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RESIDUAL SOLUBILITY OF FUCHSIAN GROUPS

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Abstract. The derived series for all co-compact non-perfect Fuchsian groups are investigated. These groups are residually finite and residually soluble. The intersection of the derived series for these groups is the identity. We will show that if Γ is not perfect, then the number of terms in the derived series up to and including the first surface group cannot exceed 4. We then use this result to compute the derived series of some important general triangle groups.

1. Introduction

A co-compact Fuchsian group $\Gamma(\gamma; m_1, m_2, \ldots, m_r)$ as usual is a discrete subgroup of $PSL(2, R)$, e.g., the group of linear fractional transformations of unit determinant and compact orbit space. It is well known that all Fuchsian groups are residually finite, i.e., the intersection of their subgroups of finite index is trivial. In this paper we study the derived series for all co-compact Fuchsian groups in detail. We classify these series according to their soluble length. Finally, we compute the terms in their derived series up to and including the first torsion-free subgroup (i.e., surface group). Our result is based upon a theorem of D. Singerman [6], a theorem of Hoare-Karrass-Solitar [3], and the notion of the *l.c.m. condition* (least common multiple condition), which is a restriction on the periods m_i, first introduced by Harvey and Maclachlan [2], [4]. Suppose Γ is any non-perfect co-compact Fuchsian group. Let $\ell(\Gamma)$ denote the smallest number of terms up to the first torsion-free subgroup in its derived series including Γ itself; we call $\ell(\Gamma)$ the *drive length* of Γ.

In Section 2 we recall some basic terminology and existing results. In Section 3 we show that the drive length $\ell(\Gamma)$ is bounded by 4. Since $N = \ell(\Gamma)$ is the least positive integer such that the N-th derived group $\Gamma^{(N)}$ of the Fuchsian group Γ is torsion-free, this implies that the smooth factor group $\Gamma/\Gamma^{(N)}$ also has drive length not exceeding 4. In Section 4 we apply the results to the $(2, 3, n)$ triangle groups.

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2. Basic terminology and existing results

Definition 2.1. *A co-compact Fuchsian group has the following presentation:*

$$\Gamma(S) = \langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma \mid x_1^{m_1}, x_2^{m_2}, \ldots, x_r^{m_r}, \prod_{i=1}^{r} x_i \prod_{j=1}^{\gamma} [a_j, b_j] \rangle, \tag{2.1}$$

where

$$S = (\gamma; m_1, m_2, \ldots, m_r), \quad r \geq 0, \gamma \geq 0, m_i \geq 2, \tag{2.2}$$

is called the signature of the Fuchsian group Γ*, the integers* m_i *are called the periods and the number* γ *is called the orbit genus; see* [1], [3], [4], [5], [7], [8].

Definition 2.2. *The rational number*

$$\chi(S) = 2(1-\gamma) + \sum_{i=1}^{r} \left(\frac{1}{m_i} - 1 \right) \tag{2.3}$$

is called the Euler characteristic of the signature S*. If* Γ_1 *is a subgroup of finite index in* $\Gamma(S)$*, then there is another signature* S_1 *and a Fuchsian group* $\Gamma_1 = \Gamma(S_1)$ *such that the index formula*

$$[\Gamma : \Gamma_1] = \chi(S_1)/\chi(S) \tag{2.4}$$

is satisfied. This equation is classical and is called the Riemann-Hurwitz relation.

Definition 2.3. *A homomorphism* $\Phi : \Gamma(S) \longmapsto G$ *onto a finite group* G *which preserves all the periods of* Γ*, i.e., for every* x_i *of order* m_i *the order of* $\Phi(x_i)$ *is precisely* m_i*, is called smooth. If* Φ *is smooth, then* $\mathrm{Ker}(\Phi)$ *is torsion free and is called a Fuchsian surface group.*

Definition 2.4. *If the periods* m_i *of* Γ *satisfy the restriction that each* m_l *divides the least common multiple of the periods* $m_1, m_2, \ldots, m_{l-1}, m_{l+1}, \ldots, m_r$*, then* Γ *is said to satisfy the* l.c.m. condition*. This condition is equivalent to the following condition: The least common multiple of any* $r-1$ *of the periods* $m_1, m_2, \ldots, m_r$ *is equal to the least common multiple of all of these periods.*

Theorem 2.5. *Suppose the Fuchsian group* Γ *has signature*

$$S = (\gamma; m_1, m_2, \ldots, m_r). \tag{2.5}$$

Then its derived group $\Gamma' = [\Gamma, \Gamma]$ *is torsion-free if and only if the periods* m_i *satisfy the l.c.m. condition.*

Proof. See Maclachlan [5, p. 701]. □

THEOREM 2.6. *Suppose the Fuchsian group Γ has signature*

$$S = (\gamma; m_1, m_2, \dots, m_r). \tag{2.6}$$

Then the group Γ contains a subgroup Γ_1 of index N with signature

$$S_1 = (\gamma_1; n_{11}, n_{12}, \dots, n_{1\rho_1}, \dots, n_{r1}, \dots, n_{r\rho_r}) \tag{2.7}$$

if and only if the following are satisfied:

(a) *There exists a finite permutation group G that is transitive on N points and an epimorphism $\theta : \Gamma(S) \longmapsto G$ satisfying the condition that the permutation $\theta(x_j)$ has precisely ρ_j cycles of the following lengths:*

$$\frac{m_j}{n_{j1}}, \frac{m_j}{n_{j2}}, \dots, \frac{m_j}{n_{j\rho_j}}. \tag{2.8}$$

(b) *Γ_1 has index N in Γ and $[\Gamma : \Gamma_1] = \chi(S_1)/\chi(S) = N$.*

Proof. See Singerman [6, p. 320]. □

3. Drive length of co-compact Fuchsain groups

In this section we investigate the derived series for a general finitely generated co-compact Fuchsian group with the above presentation and find a bound for the soluble length. As is well known, the Euler characteristic $\chi(S)$ is negative for all such groups. Therefore, if $\gamma = 0$ and $r \leq 2$, then Γ is not Fuchsian because $\chi(S) > 0$.

In the following theorem, we consider separately the cases when the orbit genus γ is non-zero, i.e., when $\gamma \geq 1$, and when $\gamma = 0$.

THEOREM 3.1. *Suppose a co-compact Fuchsian group Γ of orbit genus γ has the following presentation:*

$$\Gamma(S) = \langle x_1, \dots, x_r, a_1, b_1, \dots, a_\gamma, b_\gamma \mid x_1^{m_1}, x_2^{m_2}, \dots, x_r^{m_r}, \prod_{i=1}^{r} x_i \prod_{j=1}^{\gamma} [a_j, b_j] \rangle, \quad r \geq 3. \tag{3.1}$$

Let $\ell(\Gamma)$ denote the drive length of Γ and $\Gamma \triangleright \Gamma' \triangleright \Gamma'' \triangleright \cdots \triangleright \Gamma^{(k)} \dots$ be its derived series. Then we have the following two cases:

(1) *If $\gamma \geq 1$, then either $\Gamma' = [\Gamma, \Gamma]$ is torsion-free or Γ'', the second term in the derived series of Γ, is torsion-free (in fact, a free group). Therefore, in this case the soluble length satisfies $\ell(\Gamma) \leq 3$.*

(2) *If $\gamma = 0$, then the soluble length satisfies $\ell(\Gamma) \leq 4$, i.e., the smooth factor group $\Gamma/\Gamma^{\ell(\Gamma)}$ has soluble length not exceeding 4.*

Proof. (1) First, if $\gamma \geq 1$, then $\Gamma'' = [\Gamma', \Gamma']$ has infinite index in Γ'. Clearly, if the periods m_i satisfy the l.c.m. condition, then, by Theorem 2.5, Γ' itself is torsion-free and $\ell(\Gamma) = 2$. Otherwise, by the theorem of Hoare-Karrass-Solitar [3], Γ' is a free product of (finite and/or infinite) cyclic groups. It is then trivial that Γ'' is torsion-free (in fact, a free group). This implies $\ell(\Gamma) = 3$. Therefore in case (1) we have $\ell(\Gamma) \leq 3$.

(2) On other hand, if $\gamma = 0$, then, since Γ is not perfect, it is residually soluble, i.e., the intersection $\bigcap_{k=0}^{\infty} \Gamma^{(k)}$ of the derived series given above is the identity. Once again, if the periods m_i satisfy the l.c.m. condition, then, by Theorem 2.5, Γ' is torsion-free, in which case $\ell(\Gamma) = 2$. Otherwise, there exists at least one prime p which divides at least two of the periods. Without loss of generality (using the fact that a transformation can always be used to change the order of these periods) we may assume that $p|m_1$ and $p|m_2$. Thus Γ has the following presentation:

$$\Gamma(S) = \langle x_1, x_2, \ldots, x_r \mid x_1^{pm_1'} = x_2^{pm_2'} = x_3^{m_3} = \cdots = x_r^{m_r} = \prod_{j=1}^{r} x_j = 1 \rangle. \tag{3.2}$$

Let

$$\Phi : \Gamma(S) \longmapsto Z_p \cong \langle u \mid u^p = 1 \rangle \tag{3.3}$$

be a homomorphism defined from Γ onto Z_p given by the following equations:

$$\Phi(x_1) = u, \Phi(x_2) = u^{-1}, \Phi(x_j) = 1 \text{ for } 3 \leq j \leq r. \tag{3.4}$$

It is easy to show that Φ is a homomorphism.

Next, by Theorem 2.6, $\Gamma_1 = \Gamma(S') = \mathrm{Ker}(\Phi)$ has signature

$$S' = (\gamma_1; m_1', m_2', \overbrace{m_3, \ldots, m_3}^{p \text{ times}}, \overbrace{m_4, \ldots, m_4}^{p \text{ times}}, \ldots, \overbrace{m_r, \ldots, m_r}^{p \text{ times}}), \tag{3.5}$$

where each m_j, $3 \leq j \leq r$, is repeated p times. Now, since Γ_1 is a proper subgroup of finite index in the Fuchsian group Γ, it must be Fuchsian, and so the terms in its derived series are all proper subgroups of the corresponding terms in the derived series of Γ. Therefore, the intersection of all terms in the derived series of Γ_1, i.e., $\bigcap_{k=0}^{\infty} \Gamma_1^{(k)}$, is a subgroup of the intersection $\bigcap_{k=0}^{\infty} \Gamma^{(k)}$, which is the identity. Therefore, $\bigcap_{k=0}^{\infty} \Gamma_1^{(k)}$ must also be the identity, and so Γ_1 is not perfect. Next, since Z_p is abelian, clearly $\Gamma' \subseteq \Gamma_1$, which implies that $\Gamma'' \subseteq \Gamma_1'$. Thus, if the set $m_1', m_2', m_3, \ldots, m_r$ satisfies the l.c.m. condition, then Γ_1', and hence Γ'', is torsion-free. This implies $\ell(\Gamma) = 3$. Using the Riemann-Hurwitz relation it can be shown that γ_1, the new orbit genus, is zero. Therefore, Γ_1 has the following presentation:

$$\Gamma_1 = \langle z_1, z_2, y_{31}, \ldots, y_{3p}, \ldots, y_{rp} \mid z_1^{m_1'}, z_2^{m_2'}, y_{31}^{m_3}, \ldots, y_{3p}^{m_3}, \ldots, y_{r1}^{m_r}, \ldots, y_{rp}^{m_r}, (z_1 z_2) \prod_{i=3}^{r} \prod_{j=1}^{p} y_{ij} \rangle. \tag{3.6}$$

Next, since $r \geq 3$, there exists a second prime q which does not divide m_1' and m_2', but which does divide m_j for some $j \geq 3$. Again, without loss of generality, we may assume $q|m_3$, i.e., $m_3 = qm_3'$. Let

$$\Psi : \Gamma_1 \longmapsto Z_q \cong \langle v \mid v^q = 1 \rangle \tag{3.7}$$

be a homomorphism from Γ_1 onto Z_q given by the following equations:

$$\begin{cases} \Psi(z_1) = 1, \Psi(z_2) = 1; \\ \Psi(y_{31}) = v, \Psi(y_{32}) = v^{-1}; \\ \Psi(y_{3j}) = \Psi(y_{kl}) = 1; \quad 3 \leq j \leq p, 4 \leq k \leq r, 1 \leq l \leq p. \end{cases} \tag{3.8}$$

Letting the symmetric group S_q play the role of the permutation group in Theorem 2.6, then by the result of that theorem $\Gamma_2 = \mathrm{Ker}(\Psi)$ has signature
(3.9)

$$S'' = (\gamma_2; \overbrace{m_1', \ldots, m_1'}^{q \text{ times}} \overbrace{m_2', \ldots, m_2'}^{q \text{ times}}, m_3', m_3', \overbrace{m_3, \ldots, m_3}^{q(p-2) \text{ times}}, \ldots, \overbrace{m_r, \ldots, m_r}^{pq \text{ times}}),$$

where m_1' and m_2' are each repeated q times, m_3' appears only twice, m_3 is repeated $q(p-2)$ times, and finally each m_j, $j \geq 4$, is repeated pq times. Once again, the Riemann-Hurwitz relation can be used to show that the new orbit genus γ_2 is zero. As before, since Z_q is abelian, we have $\Gamma_1' \subseteq \Gamma_2$. Moreover, we know that $\Gamma'' \subseteq \Gamma_1'$. Hence $\Gamma'' \subseteq \Gamma_2$, and finally $\Gamma''' \subseteq \Gamma_2'$.

The periods in S'' are each repeated at least twice. Therefore they must satisfy the l.c.m. condition. Thus, by Theorem 2.5, Γ_2' is torsion-free, and hence from the above inclusions Γ''' is also torsion-free. Therefore, in this case we have $\ell(\Gamma) = 4$. Thus for all cases we have $\ell(\Gamma) \leq 4$. This completes the proof of the theorem. □

4. Drive lengths of the $(2, 3, n)$ triangle groups

As an application of our result we compute the derived series for all triangle groups of the form $(0; 2, 3, n)$, for any value of the integer n.

Example 4.1. Let Γ be a Fuchsian group with signature $S = (0; 2, 3, n)$. Then Γ has the following presentation in generators and relations:

$$\Gamma(S) = \langle x, y \mid x^2 = y^3 = (xy)^n = 1 \rangle. \tag{4.1}$$

The factor group with respect to the first derived group in the new generators $x' = x\Gamma'$ and $y' = y\Gamma'$ and their relations is given as follows:

$$\Gamma/\Gamma' = \langle x', y' \mid x'^2 = y'^3 = (x'y')^n = 1; x'y' = y'x' \rangle. \tag{4.2}$$

We now consider the following four cases:

(a) If $n \equiv \pm 1 \bmod 6$, then $(n,2) = (n,3) = 1$ (i.e., n is relatively prime to 2 and 3), so the above relations yield $x' = y' = 1$. Thus, $\Gamma' = \Gamma$, and therefore Γ is perfect, and all the terms in the series are equal. For this case we may write $\ell(\Gamma) = \infty$.

(b) If $n \equiv \pm 2 \bmod 6$, then $n = 2m$ with $(m,3) = 1$, and we have $y' = 1$. Thus we have $\Gamma/\Gamma' \cong Z_2$, and in this case Γ has the following derived series:

$$\Gamma(0;2,3,2m) \supset \Gamma(0;3,3,m) \supset \Gamma(0;m,m,m) \supset \Gamma\left(\frac{(m-1)(m-2)}{2};\rightharpoondown\right). \tag{4.3}$$

The corresponding factor groups are Z_2, Z_3 and $Z_m \oplus Z_m$. There are infinitely many automorphism groups covered by Γ which are residually soluble (Γ''' and all the terms following Γ''' in the series). In this case we have $\ell(\Gamma) = 4$.

(c) If $n \equiv \pm 3 \bmod 6$, then $n = 3m$ with $(m,2) = 1$, and we have $x' = 1$. Thus, $\Gamma/\Gamma' \cong Z_3$, and Γ has the following derived series:

$$\Gamma(0;2,3,3m) \supset \Gamma(0;2,2,2,m) \supset \Gamma(0;m,m,m,m) \supset \Gamma(m^3 - 2m^2 + 1;\rightharpoondown). \tag{4.4}$$

The corresponding factor groups are $Z_3, Z_2 \oplus Z_2$ and $Z_m \oplus Z_m \oplus Z_m$ Again there are infinitely many automorphism groups covered by Γ which are residually soluble (Γ''' and all the terms following Γ''' in the series). In this case we again have $\ell(\Gamma) = 4$.

(d) If $n \equiv 0 \bmod 6$, then $n = 6m$, for all $m \in \mathbf{N}$. In this final case we have $x'^2 = y'^3 = 1$. Thus $\Gamma/\Gamma' \cong Z_6$. Using Schreier's method one can show that Γ' is a Fuchsian group with the following presentation in generators and relations:

$$\Gamma' = \langle z = (xy)^6, a_1 = y^{-1}xyx, b_1 = xyxy^{-1} \mid z^m = z[a_1,b_1] = 1\rangle. \tag{4.5}$$

But the above presentation implies that Γ' is of index 6 in Γ and has signature $(1;m)$, a fact that can also be checked by Singerman's theorem. The second derived group Γ'' is then of infinite index in Γ. Hence, by the theorem of Hoare-Karrass-Solitar [3], any normal subgroup of infinite index in a Fuchsian group must be a free product of cyclic groups. Using that theorem we can obtain the following derived series:

$$\Gamma(0;2,3,6m) \supset \Gamma' = \Gamma(1;m) \supset \Gamma'' \supset \dots, \tag{4.6}$$

where Γ' is a free product of finite and infinite cyclic groups and Γ'' is a free group. The corresponding factor groups are $Z_6, Z \oplus Z$. Thus, in this case $\ell(\Gamma) = 3$.

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