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NONLINEAR SCHRÖDINGER EQUATION WITH A RANDOM POTENTIAL

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Dedicated to J. Doob

Abstract. In this paper we describe some recent developments on the problems of existence of quasi-periodic and almost-periodic solutions and diffusion for nonlinear Schrödinger equations with a random potential on the lattice and in the continuum.

1. Introduction

We discuss progress on disordered systems with many degrees of freedom, going back to the work of Fröhlich, Spencer and Wayne [FSW].

Take the one-dimensional case $d = 1$, and consider the nonlinear lattice Schrödinger equation

$$i\dot{q}_j + V_j q_j + \varepsilon(q_{j-1} + q_{j+1}) + \delta q_j |q_j|^2 = 0, \tag{1.1}$$

where $\{V_j \mid j \in \mathbb{Z}\}$ is a random potential.

More generally, one can consider models with finite range interactions (systems of coupled harmonic oscillators)

$$i\dot{q}_j = \frac{\partial H}{\partial \bar{q}_j},$$

with

$$H(q, \bar{q}) = \sum V_j |q_j|^2 + \varepsilon \sum_j (\bar{q}_j q_{j+1} + q_j \bar{q}_{j+1}) + Re \sum_j \lambda_j \prod_{k \in S_j} q_k^{n_k} \bar{q}_k^{n'_k}, \tag{1.2}$$

where

$$S_j \subset [j - C, j + C],$$
$$\sum n_k = \sum n'_k, \quad 4 \leq \sum (n_k + n'_k) < C.$$

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The case $\delta = 0$ in (1.1). This corresponds to 1D random Schrödinger operators.

Dynamical localization. This means that $\sup_t D(t) < \infty$, where

$$D(t) = \left(\sum j^2 |q_j(t)|^2 \right)^{1/2} \tag{1.3}$$

is the diffusion.

Problem. What can be said about diffusion in the nonlinear models? In particular, what about nonlinear dynamical localization?

Theorem 1.1 ([FSW]). *Consider* (1.2) *with*

$$\varepsilon = 0, \quad \lambda_j = \delta \quad \textit{small.}$$

For a typical realization of the $\{V_j\}$*, there is an 'abundance' of invariant tori of full dimension (i.e., almost periodic solutions on the full set of frequencies) with action variables satisfying*

$$I_j < e^{-|j|^{1+\delta}}. \tag{1.4}$$

The following problems arise:

(a) Can one replace (1.4) by an exponential decay

$$I_j < e^{-c|j|}?$$

This was solved affirmatively by J. Pöschel [Po]. We may go up to decay

$$I_j < \exp\left\{-(\log|j|)^{1+\delta}\right\},$$

but the case of polynomial decay rate remains open.

(b) Suppose $\varepsilon \neq 0$ in (1.1) and (1.2). Construct
 - time periodic solutions (1 dimensional tori),
 - quasi-periodic solutions (finite dimensional tori),
 - almost-periodic solutions (on a full set of frequencies).

(c) What may be said about the growth of $D(t)$ in general?

(d) Is there an analog of the [FSW] theorem when the interactions are not finite? The natural model here is the nonlinear Schrödinger equation on $\mathbb{R}/\mathbb{Z} = \mathbb{T}$,

$$iu_t + u_{xx} + Mu + \frac{\partial}{\partial \bar{u}} P|u|^2 = 0,$$

with periodic boundary conditions and

$$Mq = \sum_{j \in \mathbb{Z}} V_j \hat{q}(j) e^{ijx}$$

(a random Fourier multiplier).

2. Quasi-periodic solutions

The case of stationary and periodic solutions was solved in work by Albanese and Fröhlich [AF] and Albanese, Fröhlich and Spencer [AFS].

The construction of quasi-periodic solutions was given in recent work by the author and W-M. Wang ([BW1], [BW2]). It is based on the Lyapounov decomposition in P- and Q-equations and the use of the Newton method to solve the P-equations.

The linearized equation has the form

$$(\underbrace{n\cdot\omega+V_j}_{\substack{\downarrow\\ \text{diagonal}}}+\varepsilon\Delta_{(j)})\hat{q}_j(n)+\underbrace{\delta S}_{\substack{\downarrow\\ \text{Toeplitz in } n\\ \text{rapid decay in } j}}.$$

The control of the Green's functions uses recent methods from quasi-periodic localization (see [BGS] and [B1]), based on subharmonicity and the quantitative theory of semi-algebraic sets.

The basic difficulty in these problems is the presence of large sets of singular sites.

Following [FSW] the result applies in particular to the Landau-Lipschitz equation for classical spin waves with random forcing,

$$\dot{S}_j = S_j\times(\Delta S)_j + KV_j(\vec{f}\times S_j),\quad j\in\mathbb{Z}^d,$$

where $S_j\in\mathbb{R}^3$, $|S|=1$ and K is large.

3. Diffusion problems

Theorem 3.1 ([BW3]). *Assume in* (1.2)

$$|\lambda_j|<\varepsilon|j|^{-\tau},$$

where $\tau>0$ is arbitrary and $\varepsilon<\varepsilon(\tau,\kappa)$. Then

$$D(t)<t^{\kappa}\ \textit{for}\ t\to\infty.$$

The proof uses randomness and Nekhoroshev type methods to construct 'energy barriers'. Notice that here estimates are obtained for all times t, while the usual Nekhoroshev theory only leads to control for finite time span $|t|<T_\varepsilon$.

4. Invariant tori of full dimension for nonlinear Schrödinger equations with periodic boundary conditions

Consider the equation

$$iu_t+u_{xx}+Mu+\varepsilon\frac{\partial}{\partial\bar{u}}P(|u|^2)=0 \text{ on } \mathbb{T}, \tag{4.1}$$

with

$$Mu = \sum_{n\in\mathbb{Z}} V_n \hat{u}(n) e^{2\pi inx},$$
$$V_n \in [-1,1] \text{ random.}$$

There is the following analogue of the theorems of Fröhlich, Spencer, and Wayne [FSW] and Pöschel [Po]

THEOREM 4.1 ([B2]). *There is an abundance of invariant tori of full dimension with action variable decay*

$$I_n \sim e^{-|n|^\alpha} \qquad (0 < \alpha \le 1). \tag{4.2}$$

REMARKS. (i) The proof is based on usual KAM scheme.

(ii) Expanding (4.1) into a Fourier series,

$$i\dot{q}_n + (n^2 + V_n)q_n + \varepsilon \frac{\partial}{\partial \bar{q}_n} \int_{\mathbb{T}} P\left(\left|\sum q_n e^{inx}\right|^2\right) = 0, \tag{4.3}$$

where $q_n = \hat{u}(n)$, and

$$\int P\left(\left|\sum q_n e^{inx}\right|^2\right)$$

is expressed in monomials of the form

$$q_{n_1}\, \bar{q}_{n_2}\, q_{n_3}\, \bar{q}_{n_4} \cdots q_{n_{2s-1}}\, \bar{q}_{n_{2s}}, \tag{4.4}$$

which are not short range, but satisfy

$$n_1 - n_2 + n_3 - n_4 + \cdots + n_{2s-1} - n_{2s} = 0 \tag{4.5}$$

and, if resonant,

$$n_1^2 - n_2^2 + \cdots - n_{2s}^2 = 0. \tag{4.6}$$

Let $n_1^* \ge n_2^* \ge \cdots$ be the decreasing rearrangement of the modes in (4.4). Then

$$n_1^* \le n_2^* + n_3^* + \cdots \qquad \text{by (4.5)} \tag{4.7}$$

and also

$$\sum |n_j|^{1/2} \ge 2(n_1^*)^{1/2} + \frac{1}{4}\left[(n_3^*)^{1/4} + (n_4^*)^{1/4} + \cdots\right]. \tag{4.8}$$

In the case of a resonant monomial with $n_1 \ne n_2$, (4.5) and (4.6) imply

$$|n_1| + |n_2| < 2\left((n_3^*)^2 + (n_4^*)^2 + \cdots\right). \tag{4.9}$$

The normal forms are expressed in series in q_n and $\bar{q}_n$ of the form

$$H = \sum_{\ell,k,k'} B_{\ell,k,k'} \prod_n |q_n(0)|^{2\ell_n}\, q_n^{k_n}\, \bar{q}_n^{k'_n} \tag{4.10}$$

with coefficients satisfying an estimate

$$B_{\ell,k,k'} < \exp\left\{\rho\left(\sum |n_j|^{1/2} - 2(n_1^*)^{1/2}\right)\right\}. \tag{4.11}$$

and where ρ is increasing slightly along the iteration.

One needs to analyze the effect of Poisson-brackets and small divisors with respect to this norm. Thus

$$\{F_1, H_2\} = \frac{1}{i}\sum\left[\frac{\partial F_1}{\partial q_n}\frac{\partial H_2}{\partial \bar{q}_n} - \frac{\partial F_1}{\partial \bar{q}_n}\frac{\partial H_2}{\partial q_n}\right],$$

where

$$F_1 = \sum B^{(1)}_{\ell,k,k'}\frac{1}{\sum_n (k_n - k_n')\lambda_n}\prod_n |q_n(0)|^{2\ell_n} q_n^{k_n}\,\bar{q}_n^{k_n'}$$

originates from a Hamiltonian function H_1 of the form (4.10) and where $\lambda_n = n^2 + V_n$.

The expression $\sum(k_n - k_n')\lambda_n$ constitutes the 'small divisor'.

Considering

$$\frac{\partial}{\partial q_n}\left(\prod q_{n_{2s-1}}\,\bar{q}_{n_{2s}}\right)\frac{\partial}{\partial \bar{q}_n}\left(\prod q_{\nu_{2s'-1}}\,\bar{q}_{\nu_{2s'}}\right), \qquad n \in \{n_{2s-1}\} \cap \{\nu_{2s'}\},$$

we see that according to (4.10), the admissible weight is (assuming $n_1^* \geq \nu_1^*$)

$$\exp\left\{\rho\left(\sum |n_s|^{1/2} + \sum |\nu_{s'}|^{1/2} - 2|n|^{1/2} - 2|n_1^*|^{1/2}\right)\right\},$$

which is at least the product of the weights

$$\exp\left\{\rho\left(\sum |n_s|^{1/2} - 2|n_1^*|^{1/2}\right)\right\}\ \exp\left\{\rho\left(\sum |\nu_{s'}|^{1/2} - 2|\nu_1^*|^{1/2}\right)\right\},$$

since $|n| \leq \nu_1^*$.

By (4.7), if we increase ρ to $\rho + \varepsilon$, there is moreover an extra saving of

$$\exp\left\{-\frac{\varepsilon}{4}\left((n_3^*)^{1/2} + (n_4^*)^{1/2} + \cdots\right)\right\} \ll |\lambda_{n_1} - \lambda_{n_2} + \lambda_{n_3} - \lambda_{n_4} + \cdots|. \tag{4.12}$$

Indeed, if $n_1 \neq n_2$, then by (4.9), the left side of (4.12) is

$$< \exp\left\{-\frac{\varepsilon}{10}\sum |n_j|^{1/4}\right\} \leq \exp\left\{-\frac{\varepsilon}{10}\sum_n (|k_n| + |k_n'|)|n|^{1/4}\right\} \ll \prod \frac{1}{1 + |k_n - k_n'|^2 n^4} \ll \left|\sum (k_n - k_n')\lambda_n\right|.$$

If $n_1 = n_2$, then $\lambda_{n_1} = \lambda_{n_2}$, and there is no problem.

(iii) Observe that the implication (4.5)+(4.6) $\Rightarrow$ (4.9) is a one dimensional arithmetic feature. This is the main difficulty to extend the result described in this section to nonlinear Schrödinger equations in dimension 2 or higher.

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