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ADDITIVE PROCESSES AND STOCHASTIC INTEGRALS

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To the memory of J. L. Doob

ABSTRACT. Stochastic integrals of nonrandom $(l\times d)$-matrix-valued functions or nonrandom real-valued functions with respect to an additive process X on $\mathbb{R}^d$ are studied. Here an additive process means a stochastic process with independent increments, stochastically continuous, starting at the origin, and having cadlag paths. A necessary and sufficient condition for local integrability of matrix-valued functions is given in terms of the Lévy–Khintchine triplets of a factoring of X. For real-valued functions explicit expressions of the condition are presented for all semistable Lévy processes on $\mathbb{R}^d$ and some selfsimilar additive processes. In the last part of the paper, existence conditions for improper stochastic integrals $\int_0^{\infty-} f(s)dX_s$ and their extensions are given; the cases where $f(s) \asymp s^{\beta}e^{-cs^{\alpha}}$ and where $f(s)$ is such that $s=\int_{f(s)}^{\infty} u^{-2}e^{-u}du$ are analyzed.

1. Introduction

By an additive process $X=\{X_t\colon t\geqslant 0\}$ on $\mathbb{R}^d$ we mean an $\mathbb{R}^d$-valued stochastic process with independent increments, stochastically continuous, starting at the origin, and having cadlag paths. A Lévy process is an additive process with stationary increments. Stochastic integrals of nonrandom functions with respect to additive processes on $\mathbb{R}$ and the class of locally integrable functions were studied by Urbanik and Woyczyński [21], Rajput and Rosinski [11], and Kwapień and Woyczyński [7]. Rosiński [13] extended some results to the Banach space setting. Continuing the paper [17], we study, for additive processes X on $\mathbb{R}^d$, stochastic integrals $\int_B F(s)dX_s$ of nonrandom $(l\times d)$-matrix-valued functions $F(s)$ and improper stochastic integrals $\int_0^{\infty-} f(s)dX_s$ of nonrandom real-valued function $f(s)$. Using the Lévy–Khintchine triplets of a factoring of X, we give a description of the class $\mathbf{L}_{l\times d}(X)$ of locally X-integrable $(l\times d)$-matrix-valued functions F for the integral $\int_B F(s)dX_s$, which generalizes the results of [21], [11], [7] in the case $l=d=1$ and the results

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of [13] that deal with the case in which the distribution ρ_s in the factoring does not depend on s. As a special case, a description of the class $\mathbf{L}(X)$ of locally X-integrable real-valued functions f for the integral $\int_B f(s)dX_s$ is given, which extends the Musielak–Orlicz space type characterization in the papers mentioned above. In some examples, including all stable and semistable Lévy processes on $\mathbb{R}^d$ and the selfsimilar additive processes associated with stable and Γ-distributions, we find explicit necessary and sufficient conditions for f to belong to $\mathbf{L}(X)$. Then we analyze improper stochastic integrals, paying special attention, with f fixed, to the class of Lévy processes $X^{(\mu)}$ for which $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ is definable (the superscript μ in $X^{(\mu)}$ denotes the distribution at time 1). The example of the function $f(s)$ which is the inverse function of $s = g(r) = \int_r^\infty u^{-2}e^{-u}du$ shows that it is meaningful to consider some extensions of the notion of improper integrals: essential improper integrals and compensated improper integrals. This example and the case where $f(s)$ is asymptotically close to $s^\beta e^{-cs^\alpha}$ ($\alpha > 0$ and β real) for large s are fully examined.

Some related results are as follows. Applications of some improper stochastic integrals to Q-semi-selfsimilar additive processes and semi-stationary Ornstein–Uhlenbeck type processes are given in [10]. Distributions of some improper stochastic integrals related to Thorin and Goldie–Steutel–Bondesson classes are studied in [1]. Applications of the results in this paper to infinitely divisible Wald couples introduced by Roynette and Yor [14] will be given in another paper. In the case of a special type of random integrands, the improper integrals with respect to Lévy processes are studied by Erickson and Maller [3].

In Section 2 we recall the results on factorings and stochastic integrals for natural additive processes. Section 3 deals with conditions for the membership of $\mathbf{L}_{l\times d}(X)$, while Section 4 considers $\mathbf{L}(X)$ and examples. In Section 5 improper stochastic integrals are studied. (Added in revision: The material in Section 5 is further developed in the papers [18] and [19].)

2. Preliminaries on factorings and stochastic integrals

The factoring structure of additive processes and stochastic integrals of nonrandom functions with respect to them were studied by Rajput and Rosinski [11], Kwapień and Woyczyński [7], and Sato [17]. We review those results in the formulation given in [17].

We define an additive process in law by dropping the assumption of cadlag paths in the definition of an additive process, as in [16]. A Lévy process in law is similarly defined. All definitions and results in this paper remain true if we replace an additive (or Lévy) process by an additive (or Lévy) process in law. That is, our discussion is not related to cadlag property. On the other hand, any additive (or Lévy) process in law has an additive (or Lévy) process modification.

The characteristic function of a distribution μ on $\mathbb{R}^d$ is denoted by $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$; $ID(\mathbb{R}^d)$ is the class of infinitely divisible distributions on $\mathbb{R}^d$; $\mathcal{B}_0(\mathbb{R}^d)$ is the class of Borel sets B in $\mathbb{R}^d$ satisfying $\inf_{x\in B}|x| > 0$; $\mathcal{B}^0_{[0,\infty)}$ is the class of bounded Borel sets in $[0,\infty)$; δ_a is the distribution concentrated at point a; $\mathcal{L}(X)$ is the distribution of a random element X; p-lim stands for limit in probability; $\mathbf{S}_d^+$ is the class of $d\times d$ symmetric nonnegative-definite matrices; $\operatorname{tr} A$ is the trace of $A \in \mathbf{S}_d^+$; $\mathbf{M}_{l\times d}$ is the class of $l \times d$ real matrices; $I_{d\times d}$ is the $d\times d$ identity matrix; $\mathbb{R}^d$ is the d-dimensional Euclidean space with the canonical norm $|x|$ and the canonical inner product $\langle x, y\rangle$. An element of $\mathbb{R}^d$ is understood to be a column d-vector. For $U \in \mathbf{M}_{l\times d}$, U' denotes the transpose of U; thus $\langle z, Ux\rangle = \langle U'z, x\rangle$ for $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^l$. The norm of $U \in \mathbf{M}_{l\times d}$ is $\|U\| = \max_{|x|\leqslant 1}|Ux|$. If $\mu \in ID(\mathbb{R}^d)$, then the unique continuous function $\psi(z)$ on $\mathbb{R}^d$ with $\psi(0) = 0$ such that $\widehat{\mu}(z) = e^{\psi(z)}$ is called the cumulant function of μ and written as $C_\mu(z)$. If $\mu = \mathcal{L}(X)$, we write $C_X(z) = C_\mu(z)$. We use the Lévy–Khintchine triplet (A,ν,γ) of $\mu \in ID(\mathbb{R}^d)$ in the sense that

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} g(z,x)\nu(dx) + i\langle\gamma, z\rangle, \tag{2.1}$$

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - \frac{i\langle z,x\rangle}{1+|x|^2}, \tag{2.2}$$

where A is in $\mathbf{S}_d^+$, called the Gaussian covariance matrix of μ, ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$, called the Lévy measure of μ, and γ is an element of $\mathbb{R}^d$, called the location parameter of μ. Given an additive process $X = \{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$, we let $\mu_t = \mathcal{L}(X_t)$ and write (A_t, ν_t, γ_t) for the Lévy–Khintchine triplet of μ_t.

DEFINITION 2.1. An additive process X on $\mathbb{R}^d$ is said to be *natural* if γ_t is locally of bounded variation in t on $[0,\infty)$.

The location parameter γ_t depends on our choice of the integrand in the expression (2.1), but we can prove that the definition of naturalness does not depend on the choice.

PROPOSITION 2.2. *Let X be an additive process on $\mathbb{R}^d$. Then, for every $B \in \mathcal{B}^0_{[0,\infty)}$, there are a unique $A_B \in \mathbf{S}_d^+$ and a unique measure ν_B on $\mathbb{R}^d$ such that A_B and $\nu_B(D)$ for any $D \in \mathcal{B}_0(\mathbb{R}^d)$ are countably additive with respect to $B \in \mathcal{B}^0_{[0,\infty)}$ and satisfy $A_{[0,t]} = A_t$ and $\nu_{[0,t]} = \nu_t$. If, moreover, $\{X_t\}$ is natural, then there is a unique $\gamma_B \in \mathbb{R}^d$ such that γ_B is countably additive with respect to $B \in \mathcal{B}^0_{[0,\infty)}$ and $\gamma_{[0,t]} = \gamma_t$. For any natural additive process* *define*

$$\sigma^\circ(B) = \operatorname{tr} A_B + \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\, \nu_B(dx) + (\operatorname{var}\gamma)_B \quad \textit{for } B \in \mathcal{B}^0_{[0,\infty)},$$

using the measure $(\operatorname{var}\gamma)_B$ *induced by the variation function* $(\operatorname{var}\gamma)_t$ *of* γ_t*; then* σ° *is an atomless locally finite measure on* $[0,\infty)$.

Definition 2.3. Let X be an additive process on $\mathbb{R}^d$. A pair $(\{\rho_s\colon s \geqslant 0\}, \sigma)$ is called a *factoring* of X if the following five conditions are satisfied:

(1) σ is a locally finite atomless measure on $[0,\infty)$,
(2) $\rho_s \in ID(\mathbb{R}^d)$ for all $s \in [0,\infty)$,
(3) $C_{\rho_s}(z)$ is measurable (that is, Borel measurable) in s for each $z \in \mathbb{R}^d$,
(4) $\int_0^t |C_{\rho_s}(z)|\, \sigma(ds) < \infty$ for all $t \in [0,\infty)$ and $z \in \mathbb{R}^d$,
(5) it holds that

$$\widehat{\mu}_t(z) = \exp \int_0^t C_{\rho_s}(z)\sigma(ds) \quad \text{for all } t \in [0,\infty) \text{ and } z \in \mathbb{R}^d.$$

Proposition 2.4. *Let* $(\{\rho_s\}, \sigma)$ *be a factoring of an additive process* X *on* $\mathbb{R}^d$*. Denote by* $(A^{\rho_s}, \nu^{\rho_s}, \gamma^{\rho_s})$ *the triplet of* ρ_s*. Then,*

(1) A^{ρ_s}*,* γ^{ρ_s}*, and* $\nu^{\rho_s}(B)$ *for any* $B \in \mathcal{B}_0(\mathbb{R}^d)$ *are measurable in* s*,*
(2) *for all* $t \in [0,\infty)$

$$\int_0^t \left(\operatorname{tr}(A^{\rho_s}) + \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu^{\rho_s}(dx) + |\gamma^{\rho_s}| \right) \sigma(ds) < \infty,$$

(3) $A_t = \int_0^t A^{\rho_s}\sigma(ds)$*,* $\nu_t(D) = \int_0^t \nu^{\rho_s}(D)\sigma(ds)$ *for all* $D \in \mathcal{B}_0(\mathbb{R}^d)$*, and* $\gamma_t = \int_0^t \gamma^{\rho_s}\sigma(ds)$*,*
(4) *it holds that* $C_{\mu_t}(z) = \int_0^t C_{\rho_s}(z)\,\sigma(ds)$*, which is continuous.*

Proposition 2.5. *An additive process is natural if and only if it has a factoring.*

Definition 2.6. For a natural additive process X on $\mathbb{R}^d$ the measure σ° defined in Proposition 2.2 is called the *canonical measure* of X. A pair $(\{\rho_s\}, \sigma)$ is called a *canonical factoring* of X if it is a factoring of X such that $\sigma = \sigma^\circ$.

Proposition 2.7. *Let* X *be a natural additive process on* $\mathbb{R}^d$*. Then there exists a canonical factoring of* X*. It is unique in the sense that, if* $(\{\rho^\circ_s\}, \sigma^\circ)$ *is a canonical factoring of* X*, then* ρ°_s *is uniquely determined for* σ°*-a. e.* s*; moreover,*

$$\operatorname*{esssup}_{s\in[0,\infty)} \sup_{|z|\leqslant a} |C_{\rho^\circ_s}(z)| < \infty \qquad \textit{for } a \in (0,\infty),$$

$$\operatorname*{esssup}_{s\in[0,\infty)}\left(\operatorname{tr}(A^{\rho_s^\circ})+\int_{\mathbb{R}^d}(|x|^2\wedge 1)\nu^{\rho_s^\circ}(dx)+|\gamma^{\rho_s^\circ}|\right)<\infty\,,$$

where the essential suprema are relative to σ°.

PROPOSITION 2.8. *If* $\{\rho_s\colon s\geqslant 0\}$ *and* σ *satisfy conditions* (1), (2) *of Definition* 2.3 *and* (1), (2) *of Proposition* 2.4, *then* $(\{\rho_s\},\sigma)$ *is a factoring of some natural additive process* X *on* $\mathbb{R}^d$.

PROPOSITION 2.9. *An additive process is natural if and only if it is a semimartingale at the same time.*

This is essentially a result found in Jacod and Shiryaev [4].

DEFINITION 2.10. A class $\{X(B)\colon B\in\mathcal{B}^0_{[0,\infty)}\}$ of $\mathbb{R}^d$-valued random variables is called an *independently scattered random measure* if

(1) for any sequence $B_1,B_2,\dots$ of disjoint sets in $\mathcal{B}^0_{[0,\infty)}$ with $\bigcup_{n=1}^\infty B_n\in\mathcal{B}^0_{[0,\infty)}$, $\sum_{n=1}^\infty X(B_n)$ converges a. s. and equals $X(\bigcup_{n=1}^\infty B_n)$ a. s.,

(2) for any finite sequence $B_1,\dots,B_n$ of disjoint sets in $\mathcal{B}^0_{[0,\infty)}$, $X(B_1)$, $\dots,X(B_n)$ are independent,

(3) $X(\{a\})=0$ a. s. for every $a\in[0,\infty)$.

PROPOSITION 2.11. *Any independently scattered random measure* $\{X(B)\colon B\in\mathcal{B}^0_{[0,\infty)}\}$ *induces a natural additive process in law* $X_t=X([0,t])$. *Conversely, any natural additive process in law is induced from a unique independently scattered random measure. For any* $B\in\mathcal{B}^0_{[0,\infty)}$, $\mathcal{L}(X(B))$ *is infinitely divisible with triplet coinciding with* (A_B,ν_B,γ_B) *in Proposition 2.2. For any factoring* $(\{\rho_s\},\sigma)$ *of* $\{X_t\}$ *and any* $B\in\mathcal{B}^0_{[0,\infty)}$,

$$C_{X(B)}(z)=\int_B C_{\rho_s}(z)\sigma(ds).$$

EXAMPLE 2.12. Let $X=\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Then it is natural and it has a factoring given by $\rho_s=\mathcal{L}(X_1)$ for all $s\geqslant 0$ with σ being the Lebesgue measure on $[0,\infty)$.

EXAMPLE 2.13. Let $Q\in\mathbf{M}_{l\times d}$ be such that all of its eigenvalues have positive real parts. A stochastic process $X=\{X_t\}$ on $\mathbb{R}^d$ is called Q-selfsimilar if, for each $a>0$, $\{X_{at}\}$ and $\{a^QX_t\}$ have common finite-dimensional marginal distributions. Here $a^Q=\sum_{n=0}^\infty(n!)^{-1}(\log a)^nQ^n$. If X is a Q-selfsimilar additive process on $\mathbb{R}^d$, then it is natural.

In the rest of this section, let $X=\{X_t\}$ be a natural additive process on $\mathbb{R}^d$ and let $\{X(B)\colon B\in\mathcal{B}^0_{[0,\infty)}\}$ be the corresponding $\mathbb{R}^d$-valued independently scattered random measure. Let $(\{\rho_s\colon s\geqslant 0\},\sigma)$ be a factoring of $\{X_t\}$ and let σ° be the canonical measure of $\{X_t\}$.

Definition 2.14. An $\mathbf{M}_{l\times d}$-valued function F on $[0,\infty)$ is called a *simple function* if $F(s)=\sum_{j=1}^{n} 1_{B_j}(s)R_j$ for some n, where $B_1,\dots,B_n$ are disjoint Borel sets in $[0,\infty)$ and $R_1,\dots,R_n\in\mathbf{M}_{l\times d}$. If F is a simple function of this form, then we define the integral of F over $B\in\mathcal{B}^0_{[0,\infty)}$ with respect to X as

$$\int_B F(s)dX_s=\sum_{j=1}^{n} R_j\, X(B\cap B_j).$$

Proposition 2.15. *Suppose that F is a measurable $\mathbf{M}_{l\times d}$-valued function and that there is a sequence of simple functions F_n, $n=1,2,\dots$, such that*

(1) $F_n(s)\to F(s)$ $\quad\sigma^\circ$*-a. e., and*

(2) *for every $B\in\mathcal{B}^0_{[0,\infty)}$, the sequence $\int_B F_n(s)dX_s$ is convergent in probability as $n\to\infty$.*

Suppose that there is another sequence of simple functions G_n, $n=1,2,\dots$, satisfying (1) *and* (2) *with G_n in place of F_n. Then, for every $B\in\mathcal{B}^0_{[0,\infty)}$,*

$$\operatorname*{p-lim}_{n\to\infty}\int_B F_n(s)dX_s=\operatorname*{p-lim}_{n\to\infty}\int_B G_n(s)dX_s,\quad \textit{a. s.}$$

The proof uses the Nikodým theorem as in [2] and [21].

Definition 2.16. An $\mathbf{M}_{l\times d}$-valued function F on $[0,\infty)$ is said to be *locally X-integrable* if it is measurable and if there is a sequence of simple functions F_n, $n=1,2,\dots$, satisfying (1) and (2) of Proposition 2.15. The class of locally X-integrable $\mathbf{M}_{l\times d}$-valued functions is denoted by $\mathbf{L}_{l\times d}(X)$. If $F\in\mathbf{L}_{l\times d}(X)$, then we define

$$\int_B F(s)dX_s=\operatorname*{p-lim}_{n\to\infty}\int_B F_n(s)dX_s,\qquad B\in\mathcal{B}^0_{[0,\infty)}\,.$$

Note that this is $\mathbb{R}^l$-valued. Sometimes we write $\int_0^t F(s)dX_s$ for $\int_{[0,t]}F(s)dX_s$. The class of real-valued functions f such that $fI_{d\times d}\in\mathbf{L}_{d\times d}(X)$ is denoted by $\mathbf{L}(X)$ and we write $\int_B f(s)dX_s$ for $\int_B f(s)I_{d\times d}dX_s$.

It can be proved from Proposition 2.15 and Definition 2.16 that the classes $\mathbf{L}_{l\times d}(X)$ and $\mathbf{L}(X)$ are linear spaces and that $\int_B F(s)dX_s$ and $\int_B f(s)dX_s$ are linear in F and f, respectively.

Proposition 2.17. *Let $F\in\mathbf{L}_{l\times d}(X)$ and let $Y(B)=\int_B F(s)dX_s$. Then $\{Y(B)\colon B\in\mathcal{B}^0_{[0,\infty)}\}$ is an $\mathbb{R}^l$-valued independently scattered random measure, and*

$$\int_0^t |C_{\rho_s}(F(s)'z)|\,\sigma(ds)<\infty\quad \textit{for } t\in(0,\infty),\ z\in\mathbb{R}^l,\tag{2.3}$$

$$C_{Y(B)}(z)=\int_B C_{\rho_s}(F(s)'z)\sigma(ds)\quad \textit{for } B\in\mathcal{B}^0_{[0,\infty)},\ z\in\mathbb{R}^l.\tag{2.4}$$

PROPOSITION 2.18. *Let $F \in \mathbf{L}_{l\times d}(X)$. Then, for any Borel set E in $[0,\infty)$, $1_E(s)F(s)$ is in $\mathbf{L}_{l\times d}(X)$ and*

$$\int_B 1_E(s)F(s)dX_s = \int_{B\cap E} F(s)dX_s \quad \text{for } B \in \mathcal{B}^0_{[0,\infty)}.$$

Proof. Let $F_n(s)$ be the simple functions in the definition of $\int_B F(s)dX_s$. Then we can prove that $1_E(s)F_n(s)$ are simple functions and conditions (1) and (2) in Proposition 2.15 are satisfied with $1_E(s)F_n(s)$ and $1_E(s)F(s)$ in place of $F_n(s)$ and $F(s)$. Thus $1_E(s)F(s)$ belongs to $\mathbf{L}_{l\times d}(X)$. The equality asserted follows from the additivity of the integral with respect to integrands and to sets and from (2.4). □

For each $s \geqslant 0$ and $U \in \mathbf{M}_{l\times d}$, denote by ρ_s^U the distribution on $\mathbb{R}^l$ with characteristic function $\widehat{\rho}_s(U'z)$. The triplet $(A^{\rho_s^U}, \nu^{\rho_s^U}, \gamma^{\rho_s^U})$ of ρ_s^U is given by

$$A^{\rho_s^U} = UA^{\rho_s}U', \tag{2.5}$$

$$\nu^{\rho_s^U}(D) = \int_{\mathbb{R}^d} 1_D(Ux)\,\nu^{\rho_s}(dx) \quad \text{for } D \in \mathcal{B}_0(\mathbb{R}^d), \tag{2.6}$$

$$\gamma^{\rho_s^U} = U\gamma^{\rho_s} + \int_{\mathbb{R}^d} Ux\left(\frac{1}{1+|Ux|^2} - \frac{1}{1+|x|^2}\right)\nu^{\rho_s}(dx). \tag{2.7}$$

Sometimes we write $(A\{\rho_s^U\}, \nu\{\rho_s^U\}, \gamma\{\rho_s^U\})$ for this triplet.

COROLLARY 2.19. *Let F and $Y(B)$ be as in Proposition 2.17. Let $Y_t = Y([0,t]) = \int_0^t F(s)dX_s$. Then $Y = \{Y_t\}$ is a natural additive process in law on $\mathbb{R}^l$ with a factoring $(\{\rho_s^{F(s)}\}, \sigma)$. We have, for any $t \in (0,\infty)$,*

$$\int_0^t (\operatorname{tr} A^{\rho_s^{F(s)}})\,\sigma(ds) < \infty, \tag{2.8}$$

$$\int_0^t \sigma(ds) \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu^{\rho_s^{F(s)}}(dx) < \infty, \tag{2.9}$$

$$\int_0^t |\gamma^{\rho_s^{F(s)}}|\sigma(ds) < \infty. \tag{2.10}$$

The triplet $(A^{Y(B)}, \nu^{Y(B)}, \gamma^{Y(B)})$ of $Y(B)$ is given by

$$A^{Y(B)} = \int_B A^{\rho_s^{F(s)}}\sigma(ds) = \int_B F(s)A^{\rho_s}F(s)'\sigma(ds), \tag{2.11}$$

$$\nu^{Y(B)}(D) = \int_B \nu^{\rho_s^{F(s)}}(D)\sigma(ds) = \int_B \sigma(ds)\int_{\mathbb{R}^d} 1_D(F(s)x)\,\nu^{\rho_s}(dx), \tag{2.12}$$

$$\begin{aligned}\gamma^{Y(B)} &= \int_B \gamma^{\rho_s^{F(s)}}\sigma(ds) \\ &= \int_B \left(F(s)\gamma^{\rho_s} + \int_{\mathbb{R}^d} F(s)x\left(\frac{1}{1+|F(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu^{\rho_s}(dx)\right)\sigma(ds).\end{aligned} \tag{2.13}$$

This is a consequence of Propositions 2.4 and 2.17 combined. As this corollary says, a natural additive process in law X on $\mathbb{R}^d$ and a function $F \in \mathbf{L}_{l\times d}(X)$ give a natural additive process in law Y on $\mathbb{R}^l$. Up to modifications, this is a transformation from a natural additive process to a natural additive process.

Definition 2.20. If $F \in \mathbf{L}_{l\times d}(X)$ and if $\int_0^t F(s)dX_s$ converges in probability as $t \to \infty$, then the limit is denoted by $\int_0^{\infty-} F(s)dX_s$ and we say that *the improper integral $\int_0^{\infty-} F(s)dX_s$ is definable.*

Proposition 2.21. *The definability of the improper integral $\int_0^{\infty-} F(s)dX_s$ is equivalent to almost sure convergence of the additive process modification $\{\widetilde{Y}_t\}$ of $Y_t = \int_0^t F(s)dX_s$ as $t \to \infty$. It is also equivalent to its convergence in distribution.*

3. Conditions for local integrability

Let $X = \{X_t\}$ be a natural additive process on $\mathbb{R}^d$. Let $(\{\rho_s\}, \sigma)$ be a factoring of X and $(A^{\rho_s}, \nu^{\rho_s}, \gamma^{\rho_s})$ the triplet of ρ_s. Denote

$$\varphi_0(s) = \operatorname{tr} A^{\rho_s} + \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu^{\rho_s}(dx) + |\gamma^{\rho_s}|. \tag{3.1}$$

The property (2) of Proposition 2.4 says that

$$\int_0^t \varphi_0(s)\sigma(ds) < \infty \qquad \text{for } 0 < t < \infty. \tag{3.2}$$

As in the previous section, for $U \in \mathbf{M}_{l\times d}$, we denote by ρ_s^U the distribution on $\mathbb{R}^l$ with characteristic function $\widehat{\rho}_s(U'z)$ and by $(A^{\rho_s^U}, \nu^{\rho_s^U}, \gamma^{\rho_s^U})$ the triplet of ρ_s^U, which has the expression (2.5)–(2.7). Define

$$\varphi(s, U) = \operatorname{tr} A^{\rho_s^U} + \int_{\mathbb{R}^l} (|x|^2 \wedge 1)\nu^{\rho_s^U}(dx) + |\gamma^{\rho_s^U}|. \tag{3.3}$$

The following theorem is a main result of this section.

Theorem 3.1. *Let F be an $\mathbf{M}_{l\times d}$-valued measurable function on $[0, \infty)$. Then $F \in \mathbf{L}_{l\times d}(X)$ if and only if*

$$\int_0^t \varphi(s, F(s))\sigma(ds) < \infty \qquad \textit{for } 0 < t < \infty. \tag{3.4}$$

Before proving this theorem, we study properties of $\varphi(s, U)$.

Proposition 3.2. *Let $s \geqslant 0$ and $U, V \in \mathbf{M}_{l\times d}$.*

(i) *$\varphi(s, U) = \varphi(s, -U) \geqslant 0$ and $\varphi(s, 0) = 0$.*
(ii) *$\varphi(s, U)$ is continuous in U.*

(iii) $\varphi(s,cU)\leqslant(3/2)(c^2\vee1)\varphi(s,U)$ *for* $c\in\mathbb{R}$.
(iv) $\varphi(s,U+V)\leqslant(7/2)[\varphi(s,U)+\varphi(s,V)]$.
(v) *We have*

$$\varphi(s,U)\leqslant(3/2)(\|U\|^2+1)\varphi_0(s) \tag{3.5}$$

and

$$\varphi(s,U)\leqslant\|U\|^2\ \operatorname{tr}A^{\rho_s}+2^{-1}\|U\|^2\int_{\mathbb{R}^d}(|x|^2\wedge1)\nu^{\rho_s}(dx) \tag{3.6}$$
$$+\int_{\mathbb{R}^d}(|Ux|^2\wedge1)\nu^{\rho_s}(dx)+\int_{\mathbb{R}^d}(|Ux|\wedge2^{-1})(|x|^2\wedge1)\nu^{\rho_s}(dx)+\|U\|\,|\gamma^{\rho_s}|.$$

Proof. The proof of (i) and (ii) is easy and omitted. Sometimes we will not explicitly write the measure $\nu^{\rho_s}(dx)$ in the integral.

(iii) Let $c\geqslant0$. We have

$$|\gamma^{\rho_s^{cU}}|\leqslant c|\gamma^{\rho_s^U}|+\int|cUx|\frac{\big||Ux|^2-|cUx|^2\big|}{(1+|cUx|^2)(1+|Ux|^2)}$$

and $|cUx|/(1+|cUx|^2)\leqslant1/2$. Thus the integral is $\leqslant(|1-c^2|/2)\int(|Ux|^2\wedge1)$, and

$$\begin{aligned}\varphi(s,cU)&\leqslant c^2\operatorname{tr}(UA^{\rho_s}U')+\left((c^2\vee1)+\frac{|1-c^2|}{2}\right)\int(|Ux|^2\wedge1)+c|\gamma^{\rho_s^U}|\\&\leqslant(3/2)(c^2\vee1)\varphi(s,U).\end{aligned}$$

(iv) We have $\operatorname{tr}A^{\rho_s^{U+V}}\leqslant2\operatorname{tr}A^{\rho_s^U}+2\operatorname{tr}A^{\rho_s^V}$ and $\int(|Ux+Vx|^2\wedge1)\leqslant2\int(|Ux|^2\wedge1)+2\int(|Vx|^2\wedge1)$. Further

$$\begin{aligned}\gamma^{\rho_s^{U+V}}&=U\gamma^{\rho_s}+V\gamma^{\rho_s}+\int Ux\left(\frac{1}{1+|Ux|^2}-\frac{1}{1+|x|^2}\right)\\&\quad+\int Vx\left(\frac{1}{1+|Vx|^2}-\frac{1}{1+|x|^2}\right)+\int J,\end{aligned}$$

where

$$J=\frac{Ux+Vx}{1+|Ux+Vx|^2}-\frac{Ux}{1+|Ux|^2}-\frac{Vx}{1+|Vx|^2}.$$

We have $|J|\leqslant3/2$, since each term has norm not exceeding $1/2$. On the other hand,

$$J=Ux(f(|Ux+Vx|)-f(|Ux|))+Vx(f(|Ux+Vx|)-f(|Vx|)),$$

where $f(r)=(1+r^2)^{-1}$. Since $|f(r)-f(s)|\leqslant(\sqrt{3}/2)^3|r-s|$ for all r and s,

$$|J|\leqslant2(\sqrt{3}/2)^3|Ux|\,|Vx|\leqslant(\sqrt{3}/2)^3(|Ux|^2+|Vx|^2).$$

Hence we get

$$|\gamma^{\rho_s^{U+V}}|\leqslant|\gamma^{\rho_s^U}|+|\gamma^{\rho_s^V}|+\frac{3}{2}\int(|Ux|^2\wedge1)+\frac{3}{2}\int(|Vx|^2\wedge1).$$

This completes the proof of (iv).

(v) For any $A \in \mathbf{S}_d^+$ and $U \in \mathbf{M}_{l\times d}$, $\operatorname{tr}(UAU') \leqslant \|U\|^2 \operatorname{tr} A$. Thus $\operatorname{tr} A^{\rho_s^U} \leqslant \|U\|^2 \operatorname{tr} A^{\rho_s}$. We have

$$\begin{aligned}
&\int (|Ux|^2 \wedge 1)\nu^{\rho_s}(dx) \leqslant (\|U\|^2 \vee 1) \int (|x|^2 \wedge 1)\nu^{\rho_s}(dx),\\
&|\gamma^{\rho_s^U}| \leqslant \|U\|\,|\gamma^{\rho_s}| + \int |Ux| \frac{|Ux|^2 + |x|^2}{(1+|Ux|^2)(1+|x|^2)}\\
&\qquad \leqslant \|U\|\,|\gamma^{\rho_s}| + \frac{\|U\|^2}{2} \int \frac{|x|^2}{1+|x|^2} + \int (|Ux| \wedge 2^{-1}) \frac{|x|^2}{1+|x|^2}\\
&\qquad \leqslant \|U\|\,|\gamma^{\rho_s}| + \frac{\|U\|^2+1}{2} \int \frac{|x|^2}{1+|x|^2}.
\end{aligned}$$

These estimates together yield (3.5). Looking back at this proof, we see also the estimate (3.6). $\square$

Let us denote by $\mathbf{L}^\sharp_{l\times d}(X)$ the class of $\mathbf{M}_{l\times d}$-valued measurable functions F that satisfy (3.4). This class is a linear space by virtue of (iii) and (iv) of Proposition 3.2. It contains all simple functions by virtue of (3.2) and (3.5).

PROPOSITION 3.3. *Let $F \in \mathbf{L}^\sharp_{l\times d}(X)$ and let F_n, $n = 1, 2, \ldots,$ be $\mathbf{M}_{l\times d}$-valued measurable functions. Suppose that there are $E_n \in \mathcal{B}^0_{[0,\infty)}$ and $\varepsilon_n > 0$ such that $E_n \uparrow [0,\infty)$, $\varepsilon_n \downarrow 0$, $F_n(s) = 0$ for $s \in E_n^c$, and $\|F_n(s) - F(s)\| \leqslant \varepsilon_n$ for $s \in E_n$. Then $F_n \in \mathbf{L}^\sharp_{l\times d}(X)$, $n = 1, 2, \ldots,$ and*

$$\int_0^t \varphi(s, F_n(s) - F(s))\sigma(ds) \to 0 \quad \textit{for } 0 < t < \infty. \tag{3.7}$$

Proof. Using Proposition 3.2 (iv) and (3.5), we get $F_n \in \mathbf{L}^\sharp_{l\times d}(X)$. Further,

$$\begin{aligned}
&\int_0^t \varphi(s, F_n(s) - F(s))\sigma(ds)\\
&= \int_{[0,t]\cap E_n} \varphi(s, F_n(s) - F(s))\sigma(ds) + \int_{[0,t]\cap E_n^c} \varphi(s, F(s))\sigma(ds) = I_1 + I_2.
\end{aligned}$$

We see $I_2 \to 0$ since $E_n^c \downarrow \emptyset$ and $\int_0^t \varphi(s, F(s))\sigma(ds) < \infty$. From (3.6) we see

$$\begin{aligned}
I_1 \leqslant \int_{[0,t]\cap E_n} \Bigg[&\varepsilon_n^2 \operatorname{tr} A^{\rho_s} + 2^{-1}\varepsilon_n^2 \int (|x|^2 \wedge 1) + \int ((\varepsilon_n |x|)^2 \wedge 1)\\
&+ \int ((\varepsilon_n|x|) \wedge 2^{-1})(|x|^2 \wedge 1) + \varepsilon_n |\gamma^{\rho_s}| \Bigg] \sigma(ds).
\end{aligned}$$

Since $\varepsilon_n \downarrow 0$, we get $I_1 \to 0$, using the dominated convergence theorem. $\square$

Proof of Theorem 3.1. The theorem claims that $\mathbf{L}_{l\times d}(X) = \mathbf{L}^{\sharp}_{l\times d}(X)$. We know $\mathbf{L}_{l\times d}(X) \subset \mathbf{L}^{\sharp}_{l\times d}(X)$ from Corollary 2.19. To show the converse inclusion, assume that $F \in \mathbf{L}^{\sharp}_{l\times d}(X)$. Let $E_n = [0,n] \cap \{s\colon \|F(s)\| \leqslant n\}$ and let $\varepsilon_n \downarrow 0$. Choose $\mathbf{M}_{l\times d}$-valued simple functions $F_n(s)$ such that $F_n(s) = 0$ for $s \in E_n^c$ and the (j,k)-components satisfy $|F_n(s)_{jk} - F(s)_{jk}| \leqslant \varepsilon_n (ld)^{-1/2}$ for $s \in E_n$. Then $F_n \in \mathbf{L}^{\sharp}_{l\times d}(X)$ and $\|F_n(s) - F(s)\| \leqslant \varepsilon_n$ for $s \in E_n$. Applying Proposition 3.3, we get (3.7). Hence, by Proposition 3.2 (iv),

$$\int_0^t \varphi(s, F_n(s) - F_m(s))\sigma(ds) \to 0, \qquad n, m \to \infty.$$

Recalling (3.3) and using (2.11)–(2.13), we get

$$\mathcal{L}\left(\int_B (F_n(s) - F_m(s))dX_s\right) \to \delta_0, \qquad n, m \to \infty.$$

This means that $\int_B F_n(s)dX_s$ is convergent in probability as $n \to \infty$. Hence $F \in \mathbf{L}_{l\times d}(X)$. The proof is complete. □

The following two propositions are consequences of Theorem 3.1, Proposition 3.2, and Proposition 2.7.

PROPOSITION 3.4. *Suppose that F is an $\mathbf{M}_{l\times d}$-valued measurable function satisfying one of the following two conditions:*

$$\|F(s)\| \text{ is locally bounded on } [0,\infty), \tag{3.8}$$

$$\int_0^t \|F(s)\|^2\sigma(ds) < \infty \text{ for } 0 < t < \infty \text{ and } (\{\rho_s\}, \sigma) \text{ is canonical.} \tag{3.9}$$

Then $F \in \mathbf{L}_{l\times d}(X)$.

PROPOSITION 3.5. *Let F_n, $n = 1, 2, \dots$, be $\mathbf{M}_{l\times d}$-valued measurable functions such that $g(s) = \sup_n \|F_n(s)\|$ satisfies one of the following two conditions:*

$$g(s) \text{ is locally bounded on } [0,\infty), \tag{3.10}$$

$$\int_0^t g(s)^2\sigma(ds) < \infty \text{ for } 0 < t < \infty \text{ and } (\{\rho_s\}, \sigma) \text{ is canonical.} \tag{3.11}$$

Suppose that $F_n(s) \to F(s)$, σ-a. e., for some $F(s)$. Then $F_n, F \in \mathbf{L}_{l\times d}(X)$ and the convergence (3.7) *holds.*

REMARK 3.6. The condition (3.9) is the best possible in the following sense. Let $\{B_t\}$ be Brownian motion on $\mathbb{R}^d$. Given a locally finite atomless measure σ on $[0,\infty)$, define $X_t = B_{\sigma([0,t])/d}$. Then $X = \{X_t\}$ is a natural additive process with canonical measure σ and $\mathbf{L}_{l\times d}(X)$ is the totality of F such that $\int_0^t \|F(s)\|^2\sigma(ds) < \infty$ for $0 < t < \infty$.

The two kinds of convergence in $\mathbf{L}_{l\times d}(X)$ are related as follows.

Proposition 3.7. *Let F_n, $n=1,2,\ldots$, and F be in $\mathbf{L}_{l\times d}(X)$. If* (3.7) *holds, then*

$$\int_B F_n(s)dX_s \to \int_B F(s)dX_s \text{ in probability for all } B\in\mathcal{B}^0_{[0,\infty)}. \tag{3.12}$$

If (3.12) *holds and $F_n(s)\to F(s)$, σ-a. e., then* (3.7) *holds.*

Proof. If (3.7) holds, then $\mathcal{L}\left(\int_B (F_n(s)-F(s))dX_s\right)\to\delta_0$ and (3.12) holds. To see the converse, assume (3.12) and $F_n(s)\to F(s)$, σ-a. e. Then, $\int_B \operatorname{tr}(A\{\rho_s^{F_n(s)-F(s)}\})\sigma(ds)$, $\int_B\sigma(ds)\int_{\mathbb{R}^l}(|x|^2\wedge 1)\nu\{\rho_s^{F_n(s)-F(s)}\}(dx)$, and $\int_B \gamma\{\rho_s^{F_n(s)-F(s)}\}\sigma(ds)$ tend to 0 for any $B\in\mathcal{B}^0_{[0,\infty)}$. Since $F_n-F\in\mathbf{L}_{l\times d}(X)$, we have $\int_0^t|\gamma\{\rho_s^{F_n(s)-F(s)}\}|\sigma(ds)<\infty$. Using the Vitali–Hahn–Saks theorem (p. 158 of [2]), we see that $\left\{\gamma\{\rho_s^{F_n(s)-F(s)}\}, s\in[0,t]\right\}$ is uniformly integrable with respect to σ. Since $\gamma\{\rho_s^{F_n(s)-F(s)}\}\to 0$, σ-a. s., it follows that $\int_{[0,t]}|\gamma\{\rho_s^{F_n(s)-F(s)}\}|\sigma(ds)\to 0$. Therefore (3.7) is true. □

Let us consider iteration of the transformation of additive processes by stochastic integrals.

Theorem 3.8. *Let $F\in\mathbf{L}_{l\times d}(X)$ and let $Y=\{Y_t\}$ be the additive process (in law) on $\mathbb{R}^l$ defined by $Y_t=\int_0^t F(s)dX_s$. Let G be an $\mathbf{M}_{m\times l}$-valued measurable function. Then $G\in\mathbf{L}_{m\times l}(Y)$ if and only if $GF\in\mathbf{L}_{m\times d}(X)$. If $G\in\mathbf{L}_{m\times l}(Y)$, then*

$$\int_B G(s)dY_s=\int_B G(s)F(s)dX_s \quad \text{for } B\in\mathcal{B}^0_{[0,\infty)}. \tag{3.13}$$

Proof. Denote $\eta_s=\rho_s^{F(s)}$. Then $(\{\eta_s\},\sigma)$ is a factoring of Y (Corollary 2.19). Define, for $V\in\mathbf{M}_{m\times l}$,

$$\varphi^Y(s,V)=\operatorname{tr}A^{\eta_s^V}+\int_{\mathbb{R}^m}(|x|^2\wedge 1)\nu^{\eta_s^V}(dx)+|\gamma^{\eta_s^V}|.$$

Since $\widehat{\eta_s^V}(z)=\widehat{\eta}_s(V'z)$, we see that $\eta_s^V=\rho_s^{VF(s)}$, and hence $\varphi^Y(s,V)=\varphi(s,VF(s))$. Now Theorem 3.1 for X and Y shows that $G\in\mathbf{L}_{m\times l}(Y)$ if and only if $GF\in\mathbf{L}_{m\times d}(X)$. If G is a simple function, then (3.13) is proved by Proposition 2.18. For a general G in $\mathbf{L}_{m\times l}(Y)$, let $E_n=[0,n]\cap\{s\colon \|F(s)\|\leqslant n,\ \|G(s)\|\leqslant n\}$, choose simple functions $G_n(s)$ such that $G_n(s)=0$ for $s\in E_n^c$ and $\|G_n(s)-G(s)\|\leqslant n^{-2}$ for $s\in E_n$, and use Proposition 3.3. □

Let us introduce a partial order in $\mathbf{M}_{l\times d}$ to give another characterization.

DEFINITION 3.9. Let U, $V \in \mathbf{M}_{l\times d}$. We say that $V \prec U$ if $|Vx| \leqslant |Ux|$ for all $x \in \mathbb{R}^d$. (Hence, $V \prec U$ if and only if there exists $C \in \mathbf{M}_{l\times l}$ with $\|C\| \leqslant 1$ such that $V = CU$.)

We define

$$\widetilde{\varphi}(s,U) = \operatorname{tr} A^{\rho_s^U} + \int_{\mathbb{R}^l} (|x|^2 \wedge 1)\nu^{\rho_s^U}(dx) + \sup_{V\prec U} |\gamma^{\rho_s^V}|. \tag{3.14}$$

Then we can prove the following four propositions. The proofs are omitted.

PROPOSITION 3.10.

(i) *If $V \prec U$, then $\widetilde{\varphi}(s,V) \leqslant \widetilde{\varphi}(s,U)$.*
(ii) *We have $\varphi(s,U) \leqslant \widetilde{\varphi}(s,U) \leqslant \frac{3}{2}\varphi(s,U)$.*

PROPOSITION 3.11. *Let F be an $\mathbf{M}_{l\times d}$-valued measurable function on $[0,\infty)$. Then $F \in \mathbf{L}_{l\times d}(X)$ if and only if*

$$\int_0^t \widetilde{\varphi}(s,F(s))\sigma(ds) < \infty \qquad \text{for } 0 < t < \infty. \tag{3.15}$$

PROPOSITION 3.12. *If $F \in \mathbf{L}_{l\times d}(X)$ and if $G(s)$ is an $\mathbf{M}_{l\times d}$-valued measurable function satisfying $G(s) \prec F(s)$ for σ-a. e. s, then $G \in \mathbf{L}_{l\times d}(X)$.*

PROPOSITION 3.13. *Let $G \in \mathbf{L}_{l\times d}(X)$. Let F_n, $n = 1, 2, \dots$, be $\mathbf{M}_{l\times d}$-valued measurable functions such that $F_n(s) \prec G(s)$, σ-a. e. Suppose that $F_n(s) \to F(s)$, σ-a. e., for some $F(s)$. Then $F_n, F \in \mathbf{L}_{l\times d}(X)$ and* (3.7) *holds.*

A description of $\mathbf{L}_{l\times d}(X)$ directly in terms of the cumulant function of ρ_s is of some interest.

THEOREM 3.14. *An $\mathbf{M}_{l\times d}$-valued measurable function $F(s)$ is in $\mathbf{L}_{l\times d}(X)$ if and only if, for any $0 < t < \infty$, there is $0 < a_t < \infty$ such that*

$$\int_0^t \sup_{|z|\leqslant a_t} |C_{\rho_s}(F(s)'z)|\,\sigma(ds) < \infty. \tag{3.16}$$

Proof. Assume that $F \in \mathbf{L}_{l\times d}(X)$. Since any $\mu \in ID(\mathbb{R}^d)$ with triplet (A,ν,γ) satisfies

$$|C_\mu(z)| \leqslant \frac{1}{2}(\operatorname{tr} A)|z|^2 + 3(1+|z|^2)\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) + |\gamma|\,|z|,$$

we get (3.16) for arbitrary a_t from Theorem 3.1.

Conversely, assume the existence of a_t satisfying (3.16). We have

$$\int_0^t \sup_{|z|\leqslant a_t} (-\operatorname{Re} C_{\rho_s}(F(s)'z))\,\sigma(ds) < \infty. \tag{3.17}$$

Hence

$$\int_0^t \sup_{|z|\leqslant a_t} \langle z, A^{\rho_s^{F(s)}} z\rangle \sigma(ds) < \infty.$$

It follows that $\int_0^t \operatorname{tr}(A^{\rho_s^{F(s)}})\sigma(ds) < \infty$. We have

$$\int_0^t \left[\sup_{|z|\leqslant a_t} \int_{\mathbb{R}^l} (1-\cos\langle z, x\rangle)\nu^{\rho_s^{F(s)}}(dx)\right] \sigma(ds) < \infty$$

from (3.17). Thus, for $|z| \leqslant a_t$, we have

$$\int_0^t \sigma(ds) \int_0^1 dv \int_{\mathbb{R}^l} (1-\cos\langle vz, x\rangle)\nu^{\rho_s^{F(s)}}(dx) < \infty,$$

that is,

$$\int_0^t \sigma(ds) \int_{\mathbb{R}^l} \left(1 - \left|\frac{\sin\langle z, x\rangle}{\langle z, x\rangle}\right|\right) \nu^{\rho_s^{F(s)}}(dx) < \infty.$$

Since $1 - |(\sin\theta)/\theta| \geqslant (3\pi)^{-1}(\theta^2 \wedge 1)$ for all θ, we have $\int_0^t \sigma(ds) \int_{\mathbb{R}^l} (|\langle z, x\rangle|^2 \wedge 1)\nu^{\rho_s^{F(s)}}(dx) < \infty$ and hence $\int_0^t \sigma(ds) \int_{\mathbb{R}^l} (|x|^2 \wedge 1)\nu^{\rho_s^{F(s)}}(dx) < \infty$. We have

$$\int_0^t |\operatorname{Im} C_{\rho_s}(F(s)'z)|\sigma(ds) < \infty \qquad \text{for all } z \text{ with } |z| \leqslant a_t \tag{3.18}$$

from (3.16), that is,

$$\int_0^t \left| \langle \gamma^{\rho_s^{F(s)}}, z\rangle + \int_{\mathbb{R}^l} \left(\sin\langle z, x\rangle - \frac{\langle z, x\rangle}{1+|x|^2}\right) \nu^{\rho_s^{F(s)}}(dx)\right| \sigma(ds) < \infty.$$

It follows that $\int_0^t |\langle \gamma^{\rho_s^{F(s)}}, z\rangle|\, \sigma(ds) < \infty$ and hence $\int_0^t |\gamma^{\rho_s^{F(s)}}|\sigma(ds) < \infty$. Now we have (3.4). Hence $F \in \mathbf{L}_{l\times d}(X)$ from Theorem 3.1. □

REMARK 3.15. The proof of Theorem 3.14 shows that an $\mathbf{M}_{l\times d}$-valued measurable function $F(s)$ belongs to $\mathbf{L}_{l\times d}(X)$ if and only if, for any $0 < t < \infty$, there is $0 < a_t < \infty$ such that (3.17) and (3.18) hold.

REMARK 3.16. Let $X = \{X_t\}$ be a natural additive process on $\mathbb{R}^d$ with $d \geqslant 2$ and let $F(s) = (F_{jk}(s))$ be an $\mathbf{M}_{l\times d}$-valued function. Then, for each k, the kth component $X^k = \{X_t^k\}$ of X is a natural additive process on $\mathbb{R}$. If $F_{jk} \in \mathbf{L}_{1\times 1}(X^k)$ for all j and k, then $F \in \mathbf{L}_{l\times d}(X)$ and

$$\int_B F(s)dX_s = \left(\sum_{k=1}^d \int_B F_{jk}(s)dX_s^k\right)_{1\leqslant j\leqslant l} \qquad \text{for } B \in \mathcal{B}^0_{[0,\infty)}. \tag{3.19}$$

To see this, it is enough to choose for each j and k a sequence of simple functions $F_{j,k,n}$ and to consider $\mathbf{M}_{l\times d}$-valued simple functions $F_n = (F_{j,k,n})$ as in the proof of Theorem 3.1. However, *even if* $F \in \mathbf{L}_{l\times d}(X)$*, the components* F_{jk} *do not necessarily belong to* $\mathbf{L}_{1\times 1}(X^k)$. Thus we cannot always write (3.19). For a simple example, let $l = d = 2$, let X be a Lévy process on

$\mathbb{R}^2$ satisfying $X_t^1 = X_t^2$, and let $F(s) = \begin{pmatrix} f(s) & -f(s) \\ 0 & 1 \end{pmatrix}$, with f being a measurable function. Then it follows from Definition 2.16 that $F \in \mathbf{L}_{2\times 2}(X)$ and $\int_0^t F(s)dX_s = \binom{0}{X_t^2}$ while f is not necessarily in $\mathbf{L}_{1\times 1}(X^1)$.

4. Examples of locally integrable functions

Let $X = \{X_t\}$ be a natural additive process on $\mathbb{R}^d$ with a factoring $(\{\rho_s\}, \sigma)$ as in the preceding section. We denote by $\mathbf{L}(X)$ the class of real-valued functions f such that $fI_{d\times d} \in \mathbf{L}_{d\times d}(X)$ as in Definition 2.16. We write $\int_B f(s)dX_s$ for $\int_B f(s)I_{d\times d}dX_s$. The structure of the class $\mathbf{L}(X)$ is simpler than that of $\mathbf{L}_{l\times d}(X)$.

When u is a real number, we write $\varphi(s,u)$ for $\varphi(s, uI_{d\times d})$ and ρ_s^u for $\rho_s^{uI_{d\times d}}$. Thus

$$\begin{aligned}\varphi(s,u) &= \operatorname{tr} A^{\rho_s^u} + \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu^{\rho_s^u}(dx) + |\gamma^{\rho_s^u}| \\ &= u^2 \operatorname{tr} A^{\rho_s} + \int_{\mathbb{R}^d} (|ux|^2 \wedge 1)\nu^{\rho_s}(dx) \\ &\quad + \left| u\gamma^{\rho_s} + u \int_{\mathbb{R}^d} x \left(\frac{1}{1+|ux|^2} - \frac{1}{1+|x|^2} \right) \nu^{\rho_s}(dx) \right|.\end{aligned}$$

Proposition 4.1. *Let $f \in \mathbf{L}(X)$. Then $\int_0^t \varphi(s, f(s))\sigma(ds) = 0$ for all $0 < t < \infty$ if and only if $f(s) = 0$, σ°-a. e., where σ° is the canonical measure of X.*

We omit the proof. The latter condition is equivalent to saying that $f(s) = 0$, σ-a. e. on $T = \{s \in [0,\infty) \colon \varphi_0(s) > 0\}$, because, for any Borel set B,

$$d^{-1} \int_B \varphi_0(s)\sigma(ds) \leqslant \sigma^\circ(B) \leqslant d^{1/2} \int_B \varphi_0(s)\sigma(ds).$$

Proposition 4.2. *If $f_n \in \mathbf{L}(X)$, $n = 1, 2, \ldots$, satisfy*

$$\int_0^t \varphi(s, f_n(s) - f_m(s))\sigma(ds) \to 0 \text{ as } n, m \to \infty \text{ for } 0 < t < \infty, \tag{4.1}$$

then there is $f \in \mathbf{L}(X)$ such that

$$\int_0^t \varphi(s, f_n(s) - f(s))\sigma(ds) \to 0 \text{ as } n \to \infty \text{ for } 0 < t < \infty, \tag{4.2}$$

and that, for some subsequence $\{f_{n'}\}$ of $\{f_n\}$, $f_{n'}(s) \to f(s)$ σ°-a. e. s.

This proposition can be proved by standard techniques. The following fact is straightforward.

Proposition 4.3. *Define*

$$\widetilde{\varphi}(s,u) = u^2 \operatorname{tr} A^{\rho_s} + \int_{\mathbb{R}^d} (|ux|^2 \wedge 1)\nu^{\rho_s}(dx) + \sup_{v\in\mathbb{R},\, |v|\leqslant|u|} |\gamma^{\rho_s^v}| \tag{4.3}$$

for $s \geqslant 0$ and $u \in \mathbb{R}$. Then, as a function of u, $\varphi(s,u)$ is continuous and even, vanishes at 0, and increases on $[0,\infty)$. A real-valued measurable function f belongs to $\mathbf{L}(X)$ if and only if

$$\int_0^t \widetilde{\varphi}(s, f(s))\sigma(ds) < \infty \qquad \text{for } 0 < t < \infty. \tag{4.4}$$

We note that, for each t, the class of f for which $\int_0^t \widetilde{\varphi}(s, f(s))\sigma(ds)$ is finite is a Musielak–Orlicz space (cf. [7]). In the case $d = 1$, this fact is due to Rajput and Rosinski [11] and Kwapień and Woyczyński [7].

Let us give an explicit description of $\mathbf{L}(X)$ for some typical processes.

Example 4.4. Let X be a compound Poisson process on $\mathbb{R}^d$. Then $\mathbf{L}(X)$ is the class of all measurable functions on $[0,\infty)$.

Example 4.5. Let X be a strictly α-stable Lévy process or a strictly α-semistable Lévy process on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$. Let f be a measurable function on $[0,\infty)$. Then $f \in \mathbf{L}(X)$ if and only if

$$\int_0^t |f(s)|^\alpha ds < \infty \qquad \text{for } 0 < t < \infty.$$

(Use the descriptions of the characteristic functions of semistable Lévy processes in [16], Theorems 14.3, 14.7, and Proposition 14.9.)

Example 4.6. Stable (resp. semistable) processes which are not strictly stable (resp. semistable) are called second-class stable (resp. semistable) as in [20]. Let X be a second-class α-stable Lévy process or a second-class α-semistable Lévy process on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$. Let f be a measurable function on $[0,\infty)$.

(i) Let $0 < \alpha < 1$. Then $f \in \mathbf{L}(X)$ if and only if

$$\int_0^t |f(s)| ds < \infty \qquad \text{for } 0 < t < \infty.$$

(ii) Let $\alpha = 1$. Then $f \in \mathbf{L}(X)$ if and only if

$$\int_0^t |f(s)| \log^+ |f(s)| ds < \infty \qquad \text{for } 0 < t < \infty,$$

where $\log^+ |u| = (\log |u|) \vee 0$.

(iii) Let $1 < \alpha \leqslant 2$. Then $f \in \mathbf{L}(X)$ if and only if

$$\int_0^t |f(s)|^\alpha ds < \infty \qquad \text{for } 0 < t < \infty.$$

EXAMPLE 4.7. Let X be the Γ-process on $\mathbb{R}$. Then $\mathbf{L}(X)$ is the class of measurable functions f satisfying

$$\int_0^t \log^+ |f(s)| ds < \infty \quad \text{for } 0 < t < \infty.$$

The class of Q-selfsimilar additive processes on $\mathbb{R}^d$ introduced in Sato [15] is an interesting class of additive processes on $\mathbb{R}^d$ (Example 2.13). For a characterization of this class by Q-selfdecomposable distributions, see [15]. When $Q = hI_{d\times d}$ with $h > 0$, we get the usual h-selfsimilarity and the usual selfdecomposability. See [16].

EXAMPLE 4.8. Let μ be a strictly α-stable distribution on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$. Let $h > 0$ and let $Z = \{Z_t\}$ be the h-selfsimilar additive process on $\mathbb{R}^d$ with $\mathcal{L}(Z_1) = \mu$. Then a measurable function f is in $\mathbf{L}(Z)$ if and only if

$$\int_0^t |f(s)|^\alpha s^{\alpha h - 1} ds < \infty \quad \text{for } 0 < t < \infty.$$

If $h \neq 1/\alpha$, here appears a phenomenon different from the case of Lévy processes; namely, this condition requires a special property of $f(s)$ in a neighborhood of $s = 0$.

EXAMPLE 4.9. Let μ be a second-class α-stable distribution on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$ and let $h > 0$. Let $Z = \{Z_t\}$ be the h-selfsimilar additive process on $\mathbb{R}^d$ with $\mathcal{L}(Z_1) = \mu$. Recall that, even in the case $h = 1/\alpha$, Z is not a Lévy process. Let f be a measurable function.

(i) Let $\alpha \neq 1$. Then $f \in \mathbf{L}(Z)$ if and only if

$$\int_0^t |f(s)|^\alpha s^{\alpha h - 1} ds < \infty \text{ and } \int_0^t |f(s)|\, s^{h-1} ds < \infty \quad \text{for } 0 < t < \infty.$$

(ii) Let $\alpha = 1$. Then $f \in \mathbf{L}(Z)$ if and only if

$$\int_0^t |f(s)| s^{h-1} ds < \infty \text{ and } \int_0^t |f(s)| \big|\log |s^h f(s)|\big| s^{h-1} ds < \infty$$

for $0 < t < \infty$.

EXAMPLE 4.10. Let μ be Γ-distribution with arbitrary parameters and let $h > 0$. Let $Z = \{Z_t\}$ be the h-selfsimilar additive process on $\mathbb{R}$ with $\mathcal{L}(Z_1) = \mu$. A measurable function f is in $\mathbf{L}(Z)$ if and only if

$$\int_{[0,\varepsilon]\cap\{s\colon |f(s)|s^h \leqslant 1\}} s^{-1} |f(s)| s^h ds + \int_{[0,\varepsilon]\cap\{s\colon |f(s)|s^h > 1\}} s^{-1} ds < \infty$$

for some $\varepsilon > 0$.

5. Definability of improper stochastic integrals

Given a natural additive process X on $\mathbb{R}^d$ and a nonrandom function $f \in \mathbf{L}(X)$, we are interested whether the improper integral $\int_0^{\infty-} f(s)dX_s$ is definable, that is, whether $\int_0^{\infty-} f(s)I_{d\times d}dX_s$ is definable in the sense of Definition 2.20. Examples suggest that it is meaningful to extend the notion of the definability of improper integrals in two ways. In the following, $X = \{X_t\}$ denotes a natural additive process on $\mathbb{R}^d$ with a factoring $(\{\rho_s\}, \sigma)$, $X^{(\mu)} = \{X_t^{(\mu)}\}$ denotes a Lévy process on $\mathbb{R}^d$ with $\mathcal{L}(X_1^{(\mu)}) = \mu$, and (A, ν, γ) denotes the triplet of μ.

Definition 5.1. Let $f \in \mathbf{L}(X)$. We say that *the essential improper integral of f with respect to X* is definable if there is a nonrandom $\mathbb{R}^d$-valued function q_t on $[0,\infty)$ such that $\int_0^t f(s)dX_s - q_t$ is convergent in probability as $t \to \infty$.

The term *essential* follows Loève [9], Section 18.2, where he used the words *essentially convergent* and *essentially divergent* for series of independent random variables. Earlier Lévy [8], Section 43, used the words *réductible à une série convergente* and *essentiellement divergente*.

Definition 5.2. Let $f \in \mathbf{L}(X^{(\mu)})$. We say that *the compensated improper integral of f with respect to the Lévy process $X^{(\mu)}$* is definable if there is a nonrandom vector $q \in \mathbb{R}^d$ such that $f \in \mathbf{L}(X^{(\mu*\delta_{-q})})$ and that $\int_0^{\infty-} f(s)d(X_s^{(\mu*\delta_{-q})})$ is definable.

We use the word *compensated* because, usually, q is such that $\mu * \delta_{-q}$ has mean 0.

Remark 5.3. Suppose that $f \in \mathbf{L}(X^{(\mu)})$ and that $\int_0^t |f(s)|ds < \infty$ for all $t \in (0,\infty)$. Then, definability of the compensated improper integral of f with respect to $X^{(\mu)}$ implies definability of the essential improper integral, since

$$\int_0^t f(s)dX_s^{(\mu*\delta_{-q})} = \int_0^t f(s)d(X_s^{(\mu)} - sq) = \int_0^t f(s)dX_s^{(\mu)} - \int_0^t f(s)qds.$$

Lemma 5.4. *Let $Y = \{Y_t\}$ be an additive process on $\mathbb{R}^d$ with triplet $(A^{Y_t}, \nu^{Y_t}, \gamma^{Y_t})$ and let q_t be an $\mathbb{R}^d$-valued function on $[0,\infty)$. Then $\operatorname{p-lim}_{t\to\infty}(Y_t - q_t)$ exists if and only if $\sup_t \operatorname{tr} A^{Y_t} < \infty$, $\sup_t \int_{\mathbb{R}^d}(|x|^2\wedge 1)\nu^{Y_t}(dx) < \infty$, and $\lim_{t\to\infty}(\gamma^{Y_t} - q_t)$ exists in $\mathbb{R}^d$. If $\operatorname{p-lim}_{t\to\infty}(Y_t - q_t)$ exists, then its distribution is in $ID(\mathbb{R}^d)$ and has triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ given by $\widetilde{A} = \lim_{t\to\infty} A^{Y_t}$, $\widetilde{\nu}(D) = \lim_{t\to\infty}\nu^{Y_t}(D)$ for $D \in \mathcal{B}_0(\mathbb{R}^d)$, and $\widetilde{\gamma} = \lim_{t\to\infty}(\gamma^{Y_t} - q_t)$.*

To see this lemma, use [16], Theorem 8.7, together with the argument in the proof of Theorem 9.8. Now, recalling Corollary 2.19 for $F(s) = f(s)I_{d\times d}$, we can show the following three propositions.

PROPOSITION 5.5. *Let $f \in \mathbf{L}(X)$ and $Y_t = \int_0^t f(s)dX_s$. Then $\int_0^{\infty-} f(s)dX_s$ is definable if and only if the following conditions are satisfied:*

$$\int_0^\infty f(s)^2(\operatorname{tr} A^{\rho_s})\sigma(ds) < \infty, \tag{5.1}$$

$$\int_0^\infty \sigma(ds)\int_{\mathbb{R}^d}(|f(s)x|^2 \wedge 1)\nu^{\rho_s}(dx) < \infty, \tag{5.2}$$

$$\gamma^{Y_t} \text{ is convergent in } \mathbb{R}^d \text{ as } t \to \infty. \tag{5.3}$$

If $\int_0^{\infty-} f(s)dX_s$ is definable, then its distribution has triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ given by

$$\widetilde{A} = \int_0^\infty f(s)^2(\operatorname{tr} A^{\rho_s})\sigma(ds), \tag{5.4}$$

$$\widetilde{\nu}(D) = \int_0^\infty \sigma(ds)\int_{\mathbb{R}^d} 1_D(f(s)x)\nu^{\rho_s}(dx) \quad \text{for } D \in \mathcal{B}_0(\mathbb{R}^d), \tag{5.5}$$

$$\widetilde{\gamma} = \lim_{t\to\infty} \gamma^{Y_t}. \tag{5.6}$$

PROPOSITION 5.6. *Let $f \in \mathbf{L}(X)$. Then the essential improper integral of f with respect to X is definable if and only if* (5.1) *and* (5.2) *hold.*

PROPOSITION 5.7. *Let $f \in \mathbf{L}(X^{(\mu)})$. Then the compensated improper integral of f with respect to $X^{(\mu)}$ is definable if and only if*

$$\int_0^\infty f(s)^2(\operatorname{tr} A)ds < \infty, \tag{5.7}$$

$$\int_0^\infty ds\int_{\mathbb{R}^d}(|f(s)x|^2 \wedge 1)\nu(dx) < \infty, \tag{5.8}$$

and there is $q \in \mathbb{R}^d$ such that

$$\int_0^t \left| f(s)(\gamma - q) + f(s)\int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right| ds < \infty \quad \text{for } t \in (0,\infty) \tag{5.9}$$

and

$$\int_0^t \left(f(s)(\gamma - q) + f(s)\int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right) ds \quad \text{is convergent in } \mathbb{R}^d \text{ as } t \to \infty. \tag{5.10}$$

REMARK 5.8. Let us say that *the symmetrized improper integral of f with respect to X* is definable if $\int_0^{\infty-} f(s)d(X_s - X_s^\sharp)$ is definable, where $X^\sharp$ is an independent copy of X. Then, the symmetrized improper integral is definable if and only if the essential improper integral is definable. The limit in the symmetrized improper integral is the symmetrization of the limit in the essential improper integral.

Now, given a function f, we are interested in the three classes of Lévy processes in law on $\mathbb{R}^d$ defined, in terms of the distributions at time 1, by

$$\begin{aligned}
\mathfrak{D}[f,\mathbb{R}^d] &= \{\mu \in ID(\mathbb{R}^d) \colon \int_0^{\infty-} f(s)dX_s^{(\mu)} \text{ is definable}\},\\
\mathfrak{D}_{\mathrm{c}}[f,\mathbb{R}^d] &= \{\mu \in ID(\mathbb{R}^d) \colon \text{the compensated improper integral of } f\\
&\qquad \text{with respect to } X^{(\mu)} \text{ is definable}\},\\
\mathfrak{D}_{\mathrm{es}}[f,\mathbb{R}^d] &= \{\mu \in ID(\mathbb{R}^d) \colon \text{the essential improper integral of } f \text{ with}\\
&\qquad \text{respect to } X^{(\mu)} \text{ is definable}\}.
\end{aligned}$$

How these classes depend on the choice of f and what is the description of the class of the distributions of improper integrals for given f are interesting subjects. Here we give several typical examples.

Another direction of research emphasized by Jurek [6] is, given a class of distributions, to seek its stochastic integral representations.

Example 5.9. Consider the case where $f(s) = e^{-s}$. It is well-known that $\mathfrak{D}[f,\mathbb{R}^d]$ is identical with the class of $\mu \in ID(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} \log^+ |x|\, \nu(dx) < \infty, \tag{5.11}$$

where ν is the Lévy measure of μ. The class of all $\mathcal{L}(\int_0^{\infty-} e^{-s} dX_s^{(\mu)})$ with $\mu \in \mathfrak{D}[e^{-s},\mathbb{R}^d]$ coincides with the class of all selfdecomposable distributions on $\mathbb{R}^d$. This gives also a characterization of stationary processes of Ornstein–Uhlenbeck type. See [12], [16] for references. In this case we have

$$\mathfrak{D}[f,\mathbb{R}^d] = \mathfrak{D}_{\mathrm{c}}[f,\mathbb{R}^d] = \mathfrak{D}_{\mathrm{es}}[f,\mathbb{R}^d]. \tag{5.12}$$

Example 5.10. Let $f(s) = e^{-e^s}$. Then we can prove that $\mathfrak{D}[f,\mathbb{R}^d]$ is the class of $\mu \in ID(\mathbb{R}^d)$ satisfying

$$\int_{|x|>e} \log\log |x| \nu(dx) < \infty.$$

We also have (5.12) in this case.

Let

$$g_0(r) = \int_r^\infty u^{-1} e^{-u} du \quad \text{and} \quad g_1(r) = \int_r^\infty u^{-2} e^{-u} du$$

and, for $j = 0, 1$, let $f_j(s)$ be the inverse function of $g_j(r)$ (that is, $s = g_j(r)$ if and only if $r = f_j(s)$). Thus $f_j(s)$ strictly decreases from ∞ to 0 for $s \in (0, \infty)$.

EXAMPLE 5.11. The function $g_0(r)$ is $\sim r^{-1}e^{-r}$ as $r \to \infty$ and $\sim \log(1/r)$ as $r \downarrow 0$; the function $f_0(s)$ is $\sim ce^{-s}$ with some constant $c > 0$ as $s \to \infty$ and $\sim \log(1/s)$ as $s \downarrow 0$. The class $\mathfrak{D}[f_0, \mathbb{R}^d]$ is the class of $\mu \in ID(\mathbb{R}^d)$ satisfying (5.11). The class of all $\mathcal{L}(\int_0^{\infty-} f_0(s)dX_s^{(\mu)})$ with $\mu \in \mathfrak{D}[f_0, \mathbb{R}^d]$ coincides with the Thorin class $T(\mathbb{R}^d)$. The proof is given in [1]. Also in this case (5.12) holds with $f = f_0$.

The following fact shows that each of the notions of essential improper integral and compensated improper integral has its own significance.

THEOREM 5.12.

(i) *We have $\mu \in \mathfrak{D}[f_1, \mathbb{R}^d]$ if and only if*

$$\int_{|x|>1} |x|\, \nu(dx) < \infty, \tag{5.13}$$

$$\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 r^{-1}e^{-r} \left(\int_{|x|>1} x \frac{r^2|x|^2}{1 + r^2|x|^2}\, \nu(dx) \right) dr \text{ exists in } \mathbb{R}^d, \tag{5.14}$$

$$\gamma = -\int_{\mathbb{R}^d} x \frac{|x|^2}{1 + |x|^2}\, \nu(dx). \tag{5.15}$$

(ii) *We have $\mu \in \mathfrak{D}_{\mathrm{c}}[f_1, \mathbb{R}^d]$ if and only if* (5.13) *and* (5.14) *hold.*

(iii) *We have $\mu \in \mathfrak{D}_{\mathrm{es}}[f_1, \mathbb{R}^d]$ if and only if* (5.13) *holds.*

(iv) *We have*

$$\mathfrak{D}[f_1, \mathbb{R}^d] \subsetneqq \mathfrak{D}_{\mathrm{c}}[f_1, \mathbb{R}^d] \subsetneqq \mathfrak{D}_{\mathrm{es}}[f_1, \mathbb{R}^d]. \tag{5.16}$$

(v) *Suppose that $\int_{\mathbb{R}^d} |x| \log^+ |x|\, \nu(dx) < \infty$. Then μ is always in $\mathfrak{D}_{\mathrm{c}}[f_1, \mathbb{R}^d]$; it is in $\mathfrak{D}[f_1, \mathbb{R}^d]$ if and only if it satisfies* (5.15).

(vi) *Suppose that, with $0 < \beta \leqslant 1$,*

$$\nu(D) = \int_S \lambda(d\xi) \int_2^\infty 1_D(u\xi) \frac{du}{u^2(\log u)^{1+\beta}} \quad \text{for } D \in \mathcal{B}_0(\mathbb{R}^d), \tag{5.17}$$

where λ is a finite measure on the unit sphere S satisfying $\int_S \xi\lambda(d\xi) \neq 0$. Then $\mu \in \mathfrak{D}_{\mathrm{es}}[f_1, \mathbb{R}^d]$ but $\mu \notin \mathfrak{D}_{\mathrm{c}}[f_1, \mathbb{R}^d]$.

REMARK 5.13. Note that (5.13) is equivalent to saying that $\int_{\mathbb{R}^d} |x|\, \mu(dx) < \infty$; (5.15) is equivalent to $\int_{\mathbb{R}^d} x\, \mu(dx) = 0$. (Added in revision: The condition (5.14) can be replaced by the condition that

$$\lim_{a \to \infty} \int_{|x|>1} x \log(|x| \wedge a)\, \nu(dx) \text{ exists in } \mathbb{R}^d.$$

See [18].)

Proof of Theorem 5.12. The function $g_1(r)$ is $\sim r^{-2}e^{-r}$ as $r \to \infty$ and $\sim r^{-1}$ as $r \downarrow 0$; $f_1(s)$ is $\sim s^{-1}$ as $s \to \infty$ and $\sim \log(1/s)$ as $s \downarrow 0$. It follows that $f_1 \in \mathbf{L}(X^{(\mu)})$ for all μ (Proposition 3.4).

(i) The condition for $\mu \in \mathfrak{D}[f_1, \mathbb{R}^d]$ is given by (5.7), (5.8) with $f = f_1$, and (5.10) with $q = 0$ for $f = f_1$. Among them, (5.7) always holds. We have

$$\begin{aligned}\int_0^\infty ds \int_{\mathbb{R}^d} (|f_1(s)x|^2 \wedge 1)\nu(dx) &= \int_0^\infty r^{-2}e^{-r}dr \int_{\mathbb{R}^d} (|rx|^2 \wedge 1)\nu(dx) \\ &= \int_0^\infty e^{-r}dr \int_{|x|\leqslant 1/r} |x|^2\nu(dx) + \int_0^\infty r^{-2}e^{-r}dr \int_{|x|>1/r} \nu(dx) \\ &= I_1 + I_2, \qquad \text{(say)}.\end{aligned}$$

Since $I_1 = \int_{\mathbb{R}^d} |x|^2(1 - e^{-1/|x|})\nu(dx)$ and $I_2 = \int_{\mathbb{R}^d} g_1(1/|x|)\nu(dx)$, we see that (5.8) holds if and only if (5.13) holds. Under condition (5.13), condition (5.10) with $q = 0$ is equivalent to the existence in $\mathbb{R}^d$ of

$$\lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 r^{-1}e^{-r} \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|rx|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right) dr, \tag{5.18}$$

since

$$\begin{aligned}&\int_0^t \left(f_1(s)\gamma + f_1(s)\int_{\mathbb{R}^d} x \left(\frac{1}{1+|f_1(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right) ds \\ &\qquad = \int_{f_1(t)}^\infty r^{-1}e^{-r} \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|rx|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right) dr\end{aligned}$$

and

$$\int_1^\infty r^{-1}e^{-r} \left(|\gamma| + \int_{\mathbb{R}^d} |x| \left|\frac{1}{1+|rx|^2} - \frac{1}{1+|x|^2}\right| \nu(dx)\right) dr < \infty.$$

Now we see that, under (5.13), condition (5.10) with $q = 0$ implies (5.15) because, if (5.15) does not hold, then (5.18) does not exist, since

$$\int_{\mathbb{R}^d} x \left(\frac{1}{1+|rx|^2} - \frac{1}{1+|x|^2}\right)\nu(dx) \to \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2}\nu(dx), \qquad r \downarrow 0.$$

If (5.15) holds, then

$$\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|rx|^2} - \frac{1}{1+|x|^2}\right)\nu(dx) = -\int_{\mathbb{R}^d} x\frac{|rx|^2}{1+|rx|^2}\nu(dx)$$

and

$$\int_0^1 r^{-1}e^{-r}dr \int_{|x|\leqslant 1} \frac{r^2|x|^3}{1+r^2|x|^2}\nu(dx) \leqslant \int_0^1 re^{-r}dr \int_{|x|\leqslant 1} |x|^3\nu(dx) < \infty.$$

Hence, under (5.13), condition (5.10) with $q = 0$ is equivalent to (5.14) combined with (5.15). This finishes the proof of (i).

(ii) Suppose that $\mu \in \mathfrak{D}_{\mathrm{c}}[f_1, \mathbb{R}^d]$. This means that $\mu * \delta_{-q} \in \mathfrak{D}[f_1, \mathbb{R}^d]$ for some $q \in \mathbb{R}^d$. Hence, by (i), this means (5.13) and (5.14) together with

$$\gamma - q = -\int_{\mathbb{R}^d} x \frac{|x|^2}{1+|x|^2} \nu(dx). \tag{5.19}$$

Conversely, if (5.13) and (5.14) are satisfied, then we can find q satisfying (5.19) and thus $\mu * \delta_{-q} \in \mathfrak{D}[f_1, \mathbb{R}^d]$.

(iii) In order that $\mu \in \mathfrak{D}_{\mathrm{es}}[f_1, \mathbb{R}^d]$, it is necessary and sufficient that (5.7) and (5.8) hold for $f = f_1$. See Proposition 5.6. As is seen in the proof of (i), (5.1) is always satisfied, and (5.2) is satisfied if and only if (5.13) is true.

(iv) Except for the strictness of inclusions, (5.16) follows from (i), (ii), and (iii). The strictness is shown by (v) and (vi) to be proved below.

(v) The property (5.13) is evident. Notice that (5.14) also holds, as

$$\int_0^1 r^{-1} e^{-r} dr \int_{|x|>1} \frac{r^2 |x|^3}{1+r^2|x|^2} \nu(dx) \leqslant \int_{|x|>1} |x| \nu(dx) \int_0^{|x|} \frac{udu}{1+u^2} < \infty$$

since $\int_0^{|x|} (1+u^2)^{-1} udu \sim \log |x|$ as $|x| \to \infty$. Now use (i) and (ii).

(vi) We have $\int_{|x|>1} |x|\, \nu(dx) < \infty$ since $\beta > 0$. For $0 < \varepsilon < 1$ we have

$$\int_\varepsilon^1 r^{-1} e^{-r} dr \int_{|x|>1} x \frac{r^2|x|^2}{1+r^2|x|^2}\, \nu(dx) = c_\varepsilon \int_S \xi \lambda(d\xi),$$

where

$$0 < c_\varepsilon = \int_2^\infty \frac{du}{u(\log u)^{1+\beta}} \int_\varepsilon^1 \frac{r^2u^2}{1+r^2u^2} r^{-1} e^{-r} dr < \infty.$$

However, since $\beta \leqslant 1$,

$$\lim_{\varepsilon \downarrow 0} c_\varepsilon = \int_2^\infty \frac{du}{u(\log u)^{1+\beta}} \int_0^u \frac{ve^{-v/u} dv}{1+v^2} = \infty,$$

as $\int_0^u (1+v^2)^{-1} v e^{-v/u} dv \sim \log u$. Thus (5.14) is not satisfied. $\square$

Remark 5.14. (i) Assume that ν is symmetric. Then $\mu \in \mathfrak{D}[f_1, \mathbb{R}^d]$ if and only if (5.13) holds and $\gamma = 0$. Further, $\mu \in \mathfrak{D}_{\mathrm{c}}[f_1, \mathbb{R}^d]$ if and only if $\mu \in \mathfrak{D}_{\mathrm{es}}[f_1, \mathbb{R}^d]$.

(ii) Let $Y_t = \int_0^t f_1(s) dX_s^{(\mu)}$. If $\mu \in \mathfrak{D}[f_1, \mathbb{R}^d]$, then $E|Y_t| < \infty$ and

$$EY_t = \left(\gamma + \int_{\mathbb{R}^d} x \frac{|x|^2}{1+|x|^2} \nu(dx) \right) \int_{f_1(t)}^\infty r^{-1} e^{-r} dr = 0.$$

But, even if $\mu \in \mathfrak{D}[f_1, \mathbb{R}^d]$, it is possible that $E \left| \int_0^{\infty-} f_1(s) dX_s^{(\mu)} \right| = \infty$; this will be shown in another paper.

(iii) Consider the case (vi) of Theorem 5.12 and choose q_t such that $Y_t - q_t$ is convergent in probability as $t \to \infty$. Then

$$E(Y_t - q_t) = \gamma^{Y_t} - q_t + c_t \int_S \xi\lambda(d\xi),$$

where $c_t = \int_{f_1(t)}^{\infty} re^{-r}dr \int_2^{\infty}(1+r^2u^2)^{-1}u(\log u)^{-1-\beta}du \to \infty$ as $t \to \infty$ while $\gamma^{Y_t} - q_t$ tends to some finite vector in $\mathbb{R}^d$. Thus $E(Y_t - q_t)$ approaches the point at infinity nearly in the direction of $\int_S \xi\lambda(d\xi)$. The modification of Y_t to $Y_t - q_t$ is not centering.

Let us describe $\mathfrak{D}[f,\mathbb{R}^d]$, $\mathfrak{D}_{\mathrm{c}}[f,\mathbb{R}^d]$, and $\mathfrak{D}_{\mathrm{es}}[f,\mathbb{R}^d]$ in some situation including Examples 5.9 and 5.11. For two functions f and g which are positive for all large s we write $f(s) \asymp g(s)$, $s \to \infty$, if there are positive constants a_1 and a_2 such that $a_1 g(s) \leqslant f(s) \leqslant a_2 g(s)$ for all large s.

Theorem 5.15. *Suppose that f is a locally square-integrable function and that there are constants $\alpha > 0$, $\beta \in \mathbb{R}$, and $c > 0$ such that*

$$f(s) \asymp s^{\beta}e^{-cs^{\alpha}}, \qquad s \to \infty. \tag{5.20}$$

Then

$$\mathfrak{D}[f,\mathbb{R}^d] = \left\{\mu \in ID(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} (\log^+|x|)^{1/\alpha}\nu(dx) < \infty\right\}, \tag{5.21}$$

where ν is the Lévy measure of μ, and

$$\mathfrak{D}_{\mathrm{es}}[f,\mathbb{R}^d] = \mathfrak{D}_{\mathrm{c}}[f,\mathbb{R}^d] = \mathfrak{D}[f,\mathbb{R}^d]. \tag{5.22}$$

Proof. Step 1. Assume that, for some $s_0 > 0$, $f(s) = as^{\beta}e^{-cs^{\alpha}}$ for $s \geqslant s_0$. We have $f'(s) = f(s)s^{-1}(\beta - c\alpha s^{\alpha}) < 0$ for large s. So we assume that s_0 is chosen so that $f'(s) < 0$ for $s \geqslant s_0$. Let $r_0 = f(s_0)$. We further assume that s_0 is so big that $r_0 < 1$. Let $s = g(r)$, $0 < r \leqslant r_0$, be the inverse function of $r = f(s)$, $s_0 \leqslant s < \infty$. Then

$$\lim_{r\downarrow 0} \frac{g(r)}{(-\log r)^{1/\alpha}} = \lim_{s\uparrow\infty} \frac{s}{(-\log f(s))^{1/\alpha}} = c^{-1/\alpha}. \tag{5.23}$$

Let us prove (5.21). We will check (5.7)–(5.10) with $q = 0$. Since f is locally square-integrable, it is enough to check these conditions with the integrals having lower limit s_0 (see Proposition 3.4). Condition (5.9) with $q = 0$ holds since $f \in \mathbf{L}(X^{(\mu)})$ for all μ. Condition (5.7) evidently holds.

Concerning (5.8),

$$\begin{aligned}\int_{s_0}^{\infty} ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\nu(dx) \\ = \int_{s_0}^{\infty} ds \int_{|f(s)x|\leqslant 1} |f(s)x|^2\nu(dx) + \int_{s_0}^{\infty} ds \int_{|f(s)x|>1} \nu(dx).\end{aligned}$$

Let I_1 and I_2 be the first and the second term. We have $I_2 < \infty$ if and only if

$$\int_{\mathbb{R}^d} (\log^+ |x|)^{1/\alpha} \nu(dx) < \infty, \tag{5.24}$$

since $I_2 = \int_{|x|>1/r_0} (g(1/|x|) - s_0)\nu(dx)$. We have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^d} |x|^2 \nu(dx) \int_{g(1/|x|)\vee s_0}^{\infty} f(s)^2 ds \\ &= \int_{|x|>1/r_0} |x|^2 \nu(dx) \int_{g(1/|x|)}^{\infty} f(s)^2 ds + \int_{|x|\leqslant 1/r_0} |x|^2 \nu(dx) \int_{s_0}^{\infty} f(s)^2 ds \\ &= I_{1,1} + I_{1,2} \quad \text{(say)}. \end{aligned}$$

Clearly $I_{1,2} < \infty$. If $\alpha \geqslant 1$, then

$$\lim_{r\downarrow 0} \frac{1}{r^2} \int_{g(r)}^{\infty} f(s)^2 ds = \lim_{r\downarrow 0} \frac{s}{2(c\alpha s^\alpha - \beta)} = \begin{cases} 0 & (\alpha > 1), \\ (2c\alpha)^{-1} & (\alpha = 1), \end{cases}$$

and thus $\int_{g(1/|x|)}^{\infty} f(s)^2 ds = O(|x|^{-2})$ as $|x| \to \infty$, which implies $I_{1,1} < \infty$. If $0 < \alpha < 1$, then, denoting $p = (1/\alpha) - 1$, we have

$$\lim_{r\downarrow 0} \frac{1}{r^2(-\log r)^p} \int_{g(r)}^{\infty} f(s)^2 ds = (2c\alpha c^p)^{-1}$$

and thus $I_{1,1} \leqslant \text{const} \int_{|x|>1/r_0} (\log |x|)^p \nu(dx) < \infty$ whenever (5.24) holds.

Condition (5.10) with $q = 0$ is satisfied if

$$\int_{s_0}^{\infty} \left| f(s)\gamma + f(s) \int_{\mathbb{R}^d} x \left(\frac{1}{1 + |f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| ds < \infty. \tag{5.25}$$

Condition (5.25) holds true if

$$\int_{s_0}^{\infty} \left| f(s) \int_{\mathbb{R}^d} x \left(\frac{1}{1 + |f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| ds < \infty, \tag{5.26}$$

since we have $\int_{s_0}^{\infty} f(s) ds < \infty$. Further, (5.26) is true if

$$\int_{s_0}^{\infty} f(s) ds \int_{\mathbb{R}^d} \frac{|x|^3 \nu(dx)}{(1 + |f(s)x|^2)(1+|x|^2)} < \infty. \tag{5.27}$$

We claim that (5.27) is true whenever (5.24) holds. Define $h(u)$ as

$$h(u) = \int_{s_0}^{\infty} \frac{f(s)u}{1 + f(s)^2 u^2} ds = \int_0^{r_0} \frac{usdr}{(1 + r^2u^2)(c\alpha s^\alpha - \beta)}.$$

Then the iterated integral in (5.27) equals $\int_{\mathbb{R}^d} |x|^2 (1+|x|^2)^{-1} h(|x|)\nu(dx)$. If $\alpha \geqslant 1$, then, since $s(c\alpha s^\alpha - \beta)^{-1}$ is bounded and since $\int_0^{r_0} (1 + r^2u^2)^{-1} u dr \leqslant \int_0^{\infty} (1 + r^2)^{-1} dr < \infty$, we see that $h(u)$ is bounded and (5.27) holds. Suppose that $0 < \alpha < 1$. Notice from (5.23) that $s^{\alpha-1} \sim \text{const}\,(-\log r)^{-p}$ with $p = (1/\alpha) - 1$ and that

$$h(u) \leqslant b \int_0^{r_0} \frac{u}{1+r^2u^2}(-\log r)^p dr = b(\log u)^{1/\alpha} \int_{(-\log r_0)/(\log u)}^{\infty} \frac{u^{1-y}y^p dy}{1+u^{2-2y}}$$

with a constant b. The last integral is the sum of the integral from 1 to ∞ and that from $(-\log r_0)/(\log u)$ to 1, each of which can be shown to tend to 0 as $u \to \infty$. It follows that $h(|x|) = o((\log |x|)^{1/\alpha})$, $|x| \to \infty$, when $0 < \alpha < 1$. Hence (5.27) is true whenever (5.24) holds.

Now the proof of (5.21) is complete. We also see (5.22) from this proof, using Proposition 5.6 and Remark 5.3.

Step 2. Let us prove the theorem in general. We have $f_1(s) \leqslant f(s) \leqslant f_2(s)$ for $s \geqslant s_0$, where $f_j(s) = a_j s^\beta e^{-cs^\alpha}$, $j = 1, 2$, with positive constants a_1, a_2. Use the results in Step 1. It is evident that f satisfies (5.7). We have

$$\int_{s_0}^{\infty} ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\nu(dx) \leqslant \int_{s_0}^{\infty} ds \int_{\mathbb{R}^d} (|f_2(s)x|^2 \wedge 1)\nu(dx)$$

and the reverse inequality with f_1 in place of f_2. Hence f satisfies (5.8) if and only if (5.24) holds. We have also (5.27) if (5.24) holds, since

$$\int_{s_0}^{\infty} \frac{f(s)u}{1+f(s)^2u^2} ds \leqslant \int_{s_0}^{\infty} \frac{f_2(s)u}{1+f_1(s)^2u^2} ds = \frac{a_2}{a_1} \int_{s_0}^{\infty} \frac{f_1(s)u}{1+f_1(s)^2u^2} ds.$$

Thus the assertion (5.21) is shown. The second assertion (5.22) is proved similarly. □

The integral in the representation of Jurek [5] for the nested subclasses L_m, $m = 0, 1, \ldots$, can be viewed as a special case of Theorem 5.15, where $\alpha = 1/(m+1)$ and $\beta = 0$.

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