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DECREASING DILATATION CAN INCREASE DIMENSION

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ABSTRACT. We answer a question of Cui and Zinsmeister by constructing a quasiconformal map f of the disk to itself with dilatation μ such that map corresponding to dilatation $1/2\mu$, maps the circle to a curve of dimension > 1.

1. Introduction

In this note we answer a question raised by Cui and Zinsmeister in [10]: is the set of chord-arc curves starlike in a certain sense? To state their question more precisely, we need a few definitions. A positive measure ν on the unit disk, $\mathbb{D}$, is a Carleson measure if there is a $C < \infty$ so that $\nu(B(x,r) \cap \mathbb{D}) \leq Cr$ for every ball centered at a point $x \in \mathbb{T} = \partial\mathbb{D}$. We let

$$CM(\mathbb{D}) = \left\{ \mu \in L^\infty(\mathbb{D}) : \frac{|\mu(z)|^2 dxdy}{1-|z|} \text{ is a Carleson measure } \right\}.$$

Suppose G is a Fuchsian group acting on $\mathbb{D}$ and let

$$M(G) = \left\{ \mu \in L^\infty(\mathbb{D}) : \|\mu\|_\infty < 1 \text{ and } \forall g \in G, \mu = \frac{\overline{g}'}{g'} \mu \circ g \right\},$$
$$\mathcal{M}(G) = M(G) \cap CM(\mathbb{D}).$$

Note that $G_1 \subset G_2$ implies $M(G_2) \subset M(G_1)$. When G is trivial (just the identity) we denote these spaces by $M(1)$ and $\mathcal{M}(1)$. If $\mu \in M(1)$ then there is a quasiconformal map f^μ of $\mathbb{D}$ to itself, fixing $1, -1, i$ with dilatation μ, i.e., $f_z = \mu f_{\overline{z}}$. Two elements μ, ν are equivalent if $f^\mu|_\mathbb{T} = f^\nu|_\mathbb{T}$ and we write $\mu \sim \nu$ in this case. Similarly, there is a quasiconformal map f_μ of the plane to itself which has dilatation μ inside $\mathbb{D}$ and which is conformal outside $\overline{\mathbb{D}}$. This map sends $\mathbb{T}$ to some quasicircle $\Gamma = f_\mu(\mathbb{T})$. Quasicircles have a geometric characterization in terms of the Ahlfors 3-point condition (the smaller arc

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between any two points $x, y \in \Gamma$ having diameter bounded by $C|x-y|$; see [1]).

The curves Γ corresponding to $\mu \in \mathcal{M}(1)$ have been geometrically characterized in [4], and despite being "almost rectifiable"(e.g., they have tangents almost everywhere with respect to harmonic measure), they need not be locally rectifiable and, indeed, may have Hausdorff dimension strictly larger than 1. However, if $\mu \in \mathcal{M}(1)$ with sufficiently small Carleson norm, then Γ is rectifiable, indeed, it is a chord-arc curve (i.e., a biLipschitz image of the circle, characterized by the shorter arc between any two points $x, y \in \Gamma$ having length bounded by $C|x-y|$).

In [10] Cui and Zinsmeister ask the following: "Let $\mu \in \mathcal{M}(1)$ be such that $f_\mu(\partial\mathbb{D})$ is a biLipschitz image of a circle or a line. Is the same true for $f_{t\mu}(\partial\mathbb{D})$, $0<t<1$?" We shall show this is false even if $f_\mu(\partial\mathbb{D})$ is a circle.

Theorem 1.1. *There is a convergence group G and a $\mu \in \mathcal{M}(G) \subset \mathcal{M}(1)$ so that f_μ maps $\mathbb{D}$ to itself, but there is a $0<t<1$ so that $\dim(f_{t\mu}(\mathbb{T})) > 1$.*

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2. The example

A Fuchsian group G is called co-compact if $\mathbb{D}/G$ is compact and cofinite if $\mathbb{D}/G$ has finite hyperbolic area. It is called divergence type if $\sum_{g\in G} 1 - |g(0)| = \infty$ and convergence type otherwise. G is called "first kind" if its limit set (the accumulation set of an orbit) is the whole circle and is called "second kind" otherwise. We have the inclusions cocompact $\subset$ cofinite $\subset$ divergence type $\subset$ first kind.

If G is a divergence type and $\mu \in M(G)$ then $f_\mu(\mathbb{T})$ is either a circle (or line) or has Hausdorff dimension strictly larger than 1. This is called Bowen's dichotomy after Rufus Bowen who proved it in the cocompact case [8]. In this paper we will only use Bowen's dichotomy in the cofinite case, [6], [9], [11]; see [3] for the proof for divergence groups. Astala and Zinsmeister showed Bowen's dichotomy fails for all convergence groups [2] (there is always deformation whose limit set is a rectifiable curve, but not a circle or line). If G is convergence type and $\mu \in M(G)$ has compact support modulo G then $\mu \in CM(G)$ (e.g., see [7]).

If $\mu \in M(G)$, and G is of the first kind, then $\Lambda_\mu = f_\mu(\mathbb{T})$ is the limit set of the Kleinian group $G_\mu = f_\mu \circ G \circ f_\mu^{-1}$. We let $\dim(G_\mu)$ denote the Hausdorff dimension of Λ_μ. Given a G and $\mu \in M(G)$ let $G(t) = G_{t\mu}$ be the Kleinian group $f_{t\mu} \circ G \circ f_{t\mu}^{-1}$.

If G is a Kleinian group with N generators and G_n is a sequence of N-generated Kleinian groups which converges to G algebraically (this means that there is a set of generators $\{g_{1,n}, \ldots, g_{N,n}\}$ for G_n which converges to a set of generators for G), then

$$\liminf_{n\to\infty} \dim(G_n) \geq \dim(G),$$

i.e., the dimension of the limit set of a Kleinian group is upper semi-continuous with respect to algebraic convergence. This is proven in [5].

The idea for the example in Theorem 1.1 is as follows. We will choose a 1-parameter family of Fuchsian groups G_r, $r \in [0, 1/4]$, and dilatations $\mu_r \in \mathcal{M}(G_r)$ so that

(1) G_0 is cofinite.
(2) G_r is convergence type for $r > 0$.
(3) μ_r has compact support modulo G_r for all r.
(4) For all $t \in [0, 1]$, $G_r(t) \to G_0(t)$ algebraically as $r \to 0$.
(5) $\mu_r \sim 0$, i.e., $G_r(1) = G_r$.
(6) For $0 < t < 1$, $G_0(t)$ is not conjugate to G_0 by Möbius transformations.

Assume (1)–(6) hold. By (2) and (3) $\mu_r \in \mathcal{M}(G_r)$ for $r > 0$. By (1), (6) and Bowen's dichotomy $\dim(G_0(t)) > 1$ for all $0 < t < 1$. By (4) and upper semi-continuity of dimension, $\dim(G_r(t)) > 1$ if $0 < t < 1$ and r is close enough to 0 (depending on t) and by (5) f_{μ_r} is the identity map on $\mathbb{T}$. Thus for any $0 < t < 1$, G_r and μ_r satisfy Theorem 1.1 if r is close enough to zero.

All that remains is to construct groups and dilatations that satisfy (1)–(6). For $0 < r < 1/4$ and $3/4 < s \leq 1$, let

$$\Omega_{r,s} = \mathbb{C} \setminus \bigcup_{n,m\in\mathbb{Z}} B\left(\frac{1}{2} + n + is\left(\frac{1}{2} + m\right), r\right),$$

and $S_{r,s} = \Omega_{r,s}/(\mathbb{Z} + is\mathbb{Z})$. See Figure 1. For $r = 0$, this is a finite area Riemann surface, a genus one torus with a single puncture, and for $r > 0$ it is a torus with a disk removed.

Let $G_{r,s}$ be the Fuchsian group on the disk which covers $S_{r,s}$. For $r = 0$ this is a cofinite group and for $r > 0$ it is a convergence group, so conditions (1) and (2) hold. Moreover, it is clear that $G_{r,s} \to G_{0,s}$ as $r \to 0$. Also note that distinct values of s give surfaces $S_{0,s}$ which are not conformally equivalent; if there were a conformal map between two of them, it would lift to a conformal map between the respective domains $\Omega_{0,s}$, and since points are removable for conformal maps, this would extend to a conformal map of the plane (hence linear) fixing infinitely many points (hence the identity).

Next, suppose W is the strip $W = \{z = x + iy : |y| < 1/4\} = \mathbb{R} \times [-1/4, 1/4]$ and suppose we have a quasiconformal map Φ of W onto itself of the form

$$\Phi(x + iy) = x + i\varphi(y), \tag{2.1}$$

for some increasing, piecewise linear, homeomorphism φ of $[-1/4, 1/4]$ to itself. Let $\mu = \mu_\Phi$ be the dilatation of Φ and assume that $t\mu$ is the dilatation

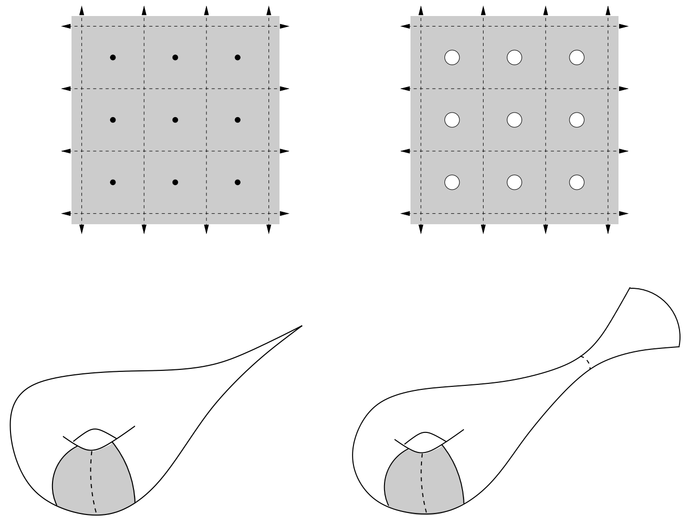

FIGURE 1. The domains $\Omega_{0,1}$ and $\Omega_{r,1}$, $r > 0$ and the corresponding quotients: a torus with a puncture (cofinite) and another with a funnel (convergence type).

of a quasiconformal mapping Φ_t from W into a substrip $\mathbb{R}\times[-\eta(t),\eta(t)]$ of the form $\Phi_t(x+iy) = x+i\varphi_t(y)$ where φ_t is also a piecewise linear homeomorphism (and is the identity if $t = 0$) and

$$\eta(t) < 1/4, \text{ for } 0 < t < 1. \tag{2.2}$$

We will show later that such a map exists.

Let $\lambda(t) = \frac{3}{4} + \eta(t)$ and define a quasiconformal map Ψ_t of the plane to itself by $\Psi_t(x + iy) = x + i\psi_t(y)$, where $\psi_t(0) = 0$ and $\psi_t'(y) = \varphi_t'(y - \lfloor y \rfloor)$ if $\text{dist}(y, \mathbb{Z}) < 1/4$ and $\psi_t'(y) = 1$ if $\text{dist}(y, \mathbb{Z}) > 1/4$. Then Ψ_0 is the identity. For $0 < t < 1$, Ψ_t takes a strip centered on a horizontal line at height n and width $1/2$ and maps it to a horizontal strip at height $n\lambda(t)$ and width $2\eta(t)$. See the shaded strips in Figure 2. Between these strips the map is conformal and linear.

The map Ψ_t is periodic in the sense that

$$\Psi_t(x + iy + n + im) = \Psi_t(x + iy) + n + im\lambda(t),$$

and so it corresponds to a well defined quasiconformal map from $S_{r,1}$ to $S_{r,\lambda(t)}$. This map, in turn, lifts to a quasiconformal map between the universal covers of these surfaces, i.e., to a quasiconformal map $f_{r,t}$ of the unit disk to itself. Moreover, if $\mu_{r,t}$ is the dilatation of this map, then $\mu_{r,t} = t\mu_{r,1}$ by construction.

Let $\mu_r = \mu_{r,1}$. Clearly μ_r is compactly supported modulo G_r, so condition (3) holds. Also note that $G_r(t) = f_{r,t} \circ G_r \circ f_{r,t}^{-1} \to G_0(t)$ as $r \to 0$, so (4)

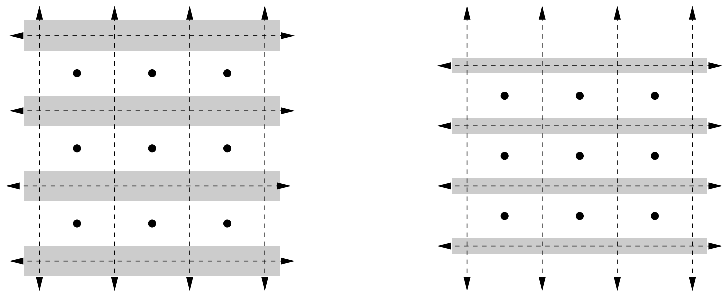

FIGURE 2. The map Ψ vertically contracts in the shaded strips and is an isometry elsewhere.

holds. Since Ψ_1 preserves $\Omega_{r,1}$ and is isotopic to the identity, $f_{r,1}$ must be the identity on $\mathbb{T}$ and so (5) holds. Finally, $G_0(t)$ is the covering group for $S_{0,\lambda(t)}$ which for $0 < t < 1$ is not conformally equivalent to $S_{0,1}$, since $\lambda(t) < 1$ for $0 < t < 1$. Thus (6) holds. This completes the proof of Theorem 1.1, except for the construction of the map φ.

To simplify notation we will assume $W = \mathbb{R} \times [0, 1]$ (then just rescale by a linear map to get the desired mapping). The particular φ we will use is very simple: for $a \in (0, 2)$ consider the piecewise linear map given by

$$\varphi(y) = \begin{cases} ay, & \text{if } 0 \le y \le 1/2; \\ (2-a)\left(y - \frac{1}{2}\right), & \text{if } 1/2 \le y \le 1. \end{cases}$$

Then φ has slope a on $(0, 1/2)$ and slope $2-a$ on $(1/2, 1)$, so $\Phi(x, y) = (x, \varphi(y))$ is quasiconformal with constant

$$K = \sup_x \max\left(\varphi', \frac{1}{\varphi'}\right) = \max\left(a, \frac{1}{a}, 2-a, \frac{1}{2-a}\right) = \max\left(\frac{1}{a}, \frac{1}{2-a}\right).$$

If $a \in (0, 1)$, then the $K = 1/a$. If $y \in (0, 1/2)$, then the dilatation μ of Φ is

$$\mu(x, y) = \frac{\partial_{\bar z}\Phi}{\partial_z \Phi} = \frac{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x + i\varphi(y))}{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x + i\varphi(y))} = \frac{1-a}{1+a}.$$

If $y \in (1/2, 1)$, then a similar computation shows

$$\mu(x, y) = \frac{1-(2-a)}{1+(2-a)} = \frac{a-1}{3-a}.$$

If $0 < t < 1$, then what map Φ_t has dilatation $t\mu$? It should have the form $(x, y) \to (x, \varphi_t(y))$ where φ_t is piecewise linear. For $y \in (0, 1/2)$ the slope s of φ_t satisfies

$$\frac{1-s}{1+s} = t\frac{1-a}{1+a} = b,$$

and solving for s gives

$$s = \frac{1-b}{1+b} = \frac{(1+a)-(1-a)t}{(1+a)+(1-a)t}.$$

Similarly, the slope of φ_t on $(1/2, 1)$ is given by

$$\frac{(3-a)-(a-1)t}{(3-a)+(a-1)t}.$$

Thus the map φ_t maps the interval $[0,1]$ to the interval $[0, \varphi_t(1)]$ where

$$\varphi_t(1) = \frac{1}{2}\left[\varphi'\left(\frac{1}{4}\right) + \varphi'\left(\frac{3}{4}\right)\right] = \frac{1}{2}\left[\frac{(1+a)-(1-a)t}{(1+a)+(1-a)t} + \frac{(3-a)-(a-1)t}{(3-a)+(a-1)t}\right]$$

Setting $a = 1/2$ gives $(15+t^2)/(15+2t-t^2)$. This equals 1 when $t = 0, 1$ and is strictly less than 1 for $t \in (0,1)$, as desired. This completes the proof of Theorem 1.1.

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