the previous ones. We recall that σ_s is the dilation operator on $(0,\infty)$ used in the definition of Boyd indices. For $A>0$, we have the following relation beween the distribution functions of $\sigma_{1/n}(f)$ and f :

$$m\{t : \left|\sigma_{1/n}f(t)\right| > A\} = m\{t : |f(nt)| > A\} = \frac{1}{n} m\{s : \ |f(s)| > A\}$$

and the decreasing rearrangement $\mu(f)$ is the right continuous inverse function of the distribution function of $|f|$.

A r.i. space $E=E(0,\infty)$ is called D-convex [K] (respectively D^*-convex; see [MSS]) in the second sense if there exists a constant $c>0$ such that for every $y_1,\ldots,y_n \in E$, $n\geq 1$,

$$\left\|\sigma_{1/n}\left(\sum_{k=1}^n \widetilde{y_k}\right)\right\|_E \leq c \max_{1\leq k\leq n} \|y_k\|_E,$$

respectively

$$\left\|\sigma_{1/n}\left(\sum_{k=1}^n \widetilde{y_k}\right)\right\|_E \geq c \inf_{1\leq k\leq n} \|y_k\|_E.$$

Here, $\widetilde{y_1},\ldots,\widetilde{y_n} \in E$ are disjointly supported and each $\widetilde{y_k}$ is equimeasurable with y_k.

The two definitions of D-convexity coincide since the distribution functions of $\sum_{i=1}^{2^n} \chi_i^{(n)} \otimes y_i$ and $\sigma_{2^{-n}} \left(\sum_{k=1}^{2^n} \widetilde{y_k} \right)$ are equal, and hence

$$\left\| \sum_{i=1}^{2^n} \chi_i^{(n)} \otimes y_i \right\|_{E([0,1]\times(0,\infty))} = \left\| \sigma_{2^{-n}} \left(\sum_{k=1}^{2^n} \widetilde{y_k} \right) \right\|_E .$$

Since E is an interpolation space between L^1 and L^∞, if E is D^*-convex in the second sense, then E is D^*-convex in the sense of Definition 1 by [MSS, Corollary 24]. Conversely, if E is D^*-convex, then $E^\times$ is D-convex in both senses, hence E is D^*-convex in the second sense by [MSS, Part 6].

Proposition 19.

(a) *If $E = E(0,\infty)$ satisfies a lower p-estimate with $1 \le p \le p_E$ (hence $p = p_E = q_E$), then E is D^*-convex.*
(b) *If $E = E(0,\infty)$ satisfies an upper q-estimate with $q_E \le q$ (hence $p_E = q_E = q$), then E is D-convex.*

Proof. By definition, if E satisfies a lower p-estimate, then $p \ge q_E$, and if E satisfies an upper q-estimate, then $q \le p_E$.

(a) Note that, for some constant $c > 0$,

$$\begin{aligned}\|y_i\|_E &\le \|\sigma_{2^n}\|_{E\to E} \|\sigma_{2^{-n}} y_i\|_E \\ &\le 2^{n/p_E} c \|\sigma_{2^{-n}} y_i\|_E \\ &\le 2^{n/p} c \|\sigma_{2^{-n}} y_i\|_E .\end{aligned}$$

Then, by the lower p-estimate,

$$\begin{aligned}\left\| \sum_{i=1}^{2^n} \chi_i^{(n)} \otimes y_i \right\|_{E([0,1]\times(0,\infty))} &\ge C \left(\sum_{i=1}^{2^n} \|\chi_i^{(n)} \otimes y_i\|^p_{E([0,1]\times(0,\infty))} \right)^{1/p} \\ &= C \left(\sum_{i=1}^{2^n} \|\sigma_{2^{-n}} y_i\|^p_{E(0,\infty)} \right)^{1/p} \\ &\ge c^{-1} C \left(2^{-n} \sum_{i=1}^{2^n} \|y_i\|_E^p \right)^{1/p} \\ &\ge c^{-1} C \left(2^{-n} \sum_{i=1}^{2^n} \|y_i\|_E \right) .\end{aligned}$$

(b) Similarly,

$$\begin{aligned}\|\sigma_{2^{-n}} y_i\|_E &\leq \|\sigma_{2^{-n}}\|_{E\to E}\|y_i\|_E \\ &\leq 2^{-n/q_E} c\|y_i\|_E \\ &\leq 2^{-n/q} c\|y_i\|_E.\end{aligned}$$

By the upper q-estimate

$$\begin{aligned}\left\|\sum_{i=1}^{2^n} \chi_i^{(n)} \otimes y_i\right\|_{E([0,1]\times(0,\infty))} &\leq C\left(\sum_{i=1}^{2^n} \|\chi_i^{(n)} \otimes y_i\|^q_{E([0,1]\times(0,\infty))}\right)^{1/q} \\ &= C\left(\sum_{i=1}^{2^n} \|\sigma_{2^{-n}} y_i\|_E^q\right)^{1/q} \\ &\leq c\, C\left(2^{-n}\sum_{i=1}^{2^n} \|y_i\|_E^q\right)^{1/q} \\ &\leq c\, C \max_{1\leq i\leq 2^n} \|y_i\|_E. \qquad \square\end{aligned}$$

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