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BIG INDECOMPOSABLE MODULES AND DIRECT-SUM RELATIONS

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Dedicated to Phillip Griffith, the master of syzygies

Abstract. A commutative Noetherian local ring $(R, \mathfrak{m})$ is said to be *Dedekind-like* provided R has Krull-dimension one, R has no non-zero nilpotent elements, the integral closure $\overline{R}$ of R is generated by two elements as an R-module, and $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$. A classification theorem due to Klingler and Levy implies that if M is a finitely generated indecomposable module over a Dedekind-like ring, then, for each minimal prime ideal P of R, the vector space M_P has dimension $0, 1$ or 2 over the field R_P. The main theorem in the present paper states that if R (commutative, Noetherian and local) has non-zero Krull dimension and is not a homomorphic image of a Dedekind-like ring, then there are indecomposable modules that are free of any prescribed rank at each minimal prime ideal.

1. Introduction

In a series of papers [14]–[16] Klingler and Levy proved the existence of tame-wild dichotomy for commutative Noetherian rings. They gave a complete classification of all finitely generated modules over Dedekind-like rings (cf. Definition 1.1) and showed that, over any ring that is not a homomorphic image of a Dedekind-like ring, the category of finite-length modules has wild representation type. A consequence of their classification is that if M is an indecomposable finitely generated module over a Dedekind-like ring R, then M_P is free of rank $0, 1$ or 2 at each minimal prime ideal P of R. The main theorem of the present paper complements this result of Klinger and Levy. We prove that if $(R, \mathfrak{m}, k)$ is a commutative local Noetherian ring of non-zero Krull dimension and R is not a homomorphic image of a Dedekind-like ring,

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then there are indecomposable modules that are free of any prescribed rank at each minimal prime.

This result was obtained in [9] for the case of a Cohen-Macaulay ring, using a direct but highly intricate construction. In [10] we gave a much simpler argument that handles all rings—Cohen-Macaulay or not—for which some power of the maximal ideal requires at least 3 generators. The remaining case, when $(R, \mathfrak{m}, k)$ is not Cohen-Macaulay and each power of $\mathfrak{m}$ is two-generated, was treated via an indirect argument using the bimodule structure of certain Ext modules. In this paper we apply the Ext argument, together with periodicity of resolutions over hypersurface rings, to give a unified treatment of the case when each power of $\mathfrak{m}$ is two-generated. Thus this paper does not rely on the technical construction in [9]. Our goal is to make the paper pretty much self-contained, though we do refer without proof to some of the results of [6], [10] and [14]–[16].

We actually obtain $\max\{|R/\mathfrak{m}|, \aleph_0\}$ pairwise non-isomorphic indecomposables of each rank. This refinement allows us, in dimension one, to obtain precise defining equations for the monoid of isomorphism classes of finitely generated modules that are free on the punctured spectrum. This generalizes the results of [6], which apply only to the Cohen-Macaulay case.

Our main theorem provides indecomposable modules that are free of specified rank at each prime P in a given finite set $\mathcal{P} \subseteq \operatorname{Spec}(R) - \{\mathfrak{m}\}$. In dimension greater than one we have to allow for the fact that if $M_P \cong R_P^{(n)}$ and Q is a prime ideal contained in P, then $M_Q \cong R_Q^{(n)}$. For $P_1, P_2 \in \mathcal{P}$ we write $P_1 \sim P_2$ if $P_1 \cap P_2$ contains a prime ideal of R (not necessarily in $\mathcal{P}$). (Of course "$\sim$" is not necessarily a transitive relation.)

Definition 1.1. The commutative, Noetherian local ring $(R, \mathfrak{m}, k)$ is *Dedekind-like* [14, Definition 2.5] provided R is one-dimensional and reduced, the integral closure $\overline{R}$ of R in the total quotient ring of R is generated by at most 2 elements as an R-module, and $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$. We call $(R, \mathfrak{m}, k)$ an *exceptional Dedekind-like ring* provided, in addition, $\overline{R}/\mathfrak{m}$ is a purely inseparable field extension of k of degree 2.

There is a global notion of Dedekind-like, which is equivalent to Noetherian and locally Dedekind-like [16, Corollary 10.7]. In this article, "Dedekind-like" always means Dedekind-like and local, except in the last section, where we take up the question of the size of finitely generated indecomposable modules over arbitrary commutative Noetherian rings.

The classification of finitely generated modules in [15] and [16] does not apply to exceptional Dedekind-like rings. The details in the exceptional case are extremely complicated and are currently being worked out by L. Klingler, G. Piepmeyer and S. Wiegand. It appears that the indecomposable modules over an exceptional Dedekind-like ring have torsion-free rank $0, 1$ or 2, as in the non-exceptional case. Thus everything in this paper would hold without the “non-exceptional” proviso. Nonetheless, since the classification of modules in the exceptional case is still a work in progress, we have decided to restrict to non-exceptional Dedekind-like rings in the second part of our main theorem below.

THEOREM 1.2 (Main Theorem). *Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring.*

(i) *Suppose R is not a homomorphic image of a Dedekind-like ring. Let $\mathcal{P}$ be a finite set of non-maximal prime ideals of R, and let n_P be a non-negative integer for each $P \in \mathcal{P}$. Assume that $n_P = n_Q$ whenever $P \sim Q$. Then there exist $|k| \cdot \aleph_0$ pairwise non-isomorphic indecomposable finitely generated R-modules X such that, for each $P \in \mathcal{P}$, the localization X_P is a free R_P-module of rank n_P.*

(ii) *Conversely, assume R is not an exceptional Dedekind-like ring, but that R is a homomorphic image of some Dedekind-like ring. If X is an indecomposable finitely generated R-module and P is a non-maximal prime, then X_P either is 0 or is isomorphic to R_P or $R_P^{(2)}$.*

It is tempting to conjecture a substantial improvement of this result in higher dimensions. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension at least two, and let $C_1, \ldots, C_t$ be the connected components of the punctured spectrum $\operatorname{Spec}(R) - \{\mathfrak{m}\}$. Given any sequence $n_1, \ldots, n_t$ of non-negative integers, is there necessarily an indecomposable R-module M such that $M_P \cong R_P^{(n_i)}$ for each i and each $P \in C_i$? Our methods do not seem to yield modules that are free on the entire punctured spectrum.

Part (ii) of the Main Theorem is an easy consequence of the classification theorem in [15]: Since the assertion is vacuous if $\dim(R) = 0$ and the hypotheses fail if $\dim(R) > 1$, we assume $\dim(R) = 1$. Let $R = D/J$, where D is a Dedekind-like ring. If D were an exceptional Dedekind-like ring, then, by assumption, J would have to be non-zero. But then R would be zero-dimensional, since exceptional Dedekind-like rings are domains. Therefore D is not exceptional, and we can apply the results in [15] and [16]. Write $P = Q/J$, where Q is a non-maximal, hence minimal, prime ideal of D. Viewing M as a D-module, we see, using [16, Corollary 16.4], that M_Q is either 0 or is isomorphic to D_Q or $D_Q^{(2)}$. Since the natural map $D_Q \to R_P$ is an isomorphism, the desired conclusion follows.

2. When some power of $\mathfrak{m}$ requires 3 or more generators

PROPOSITION 2.1. *Let $(R, \mathfrak{m}, k)$ be a commutative, Noetherian local ring for which some power $\mathfrak{m}^r$ of the maximal ideal requires at least three generators. Let $\mathcal{P}$ be a finite subset of* $\operatorname{Spec}(R) - \{\mathfrak{m}\}$*, and let n_P be a non-negative integer for each $P \in \mathcal{P}$. Assume that $n_P = n_Q$ whenever $P \sim Q$.*

Let $n_1 < \cdots < n_t$ be the distinct integers in $\{n_P \mid P \in \mathcal{P}\}$, and put $n := n_1 + \cdots + n_t$. Given any integer $q > n$, there are $|k|$ pairwise non-isomorphic indecomposable finitely generated R-modules M such that

(i) *M needs exactly $2q$ generators, and*
(ii) *$M_P \cong R_P^{(n_P)}$ for each $P \in \mathcal{P}$.*

Proof. Choose $x \in \mathfrak{m}^r - (\mathfrak{m}^{r+1} \cup (\bigcup \mathcal{P}))$, $y \in \mathfrak{m}^r - ((\mathfrak{m}^{r+1} + Rx) \cup (\bigcup \mathcal{P}))$ and $z \in \mathfrak{m}^r - ((\mathfrak{m}^{r+1} + Rx + Ry) \cup (\bigcup \mathcal{P}))$. Thus x, y and z are outside the union of the primes in $\mathcal{P}$, and their images in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ are linearly independent.

For $i = 1, \ldots, t$, let $\mathcal{P}_i = \{P \in \mathcal{P} \mid n_P = n_i\}$. Put $S_i = R - \bigcup \mathcal{P}_i$, and let K_i be the kernel of the natural map $R \to S_i^{-1}R$. We claim that $0 \in S_iS_j$ if $i \neq j$. If not, there would be a prime ideal Q disjoint from the multiplicative set S_iS_j. But then Q would be contained in $P_i \cap P_j$ for some $P_i \in \mathcal{P}_i$ and $P_j \in \mathcal{P}_j$, contradicting $P_i \not\sim P_j$. It follows that $S_i^{-1}S_j^{-1}R = 0$ if $i \neq j$, that is, $K_iS_j^{-1}R = S_j^{-1}R$ if $i \neq j$. Therefore we can choose, for each $i = 1, \ldots, t$, an element

$$\xi_i \in K_i\mathfrak{m}^{r+1} - \bigcup_{j \neq i} \Big(\bigcup \mathcal{P}_j \Big).$$

The image of ξ_i in $S_j^{-1}R$ is 0 if $i = j$ and a unit if $i \neq j$.

Let I_l denote the $l \times l$ identity matrix and 0_l the $l \times l$ zero matrix. Let $H = H_q$ be the $q \times q$ nilpotent Jordan block with 1's on the superdiagonal and 0's elsewhere. Given any element $u \in R$, put

$$\Delta = \Delta_{q,u} := (z + uy)I_q + yH_q.$$

Consider the following matrix:

$$A = A_{q,u} := \begin{bmatrix} \Xi & \Delta \\ 0_q & xI_q \end{bmatrix} \in \operatorname{Mat}_{2q \times 2q}(R), \tag{1}$$

where

$$\Xi := \begin{bmatrix} \xi_1 I_{n_1} & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 I_{n_2} & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & \xi_t I_{n_t} & 0 \\ 0 & \cdots & 0 & 0 & x^2 I_{q-n} \end{bmatrix} \in \operatorname{Mat}_{q \times q}(R).$$

We let A operate on $R^{(q)} \oplus R^{(q)}$ by left multiplication, and we put $M = M_{q,u} := \operatorname{coker}(A)$. Since the entries of A are in $\mathfrak{m}$, $M_{q,u}$ requires exactly $2q$ generators.

We now show that $M_{q,u}$ is indecomposable, and that $M_{q,u} \not\cong M_{q,u'}$ if $u, u' \in R$ and $u \not\equiv u' \pmod{\mathfrak{m}}$. Fix q, let $u, u' \in R$, and put $A' := A_{q,u'}$, $M' := M_{q,u'}$ and $\Delta' := \Delta_{q,u'}$. Let f be an arbitrary R-homomorphism from $M_{q,u}$ to

$M_{q,u'}$. We lift f to homomorphisms F and G making the following diagram commutative:

$$\begin{array}{ccccccc} R^{(q)} \oplus R^{(q)} & \xrightarrow{A} & R^{(q)} \oplus R^{(q)} & \longrightarrow & M & \longrightarrow & 0 \\ {\scriptstyle G}\Big\downarrow & & {\scriptstyle F}\Big\downarrow & & {\scriptstyle f}\Big\downarrow & & \\ R^{(q)} \oplus R^{(q)} & \xrightarrow{A'} & R^{(q)} \oplus R^{(q)} & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

When we write F and G as 2×2 block matrices, this diagram yields the equation

$$\text{(2)} \quad \begin{bmatrix} F_{11}\Xi & F_{11}\Delta + F_{12}x \\ F_{21}\Xi & F_{21}\Delta + F_{22}x \end{bmatrix} = FA = A'G$$
$$= \begin{bmatrix} \Xi G_{11} + \Delta' G_{21} & \Xi G_{12} + \Delta' G_{22} \\ G_{21}x & G_{22}x \end{bmatrix}.$$

Let stars denote the images, in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$, of elements of $\mathfrak{m}^r$. Thus $\xi_i^* = 0$ for each i, $(x^2)^* = 0$, and x^*, y^* and z^* are linearly independent over k. Let bars denote reductions modulo $\mathfrak{m}$ of elements of R and of matrices over R. Comparing the 2,2 entries of the matrix equation (2), we obtain the following equation:

$$\overline{F}_{21}(\overline{u}\overline{I}_q + \overline{H})y^* + \overline{F}_{21}z^* + \overline{F}_{22}x^* = \overline{G}_{22}x^*.$$

It follows that

$$\overline{F}_{21} = 0 \text{ and } \overline{F}_{22} = \overline{G}_{22}.$$

An examination of the 1,2 entries in (2) yields the following equation:

$$\overline{F}_{11}(\overline{u}\overline{I}_q + \overline{H})y^* + \overline{F}_{11}z^* + \overline{F}_{12}x^* = G_{22}\overline{z}^* + (\overline{u'}\,\overline{I}_q + \overline{H})\overline{G}_{22}y^*$$

It follows that

$$\text{(3)} \qquad \overline{F}_{12} = 0,\ \overline{F}_{11} = \overline{G}_{22} \text{ and } \overline{F}_{11}(\overline{u}\overline{I}_q + \overline{H}) = (\overline{u'}\ \overline{I}_q + \overline{H})\overline{G}_{22}.$$

The last two equations in (3) show that

$$\text{(4)} \qquad (\overline{u} - \overline{u'})\overline{F}_{11} = \overline{H}\ \overline{F}_{11} - \overline{F}_{11}\overline{H}.$$

Suppose now that $u \not\equiv u' \pmod{\mathfrak{m}}$. Then $\overline{u} - \overline{u'} \in k^\times$, and since $\overline{H}^q = 0$ we see, by descending induction, that $\overline{H}^i\overline{F}_{11}\overline{H}^j = 0$ for $i, j = 0, \ldots, q$. Setting $i = j = 0$, we get $\overline{F}_{11} = 0$. Since $\overline{F}_{12} = 0$ too, $\overline{F}$ is not surjective, and now Nakayama's lemma implies that f is not surjective. Since f was an arbitrary element of $\operatorname{Hom}_R(M_{q,u}, M_{q,u'})$, this shows that $M_{q,u} \not\cong M_{q,u'}$.

To prove that $M = M_{q,u}$ is indecomposable, we let $u' = u$, and we assume $f \in \operatorname{End}_R(M, M)$ is idempotent but not surjective. We will show that $f = 0$. Since $\overline{H}$ is non-derogatory, (4) implies that $\overline{F}_{11} \in k[\overline{H}]$. In particular, $\overline{F}_{11}$ is upper triangular with constant diagonal. Recall that $\overline{F}_{11} = \overline{G}_{22} = \overline{F}_{22}$ and $\overline{F}_{21} = 0$, so that $\overline{F}$ is upper triangular with constant diagonal. Since $\overline{F}$ is not surjective, it must be strictly upper-triangular. Therefore $\overline{F}^q = 0$. Then

$\operatorname{im}(f) = \operatorname{im}(f^q) \subseteq \mathfrak{m}M$, whence $1 - f$ is surjective. Since f is idempotent, $f = 0$.

It remains to prove that $S_i^{-1}M \cong (S_i^{-1}R)^{(n_i)}$ for all i. Fix an index $i \leq t$, and consider the image $\tilde{A}$ in $\mathrm{Mat}_{2q\times 2q}(S_i^{-1}R)$ of the matrix A. We recall that the $\xi_j, j \neq i$, become units in $\tilde{A}$, while ξ_i maps to 0. Also, x, y and z map to units. Using these facts, one can easily do elementary row and column operations over $S_i^{-1}R$ to show that $\tilde{A}$ is equivalent to the $2q \times 2q$ matrix B with I_{2q-n_i} in the top left corner and zeros elsewhere. Thus $S_i^{-1}M \cong \operatorname{coker}(\tilde{A}) \cong \operatorname{coker}(B) \cong (S_i^{-1}R)^{(n_i)}$ as desired. □

By item (i) in the statement of the theorem, $M_{q,u} \ncong M_{q',u'}$ if $q \neq q'$. Thus the Main Theorem is true if some power of $\mathfrak{m}$ requires at least three generators.

3. Bimodules and extensions

In this section we concoct some homological machinery to handle the more difficult case of the Main Theorem—when each power of the maximal ideal is generated by two elements.

Throughout this section let R be a commutative Noetherian ring, not necessarily local, and let A and B be module-finite R-algebras (not necessarily commutative). Let ${}_AE_B$ be an $A-B$-bimodule. We assume E is R-*symmetric*, that is, $re = er$ for $r \in R$ and $e \in E$. (Equivalently, E is a left $A \otimes_R B^{\mathrm{op}}$-module.) Furthermore we assume that E is finitely generated as an R-module. The Jacobson radical of a (not necessarily commutative) ring C is denoted by $\mathrm{J}(C)$, and the ring C is said to be *local* provided $C/\mathrm{J}(C)$ is a division ring; equivalently [7, Proposition 1.10], the set of non-units of C is closed under addition. The next result is [10, Theorem 3.2], and we refer the interested reader to [10] for its elementary proof.

THEOREM 3.1. *With notation above, let $\alpha : {}_AA \to {}_AE$ and $B_B \to E_B$ be module homomorphisms such that $\alpha(1_A) = \beta(1_B) \neq 0$. Assume A is local and $\ker(\beta) \subseteq \mathrm{J}(B)$. Then $C := \beta^{-1}(\alpha(A))$ is an R-subalgebra of B and is a local ring.*

Now we specialize the notation above. Still assuming that R is a commutative Noetherian ring, let M and N be finitely generated R-modules. Put $A := \mathrm{End}_R(M)$ and $B := \mathrm{End}_R(N)$. Note that each of the R-modules $\mathrm{Ext}^n_R(N, M)$ has a natural $A - B$-bimodule structure. Indeed, any $f \in B$ induces an R-module homomorphism $f^* : \mathrm{Ext}^n_R(N, M) \to \mathrm{Ext}^n_R(N, M)$. For $x \in \mathrm{Ext}^n_R(N, M)$ put $x \cdot f = f^*(x)$. The left A-module structure is defined similarly, and the fact that $\mathrm{Ext}^n_R(N, M)$ is a bimodule follows from the fact that $\operatorname{Ext}_R^n(\_\,,\_)$ is an additive bifunctor. Note that $\operatorname{Ext}_R^n(N,M)$ is R-symmetric, since, for $r \in R$, multiplications by r on N and on M induce the same endomorphism of $\operatorname{Ext}_R^n(N,M)$.

Put $E := \operatorname{Ext}_R^1(N,M)$, regarded as the set of equivalence classes of extensions $0 \to M \to X \to N \to 0$. Let $\alpha : {}_AA \to {}_AE$ and $\beta : B_B \to E_B$ be module homomorphisms satisfying $\alpha(1_A) = \beta(1_B) =: [\sigma]$. Then α and β are, up to signs, the connecting homomorphisms in the long exact sequences of Ext obtained by applying $\operatorname{Hom}_R(\_\,,M)$ and $\operatorname{Hom}_R(N,\_)$, respectively, to the short exact sequence σ. (When one computes Ext via resolutions one must adorn maps with appropriate $\pm$ signs, in order to ensure naturality of the connecting homomorphisms. In what follows, the choice of sign will not be important.)

Since it causes no extra effort, we phrase Lemma 3.2 and Theorem 3.3 in terms of a general *torsion theory* $(\mathcal{T},\mathcal{F})$ (cf., e.g., [8]). Then, in Corollary 3.4, we apply Theorem 3.3 with $\mathcal{T} = \{\text{modules of finite length}\}$ and $\mathcal{F} = \{\text{modules of positive depth}\} = \{\text{modules with zero socle}\}$.

An easy diagram chase establishes the following lemma, which is [10, Lemma 4.1]:

Lemma 3.2. *Let R be a commutative Noetherian ring, let M and N be finitely generated R-modules, with M torsion and N torsion-free (with respect to some torsion theory). Let A, B and E be as above, and let $\alpha : A \to E$ and $\beta : B \to E$ be module homomorphisms with $\alpha(1_A) = \beta(1_B) = [\sigma] \neq 0$. Choose a short exact sequence representing σ:*

$$(\sigma) \qquad 0 \longrightarrow M \xrightarrow{\;i\;} X \xrightarrow{\;\pi\;} N \longrightarrow 0$$

Let $\rho : \operatorname{End}_R(X) \to \operatorname{End}_R(N) = B$ be the canonical homomorphism (reduction modulo torsion). Then the image of ρ is exactly the ring $C := \beta^{-1}\alpha(A) \subseteq B$.

The next result, which is [10, Theorem 4.2], follows easily from Theorem 3.1 and Lemma 3.2:

Theorem 3.3. *Keep the notation and hypotheses of Lemma 3.2.*

(i) *Suppose C has no idempotents other than 0 and 1. If $X = U \oplus V$ (a decomposition as R-modules), then either U or V is a torsion module.*
(ii) *Suppose A is local and $\ker(\beta)$ is contained in the Jacobson radical of B. Then X is indecomposable.*

Corollary 3.4. *Let $(R,\mathfrak{m},k)$ be a commutative, Noetherian local ring, and let M be an indecomposable finitely generated R-module of finite length. Let N be a finitely generated R-module with $\operatorname{depth}(N) > 0$. Put $A := \operatorname{End}_R(M)$ and $B := \operatorname{End}_R(N)$. Suppose there exists a right B-module homomorphism $\beta : B_B \to E_B := \operatorname{Ext}_R^1(N,M)$ such that $\ker(\beta) \subseteq \mathrm{J}(B)$ (equivalently, assume there is an element $\xi \in E$ with $(0 :_B \xi) \subseteq \mathrm{J}(B)$). Let*

$0 \to M \to X \to N \to 0$ *represent* $\xi = \beta(1_B) \in E$. *Then* X *is indecomposable.*

4. Building suitable finite-length modules

To prove the Main Theorem in the remaining case, when each power of $\mathfrak{m}$ is two-generated, we need to build a sufficiently complicated indecomposable finite-length module M and then choose a suitable module N of positive depth. In this section we build the requisite finite-length modules.

The following proposition is a slightly jazzed-up version of the "Warmup" in [10]. This construction is far from new. See, for example, the papers of Higman [12], Heller and Reiner [11], and Warfield [23]. Similar constructions can be found in the classification, up to simultaneous equivalence, of pairs of matrices. (Cf. Dieudonné's discussion [3] of the work of Kronecker [17] and Weierstrass [24].)

Proposition 4.1. *Let* $(\Lambda, \mathfrak{m}, k)$ *be a commutative Noetherian local ring with* $\mathfrak{m}^2 = 0$*, let* q *be a positive integer and let* u *be a unit of* Λ*. Let* I_q *denote the* $q \times q$ *identity matrix and* H_q *the* $q \times q$ *nilpotent Jordan block (with* 1*'s on the superdiagonal and* 0*'s elsewhere). Assume* $\mathfrak{m}$ *is minimally generated by two elements* x *and* y*, let* $\Psi_{q,u} := yI_q + x(uI_q + H_q)$ *and put* $M_{q,u} := \operatorname{coker}(\Psi_{q,u})$.

(i) $M_{q,u}$ *is an indecomposable* Λ*-module requiring exactly* q *generators.*
(ii) *For every non-zero element* $t \in \mathfrak{m}$*,* $\operatorname{socle}(M_{q,u}/tM_{q,u}) \cong k^{(q)}$.
(iii) $M_{q,u} \cong M_{q',u'}$ *if and only if* $q = q'$ *and* $u \equiv u' \pmod{\mathfrak{m}}$.

Proof. Clearly $M_{q,u}$ requires exactly q generators, whence $M_{q,u} \not\cong M_{q',u'}$ if $q \neq q'$. Therefore we drop the subscripts q from now on. The "if" assertion in (iii) is clear, since $\mathfrak{m}^2 = 0$. The proofs of the "only if" assertion in (iii) and of the indecomposability of the M_u are similar to (but easier than) the proofs of the analogous assertions in Proposition 2.1. Alternatively, one can note that the associated graded modules $\operatorname{gr}_{\mathfrak{m}}(M_u)$ are among the indecomposable modules in the classification of $k[X,Y]/(X^2, XY, Y^2)$-modules, found in the references above.

To prove (ii), we drop the index u and note that $M/tM = \operatorname{coker}(\Phi)$, where $\Phi = [\,\Psi \;\; tI\,]$. Suppose first that $t = by$, where b is a unit of Λ. Elementary column operations transform Φ to the matrix $[\,xH \;\; yI\,]$. Therefore $M/tM \cong k^{(q-1)} \oplus \Lambda/(y)$, and (ii) follows. The other possibility is that $t = ax + by$, where a is a unit. In this case we can do elementary column operations to replace the superdiagonal elements of Ψ by multiples of y. Further column operations transform the matrix to the form $[\,yI \;\; xI\,]$, and we have $M/tM \cong k^{(q)}$. $\square$

If $(R, \mathfrak{m}, k)$ is Artinian and $\mathfrak{m}$ is principal, the zero-dimensional case of Cohen's Structure Theorem implies that R is a homomorphic image of a complete discrete valuation ring. Thus, if $(R, \mathfrak{m}, k)$, as in the Main Theorem, is zero-dimensional, we can apply Proposition 4.1 to the ring $R/\mathfrak{m}^2$ to get $|k|\cdot\aleph_0$ pairwise non-isomorphic indecomposable modules. Next, suppose $\dim(R) \geq 2$. Then $\mathfrak{m}$ needs three generators unless R is a two-dimensional regular local ring; and in that case $\mathfrak{m}^2$ needs three generators. By Proposition 2.1, the Main Theorem holds if $\dim(R) \geq 2$. Therefore it remains to prove the Main Theorem under the assumptions that $(R, \mathfrak{m}, k)$ is one-dimensional and each power of $\mathfrak{m}$ is at most two-generated.

Definition 4.2. A commutative, Artinian local ring $(\Lambda, \mathfrak{m}, k)$ is a *Drozd ring* provided its associated graded ring is the k-algebra $\mathrm{gr}_{\mathfrak{m}}(\Lambda) \cong k[X, Y]/(X^2, XY^2, Y^3)$. (Equivalently, $\mathfrak{m}^3 = 0$, $\mathfrak{m}$ and $\mathfrak{m}^2$ each require exactly two generators, and there is an element $t \in \mathfrak{m} - \mathfrak{m}^2$ with $t^2 = 0$.)

The main result in this section is a construction, in Proposition 4.4, of suitably complex indecomposable modules over Drozd rings. The idea of the construction below originated in work of Drozd [4] and Ringel [21]. The construction was adapted by Klingler and Levy [14] to show that the category of finite-length modules over a Drozd ring has wild representation type. Drozd rings enter the picture here because of the following result, a special case of the "Ring-theoretic Dichotomy" of Klingler and Levy [16, Theorem 14.3]:

Theorem 4.3. *Let $(\Lambda, \mathfrak{m}, k)$ be a one-dimensional local ring whose maximal ideal $\mathfrak{m}$ is generated by at most two elements. Then exactly one of the following possibilities occurs:*

(i) *Λ is a homomorphic image of a Dedekind-like ring.*
(ii) *Λ has a Drozd ring as a homomorphic image.*

Proposition 4.4 is a slight generalization of [10, Proposition 5.3]. The modification is needed to treat the case of a Cohen-Macaulay ring with multiplicity 2.

Proposition 4.4. *Let $(\Lambda, \mathfrak{m}, k)$ be a Drozd ring, and let $t, y \in \Lambda$ with $(t, y) = \mathfrak{m}$ and $t^2 = 0$. There exists a family $(M_{q,\kappa})_{q\in\mathbb{N},\kappa\in k^\times}$ of pairwise non-isomorphic indecomposable Λ-modules having the following properties:*

(i) *For all $q \in \mathbb{N}$ and $\kappa \in k^\times$ we have*

$$\frac{(0 :_{M_{q,\kappa}} (t, y^2))}{tM_{q,\kappa}} \;\cong\; k^{(q)}.$$

(ii) *For every $\xi \in \mathfrak{m}$, all $\kappa \in k^\times$, and all $q \geq 1$ the k-vectorspace*

$$\frac{(0 :_{M_{q,\kappa}} \xi) + \mathfrak{m}M_{q,\kappa}}{\mathfrak{m}M_{q,\kappa}}$$

has dimension greater than or equal to q.

Proof. Given $q \in \mathbb{N}$ and $\kappa \in k^\times$, choose $u \in \Lambda^\times$ with $u + \mathfrak{m} = \kappa$. Since $\mathfrak{m}^3 = 0$, uy^2 depends only on the coset $u + \mathfrak{m}$. Therefore we can define $M_{q,\kappa}$ to be the cokernel of the $3q \times 4q$ matrix

$$\Psi_{q,\kappa} := \begin{bmatrix} yI_q & tI_q & 0 & 0 \\ 0 & -y^2 I_q & tI_q & -yI_q \\ 0 & 0 & -(uI_q + H_q)y^2 & tI_q \end{bmatrix}$$

with H_q and I_q as in Proposition 4.1. We let $\Lambda^{(3q)} \xrightarrow{\varepsilon_{q,\kappa}} M_{q,\kappa}$ denote the quotient map.

To show that $M_{q,\kappa}$ is indecomposable and that $M_{q,\kappa} \not\cong M_{q,\kappa'}$ if $\kappa \neq \kappa'$, suppose $f : M_{q,\kappa} \to M_{q,\kappa'}$ is a Λ-homomorphism. Lift κ' to $u' \in \Lambda^\times$. As in the proof of Proposition 2.1 we obtain a commutative diagram:

$$\begin{array}{ccccc} \Lambda^{(4q)} & \xrightarrow{\Psi_{q,\kappa}} & \Lambda^{(3q)} & \longrightarrow & M_{q,\kappa} \\ {\scriptstyle G}\big\downarrow & & {\scriptstyle F}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \Lambda^{(4q)} & \xrightarrow{\Psi_{q,\kappa'}} & \Lambda^{(3q)} & \longrightarrow & M_{q,\kappa'} \end{array}$$

In principle we could proceed as in Proposition 2.1 and derive restrictions for the entries of F from the equation $F \cdot \Psi_{q,\kappa} = \Psi_{q,\kappa'} \cdot G$; instead, we consult [14, Lemma 4.8] to shorten the argument. If we let bars denote reductions modulo $\mathfrak{m}$, this lemma implies that

$$\overline{F} = \begin{bmatrix} \overline{F}_{11} & * & * \\ 0 & \overline{F}_{11} & * \\ 0 & 0 & \overline{F}_{11} \end{bmatrix},$$

where each block is a $q \times q$ matrix and $\overline{F}_{11} \cdot (u\overline{I}_q + \overline{H}_q) = (u'\overline{I}_q + \overline{H}_q) \cdot \overline{F}_{11}$.

If $\kappa \neq \kappa'$, the argument following (4) in the proof of Proposition 2.1 shows that $M_{q,\kappa} \not\cong M_{q,\kappa'}$. Of course $M_{q,\kappa}$ requires exactly $3q$ generators, so $M_{q,\kappa} \cong M_{q',\kappa'} \implies q = q'$. Thus we assume from now on that $\kappa = \kappa'$ and omit the subscripts q and κ. The proof that M is indecomposable is essentially the same as the proof of indecomposability in Proposition 2.1.

We claim that $(0 :_M (t, y^2))$ is generated by the images, under ε, of the columns of the matrix

$$\boldsymbol{\varphi} := \begin{bmatrix} tI & 0 & 0 & 0 & 0 & I \\ 0 & tI & 0 & yI & 0 & 0 \\ 0 & 0 & tI & 0 & y^2 I & -yI \end{bmatrix}$$

(where each block is $q \times q$). An easy calculation shows that both t and y^2 knock the column space of $\boldsymbol{\varphi}$ into the column space of Ψ, so the purported generators are, at least, in $(0 :_M (t, y^2))$.

To prove the claim, suppose $\boldsymbol{\alpha} \in \Lambda^{(3q)}$ and $t\boldsymbol{\alpha}$ and $y^2\boldsymbol{\alpha}$ are both in the image of Ψ. We will show that $\boldsymbol{\alpha} \in \operatorname{im}(\boldsymbol{\varphi})$.

We have

$$t\boldsymbol{\alpha} = \Psi \cdot \boldsymbol{\beta} \text{ and } y^2\boldsymbol{\alpha} = \Psi \cdot \boldsymbol{\gamma} \tag{5}$$

with $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \Lambda^{(4q)}$. Write

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \boldsymbol{\alpha}_3 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \boldsymbol{\beta}_3 \\ \boldsymbol{\beta}_4 \end{bmatrix},$$

where the $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$ are in $\Lambda^{(q)}$. The first equation in (5) yields

$$\begin{bmatrix} t\boldsymbol{\alpha}_1 \\ t\boldsymbol{\alpha}_2 \\ t\boldsymbol{\alpha}_3 \end{bmatrix} = \begin{bmatrix} y\boldsymbol{\beta}_1 + t\boldsymbol{\beta}_2 \\ -y^2\boldsymbol{\beta}_2 + t\boldsymbol{\beta}_3 - y\boldsymbol{\beta}_4 \\ -y^2(uI + H) \cdot \boldsymbol{\beta}_3 + t\boldsymbol{\beta}_4 \end{bmatrix}.$$

We can write the $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ in the form

$$\begin{aligned} \boldsymbol{\alpha}_i &= \boldsymbol{u}_{i,0} + \boldsymbol{u}_{i,1}t + \boldsymbol{u}_{i,2}y + \boldsymbol{u}_{i,3}ty + \boldsymbol{u}_{i,4}y^2, \\ \boldsymbol{\beta}_i &= \boldsymbol{v}_{i,0} + \boldsymbol{v}_{i,1}t + \boldsymbol{v}_{i,2}y + \boldsymbol{v}_{i,3}ty + \boldsymbol{v}_{i,4}y^2, \end{aligned}$$

where the entries of $\boldsymbol{u}_{i,j}$ and $\boldsymbol{v}_{i,j}$ are either units or 0. (Cf. [14, Lemma 4.2].) Since the images of t and y in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k, the equation $t\boldsymbol{\alpha}_1 = y\boldsymbol{\beta}_1 + t\boldsymbol{\beta}_2$ yields $\boldsymbol{v}_{1,0} = \boldsymbol{0}$ and $\overline{\boldsymbol{u}}_{1,0} = \overline{\boldsymbol{v}}_{2,0}$, where bars denote reduction modulo $\mathfrak{m}$. From $t\boldsymbol{\alpha}_2 = -y^2\boldsymbol{\beta}_2 + t\boldsymbol{\beta}_3 - y\boldsymbol{\beta}_4$, it follows that $\overline{\boldsymbol{u}}_{2,0} = \overline{\boldsymbol{v}}_{3,0}$ and $\boldsymbol{v}_{4,0} = \boldsymbol{0}$ and, since the socle elements ty and y^2 are linearly independent over k, that $\overline{\boldsymbol{v}}_{2,0} = -\overline{\boldsymbol{v}}_{4,2}$. From $t\boldsymbol{\alpha}_3 = -y^2(uI + H) \cdot \boldsymbol{\beta}_3 + t\boldsymbol{\beta}_4$, it follows that $\overline{\boldsymbol{u}}_{3,0} = \overline{\boldsymbol{v}}_{4,0}$ and hence that $\boldsymbol{u}_{3,0} = \boldsymbol{0}$.

Using the equation $t\boldsymbol{\alpha}_3 = -y^2(uI + H) \cdot \boldsymbol{\beta}_3 + t\boldsymbol{\beta}_4$ again, we see that $\overline{\boldsymbol{u}}_{3,2} = \overline{\boldsymbol{v}}_{4,2}$. Further, since $uI + H$ is invertible, it follows that $\boldsymbol{v}_{3,0} = \boldsymbol{0}$ and hence that $\boldsymbol{u}_{2,0} = \boldsymbol{0}$.

To summarize, we have $\overline{\boldsymbol{u}}_{3,2} = \overline{\boldsymbol{v}}_{4,2} = -\overline{\boldsymbol{v}}_{2,0} = -\overline{\boldsymbol{u}}_{1,0}$, and $\boldsymbol{u}_{2,0} = \boldsymbol{u}_{3,0} = \boldsymbol{0}$. Putting $\boldsymbol{w} := \boldsymbol{u}_{1,0}$, we have $\boldsymbol{u}_{3,2} = -\boldsymbol{w} + t\boldsymbol{\mu} + y\boldsymbol{\nu}$ for suitable $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Lambda^{(q)}$. Then

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{w} & + & t\boldsymbol{u}_{1,1} & + & y\boldsymbol{u}_{1,2} & + & ty\boldsymbol{u}_{1,3} & + & y^2\boldsymbol{u}_{1,4} \\ \boldsymbol{0} & + & t\boldsymbol{u}_{2,1} & + & y\boldsymbol{u}_{2,2} & + & ty\boldsymbol{u}_{2,3} & + & y^2\boldsymbol{u}_{2,4} \\ -y\boldsymbol{w} & + & t\boldsymbol{u}_{3,1} & + & \boldsymbol{0} & + & ty(\boldsymbol{u}_{3,3} + \boldsymbol{\mu}) & + & y^2(\boldsymbol{u}_{3,4} + \boldsymbol{\nu}) \end{bmatrix}. \tag{6}$$

From (6) it follows that $\boldsymbol{\alpha} \in \operatorname{im}(\boldsymbol{\varphi})$, as desired. This completes the proof of our claim.

It is easy to see, using the invertibility of $uI + H$, that the image of the left-most $3q \times 5q$ submatrix of $\boldsymbol{\varphi}$ is contained in $t\Lambda^{(3q)} + \operatorname{im}(\Psi)$. Letting $\gamma_1, \ldots, \gamma_q$ be the last q columns of $\boldsymbol{\varphi}$, we see that $(0 :_M (t, y^2))/tM$ is generated by $\zeta_1 := \varepsilon(\gamma_1) + tM, \ldots, \zeta_q := \varepsilon(\gamma_q) + tM$. Since $t\gamma_i,\ y\gamma_i \in t\Lambda^{(3q)} + \operatorname{im}(\Psi)$ for each i, we see that $(0 :_M (t, y^2))/tM$ is a k-vector space of dimension at most q. To complete the proof of (i), we need only show that $\zeta_1, \ldots, \zeta_q$ are linearly independent. Given a relation $\sum_{i=1}^{q} \lambda_i \zeta_i = 0$, with $\lambda_i \in \Lambda$, we have

$\sum_{i=1}^{q} \lambda_i \gamma_i \in \operatorname{im}(\Psi) + t\Lambda^{(3q)} \subseteq \mathfrak{m}\Lambda^{(3q)}$. This relation obviously forces $\lambda_i \in \mathfrak{m}$ for all i, as desired.

It remains to prove assertion (ii) of the proposition. Given $\xi \in \mathfrak{m}$, write $\xi = at + by$. Suppose first that b is a unit of Λ. For each unit vector $\mathbf{e}_i \in \Lambda^{(q)}$, put

$$\boldsymbol{\sigma}_i := \begin{bmatrix} \mathbf{e}_i \\ (\frac{a^2}{b^2}t - \frac{a}{b}y)\mathbf{e}_i \\ 0 \end{bmatrix},$$

and check that

$$\xi\boldsymbol{\sigma}_i = b\begin{bmatrix} y\mathbf{e}_i \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} t\mathbf{e}_i \\ -y^2\mathbf{e}_i \\ 0 \end{bmatrix} \in \operatorname{im}(\Psi).$$

This shows that $\varepsilon(\boldsymbol{\sigma}_i) \in (0 :_M \xi)$ for each i, and the assertion follows easily in this case.

If b is not a unit, ξ has the form $\xi = ct + dy^2$. With $\mathbf{e}_i$ as above, put

$$\boldsymbol{\tau}_i := \begin{bmatrix} \mathbf{e}_i \\ 0 \\ -y\mathbf{e}_i \end{bmatrix}.$$

Then

$$\xi\boldsymbol{\tau}_i = dy\begin{bmatrix} y\mathbf{e}_i \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} t\mathbf{e}_i \\ -y^2\mathbf{e}_i \\ 0 \end{bmatrix} - cy\begin{bmatrix} 0 \\ -y\mathbf{e}_i \\ t\mathbf{e}_i \end{bmatrix} \in \operatorname{im}(\Psi).$$

As before, the assertion follows easily. $\square$

5. When all powers of $\mathfrak{m}$ are at most 2-generated

In this section we complete the proof of the Main Theorem in the remaining case—each power of $\mathfrak{m}$ is generated by at most two elements. Recall that by Theorem 4.3 R maps onto a Drozd ring. We refer the reader to [10, Lemma 6.2] for the proof of the next result (note that $\mathrm{e}(R) = \mathrm{e}(\mathfrak{m}, R)$ denotes the multiplicity of R):

LEMMA 5.1. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring. Assume that $\mathfrak{m}$ and $\mathfrak{m}^2$ are two-generated and R/L is a Drozd ring for some ideal L. Write $\mathfrak{m} = Rt + Ry$, with $t^2 \in L$. Then $L = \mathfrak{m}^3$, and $\mathfrak{m}^r = y^{r-1}\mathfrak{m} = Rty^{r-1} + Ry^r$ for each $r \geq 1$. If, further, R is not Cohen-Macaulay, then the following also hold:*

(i) *$\mathfrak{m}^r = Ry^r$ for all $r \gg 1$. In particular,* $\mathrm{e}(R) = 1$.
(ii) *R has exactly one minimal prime ideal P. Moreover, R_P is a field and R/P is a discrete valuation ring.*
(iii) *P is a principal ideal, and $P \not\subseteq \mathfrak{m}^2$.*

PROPOSITION 5.2. *Let $(R, \mathfrak{m}, k)$ be a commutative local Noetherian ring, let P be a non-maximal prime ideal of R, and let n be any non-negative integer. Suppose there is an indecomposable finite-length R-module M such that* $\dim_k(\operatorname{socle}_R(\operatorname{Ext}^1_R(R/P, M))) \geq n$. *Then there is a short exact sequence*

$$0 \longrightarrow M \longrightarrow X \longrightarrow (R/P)^{(n)} \longrightarrow 0, \tag{7}$$

in which X is indecomposable.

Proof. Put $E_1 := \operatorname{Ext}^1_R(R/P, M)$, $N := (R/P)^{(n)}$, $A := \operatorname{End}_R(M)$, $B := \operatorname{End}_R(N) = \operatorname{Mat}_{n\times n}(R/P)$ and $E := \operatorname{Ext}^1_R(N, M) = E_1^{(n)}$. If we write elements of E as $1 \times n$ row vectors with entries in E_1, then the right B-module structure is given by matrix multiplication. Since M has finite length, A is a local ring [7, Lemmas 2.20 and 2.21].

Let $e_1, \dots, e_n$ be linearly independent elements of $\operatorname{socle}_R(E_1)$, and put $\xi := [e_1, \dots, e_n] \in E$. We claim that $(0 :_B \xi) \subseteq \mathrm{J}(B)$. For, suppose $\varphi := [a_{ij}] \in B$ with $\xi\varphi = 0$. Then $e_1 a_{1j} + \cdots + e_n a_{nj} = 0$ for each $j = 1, \dots, n$. Linear independence of the e_i now implies that $a_{ij} \in \mathfrak{m}/P$ for each i, j. Then $\varphi \in \mathrm{J}(B)$, and the claim is proved.

To complete the proof, we let (7) represent the element $\xi \in E$ and apply Corollary 3.4. □

We will divide the proof of the Main Theorem into three cases.

5.1. Case 1: R is not Cohen-Macaulay. Suppose now that $(R, \mathfrak{m}, k)$ is one-dimensional and not Cohen-Macaulay, as in the Main Theorem, and assume also that each power of $\mathfrak{m}$ is generated by at most two elements. By Theorem 4.3 and Lemma 5.1, R has a unique minimal prime ideal P; moreover, P is principal, say, $P = Rt$. Given a non-negative integer n, we seek $|k| \cdot \aleph_0$ pairwise non-isomorphic indecomposable modules X such that $X_P \cong (R/P)^{(n)}$. The proof is a slight modification of the corresponding case in [10]; we give a sketch of the argument. (See [10, Proposition 6.3 and the succeeding paragraphs] for details.)

Suppose, first, that $(0 :_R t) \subseteq \mathfrak{m}^2$. Given an arbitrary integer $q \geq \max\{1, n\}$, we apply Proposition 4.1 to $R/\mathfrak{m}^2$, getting $|k| - 1$ pairwise non-isomorphic indecomposable finite-length modules M satisfying

$$\operatorname{socle}_R(M/tM) \cong k^{(q)} \text{ and } \mathfrak{m}^2 M = 0.$$

Applying $\operatorname{Hom}_R(\_, M)$ to the short exact sequence

$$0 \longrightarrow Rt \longrightarrow R \longrightarrow R/(t) \longrightarrow 0,$$

we obtain an exact sequence

$$\operatorname{Hom}_R(R, M) \longrightarrow \operatorname{Hom}_R(Rt, M) \longrightarrow E_1 \longrightarrow 0, \tag{8}$$

where $E_1 := \operatorname{Ext}^1_R(R/P, M)$. Since $Rt \cong R/(0 :_R t)$ and $(0 :_R t)M = (0)$, the map $f \mapsto f(t)$ provides an isomorphism $\operatorname{Hom}_R(Rt, M) \cong M$. Combining this isomorphism with the usual isomorphism $\mathrm{Hom}_R(R,M) \cong M$ ($g \mapsto g(1)$), we transform (8) to the exact sequence $M \xrightarrow{t} M \to E_1 \to 0$. Thus $E_1 \cong M/tM$. Now Proposition 5.2 provides, for each M, an indecomposable module X and a short exact sequence (7). Then $X_P \cong R_P^{(n)}$. Also, since $M \cong \mathrm{H}^0_{\mathfrak{m}}(X)$ (the finite-length part of X), we see that non-isomorphic M's yield non-isomorphic X's, and the proof is complete in this case.

Next, we consider the more difficult case, when $(0 :_R t) \not\subseteq \mathfrak{m}^2$. Since R maps onto a Drozd ring by Theorem 4.3, one can show easily that $t^2 \in \mathfrak{m}^3$. Also, $t \notin \mathfrak{m}^2$ by (iii) of Lemma 5.1, so we can choose y such that $\mathfrak{m} = Rt+Ry$. To summarize, we have

$$P = Rt,\ \mathfrak{m} = Rt + Ry,\ \text{and } t^2 \in \mathfrak{m}^3. \tag{9}$$

We now complete the proof under the additional assumption that

$$t^2 = ty^2 = 0. \tag{10}$$

In this case, one checks easily that $(0 :_R t) = (t, y^2)$. Applying Proposition 4.4 to the Drozd ring $\Lambda := R/\mathfrak{m}^3$, we get $|k| \cdot \aleph_0$ indecomposable R-modules M such that $\mathfrak{m}^3 M = 0$ and the k-vector space $\frac{(0:_M(t,y^2))}{tM}$ has dimension n. Again, we obtain the exact sequence (8), and since $Rt \cong R/(t,y^2)$, we see that $\mathrm{Hom}_R(Rt,M) \cong (0 :_M (t,y^2))$, and hence $E_1 \cong \frac{(0:_M(t,y^2))}{tM}$. Thus $E_1 = \mathrm{socle}_R(E_1)$ has dimension n. As before, we can use Proposition 5.2 to produce $|k| \cdot \aleph_0$ pairwise non-isomorphic indecomposable modules X such that $X_P \cong R_P^{(n)}$.

Finally, we complete the proof when (10) is not necessarily satisfied. Since $t^2 \in \mathfrak{m}^3$ by (9), $S := R/(t^2, ty^2)$ maps onto the Drozd ring $R/\mathfrak{m}^3$. Therefore, by Theorem 4.3, S is not a homomorphic image of a Dedekind-like ring. Moreover, S is not Cohen-Macaulay, since $ty \notin Rt^2 + Rty^2$ (else $\mathfrak{m}^2$ would be principal) but $\mathfrak{m}ty \subseteq Rt^2+Rty^2$. By the argument in the previous paragraph, we obtain $|k|\cdot\aleph_0$ pairwise non-isomorphic S-modules X such that $X_Q \cong S_Q^{(n)}$, where $Q = P/(t^2, ty^2)$. Now view these modules as R-modules and note that the natural map $R_P \to S_Q$ is an isomorphism. This completes the proof of Theorem 1.2 when R is not Cohen-Macaulay.

For the rest of Section 5, we assume that $(R, \mathfrak{m}, k)$ is a one-dimensional Noetherian local Cohen-Macaulay ring such that each power of $\mathfrak{m}$ is generated by two elements, and we assume that R is not a homomorphic image of a Dedekind-like ring, equivalently (Theorem 4.3), R has a Drozd ring as a homomorphic image. By Lemma 5.1, $\Lambda := R/\mathfrak{m}^3$ is a Drozd ring. Moreover, we have the "associativity formula" (cf. [20, Theorem 14.7] or [2, Corollary 4.6.8]):

$$2 = \mathrm{e}(R) = \sum_i \mathrm{e}(R/P_i)\ell(R_{P_i}), \tag{11}$$

where the sum ranges over all minimal prime ideals P_i of R, and $\ell(R_{P_i})$ is the length of R_{P_i} as an R_{P_i}-module. Thus, R has either one or two minimal prime ideals.

5.2. Case 2: R is Cohen-Macaulay with two minimal prime ideals. Let P_1 and P_2 denote the minimal primes of R. We are given two non-negative integers n_1 and n_2, and we want to find $|k| \cdot \aleph_0$ indecomposable modules X such that $X_{P_i} \cong R_{P_i}^{(n_i)}$ for $i = 1, 2$. By (11), each R/P_i is a discrete valuation domain and R_{P_i} is a field. Since $\mathfrak{m}$ needs two generators, it follows that each $P_i \not\subseteq \mathfrak{m}^2$, so we can choose $t_i \in P_i \not\subseteq \mathfrak{m}^2$. Then $R/(t_i)$ is one-dimensional with principal maximal ideal, i.e. a discrete valuation ring; hence $P_i = Rt_i$. Suppose r is in the kernel of the diagonal map $R \to R_{P_1} \times R_{P_2}$. Then $(0 :_R r) \not\subseteq P_1 \cup P_2$, so $(0 :_R r)$ contains a non-zerodivisor. It follows that R is reduced, with total quotient ring $R_{P_1} \times R_{P_2}$ and normalization $R/P_1 \times R/P_2$. Moreover, $(0 :_R t_1) = Rt_2$ and $(0 :_R t_2) = Rt_1$.

Given any integer $n \geq \max\{n_1, n_2\}$, let $M := M_{n,\kappa}$ be one of the indecomposable Λ-modules from Proposition 4.4. Applying $\operatorname{Hom}_R(\_\,, M)$ to the short exact sequence

$$0 \longrightarrow Rt_1 \longrightarrow R \longrightarrow R/(t_1) \longrightarrow 0,$$

we obtain an exact sequence

$$\operatorname{Hom}_R(R, M) \longrightarrow \operatorname{Hom}_R(Rt_1, M) \longrightarrow E_1 \longrightarrow 0,$$

where $E_1 = \operatorname{Ext}^1_R(R/P_1, M)$. Now $\operatorname{Hom}_R(Rt_1, M) \cong (0 :_M (0 :_R t_1)) = (0 :_M t_2)$. Therefore $E_1 \cong (0 :_M t_2)/t_1 M$, and, by symmetry, $E_2 := \operatorname{Ext}^1_R(R/P_2, M) \cong (0 :_M t_1)/t_2 M$. By (ii) of Proposition 4.4, E_1 and E_2 each need at least n generators.

The rest of the proof is very similar to that of Case 1. Let $N = (R/P_1)^{(n_1)} \oplus (R/P_2)^{(n_2)}$. By the annihilator relations above, $\operatorname{Hom}_R(R/P_i, R/P_j) = 0$ if $i \neq j$. Therefore $B := \operatorname{End}_R(N) = \operatorname{Mat}_{n_1 \times n_1}(R/P_1) \times \operatorname{Mat}_{n_2 \times n_2}(R/P_2)$. Put $E := \operatorname{Ext}^1_R(N, M) = E_1^{(n_1)} \times E_2^{(n_2)}$. We regard elements of E as ordered pairs (ξ_1, ξ_2), where ξ_i is a $1 \times n_i$ row vector with entries in E_i. The right action of B on E is matrix multiplication on each of the two coordinates.

Let $e_1, \ldots, e_n \in E_1$ map to linearly independent elements of $E_1/\mathfrak{m}E_1$, and let $f_1, \ldots, f_n \in E_2$ map to linearly independent elements of $E_2/\mathfrak{m}E_2$. Consider the elements $\xi_1 := [e_1 \ \ldots \ e_{n_1}] \in E_1^{(n_1)}$ and $\xi_2 := [f_1 \ \ldots \ f_{n_2}] \in E_2^{(n_2)}$, and put $\xi := (\xi_1, \xi_2) \in E$. One checks easily that $(0 :_B \xi) \in \mathfrak{m}B \subseteq \mathrm{J}(B)$ (cf., e.g., [10, Lemma 4.4]). Corollary 3.4 now provides a short exact sequence $0 \to M \to X \to N \to 0$ with X indecomposable. This completes the proof of Theorem 1.2 when R is Cohen-Macaulay and has two minimal prime ideals.

There is one remaining case, for which we will use a very different approach.

5.3. Case 3: R is Cohen-Macaulay with one minimal prime ideal P. Given a non-negative integer n, we seek $|k| \cdot \aleph_0$ indecomposable modules X with $X_P \cong R_P^{(n)}$.

Obviously no power of $\mathfrak{m}$ can be principal, so the multiplicity of R is two. Cohen's Structure Theorem implies that R is an *abstract hypersurface*, that is, the completion $\widehat{R}$ has the form $S/(f)$, where $(S, \mathfrak{n}, k)$ is a two-dimensional regular local ring and $f \in \mathfrak{n} - \{0\}$.

Again, we consider the indecomposable Λ-modules $M := M_{n,\kappa}$ provided by Proposition 4.4. This time we will take N, the torsion-free part of the desired module X, to be a suitable direct summand of the first syzygy of M.

The next three results apply more generally to any one-dimensional abstract hypersurface.

Theorem 5.3. *Suppose $(D, \mathfrak{n}, k)$ is an abstract hypersurface of dimension 1. Let M be an indecomposable finite-length D-module whose first syzygy is isomorphic to $D^{(r)} \oplus F$, where F has no non-zero free direct summand. Let F' be an arbitrary direct summand of F. Then there is a short exact sequence*

$$0 \longrightarrow M \longrightarrow X \longrightarrow F' \longrightarrow 0, \tag{12}$$

in which X is indecomposable.

Proof. We may assume $F' \neq 0$. Put $A := \mathrm{End}_D(M)$ and $B := \mathrm{End}_D(F')$. We have a short exact sequence

$$0 \longrightarrow D^{(r)} \oplus F \longrightarrow D^{(r+s)} \longrightarrow M \longrightarrow 0, \tag{13}$$

where $s = \mathrm{rank}(F)$. Since F' is maximal Cohen-Macaulay over the Gorenstein ring D, we have $\mathrm{Ext}_D^i(F', D) = 0$ for $i > 0$ (cf. [2, Theorems 3.3.7 and 3.3.10]). Therefore, on applying the functor $\mathrm{Hom}_D(F', \_\,)$ to (13), we obtain an isomorphism

$$\mathrm{Ext}_D^1(F', M) \;\cong\; \mathrm{Ext}_D^2(F', F). \tag{14}$$

By Eisenbud's theory of matrix factorizations [5] (cf. also [26, Chapter 7]), F' has a periodic resolution with period at most 2 and with constant Betti numbers. Thus we have short exact sequences

$$0 \longrightarrow G \longrightarrow D^{(t)} \xrightarrow{\psi} F' \longrightarrow 0 \tag{15}$$

and

$$0 \longrightarrow F' \longrightarrow D^{(t)} \longrightarrow G \longrightarrow 0. \tag{16}$$

Applying $\mathrm{Hom}_D(F', \_\,)$ to (16), we get an isomorphism

$$\mathrm{Ext}_D^1(F', G) \;\cong\; \mathrm{Ext}_D^2(F', F'). \tag{17}$$

Moreover, naturality of the connecting homomorphisms in the long exact sequences of Ext implies that the isomorphisms in (14) and (17) are actually isomorphisms of right B-modules.

Next, applying $\mathrm{Hom}_D(F', \_\ )$ to (15), we get an exact sequence of right B-modules

$$\mathrm{Hom}_D(F', D^{(t)}) \xrightarrow{\psi_*} B \xrightarrow{\eta} \mathrm{Ext}^1_D(F', G) \longrightarrow 0.$$

Since F' is a direct summand of F, there is an injection of right B-modules $\mathrm{Ext}^2_D(F', F') \hookrightarrow \mathrm{Ext}^2_D(F', F)$. Composing this injection with the isomorphisms in (14) and (17), we get an injection of right B-modules $j : \mathrm{Ext}^1_D(F', G) \hookrightarrow \mathrm{Ext}^1_D(F', M)$. Putting $\beta = j\eta$, we obtain an exact sequence of right B-modules

$$\mathrm{Hom}_D(F', D^{(t)}) \xrightarrow{\psi_*} B \xrightarrow{\beta} \mathrm{Ext}^1_D(F', M)$$

We claim that $\ker(\beta)$ is contained in the Jacobson radical $\mathrm{J}(B)$ of B. To prove this, let $g \in \mathrm{Ker}(\beta) = \mathrm{im}(\psi_*)$. Then g lifts to a map $h : F' \to D^{(t)}$, with $\psi h = g$. Since F' has no non-zero free summand, $h(F') \subseteq \mathfrak{n}D^{(t)}$. This shows that $g(F') \subseteq \mathfrak{n}F'$, and the claim follows easily (cf., e.g., [10, Lemma 4.4]). The existence of the short exact sequence (12) now follows from Corollary 3.4. □

In the following, we say that a D-module M has rank s provided $M_P \cong R_P^{(s)}$ for every associated prime P of D.

Proposition 5.4. *Let $(D, \mathfrak{n}, k)$ be an abstract hypersurface of dimension 1, and assume that D has a Drozd ring Λ as a homomorphic image. Let $M := M_{n,\kappa}$ be the indecomposable Λ-module built in Proposition 4.4, and let $L := \mathrm{syz}^1_D(M)$ be the first syzygy of the D-module M. Write $L = D^{(r)} \oplus F$, where F has no non-zero free direct summand. Then* $\mathrm{rank}(F) \geq \frac{n}{e-1}$, *where $e = \mathrm{e}_D(D)$ is the multiplicity of D.*

Proof. Obviously F has a rank. Put $s := \mathrm{rank}(F)$ and $m := \mu_D(F)$ (μ = minimal number of generators required). It follows, e.g., from [13, (1.6)], that $m \leq es$. (The statement of [13, (1.6)] assumes that k is infinite. This is not a problem, since none of m, e, s is changed by the flat local base change $D \to D(X) := D[X]_{\mathfrak{n}[X]}$.) Now $\mu_D(L) = r + m = 3n - s + m$, whence $\mu_D(L) - 3n \leq (e-1)s$. Therefore it will suffice to show that $\mu_D(L) \geq 4n$. Since $\mu_D(\mathfrak{n}) = 2$, the following lemma completes the proof: □

Lemma 5.5. *Keep the notation above. There is a surjective D-homomorphism from $L = \mathrm{syz}^1_D(M)$ onto $\mathfrak{n}^{(2n)}$.*

Proof. Let χ denote the composition $D^{(3n)} \twoheadrightarrow \Lambda^{(3n)} \overset{\varepsilon}{\twoheadrightarrow} M \to 0$, so that $\ker\chi = L$, and let $\pi : D^{(3n)} \to D^{(2n)}$ be the projection onto the first two coordinates. We will show that $\pi(L) = \mathfrak{n}^{(n)} \oplus \mathfrak{n}^{(n)}$. The inclusion $\pi(L) \subseteq \mathfrak{n}^{(n)} \oplus \mathfrak{n}^{(n)}$ is obvious. For the reverse inclusion, fix $i, 1 \leq i \leq n$, and let $\mathbf{e}_i \in D^{(n)}$ be the i^{th} unit vector. Let $\tilde{t}, \tilde{y} \in \mathfrak{n}$ lift the elements $t, y \in \Lambda$ (notation as in Proposition 4.4). It will suffice to show that the four elements

$(\tilde{t}\mathbf{e}_i, 0), (\tilde{y}\mathbf{e}_i, 0), (0, \tilde{t}\mathbf{e}_i)$ and $(0, \tilde{y}\mathbf{e}_i)$ are all in $\pi(L)$. But this follows easily from the definition of the matrix $\Psi_{n,\kappa}$. $\square$

We now return to our special ring $(R, \mathfrak{m}, k)$ and the modules $M = M_{\kappa,n}$. As in Theorem 5.3, we write the first syzygy of M in the form $R^{(r)} \oplus F$, where F has no non-zero free summand. To complete the proof of the Main Theorem, it will suffice, by Theorem 5.3, to show that F has a direct summand F' of rank n. By Proposition 5.4 we know that F has rank at least n. By [22] F is isomorphic to a direct sum of ideals of R. (Cf. also [18, Theorem 2.1] for a more general statement and [1] for the analytically unramified case.) Each of these ideals must have rank 0 or 1. Therefore the desired module F' can be obtained from F by throwing out a few rank-one summands, if necessary.

6. The monoid of vector bundles

Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. By a *vector bundle* we mean a finitely generated module M such that M_P is a free R_P-module for each prime ideal $P \neq \mathfrak{m}$. We denote by R-mod the category of finitely generated R-modules and by $\mathcal{F}(R)$ the full subcategory of vector bundles. Our goal is to obtain, in Theorem 6.3, a complete set of invariants for the monoid $\mathsf{V}(\mathcal{F}(R))$ of isomorphism classes of modules in $\mathcal{F}(R)$ when $\dim(R) = 1$, where the monoid operation is given by the direct sum. (Of course $\mathcal{F}(R) = R$-mod if each R_P is a field, e.g., if R is reduced and one-dimensional, or if R is the non-Cohen-Macaulay ring given in Lemma 5.1.) The description of these monoids was worked out in [6] for the case of a one-dimensional Cohen-Macaulay ring. We will see here that the same results hold in the one-dimensional non-Cohen-Macaulay case, thanks to our Main Theorem. We refer the reader to [6, Section 1] for the relevant terminology and basic results concerning Krull monoids, divisor homomorphisms, and the class group $\mathrm{Cl}(H)$ of a Krull monoid H.

Suppose now that $(R, \mathfrak{m}, k)$ is a one-dimensional commutative Noetherian local ring. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. The Krull-Schmidt theorem implies that $\mathsf{V}(\widehat{R}\text{-mod})$ and $\mathsf{V}(\mathcal{F}(\widehat{R}))$ are free monoids, with bases consisting of the isomorphism classes of the indecomposables. In other words, $\mathsf{V}(\mathcal{F}(\widehat{R})) \cong \mathbb{N}^{(\tau)}$, the direct sum of τ copies of the additive monoid $\mathbb{N}$ of non-negative integers, where τ is the number of isomorphism classes of indecomposable vector bundles over $\widehat{R}$. It is easy to see that if M is a finitely generated R-module, then M is a vector bundle if and only if $\widehat{R} \otimes_R M$ is a vector bundle. (This follows from the faithful flatness of $R_P \to \widehat{R}_{Q_1} \times \cdots \times \widehat{R}_{Q_t}$, where P is a minimal prime of R and the Q_j are the primes of $\widehat{R}$ lying over P.) Thus the divisor homomorphism [6, Section 1.1] $\mathsf{V}(R\text{-mod}) \to \mathsf{V}(\widehat{R}\text{-mod})$ taking $[M]$ to $[\widehat{R} \otimes_R M]$ restricts to a divisor homomorphism $\mathsf{V}(\mathcal{F}(R)) \to \mathsf{V}(\mathcal{F}(\widehat{R}))$. In particular, we can regard $\mathsf{V}(\mathcal{F}(R))$ as a submonoid of $\mathsf{V}(\mathcal{F}(\widehat{R}))$. The key is to understand exactly how $\mathsf{V}(\mathcal{F}(R))$ sits inside $\mathsf{V}(\mathcal{F}(\widehat{R}))$, that is, which modules over the $\mathfrak{m}$-adic completion $\widehat{R}$ are extended from R-modules.

Proposition 6.1. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional commutative Noetherian local ring with $\mathfrak{m}$-adic completion $\widehat{R}$, and let N be a vector bundle over $\widehat{R}$. Then $N \cong \widehat{R} \otimes_R M$ for some R-module M (necessarily a vector bundle) if and only if $\operatorname{rank}_{R_P}(N_P) = \operatorname{rank}_{R_Q}(N_Q)$ whenever P and Q are minimal prime ideals of $\widehat{R}$ with $P \cap R = Q \cap R$.*

Proof. The "only if" direction is clear. For the converse, let $P_1, \dots, P_s$ be the minimal prime ideals of R, and, for each i, let n_i be the rank of N at the primes lying over P_i. Let $K = R_{P_1} \times \cdots \times R_{P_s}$, and let V be the projective K-module having rank n_i at P_i. The $K \otimes_R \widehat{R}$-module $K \otimes_R N$ is extended from the K-module V, and now [19, Theorem 3.4] implies that N is extended from an R-module. $\square$

The next result puts an upper bound on the number of non-isomorphic vector bundles. The case of a Cohen-Macaulay ring is [6, Lemma 2.3].

Proposition 6.2. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional commutative Noetherian local ring. Then $|\mathsf{V}(\mathcal{F}(R))| \le |k| \cdot \aleph_0$.*

Proof. We observe, as in the first paragraph of the proof of [6, Lemma 2.3], that each finite-length module has cardinality at most $\tau := |k| \cdot \aleph_0$, and that there are at most τ isomorphism classes of finite-length R-modules.

Let $P_1, \dots, P_s$ be the minimal prime ideals of R. Fix a vector bundle M, and let n_i be the rank of M at P_i. Since there are only countably many sequences $(n_1, \dots, n_s)$, it will suffice to show that there are at most τ non-isomorphic vector bundles with the same ranks as M at the minimal primes.

Let $K = R_{P_1} \times \cdots \times R_{P_s}$, the localization of R at the complement of the union of the minimal prime ideals. Given a vector bundle N with $\operatorname{rank}_{P_i}(N_i) = n_i$ for each i, one can choose a homomorphism $\varphi : M \to N$ such that $1_K \otimes_R \varphi$ is an isomorphism. Then $U := \ker(\varphi)$ and $V := \operatorname{coker}(\varphi)$ have finite length. Put $W := \operatorname{im}(\varphi) \cong \operatorname{coker}(U \hookrightarrow M)$. By the first paragraph, there are at most τ choices for U and V. Also, for each U, $\operatorname{Hom}_R(U, M)$ has finite length and therefore has cardinality at most τ. Therefore there are at most τ possibilities for W. Finally, the exact sequence $0 \to W \to N \to V \to 0$ and the fact that $\operatorname{Ext}^1_R(V, W)$ has finite length, and hence cardinality bounded by τ, show that there are at most τ possibilities for N. $\square$

Fix a positive integer q and an infinite cardinal τ. Let B be any $q \times \tau$ integer matrix such that each element of $\mathbb{Z}^{(q)}$ occurs τ times as a column of B. We let $\mathfrak{H}(q, \tau) := \mathbb{N}^{(\tau)} \cap \ker(B : \mathbb{Z}^{(\tau)} \to \mathbb{Z}^{(q)})$, where $\mathbb{N}$ denotes the set of non-negative integers. Finally, we put $\mathfrak{H}(0, \tau) = \mathbb{N}^{(\tau)}$. These are the monoids we will obtain as $\mathsf{V}(R\text{-mod})$ for the rings that are not Dedekind-like. Not surprisingly, the isomorphism class of the monoid $H(q,\tau)$ does not depend on how the columns of B are arranged, as long as each column is repeated τ times. (Cf. [6, Lemmas 1.1 and 2.1].)

For some Dedekind-like rings, we will obtain a different monoid. Let E be the $1\times\aleph_0$ matrix $[1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad \cdots]$, and put $\mathfrak{H}_1 := \mathbb{N}^{(\aleph_0)} \cap \ker(E : \mathbb{Z}^{(\aleph_0)} \to \mathbb{Z})$.

For a one-dimensional local ring $(R,\mathfrak{m},k)$, we define the *splitting number* $\operatorname{spl}(R)$ to be the difference $|\operatorname{Spec}(\widehat{R})| - |\operatorname{Spec}(R)|$. Thus, for example, $\operatorname{spl}(R) = 0$ means that the natural map $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ is bijective.

We can now state the main theorem of this section. For the proof, we refer the reader to the proof of [6, Theorem 2.2]. The only modification needed to eliminate the Cohen-Macaulay hypothesis is to replace Lemmas 2.3, 2.4 and 2.5 in [6] by, respectively, Proposition 6.2, Theorem 1.2 and Proposition 6.1 of this paper.

THEOREM 6.3. *Suppose* $(R,\mathfrak{m},k)$ *is a one-dimensional commutative Noetherian local ring. Let* $q := \operatorname{spl}(R)$ *be the splitting number of* R*, and let* $\tau = \tau(R) = |k| \cdot \aleph_0$.

(1) *If R is not Dedekind-like, then* $\mathsf{V}(\mathfrak{F}(R)) \cong \mathfrak{H}(q,\tau)$.
(2) *If R is a discrete valuation ring, then* $\mathsf{V}(\mathfrak{F}(R)) = \mathsf{V}(R\text{-mod}) \cong \mathbb{N}^{(\aleph_0)}$.
(3) *If R is Dedekind-like but not a discrete valuation ring, and if $q = 0$, then* $\mathsf{V}(\mathfrak{F}(R)) = \mathsf{V}(R\text{-mod}) \cong \mathbb{N}^{(\tau)}$.
(4) *If R is Dedekind-like and $q > 0$, then $q = 1$ and* $\mathsf{V}(\mathfrak{F}(R)) = \mathsf{V}(R\text{-mod}) \cong \mathbb{N}^{(\tau)} \oplus \mathfrak{H}_1$.

In every case, $\operatorname{Cl}(\mathsf{V}(\mathfrak{F}(R))) \cong \mathbb{Z}^{(q)}$.

We remark that the yet-unpublished results on the structure of modules over exceptional Dedekind-like rings have no bearing on the validity of this theorem: If R is an exceptional Dedekind-like ring, then $\operatorname{spl}(R) = 0$, and hence all that is needed is the straightforward construction of $\tau(R)$ indecomposable modules over R, given in [6, Lemma 2.6].

7. Non-local rings

In this section only, we do not assume that Dedekind-like rings are necessarily local, calling the commutative, Noetherian ring R a *(global) Dedekind-like ring* if, for each maximal ideal $\mathfrak{m}$ of R, the localization $R_{\mathfrak{m}}$ is a (local) Dedekind-like ring [16, Corollary 10.7]. If R is a (global) Dedekind-like ring such that none of the localizations of R is exceptional, and if M is a finitely generated indecomposable R-module, then the rank of M_P is at most two for every minimal prime P of R [16, Corollary 16.9]. In this section, we prove that this result fails if at least one of the localizations of R is not a homomorphic image of a Dedekind-like ring.

THEOREM 7.1. *Let R be a connected, commutative, Noetherian ring, and suppose that R is not a homomorphic image of a (global) Dedekind-like ring. Then, for every integer $n \geq 1$, there exist infinitely many indecomposable finitely generated R-modules M such that $M_P \cong R_P^{(n)}$ for each minimal prime P of R.*

Proof. We begin by fixing a maximal ideal $\mathfrak{m}$ of R such that $R_\mathfrak{m}$ is not a homomorphic image of a (local) Dedekind-like ring. If R has dimension greater than one, then we can take $\mathfrak{m}$ to be any maximal ideal of height greater than one, since (local) Dedekind-like rings have dimension one. If R has dimension one, then the existence of such a maximal ideal $\mathfrak{m}$ follows immediately from [16, Proposition 14.1 and Corollary 13.6].

Note that a Noetherian ring A is connected if and only if, for every non-empty, proper subset $\mathcal{V}$ of the set of minimal prime ideals of A, there exist a maximal ideal $\mathfrak{m}_\mathcal{V}$ of A and minimal primes $P \in \mathcal{V}$ and $Q \notin \mathcal{V}$ such that $P + Q \subseteq \mathfrak{m}_\mathcal{V}$. Thus we can find a finite list $\mathfrak{m}_1 = \mathfrak{m}, \mathfrak{m}_2, \ldots, \mathfrak{m}_t$ of maximal ideals of R such that each minimal prime of R is contained in at least one maximal ideal in the list, and such that, for every non-empty, proper subset $\mathcal{V}$ of the set of minimal prime ideals of R, there are minimal primes $P \in \mathcal{V}$ and $Q \notin \mathcal{V}$ such that $P, Q \subseteq \mathfrak{m}_i$ for some index i. Therefore, if we set $S := R - \bigcup_{i=1}^t \mathfrak{m}_i$, it follows that the localization $S^{-1}R$ is connected, with minimal primes precisely the localization of the minimal primes of R.

Suppose that we can find a finitely generated indecomposable $S^{-1}R$-module M such that $M_{S^{-1}P} \cong (S^{-1}R)^{(n)}_{S^{-1}P}$ for each minimal prime P of R. Let N be a finitely generated R-module such that $S^{-1}N \cong M$, and let $N = N_1 \oplus \cdots \oplus N_k$ be a decomposition of N into indecomposable R-modules. Since $S^{-1}N$ is indecomposable, we have $S^{-1}N_i = 0$ for all except one index i, and hence $S^{-1}N_i = M$. Then N_i is an indecomposable R-module such that $(N_i)_P \cong M_{S^{-1}P} \cong (S^{-1}R)^{(n)}_{S^{-1}P} \cong R_P^{(n)}$ for each minimal prime P of R, and the theorem is proved.

Therefore it suffices to prove the theorem under the additional hypothesis that R be semilocal. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the maximal ideals of R, where $R_{\mathfrak{m}_1}$ is not a homomorphic image of a Dedekind-like ring. Further, suppose $\mathfrak{m}_1$ has height greater than one if $\dim R > 1$. We distinguish the two cases in which R has dimension one or dimension greater than one.

Suppose first that R has dimension one. Let M_1 be an indecomposable $R_{\mathfrak{m}_1}$-module with constant rank n at the minimal primes of R contained in $\mathfrak{m}_1$ (Theorem 1.2); for $2 \leq j \leq t$, let $M_j = R_{\mathfrak{m}_j}^{(n)}$. Since R has only finitely many prime ideals, there exists, by [25, Lemma 1.11], an R-module M such that $M_{\mathfrak{m}_j} \cong M_j$ for all $j = 1, \ldots, t$. If $M = U \oplus V$, then, since $M_{\mathfrak{m}_1}$ is indecomposable, we can assume that $U_{\mathfrak{m}_1} = 0$. Since $M_{\mathfrak{m}_j}$ is $R_{\mathfrak{m}_j}$-free for $2 \leq j \leq t$, $U_{\mathfrak{m}_j}$ is $R_{\mathfrak{m}_j}$-free for all $j = 1, \ldots, t$, and it follows that U is R-projective.

Since R is connected and $U_{\mathfrak{m}_1} = 0$, it follows that $U = 0$. This shows that M is indecomposable. Since Theorem 1.2 produces infinitely many pairwise non-isomorphic indecomposable $R_{\mathfrak{m}_1}$-modules locally of constant rank n at the minimal primes of $R_{\mathfrak{m}_1}$, the theorem is proved in case R has dimension one.

Suppose instead that R has dimension greater than one, so that $\mathfrak{m}_1$ is a maximal ideal of height greater than one. Thus, either the maximal ideal of $R_{\mathfrak{m}_1}$ requires three or more generators, or $R_{\mathfrak{m}_1}$ is a regular local ring of dimension two, and the square of its maximal ideal requires three generators. Either way, let r be a positive integer such that $\mathfrak{m}_1^r/\mathfrak{m}_1^{r+1}$ is a vector space of dimension at least three over the residue field $R/\mathfrak{m}_1$. We adapt Proposition 2.1 to construct R-modules directly.

Let $\mathcal{P}$ be the set consisting of the minimal primes of R together with the remaining maximal ideals $\mathfrak{m}_2, \dots, \mathfrak{m}_t$, and choose x, y, and z as in the first sentence of the proof of Proposition 2.1, where $\mathfrak{m} = \mathfrak{m}_1$. As in that proof, given any integer $q > n$, set $\Delta := (z + y)I_q + yH_q$, and let

$$\Xi := \begin{bmatrix} 0_n & 0 \\ 0 & x^2 I_{q-n} \end{bmatrix} \in \mathrm{Mat}_{q\times q}(R).$$

Let A be the $2q \times 2q$ matrix over R defined by (1), and set $M := \mathrm{coker}(A)$. Since the images of x, y and z, in $\mathfrak{m}_1^r/\mathfrak{m}_1^{r+1}$, are linearly independent over $R/\mathfrak{m}_1$, while the image of x^2 in $\mathfrak{m}_1^r/\mathfrak{m}_1^{r+1}$ is 0, the proof of Proposition 2.1 shows that $M_{\mathfrak{m}_1}$ is indecomposable. Moreover, for $P \in \mathcal{P}$, localizing at P yields a matrix $\tilde{A}$ which is equivalent to $I_{2q-n} \oplus 0_n$ (because x, y, and z become units in R_P), and hence $M_P \cong R_P^{(n)}$.

To show that M is indecomposable, suppose $M = U \oplus V$. Since $M_{\mathfrak{m}_1}$ is indecomposable, we can assume that $U_{\mathfrak{m}_1} = 0$. For $2 \le j \le t$, $U_{\mathfrak{m}_j}$ is a direct summand of the free $R_{\mathfrak{m}_j}$-module $M_{\mathfrak{m}_j}$ and thus is free. Therefore U is R-projective; since $U_{\mathfrak{m}_1} = 0$ and R is connected, U must be zero. Thus M is indecomposable. As noted in the proof of Proposition 2.1, the localization $M_{\mathfrak{m}_1}$ of the R-module M just constructed requires exactly $2q$ generators as an $R_{\mathfrak{m}_1}$-module. Thus, by varying $q > n$, we get infinitely many pairwise non-isomorphic indecomposable R-modules locally of constant rank n at the minimal primes of R. □

We leave to the reader the minor adjustments required to obtain $|k| \cdot \aleph_0$ pairwise non-isomorphic indecomposable R modules of constant rank n, where k is the residue field at the maximal ideal $\mathfrak{m}_1$. One might be able to extend Theorem 7.1 to allow for *some* non-constant ranks at the minimal primes, but it is doubtful that one can obtain *arbitrary* ranks at the minimal primes. For example, if R has dimension one, two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, and three minimal primes P_0, P_1, and P_2, such that $P_0, P_1 \subseteq \mathfrak{m}_1$ and $P_1, P_2 \subseteq \mathfrak{m}_2$, but $P_0 \not\subseteq \mathfrak{m}_2$ and $P_2 \not\subseteq \mathfrak{m}_1$, then it is not clear that there exists an indecomposable module M of rank one at P_0 and P_2 but rank zero at P_1. Moreover, in dimension greater than one, R. Wiegand's "gluing lemma" [25, Lemma 1.11] does not apply, and it is difficult to imagine how to construct a module with arbitrary localizations at finitely many maximal ideals.

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