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NORMS OF SCHUR MULTIPLIERS

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ABSTRACT. A subset $\mathcal{P}$ of $\mathbb{N}^2$ is called Schur bounded if every infinite matrix with bounded scalar entries which is zero off of $\mathcal{P}$ yields a bounded Schur multiplier on $\mathcal{B}(\mathcal{H})$. Such sets are characterized as being the union of a subset with at most k entries in each row with another that has at most k entries in each column, for some finite k. If k is optimal, there is a Schur multiplier supported on the pattern with norm $O(\sqrt{k})$, which is sharp up to a constant. The same characterization also holds for operator-valued Schur multipliers in the cb-norm, i.e., every infinite matrix with bounded *operator* entries which is zero off of $\mathcal{P}$ yields a completely bounded Schur multiplier.

This result can be deduced from a theorem of Varopoulos on the projective tensor product of two copies of l^∞. Our techniques give a new, more elementary proof of his result.

We also consider the Schur multipliers for certain matrices which have a large symmetry group. In these examples, we are able to compute the Schur multiplier norm exactly. This is carried out in detail for a few examples including the Kneser graphs.

Schur multiplication is just the entrywise multiplication of matrices or operators in a fixed basis. These maps arise naturally as the (weak-$*$ continuous) bimodule maps for the algebra of diagonal matrices (operators). They are well-behaved completely bounded maps that play a useful role in the theory of operator algebras.

As in the case of operators themselves, the actual calculation of the norm of any specific Schur multiplier is a delicate task; and is often impossible. This has made it difficult to attack certain natural, even seemingly elementary, questions.

This study arose out of an effort to understand norms of Schur multipliers supported on certain patterns of matrix entries. The question of which patterns have the property that every possible choice of bounded scalar entries supported on the pattern yield bounded Schur multipliers was raised by Nikolskaya and Farforovskaya in [15]. We solve this problem completely. The

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answer is surprisingly elegant. The pattern must decompose into two sets, one with a bound on the number of entries in each row, and the other with a bound on the number of entries in each column. In fact, these patterns have the stronger property that if we allow operator entries supported pattern, instead of scalar entries, then they still yield completely bounded Schur multipliers.

In fact, our result may be deduced from results of Varopoulos [24], [23] and Pisier [18]. We had overlooked this work and only discovered it late in our study. Perhaps this is just as well, as we may well have stopped had we realized how close their results were to the ones we were seeking. The upshot is that we also obtain a much more elementary proof of Varopoulos' theorem. Indeed our main tool in the decomposition is an elementary, albeit powerful, combinatorial result known as the min-cut-max-flow theorem. For the other direction, we use an elementary but very clever argument of Lust-Piquard [12] to construct matrices with small norm and relatively large entries. Unfortunately we were not able to completely recover the probabilistic version of Varopoulos' Theorem. However in the pattern case, this is easy.

In Section 3, we recover results of [15] on patterns of Hankel and Toeplitz forms. The Toeplitz case is classical, and we compare the bounds from our theorem with the tighter bounds available from a deeper use of function theory.

Sections 4 and 5 deal with exact computation of the Schur norm of certain matrices that have lots of symmetry. More precisely, let G be a finite group acting transitively on a set X. We obtain an explicit formula for the Schur multiplier norm of matrices in the commutant of the action, i.e., matrices constant on each orbit of G. This uses a result of Mathias [13]. We carry this out for one nontrivial case—the adjacency matrix of the Kneser graph $K(2n+1, n)$, which has $\binom{2n+1}{n}$ vertices indexed by n-element subsets of $2n+1$, with edges between disjoint sets.

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1. Background

If $A = [a_{ij}]_{i,j \in S}$ is a finite or infinite matrix, the Schur (a.k.a. Hadamard) multiplier is the operator S_A on $\mathcal{B}(l^2(S))$ that acts on an operator $T = [t_{ij}]$ by pointwise multiplication: $S_A(T) = [a_{ij}t_{ij}]$. To distinguish from the norm on bounded operators, we will write $\|A\|_m$ for the norm of a Schur multiplier. In general it is very difficult to compute the norm of a Schur multiplier. Nevertheless, much is known in a theoretical sense about the norm. In this section, we will quickly review some of the most important results.

The following classical result owes most credit to Grothendieck. For a proof, see the books by Pisier [19, Theorem 5.1] and Paulsen [16, Theorem 8.7].

THEOREM 1.1. *For X an arbitrary set, let $S = [s_{ij}]$ be an $|X| \times |X|$ matrix with bounded entries considered as a Schur multiplier on $\mathcal{B}(l^2(X))$. Then the following are equivalent:*

(1) $\|S\|_m \leq 1$.
(2) $\|S\|_{cb} \leq 1$.
(2′) *There are contractions V and W from $l^2(X)$ to $l^2(X) \otimes l^2(Y)$ such that $S(A) = W^*(A \otimes I)V$.*
(3) *There are unit vectors x_i and y_j in $l^2(Y)$ so that $s_{ij} = x_i^* y_j$.*
(4) $\gamma_2(S) \leq 1$ *where* $\gamma_2(S) = \inf_{S=AB} \|A\|_{2,\infty} \|B\|_{1,2}$.
(5) *There are $|X| \times |X|$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ with $a_{ii} = b_{ii} = 1$ so that* $\begin{bmatrix} A & S \\ S^* & B \end{bmatrix}$ *is positive semidefinite.*

Recall that the complete bound norm of S is the norm of the inflation of S acting on operators with operator entries. The most elegant proof of (1) implies (2) is due to Smith [21]. The converse is trivial. The equivalence of (2) and (2′) is Wittstock's Theorem for representing completely bounded maps.

The equivalence of (1), (3) and (4) is due to Grothendieck. The implication (3) implies (1) is actually in Schur's original paper [20]. (3) follows from (2′) by taking $y_j = (E_{1j} \otimes I)Ve_j$ and $x_i = (E_{1i} \otimes I)We_i$. Conversely, (3) implies (2′) by taking $Ve_j = e_j \otimes y_j$ and $We_i = e_i \otimes x_i$. This condition was rediscovered by Haagerup, and became well-known as his observation. So we shall refer to these as the Grothendieck–Haagerup vectors for the Schur multiplier.

The γ_2 norm is the optimal factorization through Hilbert space of S considered as a map from l^1 to l^∞. The norm $\|A\|_{2,\infty}$ is the maximum of the 2-norm of the rows; while $\|B\|_{1,2}$ is the maximum of the 2-norm of the columns. Thus (3) implies (4) follows from $A = \sum_i e_i x_i^*$ and $B = \sum_j y_j e_j^*$. And this implication is reversible.

The equivalence of (5) is due to Paulsen, Power and Smith [17]. This follows from (3) by taking $a_{ij} = x_i^* x_j$ and $b_{ij} = y_i^* y_j$. Conversely, assume first that X is finite. Then the positive matrix P decomposes as a sum of positive rank one matrices, and thus have the form $[\bar{z}_i z_j]$ which can be seen to be a scalar version of (3). Indeed it is completely positive. Hence the sum S_P is also a completely positive Schur multiplier. Consequently

$\|S_P\|_{cb} = \|S_P(I)\| = \max\{a_{ii}, b_{ii}\} = 1$. So (2) holds. The case of general X is a routine limit argument.

The γ_2 norm is equivalent to the norm in the Haagerup tensor product $\ell^\infty(X) \otimes_h \ell^\infty(X)$, where we identify an elementary tensor $a \otimes b$ with the matrix $[a_i b_j]$. The Haagerup norm of a tensor τ is given by taking the infimum over all representations $\tau = \sum_k a_k \otimes b_k$ of

$$\Big\| \sum_k a_k a_k^* \Big\|^{1/2} \Big\| \sum_k b_k^* b_k \Big\|^{1/2}.$$

See [16, Chapter 17]. Of course, since ℓ^∞ is abelian, the order of the adjoints is irrelevant. One can see the equivalence by taking a factorization $S = AB$ from (4). Consider A as a matrix with columns $a_k \in \ell^\infty(X)$ and B as a matrix with rows $b_k \in \ell^\infty(X)$. Identify the product with the tensor $\sum_k a_k \otimes b_k$. The norm $\| \sum_k a_k a_k^* \|^{1/2}$ can be seen to be $\|A\|_{2,\infty}$ and the norm of $\| \sum_k b_k^* b_k \|^{1/2}$ to be $\|B\|_{1,2}$.

Generally, it is difficult to compute the norm of a Schur multiplier. The exception occurs when the matrix S is positive definite. Then it is a classical fact that S is a completely positive map. Consequently, $\|S\|_{cb} = \|S(I)\| = \sup_{i \in X} s_{ii}$.

Grothendieck proved another remarkable result about Schur multipliers. Recall that the projective tensor product $\ell^\infty(X) \hat{\otimes} \ell^\infty(X)$ norms a tensor τ as the infimum over representations $\tau = \sum_k a_k \otimes b_k$ of the quantity $\sum_k \|a_k\| \, \|b_k\|$. It is a surprising fact that this norm and the γ_2 or Haagerup norm are equivalent. We will need this connection to understand the relevance of work of Varopoulos. For the moment, we state this result in a way that makes a stronger connection to Schur multipliers. An elementary tensor $a \otimes b$ yields a rank one matrix $[a_i b_j]$. Thus Grothendieck's result is equivalent to:

THEOREM 1.2 (Grothendieck). *The closure of the convex hull of the rank one Schur multipliers of norm one in the topology of pointwise convergence contains the ball of all Schur multipliers of norm at most K_G^{-1}, where K_G is a universal constant.*

In terms of the projective tensor product norm for a tensor τ and the corresponding Schur multiplier S_τ, this result says that

$$K_G^{-1} \|\tau\|_{\ell^\infty(S) \hat{\otimes} \ell^\infty(S)} \le \|S_\tau\|_m \le \|\tau\|_{\ell^\infty(S) \hat{\otimes} \ell^\infty(S)}.$$

The constant K_G is not known exactly. In the complex case Haagerup [10] showed that $1.338 < K_G < 1.405$; and in the real case Krivine [11] obtained the range $[1.676, 1.783]$ and conjectured the correct answer to be $\frac{\pi}{2\log(1+\sqrt{2})}$.

We turn to the results of Varopoulos [24], [23] and Pisier [18] which relate to our work. The paper of Varopoulos [24] is famous for showing that three commuting contractions need not satisfy the von Neumann inequality. Proofs of this, including the one in the appendix of Varopoulos's paper, are generally constructive. But the argument in the main part of his paper instead establishes a result about $\ell^\infty(X)\hat{\otimes}\ell^\infty(X)$. He does not establish precise information about constants. This result was extended and sharpened by Pisier, who casts it in the language of Schur multipliers, to deal with multipliers and lacunary sets on nonamenable groups.

Consider $\{\pm 1\}^{X\times X}$ to be the space of functions from $X \times X$ to $\{1, -1\}$ with the product measure μ obtained from $p(1) = p(-1) = .5$.

THEOREM 1.3 (Varopoulos [24], [23]). *Let $S = [s_{ij}]$. The following are equivalent.*

1. *For all $\varepsilon \in \{\pm 1\}^{X\times X}$, $\|[\varepsilon_{ij}s_{ij}]\|_m < \infty$.*
2. *There exists a measurable set $Y \subset \{\pm 1\}^{X\times X}$ with $\mu(Y) > 0$ so that $\|[\varepsilon_{ij}s_{ij}]\|_m < \infty$ for all $\varepsilon \in Y$.*
3. *$S = A + B$ and there is a constant M so that*
$$\sup_i \sum_j |a_{ij}|^2 \le M^2 \quad \text{and} \quad \sup_j \sum_i |b_{ij}|^2 \le M^2.$$
4. *There is a constant M so that for every pair of finite subsets $R, C \subset X$, $\sum_{i\in R, j\in C} |s_{ij}|^2 \le M^2 \max\{|R|, |C|\}$.*

Varopoulos was not concerned with the constants. He also states the result for the projective tensor product. This is equivalent, with some change of constants, by Grothendieck's Theorem 1.2. Pisier formulates this for the Schur multiplier norm, and provides a quantitative sharpening of this result. He shows [18, Theorem 2.2] that if the average Schur multiplier norm

$$\int \|[\varepsilon_{ij}s_{ij}]\|_m \, d\mu(\varepsilon) = M,$$

then one can take the same M in (3). Our results are not quite so sharp as Pisier's, as we require a constant (Lemma 2.9) of approximately 1/4.

The constants M in the two conditions (3) and (4) are not the same. The correct relationship replaces $\max\{|R|, |C|\}$ by $|R|+|C|$, and then the constants are equal (see Lemma 2.7). So as formulated above, they are related within a factor of 2. If M is the bound in (3), it is not difficult to obtain a bound of $2M$ for (1) (see Corollary 2.6). Thus Pisier's version yields that the average Schur norm is within a factor of 2 of the maximum.

Our initial concern was with the case in which $s_{ij} \in \{0, 1\}$ and we wanted a result with integer answers. This comes down to choosing A and B to also take their values in $\{0, 1\}$. One can deduce this from Varopoulos' theorem at the expense of a factor of 2 in the estimates. See [18, Remark 2.4]. Our approach will yield integral decompositions naturally without affecting the constants.

The referee has kindly pointed out that the recent paper [14] by Neuwirth contains another description of Varopoulos's Theorem in the pattern context. He provides a concise guide to Varopoulos's papers that will lead the reader to this result.

We will also consider Schur multipliers with operator entries. If $X = [X_{ij}]$ is an infinite (or finite) matrix with operator entries, define S_X in the same way: $S_X(T) = \big[t_{ij} X_{ij}\big]$. Unlike the scalar case, in which the multiplier norm coincides with the cb-norm, it is possible [17] that $\|S_X\|_m < \|S_X\|_{cb}$. The ampliated map $S_X^{(n)}$ acts coordinatewise on $n \times n$ matrices of operators. Doing Paulsen's 'canonical shuffle', one arrives at a better formulation in which it acts on an operator with $n \times n$ matrix entries. The action is given by the formula

$$S_X^{(n)}\big(\big[A_{ij}\big] \big) = \big[A_{ij} \otimes X_{ij}\big] .$$

2. Schur bounded patterns

A *pattern* $\mathcal{P}$ is a subset of $\mathbb{N} \times \mathbb{N}$. An infinite matrix $S = [s_{ij}]$ is *supported on* $\mathcal{P}$ if $\{(i,j) : s_{ij} \neq 0\}$ is contained in $\mathcal{P}$. We let $\mathcal{S}(\mathcal{P})$ denote the set of Schur multipliers supported on $\mathcal{P}$ with scalar matrix entries $|s_{ij}| \leq 1$. Also let $\mathcal{OS}(\mathcal{P})$ denote the set of Schur multipliers S_X where $X = [X_{ij}]$ is an infinite matrix supported on $\mathcal{P}$ with operator entries X_{ij} of norm at most 1.

More generally, we will also consider Schur multipliers dominated by a given infinite matrix $A = [a_{ij}]$ with nonnegative entries. Let $\mathcal{S}(A)$ denote the set of Schur multipliers with scalar matrix entries $|s_{ij}| \leq a_{ij}$, and let $\mathcal{OS}(A)$ denote the set of Schur multipliers S_X where $X = [X_{ij}]$ is an infinite matrix with operator entries satisfying $\|X_{ij}\| \leq a_{ij}$.

Definition 2.1. Given a set of scalar (or operator valued) matrices, $\mathcal{C}$, we say that $\mathcal{C}$ is *Schur bounded* if

$$\mathfrak{s}(\mathcal{C}) := \sup_{A \in \mathcal{C}} \|S_A\|_m < \infty,$$

and we call $\mathfrak{s}(\mathcal{C})$ the Schur bound of $\mathcal{C}$. Similarly, we say that a set of operator valued matrices, $\mathcal{OC}$, is *completely Schur bounded* if

$$\mathfrak{cs}(\mathcal{OS}) := \sup_{X \in \mathcal{OC}} \|S_X\|_{cb} < \infty,$$

and we call $\mathfrak{cs}(\mathcal{OC})$ the complete Schur bound of $\mathcal{OC}$.

We say that a pattern $\mathcal{P}$ is *Schur bounded* if $\mathcal{S}(\mathcal{P})$ is Schur bounded; and say that $\mathcal{P}$ is *completely Schur bounded* if $\mathcal{OS}(\mathcal{P})$ is completely Schur bounded.

It is easy to see that if $\|A\|_m < \infty$ for all $A \in \mathcal{S}(\mathcal{P})$, then $\mathcal{S}(\mathcal{P})$ is Schur bounded. Note that if $A_{\mathcal{P}}$ is the matrix with 1s on the entries of $\mathcal{P}$ and 0s elsewhere, then $\mathcal{S}(A_{\mathcal{P}}) = \mathcal{S}(\mathcal{P})$. We will maintain a distinction because we will require *integral decompositions* when working with a pattern $\mathcal{P}$. It is also easy to see that

$$\mathfrak{s}(\mathcal{S}(A)) \le \mathfrak{s}(\mathcal{OS}(A)) \le \mathfrak{cs}(\mathcal{OS}(A))$$

and likewise for patterns.

Certain patterns $\mathcal{P}$ are easily seen to be Schur bounded, indeed even completely Schur bounded, and this is the key to our result. The following two definitions of row bounded for patterns and matrices are not parallel, as the row bound of $A_{\mathcal{P}}$ is actually the square root of the row bound of $\mathcal{P}$. Each definition seems natural for its context, so we content ourselves with this warning.

Definition 2.2. A pattern is *row bounded by k* if there are at most k entries in each row; and *row finite* if it is row bounded by k for some $k \in \mathbb{N}$. Similarly we define *column bounded by k* and *column finite*.

A nonnegative matrix $A = [a_{ij}]$ is *row bounded by L* if the rows of A are bounded by L in the l^2-norm: $\sup_{i\ge1}\sum_{j\ge1}|a_{ij}|^2 \le L^2 < \infty$. Similarly we define *column bounded by L*.

The main result of this section is:

Theorem 2.3. *For a pattern $\mathcal{P}$, the following are equivalent:*

(1) *$\mathcal{P}$ is Schur bounded.*
(2) *$\mathcal{P}$ is completely Schur bounded.*
(3) *$\mathcal{P}$ is the union of a row finite set and a column finite set.*
(4) $\displaystyle\sup_{R,C \text{ finite}} \frac{|\mathcal{P}\cap(R\times C)|}{|R|+|C|} < \infty.$

Moreover, the optimal bound m on the size of the row and column finite sets in (3) *coincides with the least integer dominating the supremum in* (4)*; and the Schur bounds satisfy*

$$\sqrt{m}/4 \le \mathfrak{s}(\mathcal{S}(\mathcal{P})) \le \mathfrak{cs}(\mathcal{OS}(\mathcal{P})) \le 2\sqrt{m}.$$

This theorem has a direct parallel for nonnegative matrices.

Theorem 2.4. *For a nonnegative infinite matrix $A = [a_{ij}]$, the following are equivalent:*

(1) *$\mathcal{S}(A)$ is Schur bounded.*
(2) *$\mathcal{OS}(A)$ is completely Schur bounded.*
(3) *$A = B + C$ where B is row bounded and C is column bounded.*
(4) $\displaystyle\sup_{R,C \text{ finite}} \frac{\sum_{i\in R, j\in C} a_{ij}^2}{|R|+|C|} < \infty.$

Moreover, the optimal bound M on the row and column bounds in (3) *coincides with the square root of the supremum M^2 in* (4)*; and the Schur bounds satisfy*

$$M/4 \le \mathfrak{s}(\mathcal{S}(A)) \le \mathfrak{cs}(\mathcal{OS}(A)) \le 2M.$$

Lemma 2.5. *If $\mathcal{P}$ is row (or column) bounded by n, then $\mathcal{P}$ is completely Schur bounded and $\mathfrak{cs}(\mathcal{OS}(\mathcal{P})) \le \sqrt{n}$.*

Likewise, if A is row (or column) bounded by L, then $\mathfrak{cs}(\mathcal{OS}(A)) \le L$.

Proof. The pattern case follows from the row bounded case for the nonnegative matrix $A = A_{\mathcal{P}}$ with $L = \sqrt{n}$. Suppose that A is row bounded by L. Let $B = \big[B_{ij}\big] \in \mathcal{OS}(A)$, namely $B_{ij} \in \mathcal{B}(\mathcal{H})$ and $\|B_{ij}\| \le a_{ij}$. We will use the isometries $\{S_i\}_{i\in\mathbb{N}}$ from $\mathcal{H}$ into $\mathcal{H}^{(\infty)}$ that carry $\mathcal{H}$ onto the ith summand of $\mathcal{H}^{(\infty)}$. Define $X_i = \sum_{j\ge1} B_{ij}S_j^*$ and $Y_i = S_i$ for $i \ge 1$. Observe that $X_iY_j = B_{ij}$, while $\|Y_j\| = 1$ and

$$\|X_i\|^2 = \|X_iX_i^*\| = \Big\| \sum_j B_{ij}B_{ij}^* \Big\| \le \sum_j a_{ij}^2 \le L^2.$$

If $T = [T_{ij}]$ is a bounded operator with matrix entries $T_{ij} \in \mathfrak{M}_n$, then

$$S_B^{(n)}(T) = \big[T_{ij} \otimes B_{ij}\big] = \operatorname{diag}(I \otimes X_i)\big[T_{ij} \otimes I\big] \operatorname{diag}(I \otimes Y_j).$$

Therefore $\|S_B(T)\| \le L\|T\|$ and so $\mathfrak{cs}(\mathcal{OS}(A)) \le L$. □

Corollary 2.6. *If $\mathcal{P}$ is the union of a set row bounded by n and a set column bounded by m, then $\mathfrak{cs}(\mathcal{OS}(\mathcal{P})) \le \sqrt{n} + \sqrt{m}$.*

Likewise, if $A = B + C$ such that B is row bounded by L and C is column bounded by M, then $\mathfrak{cs}(\mathcal{OS}(A)) \le L + M$.

We require a combinatorial characterization of sets which are the union of an n-row bounded set and an m-column bounded set. This will be a consequence of the min-cut-max-flow theorem (see [5], for example). This is an elementary result in combinatorial optimization that has many surprising consequences. For example, it has been used by Richard Haydon to give a short proof of the reflexivity of commutative subspace lattices [8]. It should be more widely known.

Lemma 2.7. *A pattern $\mathcal{P}$ is the union of a set $\mathcal{P}_r$ row bounded by m and a set $\mathcal{P}_c$ column bounded by n if and only if for every pair of finite subsets $R, C \subset \mathbb{N}$,*

$$|\mathcal{P} \cap R \times C| \le m|R| + n|C|.$$

Similarly, a matrix $A = [a_{ij}]$ with nonnegative entries decomposes as a sum $A = A_r + A_c$ where A_r is row bounded by $M^{1/2}$ and A_c is column bounded by $N^{1/2}$ if and only if for every pair of finite subsets $R, C \subset \mathbb{N}$,

$$\sum_{i\in R}\sum_{j\in C} a_{ij}^2 \le M|R| + N|C|.$$

Proof. The two proofs are essentially identical. However the decomposition of $\mathcal{P}$ must be into two disjoint subsets. This means that the decomposition

$A_{\mathcal{P}} = A_{\mathcal{P}_1} + A_{\mathcal{P}_2}$ is a split into $0, 1$ matrices. We will work with A, but will explain the differences in the pattern version when it arises.

The condition is clearly necessary.

For the converse, we first show that it suffices to solve the finite version of the problem. For $p \in \mathbb{N}$, let A_p be the restriction of A to the first p rows and columns. Suppose that we can decompose $A_p = A_{r,p} + A_{c,p}$ where $A_{r,p}$ is row bounded by $M^{1/2}$ and $A_{c,p}$ column bounded by $N^{1/2}$ for each $p \in \mathbb{N}$. Fix k so that $A_k \neq 0$. For each $p \geq k$, the set of such decompositions for A_p is a compact subset of $\mathfrak{M}_p \times \mathfrak{M}_p$. In the pattern case, we consider only $0, 1$ decompositions. The restriction to the $k \times k$ corner is also a compact set, say $\mathcal{X}_{k,p}$. Observe that this is a decreasing sequence of nonempty compact sets. Thus $\bigcap_{p \geq k} \mathcal{X}_{k,p} = \mathcal{X}_k$ is nonempty. Therefore there is a consistent choice of a decomposition $A = A_r + A_c$ so that the restriction to each $k \times k$ corner lies in $\mathcal{X}_k$ for each $k \geq 1$. In the pattern case, the entries are all zeros and ones.

So now we may assume that $A = [a_{ij}]$ is a matrix supported on $R_0 \times C_0$, where R_0 and C_0 are finite. We may also suppose that the l^2-norm of each row is greater than $M^{1/2}$ and the l^2-norm of each column is greater than $N^{1/2}$. For otherwise, we assign all of those entries in the row to A_r (or all entries in the column to A_c) and delete the row (column). Solving the reduced problem will suffice. If after repeated use of this procedure, the matrix is empty, we are done. Otherwise, we reach a reduced situation in which the l^2-norm of each row is greater than $M^{1/2}$ and the l^2-norm of each column is greater than $N^{1/2}$.

Define a graph $\mathcal{G}$ with vertices α, r_i for $i \in R_0$, c_j for $j \in C_0$, and ω. Put edges from each $r_i \in R_0$ to each $c_j \in C_0$, from α to r_i, $i \in R_0$, and from c_j to ω, $j \in C_0$. Consider a network flow on the graph in which the edge from r_i to c_j may carry a_{ij}^2 units; edges leading out of α can carry up to M units; and the edge from c_j to ω can carry $v_j - N$ units, where $v_j = \sum_{i \in R_0} a_{ij}^2$. In the pattern case, these constraints are integers.

The min-cut-max-flow theorem states that the maximal possible flow from α to ω across this network equals the minimum flow across any cut that separates α from ω. Moreover, when the data is integral, the maximal flow comes from an integral solution. A *cut* $\mathcal{X}$ is just a partition of the vertices into two disjoint sets $\{\alpha\} \cup R_1 \cup C_1$ and $\{\omega\} \cup R_2 \cup C_2$. The flow across the cut is the total of allowable flows on each edge between the two sets.

The flow across the cut $\mathcal{X}$ is

$$\begin{aligned} f(\mathcal{X}) &= \sum_{i \in R_1} \sum_{j \in C_2} a_{ij}^2 + M|R_2| + \sum_{j \in C_1} (v_j - N) \\ &= \sum_{i \in R_1} \sum_{j \in C_2} a_{ij}^2 + M|R_2| - N|C_1| + \sum_{i \in R_0} \sum_{j \in C_1} a_{ij}^2 \\ &= \sum_{i \in R_0} \sum_{j \in C_0} a_{ij}^2 - \sum_{i \in R_2} \sum_{j \in C_2} a_{ij}^2 + M|R_2| + N|C_2| - N|C_0| \end{aligned}$$

$$\geq \sum_{i\in R_0}\sum_{j\in C_0} a_{ij}^2 - N|C_0|.$$

The last inequality uses the hypothesis on A with $R = R_2$ and $C = C_2$. On the other hand, the cut separating ω from the rest has flow exactly

$$\sum_{j\in C_0}(v_j - N) = \sum_{i\in R_0}\sum_{j\in C_0} a_{ij}^2 - N|C_0|.$$

Therefore there is a network flow that achieves this maximum. In the pattern case, the solution is integral. Necessarily this will involve a flow of exactly $v_j - N$ from each $j \in C_0$ to ω. Let b_{ij} be the optimal flow from r_i to c_j. So $0 \leq b_{ij} \leq a_{ij}^2$. The flow out of each r_i equals the flow into r_i from α, whence $\sum_{j\in C_0} b_{ij} \leq M$. Since the flow into and out of each c_j are the same, we have $\sum_{i\in R_0} b_{ij} = v_j - N$.

Define the matrix $A_r = \left[\sqrt{b_{ij}}\right]$ and $A_c = \left[\sqrt{a_{ij}^2 - b_{ij}}\right]$. In the pattern case, these entries are 0 or 1. Then the rows of A_r are bounded by $M^{1/2}$. The jth column of A_c has norm squared equal to

$$\sum_{i\in R_0} a_{ij}^2 - b_{ij} = v_j - (v_j - N) = N.$$

This is the desired decomposition and it is integral for patterns. $\square$

To construct large norm Schur multipliers on certain patterns, we will make use of the following remarkable result by Françoise Lust-Piquard [12, Theorem 2]. While the method of proof is unexpected, it is both short and elementary.

Theorem 2.8 (Lust-Piquard). *Given any (finite or infinite) nonnegative matrix $X = [x_{ij}]$ satisfying*

$$\max_i \sum_j x_{ij}^2 \leq 1 \quad \text{and} \quad \max_j \sum_i x_{ij}^2 \leq 1 \text{ for all } i, j,$$

there is an operator $Y = [y_{ij}]$ so that

$$\|Y\| \leq \sqrt{6} \quad \text{and} \quad |y_{ij}| \geq x_{ij} \text{ for all } i, j.$$

The constant of $\sqrt{6}$ is not known to be the best possible; however it is optimal for a related extremal problem, as shown in an addendum to [12].

Lemma 2.9. *Let $A = [a_{ij}]$ be a nonnegative $m\times m$ matrix with $\sum_{i,j=1}^m a_{ij}^2 = m\alpha$. Then there is a Schur multiplier $S \in \mathcal{S}(A)$ such that $\|S\|_m \geq \frac{1}{2}\sqrt{\frac{\alpha}{3}}$.*

Proof. We may assume that there are no nonzero rows or columns. Let

$$r_i = \sum_{j=1}^m a_{ij}^2 \quad \text{and} \quad c_j = \sum_{i=1}^m a_{ij}^2.$$

Define
$$x_{ij} = \frac{a_{ij}}{\sqrt{r_i + c_j}}.$$
Let $X = [x_{ij}]$. The row norms of X satisfy
$$\sum_{j=1}^{m} x_{ij}^2 \leq \sum_{j=1}^{m} \frac{a_{ij}^2}{r_i} = 1;$$
and similarly the column norms are bounded by 1.

By Theorem 2.8, there is a matrix Y such that
$$\|Y\| \leq \sqrt{6} \quad \text{and} \quad |y_{ij}| \geq x_{ij} \quad \text{for all} \quad i, j.$$
Define $s_{ij} = a_{ij}x_{ij}/y_{ij}$ (where $0/0 := 0$). Then $S = [s_{ij}]$ belongs to $\mathcal{S}(A)$. Observe that
$$S(Y) = Z := [a_{ij}x_{ij}] = \left[\frac{a_{ij}^2}{\sqrt{r_i + c_j}}\right].$$
Hence $\|S\|_m \geq \|Z\|/\sqrt{6}$.

Define vectors $u = (u_i)$ and $v = (v_j)$ by
$$u_i = \Big(\frac{r_i}{m\alpha}\Big)^{1/2} \quad \text{and} \quad v_j = \Big(\frac{c_j}{m\alpha}\Big)^{1/2}.$$
Then
$$\|u\|_2^2 = \frac{1}{m\alpha} \sum_{i=1}^{m} r_i = 1$$
and similarly $\|v\|_2 = 1$. Compute
$$\|Z\| \geq u^* Z v = \frac{1}{m\alpha} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 \sqrt{\frac{r_i c_j}{r_i + c_j}}.$$
Observe that
$$\sqrt{\frac{r_i c_j}{r_i + c_j}} = \Big(\frac{1}{r_i} + \frac{1}{c_j}\Big)^{-1/2}.$$
Also
$$\begin{aligned}\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 \Big(\frac{1}{r_i} + \frac{1}{c_j}\Big) &= \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{a_{ij}^2}{r_i} + \sum_{j=1}^{m} \sum_{i=1}^{m} \frac{a_{ij}^2}{c_j} \\ &= \sum_{i=1}^{m} 1 + \sum_{j=1}^{m} 1 = 2m.\end{aligned}$$

A routine Lagrange multiplier argument shows that if $\alpha_k \geq 0$ are constants, $t_k > 0$ are variables, and $\sum_{k=1}^{m^2} \alpha_k t_k = 2m$, then $\sum_{k=1}^{m^2} \alpha_k t_k^{-1/2}$ is minimized when all t_k are equal. Hence if $\sum_{k=1}^{m^2} \alpha_k = m\alpha$,

$$\sum_{k=1}^{m^2} \alpha_k t_k^{-1/2} \geq m\alpha \Big(\frac{2m}{m\alpha}\Big)^{-1/2} = m\alpha \sqrt{\frac{\alpha}{2}}.$$

Applying this to the numbers $\frac{1}{r_i} + \frac{1}{c_j}$ yields

$$\|Z\| \geq \frac{1}{m\alpha} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}^2 \Big(\frac{1}{r_i} + \frac{1}{c_j}\Big)^{-1/2} \geq \sqrt{\frac{\alpha}{2}}.$$

Thus $\|S\|_m \geq \frac{\sqrt{\alpha}}{\sqrt{6}\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\alpha}{3}}$. □

Proof of Theorem 2.3 and Theorem 2.4. Statements (3) and (4) are equivalent by Lemma 2.7, taking $m = n$ and $M = N$. Assuming (3), Corollary 2.6 shows that $\mathcal{OS}(\mathcal{P})$ or $\mathcal{OS}(A)$ is completely Schur bounded and gives the upper bound on the complete Schur bound (2). That (2) implies (1) is trivial.

Assuming (4) in the pattern case, the supremum exceeds $m-1$; so Lemma 2.9 shows that

$$\mathfrak{s}(\mathcal{S}(\mathcal{P})) \geq \frac{\sqrt{m-1}}{2\sqrt{3}} \geq \frac{\sqrt{m}}{4}$$

for $m \geq 4$. For $m \leq 16$, $\sqrt{m}/4 \leq 1$; and 1 is also a lower bound for any pattern. For the matrix case, we use the exact supremum in Lemma 2.9, so we obtain a lower bound of $M/4$.

Conversely, if the supremum in (4) is infinite, the same argument shows that the Schur bounds of $\mathcal{S}(\mathcal{P})$ or $\mathcal{S}(A)$ are infinite. In fact, it is easy to see that this implies that $\mathcal{S}(\mathcal{P})$ or $\mathcal{S}(A)$ contains unbounded Schur multipliers. It is not difficult to produce disjoint finite rectangles $R_n \times C_n$ on which the ratio in (4) exceeds n^2. So by Lemma 2.9, we construct a Schur multiplier S_n in $\mathcal{S}(\mathcal{P})$ or $\mathcal{S}(A)$ supported on $R_n \times C_n$ with Schur norm at least $n/4$. Taking S to be defined on each rectangle as S_n and zero elsewhere, S is an unbounded Schur multiplier in this class. Thus, (1) implies (4). □

Remark 2.10 (Probabilistic considerations). Recall that μ is the probability measure on $\{1,-1\}^{\mathbb{N}^2}$ that is the product of the measure $p(1) = p(-1) = 1/2$ on each coordinate. Let $A = \big[a_{ij}\big]$ be an $\mathbb{N} \times \mathbb{N}$ matrix with non-negative entries. Suppose that there is a measurable subset $Y \subset \{\pm 1\}^{\mathbb{N}^2}$ with $\mu(Y) > 0$ and a constant C so that $\|[\varepsilon_{ij} a_{ij}]\|_m < C$ for all $\varepsilon \in Y$. It is a routine fact from measure theory that there is a finite subset F of $\mathbb{N}^2$ and a point $\eta \in \{1,-1\}^F$ so that the cylinder set $C_F = \{\varepsilon : \varepsilon(i,j) = \eta(i,j),\ (i,j) \in F\}$ satisfies

$$\mu(Y \cap C_F) > \mu(C_F)/2.$$

Let $D = C + 2\sum_{(i,j)\in F} a_{ij}$ and let

$$Z = \{\varepsilon \in \{\pm 1\}^{\mathbb{N}^2} : \|[\varepsilon_{ij} a_{ij}]\|_m < D\}.$$

Since for any $y \in Y$, the corresponding Schur multiplier is bounded by C, a rather crude estimate shows that for any $z \in \{\pm 1\}^{\mathbb{N}^2}$ which agrees with y off of the set F the Schur norm of $[z_{ij}a_{ij}]$ is bounded by D. It follows that $\mu(Z) > 1/2$.

There is nothing special about $1/2$, and indeed it follows immediately that $\|[\varepsilon_{ij}a_{ij}]\|_m < \infty$ μ–almost everywhere.

Now specialize to the case of a pattern $\mathcal{P}$, i.e., $a_{ij} \in \{0,1\}$. Since $\{\pm 1\}^{\mathbb{N}^2}$ is a group, it is easy to see that Z^2, the set of all products of two elements of Z, will equal all of $\{\pm 1\}^{\mathbb{N}^2}$. Indeed, one only need observe that for any $w = (\varepsilon_{ij})$, the two sets Z and wZ intersect, say containing $z_1 = wz_2$. Then $w = z_1 z_2$. Therefore

$$\|[\varepsilon_{ij}a_{ij}]\|_m \le D^2$$

for all choices of sign. As the points $\{\pm 1\}^{\mathbb{N}^2}$ are the extreme points of the ball of $l^\infty(\mathbb{N}^2)$, it now follows that $\mathfrak{s}(\mathcal{S}(\mathcal{P})) < \infty$.

This argument does not yield Varopoulos' Theorem for arbitrary A. We were not able to find an improvement on the duality argument in [24] which is used to deduce (4) of Theorem 2.4. Nor do we obtain Pisier's quantitative refinement about the average Schur norm being comparable (within a factor of 2) to the maximum value $\mathfrak{s}(\mathcal{S}(A))$.

Remark 2.11. One might suspect, from the $\sqrt{n}$ arising in Lemma 2.5, that if two matrices are supported on pairwise disjoint patterns, there might be an L^2 estimate on the Schur norm of the sum. This is not the case, as the following example shows.

Let $\mathbf{1} = (1,1,1,1)^t \in \mathbb{C}^4$ and $A = \mathbf{1}\mathbf{1}^* - I$. If $U = \operatorname{diag}(1, i, -1, -i)$, then the diagonal expectation is

$$\Delta(X) = S_I(X) = \frac{1}{4}\sum_{k=0}^{3} U^k X U^{*k}.$$

We use a device due to Bhatia–Choi–Davis [3]. Observe that

$$\begin{aligned} S_{A+tI}(X) &= X + (t-1)\Delta(X) \\ &= \Big(1 + \frac{t-1}{4}\Big)X + \frac{t-1}{4}\sum_{k=1}^{3} U^k X U^{*k}. \end{aligned}$$

Therefore

$$\begin{aligned} \|S_{A+tI}\|_m &\le \Big|1 + \frac{t-1}{4}\Big| + \frac{3|t-1|}{4} \\ &= \begin{cases} |t| & \text{if} \quad t \ge 1 \,\text{or}\, t \le -3 \\ \frac{1}{2}|3-t| & \text{if} \quad -3 \le t \le 1 \end{cases}. \end{aligned}$$

On the other hand, $S_{A+tI}(I) = tI$; so $\|S_{A+tI}\|_m \geq |t|$. Observe that $\frac{1}{4}\mathbf{1}\mathbf{1}^*$ is a projection. Hence $A + tI = \mathbf{1}\mathbf{1}^* + (t-1)I$ has spectrum $\{t-1, t+3\}$; and thus

$$\|A + tI\| = \max\{|t-1|, |t+3|\}.$$

So $\|A - I\| = 2$. If $-3 \leq t \leq 1$, then $S_{A+tI}(A - I) = A - tI$ has norm $|3-t|$ and so $\|S_{A+tI}\|_m \geq |3-t|/2$.

In particular, $\|S_A\|_m = \frac{3}{2}$ and $\|S_I\|_m = 1$, but

$$\|S_{A-I}\|_m = 2 > \left(\|S_A\|_m^2 + \|S_I\|_m^2\right)^{1/2}.$$

3. Hankel and Toeplitz patterns

A *Hankel pattern* is a set of the form

$$\mathcal{H}(S) = \{(i,j) : i,j \in \mathbb{N},\ i+j \in S\} \quad \text{for} \quad S \subset \mathbb{N}.$$

A *Toeplitz pattern* is a set of the form

$$\mathcal{T}(S) = \{(i,j) : i,j \in \mathbb{N}_0,\ i-j \in S\} \quad \text{for} \quad S \subset \mathbb{Z}.$$

Recall that a set $S = \{s_1 < s_2 < \dots\}$ is lacunary if there is a constant $q > 1$ so that $s_{i+1}/s_i > q$ for all $i \geq 1$.

Nikolskaya and Farforovskaya [15, Theorem 3.8] show that a Hankel pattern is Schur bounded if and only if it is a finite union of lacunary sets. Such sets are called Paley sets. They accomplish this by considering Fejér kernels and Toeplitz extensions. We give an elementary proof based on Theorem 2.3 that also yields the complete version.

Proposition 3.1 (Nikolskaya–Farforovskaya). *Consider a Hankel pattern $\mathcal{H}(S)$ of a set $S \subset \mathbb{N}$. Then the following are equivalent:*

1. *$\mathcal{H}(S)$ is Schur bounded.*
2. *$\mathcal{H}(S)$ is completely Schur bounded.*
3. *$\mathcal{H}(S)$ is the union of a row finite and a column finite set.*
4. $\sup_{k\geq 0} |S \cap (2^{k-1}, 2^k]| < \infty$.
5. *S is the union of finitely many lacunary sets.*

Proof. By Theorem 2.3, (1) and (2) are equivalent.

Let $a_k = |S \cap (2^{k-1}, 2^k]|$ for $k \geq 0$. If (3) holds, $\max_{k\geq 0} a_k = L < \infty$. So S splits into $2L$ subsets with at most one element in every second interval $(2^{k-1}, 2^k]$; which are therefore lacunary with ratio at least 2. Conversely, suppose that S is the union of finitely many lacunary sets. A lacunary set with ratio q may be split into d lacunary sets of ratio 2 provided that $q^d \geq 2$. So suppose that there are L lacunary sets of ratio 2. Then each of these sets intersects $(2^{k-1}, 2^k]$ in at most one element. Hence $\max_{k\geq 0} a_k \leq L < \infty$. Thus (3) and (4) are equivalent.

Suppose that S is the union of L sets S_i which are each lacunary with constant 2. Split each $\mathcal{H}(S_i)$ into the subsets R_i on or below the diagonal and C_i above the diagonal. Observe that R_i is row bounded by 1, and C_i is column bounded by 1. Hence (4) implies (2).

Consider the subset of $\mathcal{H}(S)$ in the first 2^k rows and columns $R_k \times C_k$. This square will contain at least $2^{k-1}a_k$ entries corresponding to the backward diagonals for $S \cap (2^{k-1}, 2^k]$, which all have more than 2^{k-1} entries. Thus

$$\sup_{k\geq 0} \frac{|\mathcal{H}(S) \cap (R_k \times C_k)|}{|R_k| + |C_k|} \geq \sup_{k\geq 0} \frac{2^{k-1}a_k}{2^k + 2^k} = \sup_{k\geq 0} \frac{a_k}{4}.$$

Hence if (3) fails, this supremum if infinite. Thus $\mathcal{H}(S)$ is not the union of a row finite and a column finite set. So (2) fails. □

The situation for Toeplitz patterns is quite different. It follows from classical results, as we explain below, and Nikolskaya and Farforovskaya outline a related proof [15, Remark 3.9]. But first we show how it follows from our theorem.

Proposition 3.2. *The Toeplitz pattern $\mathcal{T}(S)$ of any infinite set S is not Schur bounded. Further,*

$$\frac{1}{4}|S|^{1/2} \leq \mathfrak{s}(\mathcal{T}(S)) \leq \mathfrak{cs}(\mathcal{OS}(\mathcal{T}(\mathcal{S})) \leq |S|^{1/2}.$$

Proof. Since $\mathcal{T}(S)$ is clearly row bounded by $|S|$, the upper bound follows from Lemma 2.5.

Suppose that $S = \{s_1 < s_2 < \cdots < s_n\}$. Consider the $m \times m$ square matrix with upper left hand corner equal to $(s_1, 0)$ if $s_1 \geq 0$ or $(0, -s_1)$ if $s_1 < 0$. Then beginning with row $m - (s_n - s_1)$, there will be n entries of $\mathcal{T}(S)$ in each row. Thus the total number of entries is at least $n(m - (s_n - s_1))$. For m sufficiently large, this exceeds $(n-1)m$. Hence by Lemma 2.9,

$$\mathfrak{s}(\mathcal{T}(S)) \geq \frac{\sqrt{n-1}}{2\sqrt{3}} \geq \frac{\sqrt{n}}{4}$$

provided $n \geq 4$. The trivial lower bound of 1 yields the lower bound for $n < 4$. □

To see how this is done classically, we recall the following [2, Theorem 8.1]. Here, $\mathcal{T}$ denotes the space of Toeplitz operators.

Theorem 3.3 (Bennett). *A Toeplitz matrix $A = [a_{i-j}]$ determines a bounded Schur multiplier if and only if there is a finite complex Borel measure μ on the unit circle $\mathbb{T}$ so that $\hat{\mu}(n) = a_n$, $n \in \mathbb{Z}$. Moreover*

$$\|A\|_m = \|S_A|_{\mathcal{T}}\| = \|\mu\|.$$

We combine this with estimates obtained from the Khintchine inequalities.

THEOREM 3.4. *Let $(a_k)_{k\in\mathbb{Z}}$ be an l^2 sequence and let $A = [a_{i-j}]$. Then*

$$\frac{1}{\sqrt{2}}\|(a_k)\|_2 \le \mathfrak{s}(\mathcal{S}(A)) \le \|(a_k)\|_2.$$

Proof. Suppose $S \in \mathcal{S}(A)$, that is, $S = [s_{ij}]$ with $|s_{ij}| \le a_{i-j}$. Then each row of S is norm bounded by $\|(a_k)\|_2$. Hence by Lemma 2.5, $\|S\|_m \le \|(a_k)\|_2$. So $\mathfrak{s}(\mathcal{S}(A)) \le \|(a_k)\|_2$.

Conversely, let $X := \{1,-1\}^{\mathbb{Z}}$. Put the measure μ on X which is the product of measures on $\{-1,1\}$ assigning measure $1/2$ to both ± 1. For $\varepsilon = (\varepsilon_k)_{k\in\mathbb{Z}}$ in X, define $f_\varepsilon(\theta) = \sum_{k\in\mathbb{Z}} \varepsilon_k a_k e^{ik\theta}$. Then $f_\varepsilon \in L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. Hence $S_\varepsilon := S_{T_{f_\varepsilon}}$ defines a bounded Schur multiplier with

$$\|S_\varepsilon\|_m = \|f_\varepsilon\|_1 \le \|f_\varepsilon\|_2 = \|(a_k)\|_2.$$

Then we make use of the Khintchine inequality [22], [9]:

$$\frac{1}{\sqrt{2}}\|(a_k)\|_2 \le \int_X \|f_\varepsilon\|_1 \, d\mu(\varepsilon) \le \|(a_k)\|_2.$$

It follows that on average, most f_ε have L^1-norm comparable to the L^2-norm. In particular, there is some choice of ε with $\|f_\varepsilon\|_1 \ge \frac{1}{\sqrt{2}}\|(a_k)\|_2$. Thus $\mathfrak{s}(\mathcal{S}(\mathcal{A})) \ge \|S_\varepsilon\|_m \ge \frac{1}{\sqrt{2}}\|(a_k)\|_2$. $\square$

REMARK 3.5. In the case of a finite Toeplitz pattern $\mathcal{T}(S)$, say $S = \{s_1 < s_2 < \cdots < s_n\}$, $f_\varepsilon = \sum_{k=1}^n \varepsilon_k e^{is_k\theta}$. We can use the Khintchine inequality for L^∞:

$$\|(a_k)\|_2 \le \int_X \|f_\varepsilon\|_\infty \, d\mu(\varepsilon) \le \sqrt{2}\|(a_k)\|_2.$$

Thus there will be choices of ε so that $\|f_\varepsilon\|_\infty \le \sqrt{2n}$. Then note that $S_{T_{f_\varepsilon}}(T_{f_\varepsilon}) = T_{f_{\mathbf{1}}}$, where $f_{\mathbf{1}} = \sum_{k=1}^n e^{is_k\theta}$. Clearly $\|f_{\mathbf{1}}\|_\infty = f_{\mathbf{1}}(0) = n$. Thus $\|S_{T_{f_\varepsilon}}|_{\mathcal{T}(S)}\| \ge \sqrt{n/2}$.

4. Patterns with a symmetry group

Consider a finite group G acting *transitively* on a finite set X. Think of this as a matrix representation on the Hilbert space $\mathcal{H}_X$ with orthonormal basis $\{e_x : x \in X\}$. Let π denote the representation of G on $\mathcal{H}_X$ and $\mathcal{T}$ the commutant of $\pi(G)$. The purpose of this section is to compute the norm of S_T for $T \in \mathcal{T}$.

Decompose X^2 into G-orbits X_i for $0 \le i \le n$, beginning with the diagonal $X_0 = \{(x,x) : x \in X\}$. Let $T_i \in \mathcal{B}(\mathcal{H}_X)$ denote the matrix with 1s on the entries of X_i and 0 elsewhere. Then it is easy and well-known that $\mathcal{T}$ is $\operatorname{span}\{T_i : 0 \le i \le n\}$. In particular, $\mathcal{T}$ is a C*-algebra. Also observe that every element of $\mathcal{T}$ is constant on the main diagonal.

Since G acts transitively on X, $r_i := |\{y \in X : (x,y) \in X_i\}|$ is independent of the choice of $x \in X$. Thus the vector $\mathbf{1}$ of all ones is a common eigenvector for each T_i, and hence for all elements of $\mathcal{T}$, corresponding to a one-dimensional reducing subspace on which G acts via the trivial representation.

The following is an easy, general upper bound for $\|T\|_m$ for any matrix T which has been proven many times in the literature. As far as we know, the first instance is Davis [6] for the Hermitian case and Walters [25] in the general case. The latter proof is rather involved and a simplification in [4] still seems unnecessarily complicated. So we include a short easy proof here.

As usual, Δ is the expectation onto the diagonal.

Proposition 4.1. *For a matrix T,*

$$\|T\|_m \le \|\Delta(|T^*|)\|^{1/2}\,\|\Delta(|T|)\|^{1/2} = \big\|\,|T^*|\,\big\|_m^{1/2}\,\big\|\,|T|\,\big\|_m^{1/2}.$$

Proof. Use polar decomposition to factor $T = U|T|$. Define vectors $x_i = |T|^{1/2}e_i$ and $y_j = |T|^{1/2}U^*e_j$. Then

$$\langle x_i, y_j\rangle = \langle |T|^{1/2}e_i, |T|^{1/2}U^*e_j\rangle = \langle Te_i, e_j\rangle.$$

This yields a Grothendieck–Haagerup form for S_T. Now

$$\|x_i\|^2 = \langle |T|^{1/2}e_i, |T|^{1/2}e_i\rangle = \langle |T|e_i, e_i\rangle.$$

Hence $\max_i \|x_i\| = \|\Delta(|T|)\|^{1/2}$. Similarly, since $|T|^{1/2}U^* = U^*|T^*|^{1/2}$,

$$\|y_j\|^2 = \langle U^*|T^*|^{1/2}e_j, U^*|T^*|^{1/2}e_j\rangle = \langle |T^*|e_j, e_j\rangle.$$

So $\max_j \|y_j\| = \|\Delta(|T^*|)\|^{1/2}$. Therefore

$$\|T\|_m \le \max_{i,j} \|x_i\|\,\|y_j\| = \|\Delta(|T^*|)\|^{1/2}\,\|\Delta(|T|)\|^{1/2}.$$

Since $|T|$ and $|T^*|$ are positive, the Schur norm is just the sup of their diagonal entries. $\square$

Corollary 4.2. *If $T = T^*$, then $\|T\|_m \le \|\Delta(|T|)\|$.*

Remark 4.3. In general this is a strict inequality. If $T = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$, then $|T| = \begin{bmatrix} 2\sqrt{5} & \sqrt{5} \\ \sqrt{5} & \sqrt{5} \end{bmatrix}$. But $\|S_T\|_m = 4 < 2\sqrt{5}$. Indeed, take $x_1 = y_1 = 2e_1$ and $x_2 = \frac{3}{2}e_1 + \frac{\sqrt{5}}{2}e_2$ and $y_2 = \frac{3}{2}e_1 - \frac{\sqrt{5}}{2}e_2$.

The main result of this section is:

Theorem 4.4. *Let X be a finite set with a transitive action by a finite group G. If T belongs to $\mathcal{T}$, the commutant of the action of G, then for any $x_0 \in X$,*

$$\|T\|_m = \|S_T|_{\mathcal{T}}\| = |X|^{-1}\,\mathrm{Tr}(|T|) = \langle |T|e_{x_0}, e_{x_0}\rangle.$$

This result is a special case of a nice result of Mathias [13]. As far as we know, the application of Mathias' result to the case of matrices invariant under group actions has not been exploited. As Mathias's argument is short and elegant, we include it.

THEOREM 4.5 (Mathias). *If T is an $n \times n$ matrix with $\Delta(|T^*|)$ and $\Delta(|T|)$ scalar, then*

$$\|T\|_m = \frac{1}{n} \operatorname{Tr}(|T|).$$

Proof. For an upper bound, Proposition 4.1 shows that

$$\begin{aligned}\|T\|_m &\le \|\Delta(|T^*|)\|^{1/2} \, \|\Delta(|T|)\|^{1/2} \\ &= \left(\frac{1}{n} \operatorname{Tr}(|T^*|)\right)^{1/2} \left(\frac{1}{n} \operatorname{Tr}(|T|)\right)^{1/2} = \frac{1}{n} \operatorname{Tr}(|T|),\end{aligned}$$

because $|T|$ and $|T^*|$ are constant on the main diagonal, and $|T^*|$ is unitarily equivalent to $|T|$, and so has the same trace.

For the lower bound, use the polar decomposition $T = W|T|$. Let $\overline{W}$ have matrix entries which are the complex conjugates of the matrix entries of W. Write $T = [t_{ij}]$ and $W = [w_{ij}]$ as $n \times n$ matrices in the given basis. Set $\mathbf{1}$ to be the vector with n 1's. Then

$$\begin{aligned}\|T\|_m &\ge \|S_T(\overline{W})\| \ge \frac{1}{n}\langle S_T(\overline{W})\mathbf{1}, \mathbf{1}\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{w}_{ij} t_{ij} = \frac{1}{n}\sum_{j=1}^{n}\langle W^* T e_j, e_j\rangle = \frac{1}{n}\operatorname{Tr}(|T|).\end{aligned}$$

Thus $\|T\|_m = \frac{1}{n} \operatorname{Tr}(|T|)$. ☐

Proof of Theorem 4.4. We have already observed that elements of $\mathcal{T}$ are constant on the diagonal. Thus $\|T\|_m = \frac{1}{n} \operatorname{Tr}(|T|) = \langle |T| e_{x_0}, e_{x_0}\rangle$. For the rest, observe that W belongs to $\mathrm{C}^*(T)$. Hence so does $\overline{W}$ because the basis T_i of $\mathcal{T}$ has real entries. ☐

We will provide an interesting example in the next section. For now we provide a couple of more accessible ones.

EXAMPLE 4.6. Consider the action of the symmetric group $\mathfrak{S}_n$ acting on a set X with n elements in the canonical way. Then the orbits in X^2 are just the diagonal X_0 and its complement X_1. So S_{X_1} is the projection onto the off-diagonal part of the matrix.

Observe that $X_1 = \mathbf{1}\mathbf{1}^* - I$, where $\mathbf{1}$ is the vector of n ones. Since $\mathbf{1}\mathbf{1}^* = nP$, where P is the projection onto $\mathbb{C}\mathbf{1}$, $X_1 = (n-1)P - P^\perp$. Therefore we obtain a formula due to Bhatia, Choi and Davis [3]:

$$\begin{aligned}\|X_1\|_m &= \frac{1}{n}\operatorname{Tr}(|X_1|) = \frac{1}{n}\operatorname{Tr}\left((n-1)P + P^{\perp}\right)\\ &= \frac{1}{n}(n-1+n-1) = 2-\frac{2}{n}.\end{aligned}$$

EXAMPLE 4.7. Consider the cyclic group C_n acting on an n-element set, $n \geq 3$. Let U be the unitary operator given by $Ue_k = e_{k+1}$ for $1 \leq k \leq n$, working modulo n. The powers of U yields a basis for the commutant of the group action.

Consider $T = U + I$. The spectrum of U is just $\{\omega^k : 0 \leq k \leq n-1\}$ where $\omega = e^{2\pi i/n}$. Thus the spectrum of $|T|$ consists of the points

$$|1+\omega^k| = 2\left|\cos\left(\frac{k\pi}{n}\right)\right| \quad \text{for} \quad 0 \leq k \leq n-1.$$

Hence

$$\|T\|_m = \frac{1}{n}\operatorname{Tr}(|T|) = \frac{2}{n}\sum_{k=0}^{n-1}\left|\cos\left(\frac{k\pi}{n}\right)\right| = \begin{cases}\dfrac{2\cos(\frac{\pi}{2n})}{n\sin(\frac{\pi}{2n})} & n \text{ even},\\[2ex] \dfrac{2}{n\sin(\frac{\pi}{2n})} & n \text{ odd}.\end{cases}$$

Thus the limit as n tends to infinity is $\frac{4}{\pi}$. The multiplier norms for the odd cycles decrease to $\frac{4}{\pi}$, while the even cycles increase to the same limit.

EXAMPLE 4.8. Mathias [13] considers polynomials in the circulant matrices C_z given by $C_z e_k = e_{k+1}$ for $1 \leq k < n$ and $C_z e_n = z e_1$, where $|z| = 1$. This falls into our rubric because there is a *diagonal unitary* D so that $DC_zD^* = wU$ where U is the cycle in the previous example and w is any nth root of z. It is easy to see that conjugation by a diagonal unitary has no effect on the Schur norm. Thus any polynomial in C_z is unitarily equivalent to an element of $\mathrm{C}^*(U)$ via the diagonal D. Hence the Schur norm equals the normalized trace of the absolute value.

The most interesting example of this was obtained with $z = -1$ and $S_n = \sum_{k=0}^{n-1} C_{-1}^k$, which is the matrix with entries $\operatorname{sgn}(i-j)$. So the Schur multiplier defined by S_n is a finite Hilbert transform. Mathias shows that

$$\|S_n\|_m = \frac{2}{n}\sum_{j=1}^{\lfloor n/2\rfloor}\cot\frac{(2j-1)\pi}{2n}.$$

From this, he obtains sharper estimates on the norm of triangular truncation than are obtained in [1].

5. Kneser and Johnson graph patterns

In this section, we consider an interesting family of symmetric patterns which arise commonly in graph theory and combinatorial codes. The Johnson graphs $J(v,n,i)$ have $\binom{v}{n}$ vertices indexed by n element subsets of a v element set, and edges between A and B if $|A \cap B| = i$. Thus $0 \le i \le n$. We consider only $1 \le n \le v/2$ since, if $n > v/2$, one obtains the same graphs by considering the complementary sets of cardinality $v-n$. We will explicitly carry out the calculation for the Kneser graphs $K(v,n) = J(v,n,0)$, and in particular, for $K(2n+1,n)$. For more on Johnson and Kneser graphs see [7].

We obtained certain Kneser graphs from Toeplitz patterns. Take a finite subset $S = \{s_1 < s_2 < \cdots < s_{2n+1}\}$ and consider the Toeplitz pattern $\mathcal{P}$ with diagonals in S, namely $\mathcal{P} = \{(i,j) : j-i \in S\}$. Consider R to be the set of all sums of n elements from S and C to be the set of all sums of $n+1$ elements from S. Index R by the corresponding subset A of $\{1,2,\dots,2n+1\}$ of cardinality n; and likewise index each element of C by a subset B of cardinality $n+1$. Then for each entry A in R, there are exactly $n+1$ elements of C which contain it. The difference of the sums is an element of S. It is convenient to re-index C by sets of cardinality n, replacing B by its complement $\{1,2,\dots,2n+1\} \setminus B$. Then the pattern can be seen to be the Kneser graph $K(2n+1,n)$ with $\binom{2n+1}{n}$ vertices indexed by n element subsets of a $2n+1$ element set, with an edge between vertices A and B if $A \cap B = \emptyset$. In general, unfortunately, $\mathcal{P} \cap (R \times C)$ will contain more than just these entries, because two subsets of S of size $n+1$ can have the same sum.

The adjacency matrix of a graph $\mathcal{G}$ is a $v \times v$ matrix with a 1 in each entry (i,j) corresponding to an edge from vertex i to vertex j, and 0's elsewhere. This is a symmetric matrix and its spectral theory is available in the graph theory literature; see, for example, [7]. We prove the simple facts we need.

Fix (v,n) with $n \le v$ and let X denote the set of n element subsets of $\{1,\dots,v\}$. Define a Hilbert space $\mathcal{H} = \mathcal{H}_X$ as in the previous section but write the basis as $\{e_A : A \in X\}$. Observe that there is a natural action π of the symmetric group $\mathfrak{S}_v$ on X. The orbits in X^2 are

$$X_i = \{(A,B) : A, B \in X,\ |A \cap B| = i\} \quad \text{for} \quad 0 \le i \le n.$$

The matrix T_i is just the adjacency matrix of the Johnson graph $J(v,n,i)$ and, in particular, $T_n = I$.

This action has additional structure that does not hold for arbitrary transitive actions.

LEMMA 5.1. *The commutant $\mathcal{T} = \operatorname{span}\{T_i : 0 \le i \le n\}$ of $\pi(\mathfrak{S}_v)$ is abelian. Thus π decomposes into a direct sum of $n+1$ distinct irreducible representations.*

Proof. Equality with the span was observed in the last section. To see that the algebra $\mathcal{T}$ is abelian, observe that $T_i T_j = \sum_{k=0}^n a_{ijk} T_k$ where we can find the coefficients a_{ijk} by fixing any two sets $A, B \subset V$ of size n with $|A \cap B| = k$ and computing

$$a_{ijk} = |\{C \subset V : |C| = n,\ |A \cap C| = i,\ |C \cap B| = j\}.$$

This is clearly independent of the order of i and j. As $\mathcal{T}$ is abelian and $n+1$ dimensional, the representation π decomposes into a direct sum of $n+1$ distinct irreducible representations. □

COROLLARY 5.2. *$\|T_i\| = \binom{n}{i}\binom{v-n}{n-i}$ and this is an eigenvalue of multiplicity one. The spectrum of T_i contains at most $n+1$ points.*

Proof. Observe that if $|A| = n$, then the number of subsets $B \in X$ with $|A \cap B| = i$ is $\binom{n}{i}\binom{v-n}{n-i}$. Thus T_i has this many 1's in each row. Hence

$$T_i \mathbf{1} = \binom{n}{i}\binom{v-n}{n-i}\mathbf{1}.$$

Clearly T_i has nonnegative entries and is indecomposable (except for $i = n$, the identity matrix). So by the Perron–Frobenius Theorem, $\binom{n}{i}\binom{v-n}{n-i}$ is the spectral radius and $\mathbf{1}$ is the unique eigenvector; and there are no other eigenvalues on the circle of this radius. Since $T = T^*$, the norm equals spectral radius. As $\mathcal{T}$ is $n+1$ dimensional, the spectrum can have at most $n+1$ points. □

We need to identify the invariant subspaces of $\mathfrak{S}_v$ as they are the eigenspaces of T_i. The space $V_0 = \mathbb{C}\mathbf{1}$ yields the trivial representation. Define vectors associated to sets $C \subseteq \{1, \dots, v\}$ of cardinality at most n, including the empty set, by

$$v_C := \sum_{|A|=n,\ A \cap C = \emptyset} e_A.$$

Then define subspaces $V_i = \operatorname{span}\{v_C : |C| = i\}$ for $0 \le i \le n$. It is obvious that each V_i is invariant for $\mathfrak{S}_v$. Given C with $|C| = i$, we have

$$\sum_{C \subset D, |D| = i+1} v_D = (v - n - i) v_C,$$

as the coefficient of e_A counts the number of choices for the $(i+1)$st element of D disjoint from an A already disjoint from C. Therefore

$$\mathbb{C}\mathbf{1} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n.$$

So the $n+1$ subspaces $W_i = V_i \ominus V_{i-1}$ are invariant for $\mathfrak{S}_v$.

Let E_i denote the idempotent in $\mathcal{T}$ projecting onto W_i. Observe that $\mathcal{T} = \operatorname{span}\{E_i : 0 \le i \le n\}$. We need to know the dimension of these subspaces.

LEMMA 5.3. *The vectors $\{v_C : |C| = i\}$ are linearly independent. Hence* $\dim W_i = \binom{v}{i} - \binom{v}{i-1}$.

Proof. Suppose that $v_{C_0} + \sum_{|C|=i,\ C\neq C_0} \gamma_C v_C = 0$. By averaging over the subgroup of $\mathfrak{S}_v$ which fixes C_0, namely $\mathfrak{S}_i \times \mathfrak{S}_{v-i}$, we may assume that the coefficients are invariant under this action. Hence $\gamma_C = \alpha_j$ where $j = |C \cap C_0|$. So with $w_j := \sum_{|C|=i,\ |C\cap C_0|=j} v_C$, we have $\sum_{j=0}^{i} \alpha_j w_j = 0$ where $\alpha_i = 1$. We also define vectors $x_k = \sum_{|A\cap C_0|=k} e_A$, which are clearly linearly independent for $0 \le k \le i$. Compute for $0 \le j \le i$ (here A implicitly has $|A| = n$)

$$w_j = \sum_{\substack{|C|=i \\ |C\cap C_0|=j}} \sum_{A\cap C=\emptyset} e_A = \sum_{k=0}^{i-j} b_{jk} x_k$$

where the coefficients are obtained by counting, for a fixed set A with $|C_0 \cap A| = k$ and $k \le i - j$:

$$b_{jk} = |\{C : |C| = i,\ |C \cap C_0| = j,\ A \cap C = \emptyset\}| = \binom{i-k}{j}\binom{v+k-n-i}{i-j}.$$

It is evident by induction that

$$\operatorname{span}\{w_j : i - k \le j \le i\} = \operatorname{span}\{x_j : 0 \le j \le k\}.$$

So $\{v_C : |C| = i\}$ are linearly independent. $\square$

We let $T_i = \sum_{j=0}^{n} \lambda_{ij} E_j$ be the spectral decomposition of each T_i. The discussion above shows that if $|C| = j$, then v_C is contained in V_j but not V_{j-1}. Thus λ_{ij} is the unique scalar so that $(T_i - \lambda_{ij} I) v_C \in V_{j-1}$. This idea can be used to compute the eigenvalues, but the computations are nontrivial. We refer to [7, Theorem 9.4.3] for the Kneser graph $K(2n+1, n)$ which is the only one we work out in detail.

LEMMA 5.4. *The adjacency matrix for the Kneser graph $K(2n+1, n)$ has eigenvalues $(-1)^i(n+1-i)$ with eigenspaces W_i for $0 \le i \le n$.*

THEOREM 5.5. *If T is the adjacency matrix of $K(2n+1, n)$, then*

$$\|T\|_m = \|S_T|_{\mathcal{T}}\| = \frac{2^{2n}}{\binom{2n+1}{n}} = \frac{(4)(6)\dots(2n+2)}{(3)(5)\dots(2n+1)} > \frac{1}{2}\log(2n+3).$$

Proof. By Theorem 4.4 and Lemma 5.4,

$$\begin{aligned}\|T\|_m = \|\Delta(|T|)\| &= \binom{2n+1}{n}^{-1} \sum_{i=0}^{n} (n+1-i)\operatorname{Tr}(E_i) \\ &= \binom{2n+1}{n}^{-1} \sum_{i=0}^{n} (n+1-i)\left(\binom{2n+1}{i} - \binom{2n+1}{i-1}\right)\end{aligned}$$

$$
\begin{aligned}
&= \binom{2n+1}{n}^{-1} \sum_{i=0}^{n} \binom{2n+1}{i} \\
&= \binom{2n+1}{n}^{-1} \frac{1}{2} \sum_{i=0}^{2n+1} \binom{2n+1}{i} \\
&= \binom{2n+1}{n}^{-1} 2^{2n} = \frac{2^{2n} n!(n+1)!}{(2n+1)!} \\
&= \frac{2\cdot 4\cdots(2n)}{2\cdot 4\cdots(2n)} \frac{2\cdot 4\cdots(2n)\cdot(2n+2)}{1\cdot 3\cdots(2n-1)(2n+1)} \\
&= \frac{2\cdot 4\cdots(2n)\cdot(2n+2)}{1\cdot 3\cdots(2n-1)(2n+1)} \\
&= \prod_{i=0}^{n}\left(1+\frac{1}{2i+1}\right) > \frac{1}{2}\log(2n+3). \qquad \square
\end{aligned}
$$

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